Uniform estimates for polyharmonic Green functions in domains with small holes

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UNIFORM ESTIMATES FOR POLYHARMONIC GREEN FUNCTIONS IN DOMAINS WITH SMALL HOLES

HANS-CHRISTOPH GRUNAU AND FRÉDÉRIC ROBERT

Dedicated to Patrizia Pucci on the occasion of her 60th birthday.

ABSTRACT. The Green function $G_{-\Delta,\Omega}$ for the Laplacian under Dirichlet boundary conditions in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ enjoys in dimensions $n \geq 3$ the estimate:

$$0 \leq G_{-\Delta,\Omega}(x,y) \leq \frac{1}{n(n-2)e_n}|x-y|^{2-n}.$$

Here, e_n denotes the volume of the unit ball $B = B_1(0) \subset \mathbb{R}^n$. This estimate follows from the maximum principle, the construction of $G_{-\Delta,\Omega}$ and the explicit expression of a suitable fundamental solution.

When passing to the polyharmonic Green function $G_{(-\Delta)^k,\Omega}$ under Dirichlet boundary conditions almost all forms of maximum or comparison principles fail: Green function estimates become an intricate subject and, according to works of Krasovskiĭ, multiplicative constants have to be used which heavily depend on the smoothness properties of the underlying domains.

In the present paper we study a singular family of domains by removing arbitrarily small holes from a fixed smooth domain in $\mathbb{R}^n$ with $n > 2k$. We prove Green function estimates which are uniform even when the size of the hole approaches 0, i.e. when the curvature of the boundary becomes unbounded.
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1. Introduction and main results

Given an arbitrary $C^{2k,\theta}$-smooth bounded domain $\Omega \subset \mathbb{R}^n$ with exterior unit normal ν, $n > 2k \geq 2$ and $\theta \in (0,1)$, we define $G_\Omega : \overline{\Omega} \times \overline{\Omega} \setminus \{(x,x) : x \in \overline{\Omega}\} \to \mathbb{R}$ as the Green function of $(-\Delta)^k$ in the domain Ω with Dirichlet boundary condition. This means that for $f \in C^{0,\theta}(\overline{\Omega})$ the unique solution $u \in C^{2k,\theta}(\overline{\Omega})$ of the polyharmonic Dirichlet problem

$$\left\{ \begin{array}{ll} (-\Delta)^k u = f & \text{in } \Omega, \\ u = \partial_\nu u = \ldots = \partial_\nu^{(k-1)} u = 0 & \text{on } \partial\Omega \end{array} \right. \tag{1}$$

is given by

$$u(x) = \int_\Omega G_\Omega(x,y) f(y)\, dy.$$

In case of exterior domains, which will naturally arise in what follows, one has to add zero boundary conditions (i.e. decay to 0) at infinity in order to have the Green function well defined.

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We are interested in pointwise estimates for G_Ω. In the special case $k = 1$, i.e. the case of the usual Laplacian, these can be deduced by using the maximum principle. This yields that G_Ω is positive und bounded from above by the fundamental solution, i.e. for $n > 2$ and any bounded smooth domain $\Omega \subset \mathbb{R}^n$ we have

$$\forall x, y \in \Omega, x \neq y, : \qquad 0 < G_\Omega(x,y) < \frac{1}{(n-2)ne_n}|x-y|^{2-n}. \tag{2}$$

Here, e_n denotes the measure of the n-dimensional unit ball. One should observe that the constant in the right inequality is independent of Ω, even with respect to singular perturbations.

When passing to biharmonic or –more general– polyharmonic equations, i.e. the cases $k \geq 2$, the maximum principle is no longer available and positivity issues remain valid only in a very weak and modified sense. Mathematical contributions to this topic go back at least to Boggio and Hadamard [2, 7]; these papers are also fundamental for subsequent works on estimating polyharmonic Green functions. For an extensive discussion of related and more recent contributions one may see Grunau-Robert [5] and the monograph Gazzola-Grunau-Sweers [3]. There is no obvious idea how to directly prove higher order analogues to estimate (2). However, employing the general Schauder and L^p-theory developed by Agmon, Douglis, and Nirenberg [1], Krasovskiĭ [8, 9] proved that for any given bounded sufficiently smooth domain $\Omega \subset \mathbb{R}^n$, $n > 2k$, there exists $C_\Omega > 0$ such that

$$|G_\Omega(x,y)| \leq C_\Omega |x-y|^{2k-n} \text{ for all } x, y \in \Omega,\ x \neq y. \tag{3}$$

The constant C_Ω depends on $C^{2k,\theta}$-properties of the boundary $\partial\Omega$. In Krasovskiĭ's works, very general operators and boundary conditions were discussed. Applying these general results to our special polyharmonic Dirichlet problems originally required a higher degree of smoothness. However, it turns out that for our purposes, $C^{2k,\theta}$-smoothness of $\partial\Omega$ suffices. For more detailed information on this issue we refer to Theorem 2 in the appendix. Estimate (3) can also be extended to the derivatives of Green functions: For any $0 \leq r \leq 2k$, there exists $C_{\Omega,r}$ such that

$$|\nabla_y^r G_\Omega(x,y)| \leq C_{\Omega,r} |x-y|^{2k-n-r} \text{ for all } x, y \in \Omega,\ x \neq y. \tag{4}$$

Here, ∇_y^r denotes any partial derivative with respect to y of order r.

