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To cite this version:

Jørgen Bang-Jensen, Frédéric Havet. Finding good 2-partitions of digraphs I. Hereditary properties. [Research Report] RR-8867, INRIA Sophia Antipolis - I3S. 2016. hal-01279332

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ISSN 0249-6399   ISRN INRIA/RR--8867--FR+ENG

RESEARCH REPORT

N° 8867

February 2016

Project-Team Coati

Finding good 2-partitions of digraphs I. Hereditary properties

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Research Report n° 8867 — February 2016 — 16 pages

Abstract: We study the complexity of deciding whether a given digraph D has a vertex-partition into two disjoint subdigraphs with given structural properties. Let $\mathcal{H}$ and $\mathcal{E}$ denote following two sets of natural properties of digraphs: $\mathcal{H}$ ={acyclic, complete, arcless, oriented (no 2-cycle), semicomplete, symmetric, tournament} and $\mathcal{E}$ ={strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}. In this paper, we determine the complexity of deciding, for any fixed pair of positive integers k_1, k_2, whether a given digraph has a vertex partition into two digraphs D_1, D_2 such that $|V(D_i)| \geq k_i$ and D_i has property $\mathbb{P}_i$ for $i = 1, 2$ when $\mathbb{P}_1 \in \mathcal{H}$ and $\mathbb{P}_2 \in \mathcal{H} \cup \mathcal{E}$. We also classify the complexity of the same problems when restricted to strongly connected digraphs. The complexity of the problems when both $\mathbb{P}_1$ and $\mathbb{P}_2$ are in $\mathcal{E}$ is determined in the companion paper [2].

Key-words: directed graph, NP-complete, polynomial, partition, splitting digraphs, acyclic, semicomplete digraph, tournament, out-branching, feedback vertex set, 2-partition, minimum degree

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Trouver de bonnes 2-partitions des digraphes I. Propriétés héréditaires.

Résumé : Nous étudions la complexité de décider si un digraphe donné D admet une partition en deux sous-digraphes ayant des propriétés structurelles fixées. Dénotons par $\mathcal{H}$ et $\mathcal{E}$ les deux ensembles de propriétés de digraphes naturelles : $\mathcal{H}$ ={acyclique, complet, sans arcs, orienté, semicomplet, symétrique, tournoi} et $\mathcal{E}$ ={fortement connexe, connexe, degré sortant minimum au moins 1, degré entrant minimum au moins 1, semi-degré entrant minimum au moins 1, degré minimum au moins 1, avoir une arborescence sortante couvrante, avoir une arborescence entrante couvrante}. Dans ce rapport, nous déterminons la complexité de décider, pour toute paire d'entiers k_1, k_2, si un digraphie donné admet une partition en deux digraphes D_1, D_2 tels que $|V(D_i)| \geq k_i$ et D_i a la propriété $\mathbb{P}_i$ pour $i = 1, 2$ lorsque $\mathbb{P}_1 \in \mathcal{H}$ et $\mathbb{P}_2 \in \mathcal{H} \cup \mathcal{E}$. Nous classifions également la complexité des mêmes problèmes restreints aux digraphies fortement connexes. La complexité des problèmes lorsque $\mathbb{P}_1$ et $\mathbb{P}_2$ sont toutes deux dans $\mathcal{E}$ est déterminée dans le rapport suivant [2].

Mots-clés : graphe orienté, graphe dirigé, NP-complet, polynomial, partition, acyclique, digraphe semicomplet, tournoi, arborescence, ensemble d'arcs transverse, 2-partition, degré minimum

1 Introduction

A **k-partition** of a (di)graph D is a partition of $V(D)$ into k disjoint sets. Let $\mathbb{P}_1, \mathbb{P}_2$ be two (di)graph properties, then a **$(\mathbb{P}_1, \mathbb{P}_2)$-partition** of a (di)graph D is a 2-partition (V_1, V_2) where V_1 induces a (di)graph with property $\mathbb{P}_1$ and V_2 a (di)graph with property $\mathbb{P}_2$. For example a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least 1.

There are many papers dealing with vertex-partition problems on (di)graphs. Examples (from a long list) are [1, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 21, 22, 23]. Important examples for undirected graphs are bipartite graphs (those having has a 2-partition into two independent sets) and split graphs (those having a 2-partition into a clique and an independent set) [8]. It is well known and easy to show that there are linear-time algorithms for deciding whether a graph is bipartite, respectively, a split graph. The **dichromatic number** of a digraph D [17] is the minimum number k such that D has a k-partition where each set in the partition induces an acyclic digraph. This is a natural analogue of the chromatic number for undirected graphs as a graph G has chromatic number k if and only if the symmetric digraph $\overset{\leftrightarrow}{G}$, that we obtain from G by replacing every edge by a directed 2-cycle, has dichromatic number k. Contrary to the case of undirected graphs, it is already NP-complete to decide whether a digraph has dichromatic number 2 [5] (see also the proof of Theorem 4.4).

A set of vertices X in a digraph D is a **feedback vertex set** if $D - X$ is acyclic. If we wish to study feedback vertex sets with a certain property $\mathbb{P}$, this is the same as studying the ($\mathbb{P}$,acyclic)-partition problem. For example we may seek a feedback vertex set that induces an acyclic digraph and that is the (acyclic,acyclic)-partition problem which is the same as asking whether D has dichromatic number (at most) 2 and hence is NP-complete as noted above. On the other hand, if we want the feedback vertex set to be connected, we obtain the (connected, acyclic)-partition problem which is polynomial-time solvable as we show in Corollary 3.2.

In this paper and its companion paper [2] we give a complete characterization for the complexity of $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems when $\mathbb{P}_1, \mathbb{P}_2$ are one of the following properties: acyclic, complete, independent (no arcs), oriented (no directed 2-cycle), semicomplete, tournament, symmetric (if two vertices are adjacent, then they induce a directed 2-cycle), strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching. All of these 15 properties are natural properties of digraphs (as we already indicated above, symmetric digraphs correspond to undirected graphs). For each of them, it can be checked in linear time whether the given digraph has this property. Hence all the 120 distinct 2-partition problems are in NP.

Several of these 120 $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems are NP-complete and some results are surprising. For example, in [2], we show that the $(\delta^+ \geq 1, \delta \geq 1)$-partition problem is NP-complete. Some other problems are polynomial-time solvable because (under certain conditions) there are trivial $(\mathbb{P}_1, \mathbb{P}_2)$-partitions (V_1, V_2) with $|V_1| = 1$ (or $|V_2| = 1$). Therefore, in order to avoid such trivial partitions we consider **$[k_1, k_2]$-partitions**, that is, partitions (V_1, V_2) of V such that $|V_1| \geq k_1$ and $|V_2| \geq k_2$. Consequently, for each pair of above-mentioned properties and all pairs (k_1, k_2) of positive integers, we consider the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem, which consists in deciding whether a given digraph D has a $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition. When $k_1 = k_2 = 1$ we usually just write $(\mathbb{P}_1, \mathbb{P}_2)$-partition.

It might seem to be a lot of work but we are able to structure the approach in such a way that we can handle all the cases (especially most of the polynomial-time solvable ones) effectively. The results, including those from [2], are summarized in Table 1.

The paper is organized as follows. We first introduce the necessary terminology, and show that the properties in the classes $\mathcal{H}$ and $\mathcal{E}$, which we introduced in the abstract, are checkable and hereditary respectively, enumerable properties (defined below). Then in Section 3, we show that if $\mathbb{P}_{\nVdash}$ is hereditary and $\mathbb{P}_{\nvDash}$ is enumerable, then for any k_1, k_2, the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem is polynomial-time solvable. In Section 4, we determine the complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for all possible pairs $(\mathbb{P}_1, \mathbb{P}_2)$ of elements in $\mathcal{H}$. The complexity of the problem for all possible pairs $(\mathbb{P}_1, \mathbb{P}_2)$ of elements in $\mathcal{E}$ is determined in the companion paper [2]. The results are summarized in Table 1. The grey cells correspond to results proved in [2].

