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ASYMPTOTIC ANALYSIS FOR FOURTH ORDER PANEITZ EQUATIONS WITH CRITICAL GROWTH

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Abstract. We investigate fourth order Paneitz equations of critical growth in the case of n-dimensional closed conformally flat manifolds, $n \geq 5$. Such equations arise from conformal geometry and are modelized on the Einstein case of the geometric equation describing the effects of conformal changes of metrics on the Q-curvature. We obtain sharp asymptotics for arbitrary bounded energy sequences of solutions of our equations from which we derive stability and compactness properties. In doing so we establish the criticality of the geometric equation with respect to the trace of its second order terms.

In 1983, Paneitz [?] introduced a conformally covariant fourth order operator extending the conformal Laplacian. Branson and Ørsted [?], and Branson [?, ?], introduced the associated notion of Q-curvature when $n = 4$ and in higher dimensions when dealing with the conformally covariant extensions of the Paneitz operator by Graham-Jenne-Mason-Sparling. The scalar and the Q-curvatures are respectively, up to the conformally invariant Weyl's tensor in dimension four, the integrands in dimensions two and four for the Gauss-Bonnet formula for the Euler characteristic. The articles by Branson and Gover [?], Chang [?, ?], Chang and Yang [?], and Gursky [?] contain several references and many interesting material on the geometric and physics aspects associated to this notion of Q-curvature.

In what follows we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$ and consider the fourth order variational Paneitz equations of critical Sobolev growth which are written as

$$\Delta_g^2 u + b\Delta_g u + cu = u^{2^\sharp - 1} , \tag{0.1}$$

where $\Delta_g = -\mathrm{div}_g \nabla u$ is the Laplace-Beltrami operator, $b, c > 0$ are positive real numbers such that $c - \frac{b^2}{4} < 0$, u is required to be positive, and $2^\sharp = \frac{2n}{n-4}$ is the critical Sobolev exponent. Equations like (??) are modeled on the conformal equation associated to the Paneitz operator when the background metric g is Einstein. In few words, the conformal equation associated to the Paneitz operator, relating the Q-curvatures Q_g and $Q_{\tilde{g}}$ of conformal metrics on arbitrary manifolds, is written as

$$\Delta_g^2 u - \mathrm{div}_g(A_g du) + \frac{n-4}{2} Q_g u = \frac{n-4}{2} Q_{\tilde{g}} u^{2^\sharp - 1} , \tag{0.2}$$

where $\tilde{g} = u^{4/(n-4)} g$ and, if Rc_g and S_g denote the Ricci and scalar curvature of g, A_g is the smooth $(2, 0)$-tensor field given by

$$A_g = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g . \tag{0.3}$$

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When g is Einstein, so that $Rc_g = \lambda g$ for some $\lambda \in \mathbb{R}$, equation (??) can be simplified and written as

$$\Delta_g^2 u + \frac{b_n \lambda}{n-1}\Delta_g u + \frac{c_n \lambda^2}{(n-1)^2} u = \frac{n-4}{2} Q_{\tilde{g}} u^{2^\sharp - 1} , \tag{0.4}$$

where b_n and c_n are given by

$$b_n = \frac{n^2-2n-4}{2} \text{ and } c_n = \frac{n(n-4)(n^2-4)}{16} . \tag{0.5}$$

In particular, as mentioned, (??) is of the type (??). An important remark is that $c_n - \frac{b_n^2}{4} = -1$ is negative. Given $\Lambda > 0$ we let $\mathcal{S}_{b,c}^\Lambda$ be the set

$$\mathcal{S}_{b,c}^\Lambda = \left\{ u \in C^4(M), u > 0, \text{ s.t. } \|u\|_{H^2} \le \Lambda \text{ and } u \text{ solves (??)} \right\} , \tag{0.6}$$

where H^2 is the Sobolev space of functions in L^2 with two derivatives in L^2. Following standard terminology, we say that (??) is *compact* if for any $\Lambda > 0$, $\mathcal{S}_{b,c}^\Lambda$ is compact in the C^4-topology (we adopt here the bounded version of compactness, as first introduced by Schoen [?]). The stronger notion of stability we discuss in the sequel is defined as follows:

Definition 0.1. *Equation (??) is stable if it is compact and if for any $\Lambda > 0$, and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any b' and c', it holds that*

$$d_{C^4}^{\hookrightarrow}(\mathcal{S}_{b',c'}^\Lambda ; \mathcal{S}_{b,c}^\Lambda) < \varepsilon \tag{0.7}$$

as soon as $|b'-b| + |c'-c| < \delta$, where $\mathcal{S}_{b,c}^\Lambda$ and $\mathcal{S}_{b',c'}^\Lambda$ are given by (??), and where for $X, Y \subset C^4$, $d_{C^4}^{\hookrightarrow}(X;Y)$ is the pointed distance defined as the sup over the $u \in X$ of the inf over the $v \in Y$ of $\|v-u\|_{C^4}$.