The constant C_Ω in the Green function estimate (3) depends –as soon as $k > 1$– heavily on the smoothness properties of $\partial\Omega$. As long as one considers families of domains with uniform smoothness properties one may choose the same constant.

In the present article, we exhibit families of domains with unbounded curvature, namely fixed domains Ω where we punch out arbitrarily small holes. For uniform Green function estimates, (3) can no longer be used since the curvature blows-up, and so does in general the constant C_Ω. Nevertheless, we can prove the following uniform estimates.

Theorem 1. *Let Ω be a $C^{2k,\theta}$-smooth bounded domain of $\mathbb{R}^n$ with $n > 2k$, $\theta \in (0,1)$, and $k \geq 1$. We choose any point $x_0 \in \Omega$. Let ω be a further $C^{2k,\theta}$-smooth bounded domain of $\mathbb{R}^n$ containing 0. We fix a number $q \in (0,1)$. Then there exists a constant $C = C(\Omega, \omega, x_0, q) > 0$ such that that for all $\varepsilon \in (0, q\frac{d(x_0,\partial\Omega)}{\mathrm{diam}(\omega)})$, we have that*

$$|G_{\Omega_\varepsilon}(x,y)| \leq C|x-y|^{2k-n} \textit{ for all } x, y \in \Omega_\varepsilon,\ x \neq y,$$

where $\Omega_\varepsilon := \Omega \setminus \overline{\{x_0 + \varepsilon\omega\}}$. In particular, this estimate is uniform for $\varepsilon \searrow 0$.

Remark 1. (i) In Proposition 3 we extensively discuss the invariance properties of the polyharmonic operator and the corresponding Green function under Möbius transforms of $\mathbb{R}^n$. As a consequence, for any such Möbius transform J of $\mathbb{R}^n$ one has the same estimate for $|G_{J(\Omega_\varepsilon)}|$ with the same constant as for $|G_{\Omega_\varepsilon}|$ in Theorem 1 above.

(ii) In small dimensions one observes a different behaviour of the Green function. Nakai and Sario [11] discussed the biharmonic case $k = 2$ in dimension $n = 2$ with the help of energy estimates and their approach can probably be used for any $k \geq 2$ and any dimension $n < 2k$. In this small dimensions case some (in general not all) of the Dirichlet boundary conditions remain in x_0 even in the singular limit $\Omega_0 = \Omega \setminus \{x_0\}$. This phenomenon cannot be expected in large dimensions $n \geq 2k$.

It is then natural to ask whether in estimates like (4) we may also expect uniformity with respect to the family of domains $(\Omega_\varepsilon)_\varepsilon$. This, however, is not the case, even not for the Laplacian, i.e. $k = 1$. More precisely, we have the following:

Proposition 1. *Let Ω, $q \in (0, 1)$, Ω_ε, $\varepsilon > 0$, be as in Theorem 1. Then for all $1 \leq r \leq 2k$, we have that*

$$\sup_{\varepsilon\in(0,qd(x_0,\partial\Omega)/\operatorname{diam}(\omega))}\ \sup_{x,y\in\Omega_\varepsilon,\ x\neq y} |x-y|^{n-2k+r}|\nabla_y^r G_{\Omega_\varepsilon}(x,y)| = +\infty.$$

As mentioned at the beginning, one has a comparison principle for (1) in general only in the second order case, i.e. if $k = 1$. In this case, $G_\Omega > 0$ holds true for any Ω, while if $k \geq 2$ one has positivity $G_\Omega > 0$ only in very restricted classes of domains among which are balls (Boggio [2]) and small perturbations of balls (Grunau-Robert [5]). In general, however, one has sign change, i.e. $G_\Omega \not\geq 0$. Already Hadamard [7] observed that this will occur in the biharmonic case in two-dimensional annuli with very small inner radii, see also Nakai-Sario [11]. On the other hand, for fixed domains, the negative part will be "relatively" small. For more detailed information on this issue one may see Grunau-Robert [5], Gazzola-Grunau-Robert [3] and Grunau-Robert-Sweers [6]. For instance, the authors proved in [5] that for any $C^{4,\theta}$-smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n > 4$, there exists $C_\Omega > 0$ such that $\|(G_\Omega)_-\|_{L^\infty(\Omega\times\Omega)} \leq C_\Omega$, where G_Ω is the Green function for $(-\Delta)^2$ with Dirichlet boundary condition. A natural question is to ask whether one may expect uniformity of this lower bound with respect to families of domains. As shown by the following proposition, the validity of this guess is equivalent to the nonnegativity of *all* Green functions:

Proposition 2. *We assume that $n > 2k$. The two following assertions are equivalent:*

(i) *There exists $C(k, n, \theta)$ depending only on k, n, θ such that such that*

$$\|(G_\Omega)_-\|_{L^\infty(\Omega\times\Omega)} \leq C(k, n, \theta)$$

for all $C^{2k,\theta}$-smooth bounded domains $\Omega \subset \mathbb{R}^n$.