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc	P	P	P
conn.	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\mathbb{B}^+$	NPc^R	P	P	NPc	P	NPc	P	NPc	P	P	P
$\mathbb{B}^-$	NPc^R	P	NPc	P	P	P	NPc	NPc	P	P	P
$\delta \geq 1$	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\delta^+ \geq 1$	NPc^R	NPc	NPc	P	NPc	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	NPc	P	NPc	NPc	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

Properties: conn. : connected; $\mathbb{B}^+$: out-branchable; $\mathbb{B}^-$: in-branchable; $\mathbb{A}$: acyclic; $\mathbb{C}$: complete; $\mathbb{X}$: any property in 'being independent', 'being oriented', 'being semi-complete', 'being a tournament' and 'being symmetric'.
Complexities: P: polynomial-time solvable; NPc : NP-complete for all values of k_1, k_2; NPc^L : NP-complete for $k_1 \geq 2$, and polynomial-time solvable for $k_1 = 1$. NPc^R : NP-complete for $k_2 \geq 2$, and polynomial-time solvable for $k_2 = 1$.

Table 1: Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for some properties $\mathbb{P}_1, \mathbb{P}_2$.

All the NP-completeness proofs given in this paper are also valid if we restrict the input digraph to be strongly connected. However, for some partition problem with two enumerable properties, the complexity is sometimes different when we restrict to strongly connected digraphs as shown in [2]. The complexity results of the problems restricted to strongly connected digraphs are summarized in Table 2. The grey cells correspond to results proved in [2].

2 Notation and definitions

Notation follows [3]. In this paper graphs and digraphs have no parallel edges/arcs and no loops. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set A. We use $|D|$ to denote $|V(D)|$. Given an arc $uv \in A$ we say that u **dominates** v and v is **dominated** by u. If uv or vu (or both) are arcs of D, then u and v are **adjacent**. If none of the arcs exist in D, then u and v are **non-adjacent**. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple copies of the same edge (coming from directed 2-cycles). A digraph D is **connected** if $UG(D)$ is a connected graph, and the **connected components** of D are those of $UG(D)$.

A (u, v)-path is a directed path from u to v, and for two disjoint non-empty subsets X, Y of V an (X, Y)-path is a directed path which starts in a vertex $x \in X$ and ends in a vertex $y \in Y$

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{H}$
strong	NPc	**P**	NPc*	NPc*	**P**	NPc^L	NPc^L	NPc	P	P	P
conn.	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\mathbb{B}^+$	NPc*	P	P	NPc*	P	NPc^L	P	NPc^L	P	P	P
$\mathbb{B}^-$	NPc*	P	NPc*	P	P	P	NPc^L	NPc^L	P	P	P
$\delta \geq 1$	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\delta^+ \geq 1$	NPc^R	**P**	NPc^R	P	**P**	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	**P**	P	NPc^R	**P**	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	**P**	NPc^R	NPc^R	**P**	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{H}$	P	P	P	P	P	P	P	P	NPc	P	P

The legend is the same as in Table 1, but we have one more complexity type: NPc* : NP-complete for $k_1, k_2 \geq 2$, and polynomial-time solvable for $k_1 = 1$ or $k_2 = 1$. We also emphasize with a bold **P**, the problems that are polynomial-time solvable on strong digraphs and NP-complete in the general case.

Table 2: Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem on strong digraphs.

and whose internal vertices are not in $X \cup Y$. A digraph is **strongly connected** (or **strong**) if it contains a (u, v)-path for every ordered pair of distinct vertices u, v. A **strong component** of a digraph D is a maximal subdigraph of D which is strong. An **initial** (resp. **terminal**) strong component of D is a strong component X with no arcs entering (resp. leaving) X in D.

The **subdigraph induced** by a set of vertices X in a digraph D, denoted by $D\langle X\rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X. When X is a subset of the vertices of D, we denote by $D - X$ the subdigraph $D\langle V - X\rangle$. If D' is a subdigraph of D, for convenience we abbreviate $D - V(D')$ to $D - D'$.

A digraph is **acyclic** if it does not contain any directed cycles. An **oriented** graph is a digraph without directed 2-cycles. A **semicomplete** digraph is a digraph with no non-adjacent vertices and a **tournament** is a semicomplete digraph which is also an oriented graph. Finally, a **complete** digraph is a digraph in which every pair of distinct vertices induce a directed 2-cycle.

The **in-degree** (resp. **out-degree**) of v, denoted by $d_D^-(v)$ (resp. $d_D^+(v)$), is the number of arcs from $V \setminus \{v\}$ to v (resp. v to $V \setminus \{v\}$). The **degree** of v, denoted by $d_D(v)$ is given by $d_D(v) = d_D^+(v) + d_D^-(v)$. Finally the **minimum out-degree**, respectively **minimum in-degree**, **minimum degree** is denoted by $\delta^+(D)$, respectively $\delta^-(D)$, $\delta(D)$ and the **minimum semi-degree** of D, denoted by $\delta^0(D)$ is defined as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. A vertex is **isolated** if it has degree 0.

An **out-tree** rooted at the vertex s, also called an **s-out-tree** is a connected digraph T such that $d_T^-(s) = 0$ and $d_T^-(v) = 1$ for every vertex v different from s. Equivalently, for every $v \in V(T) \setminus \{s\}$ there is a unique (s, v)-path in T. The directional dual notion is the one of **in-tree**. An **in-tree** rooted at the vertex s, or **s-in-tree**, is a digraph T such that $d_T^+(s) = 0$ and $d_T^+(v) = 1$ for every vertex v different from s.

An **s-out-branching** (resp. **s-in-branching**) is a spanning s-out-tree (resp. s-in-tree). We say that a subset $X \subseteq V(D)$ is **out-branchable** (resp. **in-branchable**) if $D\langle X\rangle$ has an s-out-branching (resp. s-in-branching) for some $s \in X$.

Let D be a digraph. For a set S of vertices of D, we denote by $\text{Reach}_D^+(S)$, or simply $\text{Reach}^+(S)$ if D is clear form the context, the set of vertices that can be reached from S in D, that is, the set of vertices v for which there exists an (S, v)-path in D. Similarly, we denote by

$\mathrm{Reach}^-_D(S)$, or simply $\mathrm{Reach}^-(S)$, the set of vertices that can reach S in D, that is, the set of vertices v for which there exists a (v, S)-path in D. For sake of clarity, we write $\mathrm{Reach}^+_D(x)$ (resp. $\mathrm{Reach}^-_D(x)$) in place of $\mathrm{Reach}^+_D(\{x\})$ (resp. $\mathrm{Reach}^-_D(\{x\})$). The following lemma is well-known and easy to prove.

Lemma 2.1 *Let D be a digraph. If S is a set of vertices such that $D\langle S\rangle$ is out-branchable and $\mathrm{Reach}^+_D(S) = V(D)$, then D has an out-branching with root in S.*

2.1 Hereditary, checkable and enumerable properties

Recall the definitions of the two classes of properties $\mathcal{H}, \mathcal{E}$: $\mathcal{H}$ ={acyclic, complete, arcless, oriented, semicomplete, symmetric, tournament} and $\mathcal{E}$ ={strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, out-branchable, in-branchable}. A property $\mathbb{P}$ is **checkable** if there is a polynomial-time algorithm deciding whether a given digraph has the property $\mathbb{P}$. Observe that the fifteen properties in $\mathcal{E} \cup \mathcal{H}$ are all checkable.