The meaning of (??) is that small perturbations of b and c in (??) do not create solutions which stand far from solutions of the original equation. Stability is an important notion in view of topological arguments and degree theory. Also it has a natural translation in terms of phase stability for solitons of the fourth order Schrödinger equations introduced by Karpman [?] and Karpman and Shagalov [?] (see the remark at the end of Section ??). The main questions we ask here are:

Questions: (Q1) *describe and control the asymptotic behavior of arbitrary finite energy sequences of solutions of equations like (??).*
(Q2) *find conditions on b and c for (??) to be stable.*

By contradiction, (??) is stable if and only if for any sequences $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ of real numbers converging to b and c, and any sequence $(u_\alpha)_\alpha$ of smooth positive solutions of

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\sharp - 1} \tag{0.8}$$

such that $(u_\alpha)_\alpha$ is bounded in H^2, there holds that, up to a subsequence, $u_\alpha \to u_\infty$ in $C^4(M)$, where u_∞ is a smooth positive solution of (??). In other words, (??) is stable if we can impede bubbling for arbitrary bounded sequences in H^2 of solutions of arbitrary sequences of equations like (??), including (??) itself. In order to do so, we need sharp answers to (Q1).

As is well known, critical equations tend to be unstable (precisely because of the bubbling which is usually associated with critical equations). A consequence of Theorem ?? below is that bubbling is not only associated with the criticality of the equation but also with the geometry through the relation $b = \frac{1}{n}\mathrm{Tr}_g(A_g)$ which, see (**??**) below, characterizes the middle term of the geometric equation (**??**).

Concerning the bound on the energy we require in Definition **??**, it should be noted that we cannot expect the existence of a priori H^2-bounds for arbitrary sequences of equations like (**??**) when dealing with large coefficents b and c (like it is the case for Yamabe type equations associated with second order Schrödinger operators with large potentials). In parallel it is intuitively clear that bounded sequences in H^2 of solutions of equations like (**??**) can develop an arbitrarily large number of peaks. Summing sphere singularities in a naive way we indeed can prove, see Hebey, Robert and Wen [**?**], that for any quotient of the n-sphere, $n \geq 12$, there exist sequences $(u_\alpha)_\alpha$ and $(v_\alpha)_\alpha$ of smooth positive solutions of

$$\Delta_g^2 u + b_n \Delta_g u + c_\alpha u = u^{2^\sharp - 1}$$

such that $(u_\alpha)_\alpha$ blows up with an arbitrarily large given number k of peaks and $\|v_\alpha\|_{H^2} \to +\infty$ as $\alpha \to +\infty$, where $(c_\alpha)_\alpha$ is a sequence of smooth functions converging in the C^1-topology to c_n, and b_n and c_n are as in (**??**). In other words, illustrating the above discussion, we see that equations like (**??**) create bubbling, even multiple of cluster type (namely with blow-up points collapsing on a single point), and that there is no statement about universal a priori H^2-bounds for arbitrary solutions of arbitrary equations like (**??**). Also we see that an equation can be compact and unstable (the geometric equation is compact on quotients of the sphere). Compactness for the geometric equation in the conformally flat case has been established by Hebey and Robert [**?**], and by Qing and Raske [**?**, **?**]. The elegant geometric approach in Qing and Raske [**?**, **?**] is based on the integral representation of the solutions through the developing map under the natural assumption that the Poincaré exponent is small. Recently, Wei and Zhao [**?**] constructed blow-up examples in the non conformally flat case when $n \geq 25$.

Let $\lambda_i(A_g)_x$, $i = 1, \ldots, n$, be the g-eigenvalues of $A_g(x)$ repeated with their multiplicity. Let λ_1 be the infimum over i and x, and λ_2 be the supremum over i and x of the $\lambda_i(A_g)_x$'s. Following Hebey, Robert and Wen [**?**] we define the wild spectrum of A_g to be the interval $\mathcal{S}_w = [\lambda_1, \lambda_2]$. It was proved in [**?**] that (**??**) is stable on conformally flat manifolds when $n = 6, 7, 8$ and $b < \lambda_1$, or $n \geq 9$ and $b \notin \mathcal{S}_w$. We improve these results in different important significative directions in the present article: we add the case of dimension $n = 5$, we replace the condition $b \notin \mathcal{S}_w$ by the much weaker condition $b \neq \frac{1}{n}\mathrm{Tr}_g(A_g)$, and we accept large values of b when $n = 6, 7$. On the other hand, we leave open the question of getting similar results in the nonconformally flat case. In the above discussion, and in what follows, $\mathrm{Tr}_g(A_g) = g^{ij}A_{ij}$ is the trace of A_g with respect to g. There clearly holds that $\frac{1}{n}\mathrm{Tr}_g(A_g) \in \mathcal{S}_w$ at any point in M, and it is easily seen that