(ii) *$G_\Omega \geq 0$ for all $C^{2k,\theta}$-smooth bounded domains $\Omega \subset \mathbb{R}^n$.*

Since (ii) is false for the higher order case $k \geq 2$ (see the discussion and references in the monograph Gazzola-Grunau-Sweers [3, pp. 62/63 and 69/70]) we conclude that there is no uniform bound for negative parts of biharmonic and polyharmonic

Green functions. We emphasise that we only discuss Dirichlet boundary conditions and that positivity issues may be quite different for other boundary conditions.

Notation: In the sequel, $C(a,b,\dots)$ denotes a constant depending on $\omega,\Omega,a,b,\dots$. The same notation can be used for two different constants from line to line, and even in the same line.

2. Proofs

We start with proving Theorem 1 and proceed in several steps. In order to keep the exposition as simple as possible we shall prove the theorem for $q=\frac{1}{42}$. At the end of Section 2.3 we shall indicate how to modify the proof for larger $q<1$. Without loss of generality, we assume that $x_0=0$ so that $\Omega_\varepsilon:=\Omega\setminus\varepsilon\omega$.

2.1. **The Green function in the exterior domain $\mathbb{R}^n\setminus\omega$.** Let ω be a $C^{2k,\theta}$ domain of $\mathbb{R}^n$ such that $0\in\omega$. We define

$$\omega_0 := \mathrm{inv}(\mathbb{R}^n\setminus\overline{\omega})\cup\{0\}\ ,\text{ where }\ \mathrm{inv}:\left\{\begin{array}{ccc}\mathbb{R}^n\setminus\{0\} & \to & \mathbb{R}^n\setminus\{0\},\\ x & \mapsto & \frac{x}{|x|^2}.\end{array}\right.$$

We emphasise that the inversion inv is a special Möbius transform of $\mathbb{R}^n$ and in particular conformal. The set ω_0 is a $C^{2k,\theta}$-smooth bounded domain of $\mathbb{R}^n$ containing 0. We define

$$G_{(\varepsilon\omega)^c}(x,y):=\varepsilon^{n-2k}|y|^{2k-n}|x|^{2k-n}G_{\omega_0}(\varepsilon\,\mathrm{inv}(x),\varepsilon\,\mathrm{inv}(y)) \tag{5}$$

for all $x,y\in\mathbb{R}^n\setminus\varepsilon\omega$. The following proposition shows that this is indeed the polyharmonic Green function in $(\varepsilon\omega)^c$. In order to have this Green function well defined (and behaving well) we impose the condition on it to decay to 0 at inifinity.

Proposition 3. *For any $\varphi\in C_c^{2k}(\mathbb{R}^n\setminus\varepsilon\omega)$ such that $\partial_\nu^{(i)}\varphi=0$ on $\partial(\varepsilon\omega)$ for $i=0,\dots,(k-1)$, we have that*

$$\varphi(x)=\int_{\mathbb{R}^n\setminus\varepsilon\omega}G_{(\varepsilon\omega)^c}(x,y)(-\Delta)^k\varphi(y)\,dy \tag{6}$$

for all $x\in\mathbb{R}^n\setminus\varepsilon\omega$. Moreover, for all $0\le i\le 2k$, the derivatives with respect to y satisfy the upper bound

$$|\nabla_y^iG_{(\varepsilon\omega)^c}(x,y)|\le C|y|^{-i}\sum_{r\le i}|x|^r|x-y|^{2k-n-r}. \tag{7}$$

We remark that the Green functions in exterior domains display a completely different behaviour in small dimensions $n\le 2k$. In this case there are no bounds in $(\varepsilon\omega)^c$ which are uniform with respect to $\varepsilon\searrow 0$.

Proof. We prove the claim first for $\varepsilon=1$. Let $\varphi\in C_c^{2k}(\mathbb{R}^n\setminus\omega)$ be such that $\partial_\nu^{(i)}\varphi=0$ on $\partial\omega$ for $i=0,\dots,(k-1)$. We show that

$$\varphi(x)=\int_{\mathbb{R}^n\setminus\omega}|y|^{2k-n}|x|^{2k-n}G_{\omega_0}(\mathrm{inv}(x),\mathrm{inv}(y))(-\Delta)^k\varphi(y)\,dy \tag{8}$$

for all $x\in\mathbb{R}^n\setminus\omega$.

Indeed, inv is the composition of two sterographic projections of opposite poles, and therefore, it is conformal and the pull-back of the Euclidean metric Eucl via inv is $\mathrm{inv}^\star\,\mathrm{Eucl}=|\cdot|^{-4}\,\mathrm{Eucl}=\mu^{4/(n-2k)}\,\mathrm{Eucl}$ where $\mu(x):=|x|^{2k-n}$ for all $x\in\mathbb{R}^n\setminus\{0\}$.