A property $\mathbb{P}$ is **hereditary** if the set of digraphs having the property is closed by taking induced subdigraphs, i.e. if a digraph has the property $\mathbb{P}$, then all its induced subdigraphs also have the property $\mathbb{P}$. It is easy to see that all properties in $\mathcal{H}$ are hereditary, while e.g., being connected is not a hereditary property.

A property $\mathbb{P}$ is **enumerable** if given a digraph one can enumerate in polynomial time all its (inclusion-wise) maximal subdigraphs having property $\mathbb{P}$. In particular, this requires that the number of maximal subdigraphs of a digraph with property $\mathbb{P}$ is polynomial.

Lemma 2.2 *The following are enumerable properties: being connected, being strongly connected, being out-branchable, being in-branchable, having minimum in-degree (resp. out-degree, semi-degree, degree) at least k. In particular, all properties in $\mathcal{E}$ are enumerable.*

Proof: The first two properties are clearly enumerable: the maximal subdigraphs are the connected, respectively, the strongly connected components and those can be found in linear time.

To find the maximal subdigraphs that are out-branchable we first compute the strong components of D. Let $S_1, \dots, S_p$ be the initial strong components, that is, those with no arcs entering. Then the maximal out-branchable subdigraphs of D are the $\mathrm{Reach}^+_D(S_i)$, $1 \leq i \leq p$. Clearly these can be identified in polynomial time. The maximal subdigraphs that are in-branchable can be obtained in a similar by directional duality.

The remaining properties all deal with degrees. Here there will be at most one maximal subdigraph with the property. We illustrate this only for in-degree but the others are analogous. If $\delta^-(D) \geq k$, then D is the unique maximal subdigraph with in-degree at least k. Otherwise we may delete a vertex of in-degree less than k and continue this until the resulting digraph is either empty or we reach an induced subdigraph D' with $\delta^-(D') \geq k$. Clearly D' is the unique maximal subdigraph with in-degree at least k and we produce either this or conclude than D has no such subdigraph in time $O(|V|^2)$ (say, by using a priority queue). $\diamond$

2.2 Variants of 3-SAT used in the paper

Let us recall the definition of the 3-SAT problem(s): An instance is a boolean formula $\mathcal{F} = C_1 \wedge C_2 \wedge \dots \wedge C_m$ over the set of n boolean variables $x_1, \dots, x_n$. Each clause C_i is of the form $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ where each $\ell_{i,j}$ belongs to $\{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots \bar{x}_n\}$ and $\bar{x}_i$ is the negation of variable x_i. Our NP-completeness proofs will use reductions from the 3-SAT problem and the following variants of the 3-SAT problem: 2-IN-3-SAT, where exactly two of the three literals in each clause should be safisfied and NOT-ALL-EQUAL-3-SAT (NAE-3-SAT), where every clause must have at least one true and at least one false literal. These variants are both NP-complete [19].

In all of the NP-completeness proofs below we will use the following easy fact: for any pair of fixed integers k, k' and any given instance $\mathcal{F}$ of 3-SAT, 2-IN-3-SAT or NAE-3-SAT, we can always add new variables and clauses whose number only depends on k, k' such that the resulting formula $\mathcal{F}'$ has at least $\max\{k, k'\}$ clauses and at least $\max\{k, k'\}$ variables and $\mathcal{F}'$ is satisfiable if and only if $\mathcal{F}$ is satisfiable. In all the proofs below we may hence assume that the 3-SAT instances that we use in the reductions satisfy that $\min\{n, m\} \geq \max\{k_1, k_2\}$, where k_1, k_2 are the lower bounds on the two sides of the partition. It will be clear from the proofs that this ensures that the partitions (V_1, V_2) that we obtain from a satisfying truth assignment will always satisfy that $|V_i| \geq k_i$ for $i = 1, 2$.

For a given instance $\mathcal{F}$ of 3-SAT the bipartite incidence graph $G(\mathcal{F})$ of $\mathcal{F}$ has bipartition classes the set of variables and the set of clauses of $\mathcal{F}$ and there is an edge between variable x_i and clause C_j if C_j contains a literal on x_i. We say that $\mathcal{F}$ is a **connected** instance of 3-SAT if $G(\mathcal{F})$ is a connected graph. It is not difficult to see that all 3-SAT-variants above remain NP-complete if we also request that the instance $\mathcal{F}$ is connected: If we are given a non-connected instance of 3-SAT (resp. NAE-3-SAT, 2-IN-3-SAT) then we just need to add 2 extra clauses and at most 3 new variables so that the new instance $\mathcal{F}'$ is satisfiable if and only if $\mathcal{F}$ is and $\mathcal{F}'$ has one connected component less than $\mathcal{F}$. Thus in our proof below we may always assume that $\mathcal{F}$ is a connected instance of the relevant variant of 3-SAT.

3 Partitioning into parts with a checkable hereditary property and an enumerable property

Theorem 3.1 *Let $\mathbb{H}$ be a checkable hereditary property, $\mathbb{E}$ be an enumerable property, and let k_1 and k_2 be two positive integers. One can decide in polynomial time whether a given digraph D has a $(\mathbb{H}, \mathbb{E})$-$[k_1, k_2]$-partition.*

Proof: We shall describe a polynomial-time procedure that for any fixed set U_1 of k_1 vertices of D decides whether D has an $(\mathbb{H}, \mathbb{E})$-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1$. Then applying this algorithm to the $O(n^{k_1})$ k_1-subsets of $V(D)$, we obtain the desired algorithm.

The procedure proceeds as follows. First, we enumerate the maximal subdigraphs of $D - U_1$ with property $\mathbb{E}$. This can be done in polynomial time because $\mathbb{E}$ is enumerable. Now for each such subdigraph F, (there is a polynomial number of them), we check whether $|F| \geq k_2$ and if $D - F$ has property $\mathbb{H}$ (which can be done in polynomial time) because $\mathbb{H}$ is checkable. In the affirmative, we return 'Yes', and in the negative we proceed to the next subdigraph. If no more subdigraph remains, we return 'No'.

The above procedure clearly runs in polynomial time. To prove that it is valid we need to show that D has an $(\mathbb{H}, \mathbb{E})$-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1$ if and only if there is a maximal subdigraph F of $D - U_1$ with property $\mathbb{E}$ of order at least k_2 such that $D - F$ has property $\mathbb{P}$.

If there is a maximal subdigraph F of $D - U_1$ with property $\mathbb{E}$ of order at least k_2 such that $D - F$ has property $\mathbb{P}$, then $(V(D - F), V(F))$ is clearly an $(\mathbb{H}, \mathbb{E})$-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1$.

Conversely, assume there is an $(\mathbb{H},\mathbb{E})$-$[k_1,k_2]$-partition (V_1,V_2) with $U_1 \subseteq V_1$. Then $D\langle V_2\rangle$ has property $\mathbb{E}$ and thus is contained in a maximal subdigraph F of $D-U_1$ with property $\mathbb{E}$. Since F is a superdigraph of $D\langle V_2\rangle$ it has order at least k_2. In addition, $U_1 \subseteq V(D-F) \subseteq V_1$, so $D-F$ has the property $\mathbb{H}$, because this property is hereditary and V_1 has it. $\diamond$

One can easily check that the algorithm described in the proof of Theorem 3.1 runs in time $O(n^{k_1+c})$ for some constant c. A natural question is then to ask whether the problem could be FPT with respect to (k_1,k_2), that is, in time $f(k_1,k_2)n^c$ for some constant c and computable function f, and if not, one may ask if it can be solved in FPT time with respect to k_2 only, that is, in time $g(k_1)n^{h(k_2)}$ for some computable function g and h.

Using Theorem 3.1 we can now settle the complexity of 56 of the 120 2-partition problems we are studying.