$$\mathrm{Tr}_g(A_g) = \frac{n^2 - 2n - 4}{2(n-1)} S_g \, . \tag{0.9}$$

Let $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of solutions of (**??**). Up to a subsequence, $u_\alpha \rightharpoonup u_\infty$ weakly in H^2 for some $u_\infty \in H^2$ which solves (**??**). When $c - \frac{b^2}{4} < 0$, by the maximum principle, either $u_\infty > 0$ in M or $u_\infty \equiv 0$. In the second order case, in low dimensions (namely $n = 3, 4, 5$) we know from Druet [**?**] that we necessarily have that $u_\infty \equiv 0$ if the convergence of u_α to u_∞ is not strong (but only weak) and the u_α's solve Yamabe type equations. In the framework of question (Q1), we also address in this article the question of whether or not such type of results extend to the fourth order case when passing from Yamabe type equations to Paneitz equations like (**??**). We positively answer to this question in Theorem **??** below, the low dimensions being now $5, 6, 7$.

Theorem 0.1. *Let (M, g) be a smooth compact conformally flat Riemannian manifold of dimension $n = 5, 6, 7$ and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be sequences of real numbers converging to b and c, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of smooth positive solutions of (**??**) such that $u_\alpha \rightharpoonup u_\infty$ weakly in H^2 as $\alpha \to +\infty$. Then either $u_\alpha \to u_\infty$ strongly in any C^k-topology, or $u_\infty \equiv 0$.*

Theorem **??** answers the above mentioned question of whether or not we can have a nontrivial limit profile for blowing-up sequences of solutions of (**??**). As a remark, the geometric equation on the sphere provides in any dimension $n \geq 5$ an example of an equation like (**??**) with sequences $(u_\alpha)_\alpha$ of solutions such that $u_\alpha \not\to u_\infty$ strongly and $u_\infty \equiv 0$. Now we return to the question of the stability of (**??**). When $n = 5$ we let G be the Green's function of the fourth order Paneitz type operator $P_g = \Delta_g^2 + b\Delta_g + c$. Then

$$G_x(y) = \frac{1}{6\omega_4 d_g(x, y)} + \mu_x(y) \ , \tag{0.10}$$

where $G_x(\cdot) = G(x, \cdot)$ is the Green's function at x of P_g, ω_4 is the volume of the unit 5-sphere, and μ_x is $C^{0,\theta}$ in M for $\theta \in (0, 1)$. The mass at x of P_g is $\mu_x(x)$. Our second result states as follows.

Theorem 0.2. *Let (M, g) be a smooth compact conformally flat Riemannian manifold of dimension $n \geq 5$ and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Assume that one of the following conditions holds true:*
(i) $n = 5$ and $\mu_x(x) > 0$ for all x ,
(ii) $n = 6$ and $b \notin \mathcal{S}_w$,
(iii) $n = 8$ and $b < \frac{1}{8} \min_M Tr_g(A_g)$,
(iv) $n = 7$ or $n \geq 9$ and $b \neq \frac{1}{n} Tr_g(A_g)$ in M .
*Then for any sequences $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ of real numbers converging to b and c, and any bounded sequence $(u_\alpha)_\alpha$ in H^2 of smooth positive solutions of (**??**) there holds that, up to a subsequence, $u_\alpha \to u_\infty$ in $C^4(M)$ for some smooth positive solution u_∞ of (**??**). In particular, (**??**) is stable.*

As a direct consequence of Theorem **??**, cluster solutions and bubble towers do not exist for (**??**) when one of the conditions (i) to (iv) is assumed to hold. This includes the existence of cluster solutions or bubble towers constructed by means of perturbing (**??**) such as in (**??**).

Let G_0 be the Green's function of the geometric Paneitz operator P_0 in the left hand side of (**??**). Humbert and Raulot [**?**] proved the very nice result that in the conformally flat case, assuming that the Yamabe invariant is positive, that P_0 is positive, and that $G_0 > 0$ outside the diagonal, then the mass of G_0 is nonnegative and equal to zero at one point if and only if the manifold is conformally diffeomorphic to the sphere. A similar result was previously established by Qing and Raske [**?**, **?**] when the Poincaré exponent is small. By (i) in Theorem **??** we need to find conditions under which $\mu_x(x) > 0$ for all x, where $\mu_x(x)$ is the mass of our operator $P_g = \Delta_g^2 + b\Delta_g + c$. A third theorem we prove, based on the Humbert and Raulot [?] result, is as follows.