As a consequence, considering $(-\Delta)^k$ as the conformal operator of Graham-Jenne-Mason-Sparling for the Euclidean space (see [4]), the conformal law of the GJMS operators yields

$$((-\Delta)^k\varphi)\circ \mathrm{inv} = \mu^{-(n+2k)/(n-2k)}(-\Delta)^k(\mu(\varphi\circ \mathrm{inv})).$$

In addition, the Jacobian of inv and then the Riemannian element of volume of $\mathrm{inv}^\star\,\mathrm{Eucl}$ are

$$\mathrm{Jac}(\mathrm{inv}) = |\cdot|^{-2n} \text{ and } dv_{\mathrm{inv}^\star\,\mathrm{Eucl}} = |\cdot|^{-2n}\,dx.$$

This transformation behaviour of polyharmonic operators with respect to Möbius transforms is classical, see e.g. Loewner [10] and references therein. A convenient and easily accessible reference is also Gazzola-Grunau-Sweers [3, Lemma 6.14] .

We fix $x\in\mathbb{R}^n\setminus\overline{\omega}$ and we consider $x' := \mathrm{inv}(x)\in\omega_0\setminus\{0\}$. We define $\tilde\varphi(y) := \mu(y)\varphi\circ \mathrm{inv}(y) = |y|^{2k-n}\varphi(y/|y|^2)$ for $y\in\overline{\omega}_0\setminus\{0\}$. We find that $\tilde\varphi$ is vanishing around 0 and therefore extends smoothly to $\overline{\omega}_0$. It follows from Green's representation formula that

$$\tilde\varphi(x') = \int_{\omega_0} G_{\omega_0}(x',y)(-\Delta)^k\tilde\varphi(y)\,dy.$$

Performing the change of variable $y = \mathrm{inv}(z)$ and using the above properties yields

$$\tilde\varphi(x') = \int_{\mathbb{R}^n\setminus\omega} |z|^{n+2k}G_{\omega_0}(x',\mathrm{inv}(z))(-\Delta)^k\varphi(z)|z|^{-2n}\,dz.$$

Going back to the expression of φ yields (8).

Given α a multi-index and $j\in\{1,\dots,n\}$, there exists an homogeneous polynomial P_j^α of degree $|\alpha|+1$ such that

$$\partial^\alpha \mathrm{inv}(x)_j = \frac{P_j^\alpha(x)}{|x|^{2(|\alpha|+1)}}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$, where $|\alpha|$ is the length of the index. We fix $x,y\in\mathbb{R}^n\setminus\omega$ such that $x\neq y$. With help of the binomial formula, the derivative of order α with respect to y is such that

$$|\partial_y^\alpha G_{\omega^c}(x,y)| \le C|x|^{2k-n}\sum_{\beta\le\alpha}|y|^{2k-n-|\alpha|+|\beta|}|\partial_y^\beta(G_{\omega_0}(\mathrm{inv}(x),\mathrm{inv}(y)))|, \tag{9}$$

where we have adopted the standard order on multi-indices. For $|\beta|\ge1$, the chain rule yields

$$\partial^\beta(f\circ \mathrm{inv}) = \sum_{1\le r\le|\beta|}\sum_{I_1+\ldots+I_r=\beta}\sum_{j_1,\ldots,j_r} c_{j_1,\ldots,j_r}^{(I_1,\ldots,I_r)}\partial^{I_1}\mathrm{inv}_{j_1}\ldots\partial^{I_r}\mathrm{inv}_{j_r}(\partial^{j_1\ldots j_r}f)\circ \mathrm{inv}$$

for any function f when the derivatives make sense. The second sum is taken over all decompositions of β as a sum of r multi-indices and the $c_{j_1,\ldots,j_r}^{(I_1,\ldots,I_r)}$ are combinatorial constants which can be calculated explicitly. When restricting to suitable decompositions of β these constants are equal to 1. This formula yields

$$|\partial_y^\beta(G_{\omega_0}(\mathrm{inv}(x),\mathrm{inv}(y)))| \le C\sum_{r\le|\beta|}|y|^{-|\beta|-r}|(\nabla^r G_{\omega_0})(\mathrm{inv}(x),\mathrm{inv}(y))| \tag{10}$$

for all $\beta\le\alpha$. Here, $\nabla^r f = (\partial^\gamma f)_{|\gamma|=r}$ when this makes sense.

It follows from Krasovskiĭ [8,9] that for any $0 \leq r \leq 2k$, there exists $C = C(\omega_0 = \mathrm{inv}(\overline{\omega}^c) \cup \{0\}, r) > 0$ such that

$$|\nabla^r G_{\omega_0}(x,y)| \leq C|x-y|^{2k-n-r}$$

for all $x, y \in \omega_0$, $x \neq y$. For the sake of completeness, we refer to Theorem 2 in the appendix where we comment on an alternative to Krasovskiĭ's proof. Noting that

$$|\,\mathrm{inv}(x) - \mathrm{inv}(y)| = \frac{|x-y|}{|x| \cdot |y|} \tag{11}$$

and putting (9), (10) and (11) together yields

$$|\partial_y^\alpha G_{\omega^c}(x,y)| \leq C|y|^{-|\alpha|} \sum_{r \leq |\alpha|} |x|^r |x-y|^{2k-n-r}.$$