Corollary 3.2 *The $(\mathbb{P}_1,\mathbb{P}_2)$-partition problem is polynomial-time solvable for all choices of $\mathbb{P}_1,\mathbb{P}_2$ with $\mathbb{P}_1 \in \mathcal{H}$ and $\mathbb{P}_2 \in \mathcal{E}$* $\diamond$

4 2-partitions into parts with hereditary properties

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

4.1 The locally constrained cases

We first deal with the local conditions $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$. These can be expressed as a condition on pairs of vertices in the same part of a partition. This indicates that a reduction to 2-SAT may work, which is indeed the case.

Theorem 4.1 *Let k_1,k_2 be fixed positive integers. The $(\mathbb{P}_1,\mathbb{P}_2)-[k_1,k_2]$-partition problem is polynomial-time solvable for all $\mathbb{P}_1,\mathbb{P}_2 \in \{\mathbb{C},\mathbb{I},\mathbb{O},\mathbb{S},\mathbb{T},\mathbb{Z}\}$.*

Proof: Clearly we can assume that the input $D=(V,A)$ has at least k_1+k_2 vertices. Denote V by $V=\{v_1,v_2,\dots,v_n\}$ and build an instance of 2-SAT with variables $x_1,\dots,x_n$ and clauses depending on which problem we deal with. We shall always associate the vertex set V_1 of a partition (V_1,V_2) with the true literals in a given truth assignment. The following shows which clauses to add for the given problem:

- If $\mathbb{P}_1=\mathbb{C}$ (resp. $\mathbb{P}_2=\mathbb{C}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i,v_j do not induce a directed 2-cycle in D.
- If $\mathbb{P}_1=\mathbb{I}$ (resp. $\mathbb{P}_2=\mathbb{I}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i and v_j are adjacent in D.
- If $\mathbb{P}_1=\mathbb{O}$ (resp. $\mathbb{P}_2=\mathbb{O}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i,v_j induce a directed 2-cycle in D.
- If $\mathbb{P}_1=\mathbb{S}$ (resp. $\mathbb{P}_2=\mathbb{S}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i and v_j are not adjacent in D.
- If $\mathbb{P}_1=\mathbb{T}$ (resp. $\mathbb{P}_2=\mathbb{T}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i,v_j are not adjacent in D or they form a directed 2-cycle in D.

- If $\mathbb{P}_1 = \mathbb{Z}$ (resp. $\mathbb{P}_2 = \mathbb{Z}$), then add a clause $(\bar{x}_i \vee \bar{x}_j)$ (resp. $(x_i \vee x_j)$) whenever v_i, v_j are adjacent in D but do not induce a directed 2-cycle in D.

It is easy to check that for each of the 36 choices (15 of which are the same) of $(\mathbb{P}_1, \mathbb{P}_2)$ the corresponding formula $\mathcal{F}(D)$ is satisfiable if and only if D has a $(\mathbb{P}_1, \mathbb{P}_2)$-partition (V_1, V_2) by letting V_1 correspond to those vertices v_i for which the corresponding variable x_i is true (and conversely). Note that it is possible that $V_1 = \emptyset$ (resp. $V_2 = \emptyset$), but in this case for any vertex x $(\{x\}, V(D) \setminus \{x\})$ (resp. $(V(D) \setminus \{x\}, \{x\})$) is a $(\mathbb{P}_1, \mathbb{P}_2)$-partition because the digraph with one vertex has the property $\mathbb{P}_1$ (resp. $\mathbb{P}_2$) and $\mathbb{P}_2$ (resp. $\mathbb{P}_1$) is hereditary. The size of $\mathcal{F}(D)$ is $O(n^2)$ as every pair of vertices give rise to at most 2 clauses. Since 2-SAT is solvable in linear time in the number of variables and clauses, each of the problems can be solved in time $O(n^{k_1+k_2+2})$: We consider (at most) all possible choices (V'_1, V'_2) of k_1 vertices V'_1 that must lie in V_1 and k_2 vertices V'_2 that must lie in V_2 and for each of these (at most) $O(n^{k_1+k_2})$ choices we first set $V_i = V'_i$ and then move all other vertices that are now forced to be in V_1 or V_2 to that set (this may lead to new vertices that have to be moved etc). If this leads to a contradiction, then there is no $(\mathbb{P}_1, \mathbb{P}_2)$-partition with $V'_i \subseteq V_i$ and we continue with the next candidate for V'_1, V'_2. After this we either have a $(\mathbb{P}_1, \mathbb{P}_2)$-partition of D or D has a $(\mathbb{P}_1, \mathbb{P}_2)$-partition if and only if there is a $(\mathbb{P}_1, \mathbb{P}_2)$-partition of $D\langle V \setminus V_1 \cup V_2\rangle$. $\diamond$

4.2 $(\mathbb{A}, \mathbb{P})$-partition, $\mathbb{P} \in \{\mathbb{A}, \mathbb{C}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$

When (at least) one part is required to be acyclic, we no longer have just a local condition and, as we shall see, the problem becomes more complicated. We first show that the $(\mathbb{A}, \mathbb{C})$-$[k_1, k_2]$-partition problems are polynomial-time solvable.

Theorem 4.2 *For all positive integers k_1, k_2, the $(\mathbb{A}, \mathbb{C})$-$[k_1, k_2]$-partition problem is polynomial-time solvable.*

Proof: Given a digraph $D = (V, A)$, we form its directed complement $\overline{D} = (V, (V \times V) \setminus A)$, that is, for every ordered pair $u, v \in V$ of vertices the arc uv is in $\overline{D}$ if and only if it is not in D. Now every $(\mathbb{A}, \mathbb{C})$-partition (V_1, V_2) of D is an $(\mathbb{S}, \mathbb{I})$-partition of $\overline{D}$. The converse may not hold: if (V_1, V_2) is an $(\mathbb{S}, \mathbb{I})$-partition of $\overline{D}$, there may be directed cycles in the (oriented) subdigraph $D\langle V_1\rangle$. However, for any pair of subsets V_1, V'_1 where both $(V_1, V \setminus V_1)$ and $(V'_1, V \setminus V'_1)$ are $(\mathbb{S}, \mathbb{I})$-partitions of $\overline{D}$ we have $|V_1 \triangle V'_1| \leq 2$ because an independent set and a clique intersect in at most one vertex. Therefore we can solve the $(\mathbb{A}, \mathbb{C})$-$[k_1, k_2]$-partition problem as follows: we first check whether $\overline{D}$ has an $(\mathbb{S}, \mathbb{I})$-partition (V_1, V_2) and if so, we check whether one of the $O(n^2)$ possible 2-partitions (V'_1, V'_2) such that $|V_1 \triangle V'_1| \leq 2$ is an $(\mathbb{A}, \mathbb{C})$-$[k_1, k_2]$-partition of D. $\diamond$

In contrast, we now prove that the $(\mathbb{A}, \mathbb{P})$-$[k_1, k_2]$-partition problems are NP-complete for $\mathbb{P} \in \{\mathbb{A}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$. All our reductions use superdigraphs of the digraph $B(\mathcal{F})$ which is obtained from a given 3-SAT instance $\mathcal{F} = \mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over the set of n boolean variables $x_1, \ldots, x_n$. The digraph $B(\mathcal{F})$ is defined from $\mathcal{F}$ as follows. Let q_i denote the maximum of the number of times x_i occurs in the clauses and the number of times $\bar{x}_i$ occurs in the clauses.The vertex set of $B(\mathcal{F})$ is $(\bigcup_{i\in[n]}\{x_{i,j} | j \in [q_i]\}) \cup (\bigcup_{i\in[n]}\{y_{i,j} | j \in [q_i]\})$ and the arc set of $B(\mathcal{F})$ is the union of the arc sets of the n complete bipartite digraphs $B_1, B_2, \ldots, B_n$ where B_i has vertex set $\{x_{i,j} | j \in [q_i]\} \cup \{y_{i,j} | j \in [q_i]\}$.