Theorem 0.3. *Let (M,g) be a smooth compact conformally flat Riemannian manifold of dimension $n = 5$ with positive Yamabe invariant such that the Green's function of the geometric Paneitz operator P_0 is positive and let $b, c > 0$ be positive real numbers. We assume that $bg \leq A_g$ in the sense of bilinear forms and $c \leq \frac{1}{2}Q_g$, and in case $A_g \equiv bg$ and $c \equiv \frac{1}{2}Q_g$ simultaneously, we assume in addition that (M,g) is not conformally diffeomorphic to the standard sphere. Then the mass $\mu_x(x)$ of P_g is positive for all $x \in M$.*

Theorem **??** raises the question of the existence of 5-dimensional compact conformally flat manifolds with positive Yamabe invariant and positive Green's function for P_0. By the analysis in Qing and Raske [?] compact conformally flat manifolds of positive Yamabe invariant and of Poincaré exponent less that $\frac{n-4}{2} = \frac{1}{2}$ are such that the Green's function of P_0 is positive. Explicit examples of such manifolds are in Qing-Raske [?].

The paper is organized as follows. In Section **??** we establish sharp pointwise estimates for arbitrary sequences of solutions of (**??**). This answers (Q1). Thanks to these estimates we prove Theorem **??** in Section **??**. Trace estimates are proved to hold in Section **??**. By the estimates in Sections **??** and **??** we can prove Theorem **??** in Section **??** when $n \geq 6$. In Section **??** we prove Theorem **??** in the specific case $n = 5$. Theorem **??** provides the answer to (Q2). We prove Theorem **??** in Section **??**.

1. Pointwise estimates

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. We do not need in this section to assume that g is conformally flat. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**). Up to a subsequence, $u_\alpha \rightharpoonup u_\infty$ weakly in H^2 as $\alpha \to +\infty$. By standard elliptic theory, either $u_\alpha \to u_\infty$ in C^4 or the u_α's blow up and

$$\|u_\alpha\|_{L^\infty} \to +\infty \tag{1.1}$$

as $\alpha \to +\infty$. From now on we assume that (**??**) holds true. By Hebey and Robert [?] and Hebey, Robert and Wen [?], there holds that

$$u_\alpha = u_\infty + \sum_{i=1}^{k} B_\alpha^i + \mathcal{R}_\alpha\ , \tag{1.2}$$

where $\mathcal{R}_\alpha \to 0$ in H^2 as $\alpha \to \infty$, and the B_α^i's are bubble singularities in H^2. Such B_α^i's are given by

$$B_\alpha^i = \eta\left(d_{i,\alpha}\right)\left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_\alpha^2}{\sqrt{\lambda_n}}}\right)^{\frac{n-4}{2}}\ , \tag{1.3}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth nonnegative cutoff function with small support (less than the injectivity radius of g), $d_{i,\alpha}(\cdot) = d_g(x_{i,\alpha}, \cdot)$, $\lambda_n = n(n-4)(n^2-4)$, $k \geq 1$ is an integer, and for any i, $(x_{i,\alpha})_\alpha$ is a converging sequence of points in M and

$(\mu_{i,\alpha})_\alpha$ is a sequence of positive real numbers such that $\mu_{i,\alpha} \to 0$ as $\alpha \to +\infty$. Moreover, we also have that the following structure equation holds true: for any $i \neq j$,

$$\frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha}\mu_{j,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} \to +\infty \tag{1.4}$$

as $\alpha \to +\infty$, and that there exists $C > 0$ such that

$$r_\alpha(x)^{\frac{n-4}{2}} |u_\alpha(x) - u_\infty(x)| \leq C \tag{1.5}$$

for all α and all $x \in M$, where

$$r_\alpha(x) = \min_{i=1,\dots,k} d_g(x_{i,\alpha}, x) \ . \tag{1.6}$$

By (**??**) and (**??**) we have that $u_\alpha \rightharpoonup u_\infty$ in H^2 and $u_\alpha \to u_\infty$ in $C^0_{loc}(M\backslash\mathcal{S})$, where $\mathcal{S}$ is the set consisting of the limits of the $x_{i,\alpha}$'s. We define μ_α by

$$\mu_\alpha = \max_{i=1,\dots,k} \mu_{i,\alpha} \ . \tag{1.7}$$

There holds that $\mu_\alpha \to 0$ as $\alpha \to +\infty$ since $\mu_{i,\alpha} \to 0$ for all i as $\alpha \to +\infty$. We aim here at proving the following sharp pointwise estimates.

Proposition 1.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). There exists $C > 0$ such that, up to a subsequence,*

$$\left|\nabla^j u_\alpha\right| \leq C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j} + \|u_\infty\|_{L^\infty}\right) \tag{1.8}$$

*in M, for all $j = 0, 1, 2, 3$ and all α, where r_α is as in (**??**), and μ_α is as in (**??**).*

We split the proof of Proposition **??** into several lemmas. The first lemma we prove is a general basic result we will use further in the proof of Lemma **??**.