This proves the claim for $\varepsilon = 1$, while for arbitrary $\varepsilon > 0$ it follows from the previous reasoning and the observation that $G_{(\varepsilon\omega)^c}(x,y) := \varepsilon^{2k-n} G_{\omega^c}(x/\varepsilon, y/\varepsilon)$. □

2.2. **Control outside a small annulus.** Given $\delta \in \left(0, \frac{d(0,\partial\Omega)}{3}\right)$, we define $\eta_\delta \in C_c^\infty(\Omega)$ such that $\eta_\delta(x) = 1$ for all $x \in B_\delta(0)$ and $\eta_\delta(x) = 0$ for all $x \in \Omega \setminus B_{2\delta}(0)$. Given $\varepsilon \in (0, \frac{\delta}{2\,\mathrm{diam}(\omega)})$ and $x, y \in \Omega_\varepsilon$, we define

$$\tilde{G}_{\varepsilon,\delta}(x,y) := \eta_\delta(y) G_{(\varepsilon\omega)^c}(x,y) + (1 - \eta_\delta(y)) G_\Omega(x,y). \tag{12}$$

We get that

$$\begin{aligned}(-\Delta)^k \tilde{G}_{\varepsilon,\delta}(x,\,\cdot\,) &= \eta_\delta (-\Delta)^k G_{(\varepsilon\omega)^c}(x,\,\cdot\,) + (1-\eta_\delta)(-\Delta)^k G_\Omega(x,\,\cdot\,) \\ &\quad + \sum_{i<2k} \left(A_i(\nabla^{2k-i}\eta_\delta, \nabla^i G_{(\varepsilon\omega)^c}(x,\,\cdot\,)) + A_i(\nabla^{2k-i}(1-\eta_\delta), \nabla^i G_\Omega(x,\,\cdot\,)) \right),\end{aligned}$$

where the $A_i's$ are contractions of suitable tensors, that is bilinear forms with smooth coefficients. Therefore, for any $x \in \Omega_\varepsilon$, there exists $f_{\varepsilon,\delta,x}$ such that

$$(-\Delta)^k \tilde{G}_{\varepsilon,\delta}(x,\,\cdot\,) = \delta_x + f_{\varepsilon,\delta,x} \text{ in } \mathcal{D}'(\Omega_\varepsilon).$$

Moreover, the pointwise control (7) yields

$$|f_{\varepsilon,\delta,x}(y)| \leq C \cdot \mathbf{1}_{B_{2\delta}(0)\setminus B_\delta(0)} |x-y|^{1-n}$$

for all $x, y \in \Omega_\varepsilon$, $x \neq y$. In particular, there exists $C(\delta) > 0$ such that

$$\|f_{\varepsilon,\delta,x}\|_{L^\infty(\Omega_\varepsilon)} \leq C(\delta) \text{ for all } \varepsilon > 0 \text{ and } x \in \Omega_{\varepsilon,\delta} \tag{13}$$

where

$$\Omega_{\varepsilon,\delta} := \left(\Omega_\varepsilon \cap B_{\delta/2}(0)\right) \cup \left(\Omega_\varepsilon \setminus \overline{B_{3\delta}(0)}\right) = \Omega_\varepsilon \setminus \left(\overline{B_{3\delta}(0)} \setminus B_{\delta/2}(0)\right).$$

Then it follows from elliptic theory that for any $x \in \Omega_{\varepsilon,\delta}$, there exists $u_{x,\varepsilon,\delta} \in W_0^{k,2}(\Omega_\varepsilon)$ such that

$$\left\{ \begin{array}{ll} (-\Delta)^k u_{x,\varepsilon,\delta} = f_{\varepsilon,\delta,x} & \text{in } \Omega_\varepsilon, \\ \partial_\nu^{(i)} u_{x,\varepsilon,\delta} = 0 & \text{for all } i = 0, \ldots, k-1 \text{ on } \partial\Omega_\varepsilon. \end{array} \right. \tag{14}$$

We claim that $u_{x,\varepsilon,\delta} \in C^{2k-1}(\Omega_\varepsilon)$ for all $\varepsilon, \delta > 0$ and $x \in \Omega_{\varepsilon,\delta}$. Moreover, there exists $C(\delta) > 0$ such that

$$\|u_{x,\varepsilon,\delta}\|_{C^{2k-1}(\Omega \setminus B_{\delta/4}(0))} \leq C(\delta) \tag{15}$$

for all admissible $\varepsilon, \delta > 0$ and $x \in \Omega_{\varepsilon,\delta}$.