The choice of q_i implies that for each clause C_j we can associate a set W_j of three vertices of $B(\mathcal{F})$ so that $W_j \cap W_{j'} = \emptyset$ if $j \neq j'$. This can be done as follows: the ordering $C_1, \ldots, C_m$ of the clauses induces an ordering of the occurrences of each literal x_i or $\bar{x}_i$ in these. Hence we can construct the sets W_j, $j \in [m]$, by picking, for each clause C_i a private copy of vertices corresponding to each of its literals (the x, y vertices correspond to these), so if e.g. $C_j =$

$x_1 \vee \bar{x}_4 \vee x_7$ and these are the, respectively i'th, f'th and h'th occurrences of these literals, then we set $W_j = \{x_{1,i}, y_{4,f}, x_{7,h}\}$.

The following is just a reformulation of the corresponding 3-SAT problem:

Theorem 4.3 *Let $\mathcal{F}$ be a 3-SAT formula and let $B(\mathcal{F})$ be the corresponding bipartite digraph. Then the following holds:*

- *$B(\mathcal{F})$ has a 2-partition (V_1, V_2) such that V_1 intersects all the sets $W_1, \ldots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of 3-SAT.*
- *$B(\mathcal{F})$ has a 2-partition (V_1, V_2) such that each V_i intersects all the sets $W_1, \ldots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.* $\diamond$

Theorem 4.4 *The $(\mathbb{A}, \mathbb{P})$-$[k_1, k_2]$-partition problem is NP-complete for $\mathbb{P} \in \{\mathbb{A}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$ and every choice of positive integers k_1, k_2. This holds even when the input is restricted to strongly connected digraphs.*

Proof: All the reductions we will describe are clearly polynomial so we will not mention that below but just prove that the reductions are correct. It will also be clear from the proofs below that the partitions (V_1, V_2) that we derive from a satisfying truth assignment will always satisfy that both sides of the partition have size at least the number of variables in the given 3-SAT formula $\mathcal{F}$. Hence, by the remark we made after the definition of 3-SAT in Section 2, by choosing $\mathcal{F}$ appropriately, the partitions will have sufficiently many vertices in each side. We will thus drop the $[k_1, k_2]$ suffix of the problems below. It is easy to check that all our digraphs used in the NP-completeness proofs below are strongly connected, provided that the 3-SAT instance is connected. Hence, by the remark at the end of Section 2.2, all $(\mathbb{A}, \mathbb{P})$-partition problems with $\mathbb{P} \in \{\mathbb{A}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$ remain NP-complete when restricted to strongly connected digraphs.

$(\mathbb{A}, \mathbb{A})$ The $(\mathbb{A}, \mathbb{A})$-partition problem was proved NP-complete in [5] (by a reduction from hypergraph 2-colourability). It has also been proved to be already NP-complete for tournaments in [6]. We provide a short different proof here since we use the construction in the other proofs. It can easily be modified to prove the NP-completeness of the $(\mathbb{A}, \mathbb{A})$-partition problem for semicomplete digraphs. See Corollary 4.5

We show how to reduce NAE-3-SAT this problem. Let $\mathcal{F}$ be an instance of NAE-3-SAT with variables $x_1, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. Let $B(\mathcal{F})$ be the corresponding bipartite digraph as described above and form the digraph $D(\mathcal{F})$ by adding the arcs of m vertex disjoint directed 3-cycles on the vertex sets $W_1, \ldots, W_m$ to $B(\mathcal{F})$ (we chose an arbitrary directed 3-cycle for each W_j).

We claim that $D(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{A})$-partition if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.

Suppose first that (V_1, V_2) is an $(\mathbb{A}, \mathbb{A})$-partition of $D(\mathcal{F})$. Then for each directed 2-cycle in B_j, $j \in [n]$, we have precisely one end in V_1 and the other in V_2 so, for each $i \in [n]$, we have either $\{x_{i,j} | j \in [q_i]\} \subset V_1$ and $\{y_{i,j} | j \in [q_i]\} \subset V_2$, or $\{x_{i,j} | j \in [q_i]\} \subset V_1$ and $\{y_{i,j} | j \in [q_i]\} \subset V_1$. Now assign the value *true* to a variable x_i if the first case occurs and false if the second case occurs. As none of the m directed 3-cycles is fully contained in V_1 or V_2, this truth assignment satisfies either one or two literals of each clause.

Reciprocally, assume that a truth assignment $t : \{x_1, \ldots, x_n\} \rightarrow \{true, false\}$ satisfies one or two literals of each clause. Set $V_1 = (\bigcup_{i|t(x_i)=true}\{x_{i,j} | j \in [q_i]\}) \cup (\bigcup_{i|t(x_i)=false}\{y_{i,j} | j \in [q_i]\})$ and $V_2 = V(D(\mathcal{F})) \setminus V_1$. It is easy to check that (V_1, V_2) is an $(\mathbb{A}, \mathbb{A})$-partition of D.

$(\mathbb{A},\mathbb{I})$ We show a polynomial reduction of 2-IN-3-SAT to the $(\mathbb{A},\mathbb{I})$-partition problem. Let $\mathcal{F}$ be an instance of 2-IN-3-SAT and form the digraph $D(\mathcal{F})$ in the same way as above.

We claim that $D(\mathcal{F})$ has an $(\mathbb{A},\mathbb{I})$-partition if and only if $\mathcal{F}$ has a truth assignment which satisfies exactly two literals of each clause.

Suppose first that t is such a truth assignment. Let V_1 (resp. V_2) be the set of vertices corresponding to true (resp. false) literals, that is,

$$V_1=\left(\bigcup_{i|t(x_i)=true}\{x_{i,j}|j\in[q_i]\}\right)\cup\left(\bigcup_{i|t(x_i)=false\}}\{y_{i,j}|j\in[q_i]\}\right),\ \text{and } V_2=V(D(\mathcal{F}))\setminus V_1.$$

Then V_2 is independent since the only arcs it could potentially contain would be from vertices corresponding to literals of a clause and it contains exactly one of these. This also means that V_1 does not contain any directed 3-cycle and also no directed 2-cycle by definition of V_1 and hence $D\langle V_1\rangle$ is acyclic (the only possible directed cycles in $D\langle V_1\rangle$ are 2-cycles and 3-cycles and V_1 contains precisely one vertex of each directed 2-cycle).

Reciprocally, assume that (V_1,V_2) is an $(\mathbb{A},\mathbb{I})$-partition of $D(\mathcal{F})$. Then for each $i\in[n]$ either $\{x_{i,j}|j\in[q_i]\}\subseteq V_1$ and $\{y_{i,j}|j\in[q_i]\}\cap V_1=\emptyset$, or $\{x_{i,j}|j\in[q_i]\}\cap V_1=\emptyset$ and $\{y_{i,j}|j\in[q_i]\}\subseteq V_1$. Moreover V_1 contains precisely two vertices of each directed 3-cycle corresponding to a clause since $D\langle V_2\rangle$ is arcless. Thus by assigning the value $true$ to all variables whose corresponding $x_{i,j}$ vertices are in V_1 and $false$ to the remaining ones, we obtain the desired truth assignment.

$(\mathbb{A},\mathbb{O})$ To see that the 3-SAT problem polynomially reduces to this problem, it suffices to show that, for a given instance $\mathcal{F}$ of 3-SAT, the digraph $D(\mathcal{F})$ (defined as we did above) has an $(\mathbb{A},\mathbb{O})$-partition if and only if $\mathcal{F}$ is satisfiable. This is easy to see using the observations we have already made about the digraph $D(\mathcal{F})$: the oriented part will contain at least one vertex of each directed 3-cycle so setting a variable true if and only if the corresponding set of vertices in $D(\mathcal{F})$ are in the oriented part, we obtain a satisfying truth assignment and conversely.