Lemma 1.1. *Let $\delta > 0$ and $(g_\alpha)_\alpha$ be a sequence of Riemannian metrics in the Euclidean ball $B_0(2\delta)$ such that $g_\alpha \to \xi$ in $C^4_{loc}(B_0(2\delta))$ as $\alpha \to +\infty$, where ξ is the Euclidean metric. Let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be bounded sequences of positive real numbers such that $c_\alpha \leq \frac{b_\alpha^2}{4}$ for all α. Let $(w_\alpha)_\alpha$ be a sequence of positive functions such that*

$$\Delta^2_{g_\alpha} w_\alpha + b_\alpha \Delta_{g_\alpha} w_\alpha + c_\alpha w_\alpha = w_\alpha^{2^\sharp - 1} \tag{1.9}$$

in $B_0(2\delta)$ for all α. Assume $\|w_\alpha\|_{L^\infty(B_0(\delta))} \to +\infty$ as $\alpha \to +\infty$ and that there exists $C > 0$ such that $\|w_\alpha\|_{L^\infty(B_0(\delta)\backslash B_0(\delta/2))} \leq C$ for all α. Then

$$\int_{B_0(\delta)} w_\alpha^{2^\sharp} dx \geq (1 + o(1)) K_n^{-n/4} \ , \tag{1.10}$$

where $o(1) \to 0$ as $\alpha \to +\infty$ and K_n is the sharp constant in the Euclidean Sobolev inequality $\|u\|^2_{2^\sharp} \leq K_n \|\Delta u\|_2^2$, $u \in \dot{H}^2$. Here $\dot{H}^2$ is the completion of $C^\infty_0(\mathbb{R}^n)$ with respect to $u \to \|\Delta u\|_2$.

Proof of Lemma **??**. Let ν_α be given by

$$\nu_\alpha^{2-\frac{n}{2}} = \max_{B_0(\delta)} w_\alpha$$

and x_α be such that $w_\alpha(x_\alpha) = \nu_\alpha^{2-\frac{n}{2}}$. By assumption, $\nu_\alpha \to 0$ as $\alpha \to +\infty$ and $x_\alpha \in B_0(\delta/2)$ for $\alpha \gg 1$. Let $\tilde{w}_\alpha$ be given by

$$\tilde{w}_\alpha(x) = \nu_\alpha^{\frac{n-4}{2}} w_\alpha\left(x_\alpha + \nu_\alpha x\right) .$$

For any $R > 1$, $\tilde{w}_\alpha$ is defined in $B_0(R)$ provided $\alpha \gg 1$ is sufficiently large. Let $d_{1,\alpha}$ and $d_{2,\alpha}$ be given by

$$d_{1,\alpha} = \frac{b_\alpha}{2} + \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} \text{ and } d_{2,\alpha} = \frac{b_\alpha}{2} - \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} . \tag{1.11}$$

Then, as is easily checked, for any $R > 1$,

$$\begin{aligned} &\Delta_{\tilde{g}_\alpha}^2 \tilde{w}_\alpha + b_\alpha \nu_\alpha^2 \Delta_{\tilde{g}_\alpha} \tilde{w}_\alpha + c_\alpha \nu_\alpha^4 \tilde{w}_\alpha \\ &= \left(\Delta_{\tilde{g}_\alpha} + d_{1,\alpha}\nu_\alpha^2\right) \circ \left(\Delta_{\tilde{g}_\alpha} + d_{2,\alpha}\nu_\alpha^2\right) \tilde{w}_\alpha \\ &= \tilde{w}_\alpha^{2^\sharp - 1} \end{aligned} \tag{1.12}$$

in $B_0(R)$ for all $\alpha \gg 1$ sufficiently large, where $\tilde{g}_\alpha(x) = g_\alpha(x_\alpha + \nu_\alpha x)$. Since $0 \leq \tilde{w}_\alpha \leq 1$, it follows from (**??**) and classical elliptic theory, as developped in Gilbarg-Trudinger [**?**], that the $\tilde{w}_\alpha$'s are bounded in $C^{4,\theta}_{loc}(\mathbb{R}^n)$, $\theta \in (0,1)$. In particular, there exists $\tilde{w}$ such that, up to a subsequence, $\tilde{w}_\alpha \to \tilde{w}$ in $C^4_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. By rescaling invariance rules there also holds that $\tilde{w} \in H^2$. Passing to the limit as $\alpha \to +\infty$ in (**??**), we get that