We prove this claim. For simplicity, we define

$$((-\Delta)^{k/2}\psi)^2 := \left\{ \begin{array}{ll} ((-\Delta)^l\psi)^2 & \text{if } k=2l \text{ is } \textit{even} \\ |\nabla(-\Delta)^l\psi|^2 & \text{if } k=2l+1 \text{ is } \textit{odd}. \end{array} \right.$$

As a consequence, $u \mapsto \|(-\Delta)^{k/2}u\|_2$ is a norm on $W_0^{k,2}(\Omega_\varepsilon)$, the completion of $C_c^\infty(\Omega_\varepsilon)$ for the usual norm. Multiplying (14) by $u_{x,\varepsilon,\delta}$ and integrating by parts yields with Hölder's inequality

$$\begin{aligned} \int_\Omega ((-\Delta)^{k/2}u_{x,\varepsilon,\delta})^2\,dx &= \int_{\Omega_\varepsilon} ((-\Delta)^{k/2}u_{x,\varepsilon,\delta})^2\,dx \\ &= \int_{\Omega_\varepsilon} f_{\varepsilon,\delta,x}u_{x,\varepsilon,\delta}\,dy \leq \|f_{\varepsilon,\delta,x}\|_{\frac{2n}{n+2k}}\|u_{x,\varepsilon,\delta}\|_{\frac{2n}{n-2k}}. \end{aligned}$$

Sobolev's inequality yields the existence of $C_{n,k}>0$ such that

$$\|u\|_{\frac{2n}{n-2k}} \leq C_{n,k}\|(-\Delta)^{k/2}u\|_2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. The density of $C_c^\infty(\Omega_\varepsilon)$ in $W_0^{k,2}(\Omega_\varepsilon)$ allows to conclude that

$$\|u_{x,\varepsilon,\delta}\|^2_{\frac{2n}{n-2k}} \leq C_{n,k}^2\|f_{\varepsilon,\delta,x}\|_{\frac{2n}{n+2k}}\|u_{x,\varepsilon,\delta}\|_{\frac{2n}{n-2k}}$$

for all $\varepsilon>0$ and $x\in\Omega_{\varepsilon,\delta}$. Therefore $\|u_{x,\varepsilon,\delta}\|_{\frac{2n}{n-2k}} \leq C'(\delta)$.

It follows from elliptic theory (see for instance Agmon-Douglis-Nirenberg [1]) that for all $p>1$ and all $\delta'>0$, there exists $C(\delta')>0$ such that

$$\|u_{x,\varepsilon,\delta}\|_{W^{2k,p}(\Omega\setminus B_{\delta'}(0))} \leq C(\delta',p,\Omega)(\|f_{\varepsilon,\delta,x}\|_p + \|u_{x,\varepsilon,\delta}\|_p).$$

The claim (15) follows from this inequality, Sobolev's inequalities and iterations.

It remains to gain control of $u_{x,\varepsilon,\delta}$ in $B_{\delta'}(0)\setminus(\varepsilon\omega)$. To this end we consider $\eta_\delta u_{x,\varepsilon,\delta}$ and observe that this function solves a Dirichlet problem in the exterior domain $(\overline{\varepsilon\omega})^c$. Indeed, we have that

$$\begin{aligned} (-\Delta)^k(\eta_\delta u_{x,\varepsilon,\delta}) &= \eta_\delta(-\Delta)^k u_{x,\varepsilon,\delta} + \sum_{i<2k} A_i(\nabla^{2k-i}\eta_\delta, \nabla^i u_{x,\varepsilon,\delta}) \\ &= \eta_\delta f_{\varepsilon,\delta,x} + \sum_{i<2k} A_i(\nabla^{2k-i}\eta_\delta, \nabla^i u_{x,\varepsilon,\delta}) := \tilde{f}_{\varepsilon,\delta,x} \end{aligned}$$

where the A_i's are as above. We observe that

$$\operatorname{supp} \tilde{f}_{\varepsilon,\delta,x} \subset B_{2\delta}(0) \text{ and } \|\tilde{f}_{\varepsilon,\delta,x}\|_\infty \leq C(\delta)$$

for all $x\in\Omega_{\varepsilon,\delta}$. Since $\eta_\delta u_{x,\varepsilon,\delta}$ has compact support in $\mathbb{R}^n\setminus\varepsilon\omega$ and vanishes up to $(k-1)^{th}$ order on $\partial(\varepsilon\omega)$, Green's representation formula (6) yields

$$(\eta_\delta u_{x,\varepsilon,\delta})(z) = \int_{\mathbb{R}^n\setminus\varepsilon\omega} G_{(\varepsilon\omega)^c}(z,y)\tilde{f}_{\varepsilon,\delta,x}(y)\,dy$$

for all $z\in\mathbb{R}^n\setminus\overline{\varepsilon\omega}$. Consequently, for any $z\in B_\delta(0)\setminus(\varepsilon\omega)$, one gets

$$\begin{aligned} |u_{x,\varepsilon,\delta}(z)| &= |(\eta_\delta u_{x,\varepsilon,\delta})(z)| \leq \int_{\mathbb{R}^n\setminus\varepsilon\omega} |G_{(\varepsilon\omega)^c}(z,y)\tilde{f}_{\varepsilon,\delta,x}(y)|\,dy \\ &\leq C(\delta)\int_{B_{2\delta(0)}\setminus\varepsilon\omega} |y-z|^{2k-n}\,dz \leq C(\delta). \end{aligned}$$