$(\mathbb{A},\mathbb{T})$ We show a polynomial reduction from NAE-3-SAT to this problem. Let R be the digraph with vertex set $\{\ell_1,\ell_2,\ell_3,c_1,c_2,c_3\}$ and arc set $\{\ell_1\ell_2,\ell_2\ell_3,\ell_3\ell_1,c_1c_2,c_2c_3,c_3c_1\}\cup\{c_1\ell_1,c_2\ell_2,c_3\ell_3\}\cup\{\ell_ic_j|i,j\in[3]\}$. It is easy to check that R has an $(\mathbb{A},\mathbb{T})$-partition and for each such partition (V_1,V_2), either two of the vertices $\{\ell_1,\ell_2,\ell_3\}$ and one of the vertices $\{c_1,c_2,c_3\}$ are in the tournament part V_2 or one of the vertices $\{\ell_1,\ell_2,\ell_3\}$ and two of the vertices $\{c_1,c_2,c_3\}$ are in V_2. Note also that for $i\in[3]$, ℓ_i and c_i are in different parts of the partition as they form a directed 2-cycle.

Let $\mathcal{F}$ be an instance of NAE-3-SAT with variables $x_1,\dots,x_n$ and clauses $C_1,C_2,\dots,C_m$. Form the digraph $H(\mathcal{F})$ by adding the following to $B(\mathcal{F})$. Add vertices $(\bigcup_{j\in[m]}\{c_{j,1},c_{j,2},c_{j,3}\})$ and the arc set which is the union of the sets A_1,A_2 defined as follows:

- A_1 consists of the arcs of the m copies R_j, $j\in[m]$ where R_j is obtained by using the 3 vertices in W_j corresponding to the literals of C_j as the vertices $\{\ell_1,\ell_2,\ell_3\}$ and letting $\{c_{j,1},c_{j_2},c_{j,3}\}$ correspond to c_1,c_2,c_3.
- A_2 consists of the union of
  - $*$ all arcs of the form $x_{i,j}x_{i',j'}$, $i,i'\in[n],j\in[q_i],j'\in[q_{i'}]$, Where $i<i'$ or $i=i'$ and $j<j'$,

- ∗ all arcs of the form $y_{i,j}y_{i',j'}$, $i, i' \in [n], j \in [q_i], j' \in [q_{i'}]$, where $i < i'$ or $i = i'$ and $j < j'$ and
- ∗ all arcs of the form $x_{i,j}y_{i',j'}$, $i, i' \in [n], j \in [q_i], j' \in [q_{i'}]$, where $i < i'$ and
- ∗ all arcs of the form $x_{i,j}c_{r,s}$, $i \in [n], j \in [q_i], r \in [m], s \in [3]$
- ∗ all arcs of the form $y_{i,j}c_{r,s}$, $i \in [n], j \in [q_i], r \in [m], s \in [3]$.
- ∗ all arcs of the form $c_{r,s}c_{r',s'}$, $r, r' \in [m], s, s' \in [3]$, where $r < r'$.

Note that, by definition of A_1 and A_2, we may get a directed 2-cycle between two vertices corresponding to literals of the same clause. In that case we keep only the arc from A_1. Note also that $H(\mathcal{F})$ is in fact a semicomplete digraph.

We claim that $\mathcal{F}$ has a truth assignment which satisfies one or two literals of every clause if and only if $H(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{T})$-partition (V_1, V_2). Suppose first that t is such a truth assignment. Let V_2 consist of the union of all $x_{i,j}$ vertices such that x_i is true, all $y_{e,f}$ vertices such that x_e is false and those vertices among $c_{1,1}, c_{1,2}, c_{1,3} \ldots, c_{m,1}, c_{m,2}, c_{m,3}$ that do not form any directed 2-cycle with the chosen x, y vertices. By the definition of R and the fact that t is a valid truth assignment, for each $j \in [m]$, V_2 contains exactly three vertices of R_j. Set $V_1 = V(H(\mathcal{F})) \setminus V_2$.

Let us show that (V_1, V_2) is an $(\mathbb{A}, \mathbb{T})$-partition of $H(\mathcal{F})$. First observe that the subdigraph induced by V_2 is semicomplete as it is an induced subdigraph of the semicomplete digraph $H(\mathcal{F})$. There can be no directed 2-cycle in V_2 since, by construction (from t), the only possible directed 2-cycles would be of the form $\ell_{j,i}c_{j,i}$ for some $j \in [m], i \in [3]$ and we avoided those by the definition of V_1.

To see that $D\langle V_1\rangle$ is acyclic, first observe that, by the construction of V_2, there are no directed 2-cycles in V_1 and none of the directed 3-cycles $c_{j,1}c_{j,2}c_{j,3}c_{j,1}$, $j \in [m]$ are in V_1. Now the claim follows from the way we added arcs between literal vertices and vertices of the kind $c_{j,i}$ in the definition of A_2: there are no arcs from a $c_{j,i}$ vertex to a vertex of the kind $x_{p,q}, y_{r,s}$ and each of the subdigraphs of $H(\mathcal{F})$ induced by literal vertices, respectively the V_1 vertices of the kind $c_{a,b}$ are acyclic.

Suppose now that (V_1, V_2) is an $(\mathbb{A}, \mathbb{T})$-partition of $H(\mathcal{F})$. By construction, using the same arguments as in the previous cases, we see that for every variable x_i either all vertices of the form $x_{i,j}$ are in V_1 and those of the form $y_{i,j}$ are in V_2, or all vertices of the form $x_{i,j}$ are in V_2 and those of the form $y_{i,j}$ are in V_1. So, as in the other proofs, we get a well-defined truth assignment t by letting x_i be true precisely when all $x_{i,j}$ are in V_2. It follows from the remark on $(\mathbb{A}, \mathbb{T})$-partitions of the 6-vertex subdigraphs R_j that this truth assignment satisfies either one or two literals of each clause.

$(\mathbb{A}, \mathbb{S})$ We show a polynomial reduction from 2-IN-3-SAT to this problem. Let $\mathcal{F}$ be an instance of 2-IN-3-SAT with variables $x_1, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. Form the digraph $G(\mathcal{F})$ by adding the following vertices and arcs to $B(\mathcal{F})$: add vertices $\{x_{1,q_1+1}, y_{1,q_1+1}, \ldots, x_{n,q_n+1}, y_{n,q_n+1}\} \cup (\bigcup_{j\in[m]}\{c_{j,1}, c_{j,2}, c_{j,3}\})$ and new arcs formed by the union of A_1, A_2, A_3 defined as follows:

- $A_1 = \{x_{i,q_i+1}y_{i,q_i+1}, y_{i,q_i+1}x_{i,q_i+1} | i \in [n]\}$.
- A_2 consists of the arcs of the m directed 3-cycles $Q_j = c_{j,1}c_{j,2}c_{j,3}c_{j,1}$, $j \in [m]$ and the arcs of the m vertex-disjoint complete digraphs M_j, $j \in [m]$ on three vertices where $V(M_j) = W_j$ for $j \in [m]$. Finally, for each clause C_j, $j \in [m]$, A_2 contains six arcs from W_j to $V(Q_j)$ such that each vertex in $V(Q_j)$ receives exactly two arcs from W_j and each vertex of W_j sends exactly two arcs to Q_j.