$$\Delta^2 \tilde{w} = \tilde{w}^{2^\sharp - 1} .$$

Since $\tilde{w}(0) = \max_{\mathbb{R}^n} \tilde{w} = 1$, it follows from Lin's classification [**?**] that $\tilde{w}$ is the ground state in (**??**) below. Noting that for any $R > 1$,

$$\begin{aligned} \int_{B_0(R)} \tilde{w}_\alpha^{2^\sharp} dv_{\tilde{g}_\alpha} &= \int_{B_{x_\alpha}(R\nu_\alpha)} w_\alpha^{2^\sharp} dv_{g_\alpha} \\ &\leq (1 + o(1)) \int_{B_0(\delta)} w_\alpha^{2^\sharp} dx , \end{aligned}$$

we get from the L^∞_{loc}-convergence $\tilde{w}_\alpha \to \tilde{w}$ that

$$\liminf_{\alpha \to +\infty} \int_{B_0(\delta)} w_\alpha^{2^\sharp} dx \geq \int_{B_0(R)} \tilde{w}^{2^\sharp} dx . \tag{1.13}$$

Noting that $\tilde{w}$ is an extremal function for the sharp inequality $\|u\|_{2^\sharp}^2 \leq K_n \|\Delta u\|_2^2$, it is easily seen that $\int_{\mathbb{R}^n} \tilde{w}^{2^\sharp} dx = K_n^{-n/4}$. Letting $R \to +\infty$ in (**??**), this ends the proof of the lemma. □

From now on we let $B : \mathbb{R}^n \to \mathbb{R}$ be the ground state given by

$$B(x) = \left(1 + \frac{|x|^2}{\sqrt{\lambda_n}}\right)^{-\frac{n-4}{2}} , \tag{1.14}$$

where λ_n is as in (**??**). Given $x \in M$, $\mu > 0$ and $u : M \to \mathbb{R}$, we also define the function $R_x^\mu u$ by

$$R_x^\mu u(y) = \mu^{\frac{n-4}{2}} u\left(\exp_x(\mu y)\right) , \tag{1.15}$$

where $y \in \mathbb{R}^n$ is such that $|y| < \frac{i_g}{\mu}$, and $i_g > 0$ is the injectivity radius of (M, g). For $(u_\alpha)_\alpha$ as above, and $i \in \{1, \dots, k\}$, we define $\mathcal{S}_{i,r}, \mathcal{S}_{i,t} \subset \mathbb{R}^n$ by

$$\begin{aligned} \mathcal{S}_{i,t} &= \left\{ \lim_{\alpha \to +\infty} \frac{1}{\mu_{i,\alpha}} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}), j = 1, \dots, k \right\} \\ \mathcal{S}_{i,r} &= \left\{ \lim_{\alpha \to +\infty} \frac{1}{\mu_{i,\alpha}} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}), j \in I_i \right\} , \end{aligned} \tag{1.16}$$

where I_i is the subset of $\{1, \dots, k\}$ consisting in the j's which are such that $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})$ and $\mu_{j,\alpha} = o(\mu_{i,\alpha})$. The second lemma we prove establishes local limits for the u_α's.

Lemma 1.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). Then, up to a subsequence,*

$$R_{x_{i,\alpha}}^{\mu_{i,\alpha}} u_\alpha \to B \tag{1.17}$$

*in $C^4_{loc}(\mathbb{R}^n \backslash \mathcal{S}_{i,r})$ as $\alpha \to +\infty$ for all i, where $\mathcal{S}_{i,r}$ is as in (**??**).*

Proof of Lemma **??**. First we claim that for any i and any $K \subset\subset \mathbb{R}^n \backslash \mathcal{S}_{i,t}$, there exists $C_K > 0$ such that

$$\left| R_{x_{i,\alpha}}^{\mu_{i,\alpha}} u_\alpha \right| \leq C_K \tag{1.18}$$

in K, for all $\alpha \gg 1$ sufficiently large. Fix i and K. For any $x \in K$, and any j,

$$\begin{aligned} d_g\left(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} x), x_{j,\alpha}\right) &\geq C \left| \mu_{i,\alpha} x - \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}) \right| \\ &\geq C \mu_{i,\alpha} \left| x - \frac{1}{\mu_{i,\alpha}} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}) \right| \\ &\geq C \mu_{i,\alpha} \end{aligned}$$

by the definition of $\mathcal{S}_{i,t}$ in (**??**). Then (**??**) follows from (**??**). Also, by direct computations, using the structure equation (**??**), there holds that for any $i \neq j$, $R_{x_{i,\alpha}}^{\mu_{i,\alpha}} B_\alpha^j \to 0$ in $L^\infty_{loc}(\mathbb{R}^n \backslash \mathcal{S}_{i,r})$ as $\alpha \to +\infty$, where the B_α^i's are as in (**??**). Noting that $R_{x_{i,\alpha}}^{\mu_{i,\alpha}} B_\alpha^i(x) = \eta(\mu_{i,\alpha} x) B(x)$, where η is as in (**??**) and B is as in (**??**), we get that for any i,