This inequality combined with (15) yields

$$\|u_{x,\varepsilon,\delta}\|_{L^\infty(\Omega_\varepsilon)} \leq C(\delta) \text{ for all } x\in\Omega_{\varepsilon,\delta}. \tag{16}$$

As a consequence, we find that

$$(-\Delta)^k(\tilde{G}_{\varepsilon,\delta}(x,\,\cdot\,) - u_{x,\varepsilon,\delta}) = \delta_x \text{ weakly in } \mathcal{D}'(\Omega_\varepsilon)$$

and $\partial_\nu^{(i)}(\tilde{G}_{\varepsilon,\delta}(x,\,\cdot\,) - u_{x,\varepsilon,\delta}) = 0$ on $\partial\Omega_\varepsilon$ for all $x \in \Omega_{\varepsilon,\delta}$ and all $i = 0,\ldots,k-1$. The uniqueness of the Green function implies that

$$G_{\Omega_\varepsilon}(x,\,\cdot\,) = \tilde{G}_{\varepsilon,\delta}(x,\,\cdot\,) - u_{x,\varepsilon,\delta} \tag{17}$$

and then, using (6) and (16), we arrive at

$$|G_{\Omega_\varepsilon}(x,y)| \leq |\tilde{G}_{\varepsilon,\delta}(x,y)| + |u_{x,\varepsilon,\delta}(y)| \leq C(\Omega,\omega)|x-y|^{2k-n} + C(\delta) \tag{18}$$

$$\leq C(\delta)|x-y|^{2k-n} \tag{19}$$

for all $x \in \Omega_{\varepsilon,\delta} = (\Omega_\varepsilon \cap B_{\delta/2}) \cup (\Omega_\varepsilon \setminus \overline{B}_{3\delta}(0))$ and all $y \in \Omega_\varepsilon$.

2.3. **Conclusion of the proof of Theorem 1.** We fix $\delta_0 \in (0, \frac{d(x_0,\partial\Omega)}{21})$. We apply Section 2.2 with $\delta := \delta_0$ and to $\delta := 7\delta_0$. Since $\Omega_\varepsilon = \Omega_{\varepsilon,\delta_0} \cup \Omega_{\varepsilon,3\delta_0}$, it follows from (18) that there exists $C > 0$ such that

$$|G_{\Omega\setminus\overline{\varepsilon\omega}}(x,y)| \leq C|x-y|^{2k-n} \text{ for all } x,y \in \Omega \setminus \overline{\varepsilon\omega},\ x \neq y.$$

This proves Theorem 1 for $q = 1/42$. For $q \in (1/42, 1)$, instead of $\delta/2$, δ, 2δ, 3δ, in Section 2.2 one has to work with $\delta/(1+\sigma)$, δ, $(1+\sigma)\delta$, $(1+2\sigma)\delta$ with $\sigma > 0$ sufficiently close to 0. Alternatively one may argue that for

$$\varepsilon \in [(1/42)d(x_0,\partial\Omega)/\operatorname{diam}(\omega), qd(x_0,\partial\Omega)/\operatorname{diam}(\omega))$$

the boundaries of the Ω_ε enjoy uniform $C^{2k,\theta}$-properties so that (3) holds uniformly with respect to these ε. □

2.4. **Proof of Proposition 1.** We argue by contradiction and assume that there exist $1 \leq r \leq 2k$ and $C > 0$ such that

$$|x-y|^{n-2k+r}|\nabla_y^r G_{\Omega_\varepsilon}(x,y)| \leq C \tag{20}$$

for all $x,y \in \Omega_\varepsilon$, $x \neq y$, uniformly in $\varepsilon \to 0$. For any $x,y \in (\varepsilon^{-1}\Omega) \setminus \omega$, we define $G_\varepsilon(x,y) := \varepsilon^{n-2k}G_{\Omega_\varepsilon}(\varepsilon x, \varepsilon y)$. It follows from (12), (15), and (17) that for any $x \in \omega^c$, we have that

$$\lim_{\varepsilon\to 0} G_\varepsilon(x,y) = G_{\omega^c}(x,y)$$

in $C^0_{loc}(\mathbb{R}^n \setminus (\omega \cup \{x\}))$. Since $(-\Delta)^k G_\varepsilon(x,\,\cdot\,) = 0$ and $G_\varepsilon(x,\,\cdot\,)$ vanishes on $\partial\omega$ up to order $(k-1)$, elliptic regularity yields convergence in $C^{2k}_{loc}(\omega^c \setminus \{x\})$. Rewriting (20) for G_ε and passing to the limit $\varepsilon \to 0$ yields

$$|x-y|^{n-2k+r}|\nabla_y^r G_{\omega^c}(x,y)| \leq C \tag{21}$$

for all $x,y \in \mathbb{R}^n \setminus \omega$, $x \neq y$. We fix $x \neq 0$ and we define $G_R(z) := R^{n-2k}G_{\omega^c}(Rx,z)$ for all $z \in \omega^c$ and $R > R_0$ large enough. It follows from the explicit expression of G_{ω^c} in (5) that