- A_3 consists of the union of
    - all arcs of the form $x_{i,j}x_{i',j'}$, $i,i' \in [n], j \in [q_i+1], j' \in [q_{i'}+1]$, where $i < i'$ or $i = i'$ and $j < j'$,
    - all arcs of the form $y_{i,j}y_{i',j'}$, $i,i' \in [n], j \in [q_i+1], j' \in [q_{i'}+1]$, where $i < i'$ or $i = i'$ and $j < j'$,
    - all arcs of the form $x_{i,j}y_{i',j'}$, $i,i' \in [n], j \in [q_i+1], j' \in [q_{i'}+1]$, where $i < i'$,
    - all arcs of the form $x_{i,j}c_{r,s}$, $i \in [n], j \in [q_i+1], r \in [m], s \in [3]$, except those where $x_{i,j} \in W_r$
    - all arcs of the form $y_{i,j}c_{r,s}$, $i \in [n], j \in [q_i+1], r \in [m], s \in [3]$, except those where $y_{i,j} \in W_r$,
    - all arcs of the form $c_{r,s}c_{r',s'}$, $r,r' \in [m], s,s' \in [3]$, where $r < r'$.

We claim that $\mathcal{F}$ has a truth assignment which satisfies exactly two literals of every clause if and only if $G(\mathcal{F})$ has an $(\mathbb{A},\mathbb{S})$-partition (V_1,V_2).

Suppose first that t is such a truth assignment. Let V_2 consist of the union of all $x_{i,j}$ vertices such that v_i is true, all $y_{e,f}$ vertices such that v_e is false and the precisely m vertices $c_{1,g_1},\dots,c_{m,g_m}$ such that, for each $j \in [m]$, c_{j,g_j} is the unique vertex of Q_j which has two in-neighbours among those x,y vertices (these correspond to the two true literals of C_j). Set $V_1 = V(G(\mathcal{F}))\backslash V_2$. Let us prove that (V_1,V_2) is an $(\mathbb{A},\mathbb{S})$-partition of $G(\mathcal{F})$. First, observe that the subdigraph induced by V_2 is semicomplete: the only non-adjacent pairs of vertices in $G(\mathcal{F})$ are those containing exactly one of the vertices $c_{r,s}$ (such a vertex has precisely one non-neighbour and it is in W_r), those containing one of the vertices x_{i,q_i+1}, $i \in [n]$ and a vertex $y_{i,j}$, $j \in [q_i]$ or those containing one of the vertices y_{i,q_i+1}, $i \in [n]$ and a vertex $x_{i,j}$ $j \in [q_i]$. In the choice of V_1,V_2 above we chose V_2 so that it has no pairs of that kind.

To see that V_1 is acyclic, first note that $D\langle V_1\rangle$ has no 2-cycle since it contains exactly one vertex of each M_j and no pair $x_{i,j},y_{i,j'}$. Now it suffices to observe that, as the subdigraph $G(\mathcal{F})$ of induced by all $c_{j,i}$ vertices contains exactly m directed cycles, one for each clause and there is no arc from a $c_{j,i}$ vertex to a literal vertex, the only possible directed cycles in V_1 would be the directed 3-cycles Q_j but here we put one of the vertices in V_2.

Suppose now that (V_1,V_2) is an $(\mathbb{A},\mathbb{S})$-partition of $G(\mathcal{F})$. By construction, using the same arguments as in the previous cases, together with the fact that the vertex x_{i,q_i+1} (resp. y_{i,q_i+1}) has no neighbour in $\{y_{i,j} | j \in [q_i]\}$ (resp. $\{x_{i,j} | j \in [q_i]\}$), we see that for each $i \in [n]$ either all vertices of the form $x_{i,j}$ are in V_1 and those of the form $y_{i,j}$ are in V_2 or conversely. So, as in the other proofs, we get a well-defined truth assignment ϕ by letting x_i be true precisely when all $x_{i,j}$ are in V_2. Let us show that this truth assignment satisfies exactly two literals of each clause: Since V_1 is acyclic, for each $j \in [m]$, at least two of the vertices corresponding to the literals of C_j are in V_2 so ϕ satisfies at least two variables of each clause. To see that it cannot satisfy three literals of any clause, it suffices to notice that if all three literal vertices of some C_j were in V_2 then V_1 would contain the 3-cycle Q_j, because each vertex in Q_j has a non-neighbour in W_j. This would contradict that $D\langle V_1\rangle$ is acyclic.

$(\mathbb{A},\mathbb{Z})$ We show a polynomial reduction from 2-IN-3-SAT to the $(\mathbb{A},\mathbb{Z})$-partition problem. First consider the digraph U with vertex set $\{u_1,u_2,u_3,v_1,v_2,v_3\}$ and arc set $\{u_iu_j | i,j \in [3], i \neq j\} \cup \{v_1v_2, v_2v_3, v_3v_1\} \cup \{u_iv_i | i \in [3]\}$. It is easy to check that U has exactly three distinct $(\mathbb{A},\mathbb{Z})$-partitions : $(\{u_3,v_1,v_2\},\{u_1,u_2,v_3\})$, $(\{u_1,v_2,v_3\},\{u_2,u_3,v_1\})$, and $(\{u_2,v_3,v_1\},\{u_3,u_1,v_2\})$.

Let $\mathcal{F}$ be an instance of 2-IN-3-SAT with variables $x_1, \dots, x_n$ and clauses $C_1, C_2, \dots, C_m$. Form the digraph $K(\mathcal{F})$ by adding the following vertices and arcs to $B(\mathcal{F})$: add new vertices $\{d_{i,p} | i \in [n], p \in [4]\} \cup (\bigcup_{j\in[m]}\{v_{j,1}, v_{j,2}, v_{j,3}\})$ and the arc sets A_1, A_2, A_3 defined below.

- A_1 is the sets of arcs of the n disjoint directed 4-cycles $d_{i,1}d_{i,2}d_{i,3}d_{i,4}d_{i,1}$, $i \in [n]$.
- A_2 is the arc-disjoint union of the arcs of m copies $U_1, \dots, U_m$ of U where we identify the vertices $u_{j,1}, u_{j,2}, u_{j,3}$ of the j'th copy of U with the vertices of W_j (the $v_{j,i}$-vertices are all distinct).
- $A_3 = \bigcup_{i\in[n], j\in[q_i]}\{d_{i,1}y_{i,j}, d_{i,3}y_{i,j}, d_{i,2}x_{i,j}, d_{i,4}x_{i,j}\}$.

We claim that $K(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{Z})$-partition (V_1, V_2) if and only if $\mathcal{F}$ has a truth assignment which satisfies exactly two literals of each clause.

First assume that we have such a truth assignment ϕ. Then let V_2 contain exactly those vertices $x_{i,j}$ and $d_{i,1}, d_{i,3}$ such that x_i is true and all those vertices $y_{e,f}$ and $d_{e,2}, d_{e,4}$ such that x_e is false and the precisely m vertices $v_{1,h_1}, \dots, v_{m,h_m}$ such that for each $j \in [m]$ none of the two vertices of $W_j \cap V_2$ are adjacent to v_{j,h_j}. Set $V_1 = V(K(\mathcal{F})) \setminus V_2$. As there are no arcs from the set of $v_{j,k}$ vertices to the remaining vertices, the digraph $D\langle V_1\rangle$ is clearly acyclic (note that the $d_{i,j}$ vertices have no arcs in the part they belong to). By the way we chose v_{j,h_j} (picking exactly that vertex of U_j with no adjacency to $W_j \cap V_2$) we also have that $D\langle V_2\rangle$ is a symmetric digraph.