$$R_{x_{i,\alpha}}^{\mu_{i,\alpha}} \sum_{j=1}^k B_\alpha^j \to B \tag{1.19}$$

in $L^\infty_{loc}(\mathbb{R}^n \backslash \mathcal{S}_{i,r})$ as $\alpha \to +\infty$. Now we prove (**??**). By (**??**),

$$R_{x_{i,\alpha}}^{\mu_{i,\alpha}} u_\alpha - R_{x_{i,\alpha}}^{\mu_{i,\alpha}} \sum_{j=1}^k B_\alpha^j \to 0 \tag{1.20}$$

in $L^{2^\sharp}_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. Moreover, in any compact subset of $\mathbb{R}^n$,

$$\Delta_{g_\alpha}^2 \tilde{u}_\alpha + b_\alpha \mu_{i,\alpha}^2 \Delta_{g_\alpha} \tilde{u}_\alpha + c_\alpha \mu_{i,\alpha}^4 \tilde{u}_\alpha = \tilde{u}_\alpha^{2^\sharp - 1} \tag{1.21}$$

for $\alpha \gg 1$ sufficiently large, where $\tilde{u}_\alpha = R_{x_{i,\alpha}}^{\mu_{i,\alpha}} u_\alpha$ and $g_\alpha(x) = \left(\exp_{x_{i,\alpha}}^\star g\right)(\mu_{i,\alpha} x)$. Since $\mu_{i,\alpha} \to 0$, we have that $g_\alpha \to \xi$ in $C^4_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. By (**??**), the sequence $(\tilde{u}_\alpha)_\alpha$ is bounded in $L^\infty_{loc}(\mathbb{R}^n\backslash\mathcal{S}_{i,t})$. By (**??**) and (**??**), there also holds that

$$\lim_{\delta\to 0}\limsup_{\alpha\to+\infty}\int_{B_x(\delta)}\tilde{u}_\alpha^{2^\sharp}dx = 0 \tag{1.22}$$

for all $x\in\mathbb{R}^n\backslash\mathcal{S}_{i,r}$. By Lemma **??** we then get that the sequence $(\tilde{u}_\alpha)_\alpha$ is actually bounded in $L^\infty_{loc}(\mathbb{R}^n\backslash\mathcal{S}_{i,r})$. By (**??**) and elliptic theory it follows that $(\tilde{u}_\alpha)_\alpha$ is bounded in $C^{4,\theta}_{loc}(\mathbb{R}^n\backslash\mathcal{S}_{i,r})$, $\theta\in(0,1)$. This ends the proof of Lemma **??**. □

The following lemma establishes pointwise estimates for the u_α's. The estimates in Lemma **??** are a trace extension of the estimates (**??**). In particular, as is easily checked, (**??**) below implies (**??**).

Lemma 1.3. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 5$, and $b,c>0$ be positive real numbers such that $c-\frac{b^2}{4}<0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha\to+\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of* (**??**) *satisfying* (**??**)*. Then, up to a subsequence,*

$$r_\alpha^4\left|u_\alpha-u_\infty-\sum_{i=1}^k B_\alpha^i\right|^{2^\sharp-2}\to 0 \tag{1.23}$$

in $L^\infty(M)$ as $\alpha\to+\infty$, where r_α is as in (**??**)*, and the B_α^i's are as in* (**??**)*.*

Proof of Lemma **??**. Let $D_\alpha: M\to\mathbb{R}$ be given by

$$D_\alpha(x)=\min_{i=1,\dots,k}\left(d_g(x_{i,\alpha},x)+\mu_{i,\alpha}\right)\ . \tag{1.24}$$

We prove sligthly more than (**??**), namely that

$$D_\alpha^4\left|u_\alpha-u_\infty-\sum_{i=1}^k B_\alpha^i\right|\to 0 \tag{1.25}$$

in $L^\infty(M)$ as $\alpha\to+\infty$, where D_α is as in (**??**). Let $x_\alpha\in M$ be such that

$$\begin{aligned} &D_\alpha^4(x_\alpha)\left|u_\alpha(x_\alpha)-u_\infty(x_\alpha)-\sum_{i=1}^k B_\alpha^i(x_\alpha)\right|^{2^\sharp-2}\\ &=\max_{x\in M}D_\alpha^4(x)\left|u_\alpha(x)-u_\infty(x)-\sum_{i=1}^k B_\alpha^i(x)\right|^{2^\sharp-2}\ . \end{aligned} \tag{1.26}$$