$$\lim_{R\to+\infty} G_R(z) = G(z) := |x|^{2k-n}|z|^{2k-n}G_{\omega_0}\left(0, \frac{z}{|z|^2}\right) \tag{22}$$

in $C^0_{loc}(\mathbb{R}^n \setminus \omega)$. Since $(-\Delta)^k G_R = 0$ and G_R vanishes on $\partial\omega$ up to order $(k-1)$, elliptic regularity yields the convergence of (G_R) to G in $C^{2k}_{loc}(\omega^c)$. On the other hand, (21) may be rewritten as

$$|\nabla^r G_R(z)| \leq CR^{-r}|x - R^{-1}z|^{2k-n-r}$$

for z in a compact sudomain of $\mathbb{R}^n \setminus \omega$ and R large enough. Since $r \geq 1$, passing to the limit $R \to +\infty$ yields $\nabla^r G = 0$ in $\mathbb{R}^n \setminus \omega$, which contradicts the explicit expression (22) of G. This concludes the proof of Proposition 1. □

2.5. **Proof of Proposition 2.** Assume that (ii) does not hold. Then there exists a $C^{2k,\theta}$-smooth bouned domain $\omega_0 \subset \mathbb{R}^n$ such that G_{ω_0} attains some negative values, say at $(x_0, y_0) \in \omega_0 \times \omega_0$, $x \neq 0$. We define $\omega := (\mathrm{inv}(\mathbb{R}^n \setminus \overline{\omega_0})) \cup \{0\}$ and $\Omega_\varepsilon := \Omega \setminus \overline{\varepsilon\omega}$ where $\varepsilon > 0$ is small and Ω is a smooth bounded domain containing 0. It follows from (12), (16), and (17) that

$$\lim_{\varepsilon \to 0} \varepsilon^{n-2k} G_{\Omega_\varepsilon}(\varepsilon x, \varepsilon y) = G_{\omega^c}(x, y) = |x|^{2k-n} |y|^{2k-n} G_{\omega_0}(\mathrm{inv}(x), \mathrm{inv}(y))$$

for all $x, y \in \mathbb{R}^n \setminus \overline{\omega}$. Choosing $x := \mathrm{inv}(x_0)$ and $y := \mathrm{inv}(y_0)$ yields

$$\lim_{\varepsilon \to 0} G_{\Omega_\varepsilon}(\varepsilon x, \varepsilon y) = -\infty,$$

and then (i) does not hold. Conversely, if (ii) holds, then (i) holds. □

Appendix A. Pointwise control of the Green function for fixed domains

The following result, under stronger smoothness assumptions on Ω but at the same time in a more general context, is due to Krasovskiĭ [8, 9]:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{2k,\theta}$-smooth bounded domain of $\mathbb{R}^n$ with $n > 2k$, $\theta \in (0, 1)$, and $k \geq 1$. Let G_Ω be the Green function for $(-\Delta)^k$ with Dirichlet boundary condition. Then for all $0 \leq r \leq 2k$, there exists $C = C(\Omega, r) > 0$ such that*

$$|\nabla_y^r G_\Omega(x, y)| \leq C|x - y|^{2k-n-r} \tag{23}$$

for all $x, y \in \Omega$, $x \neq y$.

We sketch here an alternative proof.

Proof. The case $r = 0$ and $k = 2$ under the smoothness assumptions as in the theorem is treated in Grunau-Robert [5, Theorem 4] (see also Gazzola-Grunau-Sweers [3, Propositions 4.22 and 4.23] for an exposition in book form). By making the obvious changes one may check that the proof can be extended to any $k \geq 1$ and $n > 2k$. (Only the discussion of the smaller dimensions $n \leq 2k$ requires more care.) This means that there exists a constant $C(\Omega) > 0$ such that

$$|G_\Omega(x, y)| \leq C(\Omega)|x - y|^{2k-n} \tag{24}$$

for all $x, y \in \Omega$, $x \neq y$. We fix $r \geq 1$ and we prove (23) by using local elliptic estimates and rescaling arguments. We proceed as in Grunau-Gazzola-Sweers [3, Prop. 4.23] and use the following local Schauder estimate from Agmon-Douglis-Nirenberg [1, Theorem 9.3] which holds true also close to $\partial\Omega$. For any two concentric balls $B_R \subset B_{2R}$ and any polyharmonic function v on $B_{2R} \cap \Omega$ satisfying homogeneous Dirichlet boundary conditions on $B_{2R} \cap \partial\Omega$ we have

$$\|\nabla^r v\|_{L^\infty(B_R \cap \Omega)} \leq \frac{C}{R^r} \|v\|_{L^\infty(B_{2R} \cap \Omega)}. \tag{25}$$

The constant is uniform in R; the behaviour with respect to (small) R is obtained by means of scaling.

Keeping $x \in \Omega$ fixed, for any $y \in \Omega \setminus \{x\}$ we choose $R = |x - y|/4$ and apply (25) and (24) in $B_R(y) \subset B_{2R}(y)$ to $G_\Omega(x, \cdot)$. This proves (23). □

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