Conversely, let (V_1, V_2) be an $(\mathbb{A}, \mathbb{Z})$-partition. First observe that the adjacencies between vertices of the 4-cycles $d_{i,1}d_{i,2}d_{i,3}d_{i,4}d_{i,1}$, $i \in [n]$ and the variable vertices imply that, for each $i \in [n]$, either all vertices $x_{i,j}, j \in [q_i]$ are in V_2 and all vertices $y_{i,j}, j \in [q_i]$ are in V_1, or all vertices $x_{i,j}, j \in [q_i]$ are in V_1 and all vertices $y_{i,j}, j \in [q_i]$ are in V_2. This follows from the fact that we cannot have all vertices of such a 4-cycle in V_1. Hence we get a well-defined truth assignment from the partition by assigning the value *true* to x_i if the first case above occurs and *false* if the second case occurs. Now it follows from the property of the digraph U that for each $j \in [m]$ the clause C_j has exactly two true literals, namely those corresponding to those vertices of W_j that are in V_2. ◇

Corollary 4.5 *For all fixed integers k_1, k_2 the $(\mathbb{A}, \mathbb{A})$-$[k_1, k_2]$-partition problem and the $(\mathbb{A}, \mathbb{T})$-$[k_1, k_2]$-partition problem are NP-complete already for semicomplete digraphs.*

Proof: The last part was done when we proved that $(\mathbb{A}, \mathbb{T})$-partition was NP-complete as the digraph $H(\mathcal{F})$ was in fact semicomplete. To show that the $(\mathbb{A}, \mathbb{A})$-partition problem is NP-complete for semicomplete digraphs it suffices to notice that we can add arcs to the digraph $D(\mathcal{F})$ that we constructed in the proof for $(\mathbb{A}, \mathbb{A})$-partition, then we get an equivalent semicomplete instance: add the following arcs to obtain $D^S(\mathcal{F})$:

- all arcs of the form $x_{i,j}x_{i',j'}$, $i, i' \in [n], j \in [q_i], j' \in [q_{i'}]$, where $i < i'$ or $i = i'$ and $j < j'$,
- all arcs of the form $y_{i,j}y_{i',j'}$, $i, i' \in [n], j \in [q_i], j' \in [q_{i'}]$, where $i < i'$ or $i = i'$ and $j < j'$, and
- all arcs of the form $x_{i,j}y_{i',j'}$, $i, i' \in [n], j \in [q_i], j' \in [q_{i'}]$, where $i < i'$.

It is easy to check that the only directed cycles of $D^s(\mathcal{F})$ which do not contain both vertices of some 2-cycle $x_{i,j}y_{i,j'}$ are the m directed 3-cycles corresponding to the clauses. Together with the arguments used in the proof above for the $(\mathbb{A}, \mathbb{A})$-partition problem this shows that $D^s(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{A})$-partition if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT. ◇

5 Concluding remarks

In this paper, we gave polynomial-time algorithms for many $[k_1, k_2]$-partition problems for k_1 and k_2 fixed. However, the proposed algorithms are only polynomial when k_1 and k_2 are fixed and generally have a typical running time of $O(n^{\alpha \cdot k_1 + \beta \cdot k_2 + \gamma})$ for some constants α, β, γ. This means that the $[k_1, k_2]$-partition problem is in XP with respect to the parameter (k_1, k_2). A natural question is then to ask whether some of those problems can be solved in polynomial time or when this is not the case, then in FPT time when k_1 and k_2 are not fixed.

Problem 5.1 *For which pairs* $(\mathbb{P}_1, \mathbb{P}_2)$ *of properties among the ones studied in this paper and [2], does there exist an algorithm that, given a digraph* D *and two integers* k_1, k_2*, decides whether* D *admits a* $(\mathbb{P}_1, \mathbb{P}_2)$*-*$[k_1, k_2]$*-partition in polynomial time? Which ones can be solved in FPT time (i.e.* $f(k_1, k_2)n^c$*)-time with* f *a computable function and* c *a constant.*

The companion paper [2] contains a number of further problems to study, one of which concerns combinations of several of the properties from $\mathcal{H} \cup \mathcal{E}$.

References

[1] N. Alon. Splitting digraphs. *Combin. Probab. Comput.*, 15:933–937, 2006.

[2] J. Bang-Jensen, N. Cohen, and F. Havet. Finding good 2-partitions of digraphs II. Enumerable properties. Technical Report RR-8868, INRIA, February 2016.

[3] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications, 2nd Edition.* Springer-Verlag, London, 2009.

[4] J. Bensmail. On the complexity of partitioning a graph into a few connected subgraphs. *J. Combin. Optimization*, 30:174–187, 2015.

[5] D. Bokal, G. Fijavz, M. Juvan, Kayl P.M., , and Mohar B. The circular chromatic number of a digraph. *J. Graph Theory*, 46:227–240, 2004.

[6] Xujin Chen, Xiaodong Hu, and Wenan Zang. A min-max theorem on tournaments. *SIAM Journal on Computing*, 37(3):923–937, 2007.

[7] M.E. Dyer and A.M. Frieze. On the complexity of partitioning graphs into connected subgraphs. *Discrete. Appl. Math.*, 10:139–153, 1985.

[8] S. Földes and P. Hammer. Split graphs. *Congressus Numerantium*, 19:311–315, 1977.

[9] A Grigoriev and R. Sitters. Connected feedback vertex set in planar graphs. In *Graph Theoretical concepts in computer science, 35th international workshop, WG2009*, volume 5911 of *Lecture Notes in Comp. Science*, pages 143–153. Springer Verlag, Berlin, 2009.

[10] P. Hajnal. Partition of graphs with condition on the connectivity and minimum degree. *Combinatorica*, 3:95–99, 1983.

[11] D. Kühn and D. Osthus. Partitions of graphs with high minimum degree or connectivity. *J. Combinatorial Theory Ser. B*, 88:29–43, 2003.

[12] D. Kühn, D. Osthus, and T. Townsend. Proof of a tournament partition conjecture and an application to 1-factors with prescribed cycle length. *Combinatorica*, to appear.

[13] H.-O. Le, V.B. Le, and H. Müller. Splitting a graph into disjoint induced paths or cycles. *Discrete. Appl. Math.*, 131:199–212, 2003.

[14] N. Lichiardopol. Vertex-Disjoint Subtournaments of Prescribed Minimum Outdegree or Minimum Semidegree: Proof for Tournaments of a Conjecture of Stiebitz. *International J. Combinatorics*, pages Article ID 273416, 9 pages, 2012.

[15] MH Liu and BG Xu. Bipartition of graph under degree constraints. *Science China Mathematics*, 58:869–874, 2015.

[16] N. Misra, G. Philip, V. Raman, S. Saurabh, and S. Sikdar. FPT algorithms for connected feedback vertex set. *J. Combin. Optimization*, 24:131–146, 2012.

[17] V. Neumann-Lara. The dichromatic number of a digraph. *J. Combin. Theory Ser. B.*, 33:265–270, 1982.

[18] K.B. Reid. Two complementary circuits in two-connected tournaments. In *Cycles in graphs (Burnaby, B.C., 1982)*, volume 115 of *North-Holland Math. Stud.*, pages 321–334. North-Holland, 1985.

[19] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC 10)*, pages 216–226, New York, 1978. ACM.

[20] M. Stiebitz. Decomposition of graphs and digraphs. In *KAM Series in Discrete Mathematics-Combinatorics-Operations Research-Optimization 95-309*. Charles University Prague, 1995.

[21] H. Suzuki, N. Takahashi, and T. Nishizeki. A linear algorithm for bipartion of biconnected graphs. *Information Processing Letters*, 33:227–231, 1990.

[22] C. Thomassen. Graph decomposition with constraints on the connectivity and minimum degree. *J. Graph Theory*, 7:165–167, 1983.

[23] P. van't Hof, D. Paulusma, and G.J. Woeginger. Partitioning graphs into connected parts. *Theoretical Computer Science*, 410:4834–4843, 2009.

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ISSN 0249-6399