First we claim that

$$\begin{aligned} &\liminf_{\alpha\to+\infty}D_\alpha^4(x_\alpha)\left|u_\alpha(x_\alpha)-u_\infty(x_\alpha)-\sum_{i=1}^k B_\alpha^i(x_\alpha)\right|^{2^\sharp-2}>0\\ &\Longrightarrow \lim_{\alpha\to+\infty}D_\alpha(x_\alpha)^4B_\alpha^i(x_\alpha)^{2^\sharp-2}=0 \end{aligned} \tag{1.27}$$

for all i. In order to prove (**??**) we proceed by contradiction and assume that there exists $\varepsilon_0>0$ and i such that $D_\alpha(x_\alpha)^4B_\alpha^i(x_\alpha)^{2^\sharp-2}\geq\varepsilon_0$, and thus such that

$$\eta\left(d_g(x_{i,\alpha},x_\alpha)\right)\left(\frac{D_\alpha(x_\alpha)\mu_{i,\alpha}}{\mu_{i,\alpha}^2+\frac{d_g(x_{i,\alpha},x_\alpha)^2}{\sqrt{\lambda_n}}}\right)^4\geq\varepsilon_0\ . \tag{1.28}$$

By (**??**) we get that $d_g(x_{i,\alpha}, x_\alpha) \to 0$ as $\alpha \to +\infty$, that there exists $\lambda \geq 0$ such that, up to a subsequence,

$$\frac{d_g(x_{i,\alpha}, x_\alpha)}{\mu_{i,\alpha}} \to \lambda \tag{1.29}$$

as $\alpha \to +\infty$, and that

$$\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} + \frac{d_g(x_{j,\alpha}, x_\alpha)}{\mu_{i,\alpha}} \geq \varepsilon_0^{1/4} \tag{1.30}$$

for all α and j. Let y_α be given by

$$y_\alpha = \frac{1}{\mu_{i,\alpha}} \exp_{x_{i,\alpha}}^{-1}(x_\alpha) \ .$$

By (**??**), $d(y_\alpha, \mathcal{S}_{i,r}) \geq \varepsilon$ for all α, where $\varepsilon > 0$ is independent of α, while by (**??**) there holds that $|y_\alpha| \leq C$ for all α, where $C > 0$ is independent of α. We have that $D_\alpha(x_\alpha) \leq \mu_{i,\alpha}$ by (**??**). By Lemma **??** we then get that

$$D_\alpha^4(x_\alpha) \left| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - B_\alpha^i(x_\alpha) \right|^{2^\sharp - 2} \to 0 \tag{1.31}$$

as $\alpha \to +\infty$. As in the proof of Lemma **??**, using the structure equation (**??**), there also holds that $D_\alpha(x_\alpha)^4 B_\alpha^j(x_\alpha)^{2^\sharp - 2} \to 0$ as $\alpha \to +\infty$ for all $j \neq i$. Coming back to (**??**), the contradiction follows with the assumption in (**??**). This proves (**??**). Now we prove (**??**). Here again we proceed by contradiction and assume that there exists $\varepsilon_0 > 0$ such that

$$D_\alpha^4(x_\alpha) \left| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - \sum_{i=1}^k B_\alpha^i(x_\alpha) \right|^{2^\sharp - 2} \geq \varepsilon_0 \ , \tag{1.32}$$

where x_α is as in (**??**). We claim that

$$u_\alpha(x_\alpha) \to +\infty \tag{1.33}$$

as $\alpha \to +\infty$. By (**??**) and (**??**), we get (**??**) if we prove that $D_\alpha(x_\alpha) \to 0$ as $\alpha \to +\infty$. Suppose on the contrary that, up to a subsequence, $D_\alpha(x_\alpha) \to \delta$ as $\alpha \to +\infty$ for some $\delta > 0$. By (**??**) and (**??**) there holds that

$$|u_\alpha(x) - u_\infty(x)|^{2^\sharp - 2} \leq C \, |u_\alpha(x_\alpha) - u_\infty(x_\alpha)|^{2^\sharp - 2} + o(1) \tag{1.34}$$

for all $x \in B_{x_\alpha}(\delta/2)$, and all $\alpha \gg 1$. Assuming that (**??**) is false, it follows from (**??**) that the u_α's are uniformly bounded in a neighborhood of the x_α's. By elliptic theory we then get that $u_\alpha \to u_\infty$ in $L^\infty(\Omega)$, where Ω is a neighborhood of the limit of the x_α's. Hence $u_\alpha(x_\alpha) - u_\infty(x_\alpha) \to 0$ and we get a contradiction with (**??**) and (**??**). This proves (**??**). Now we let μ_α be given by $\mu_\alpha^{2-\frac{n}{2}} = u_\alpha(x_\alpha)$ and define $\tilde{u}_\alpha$ by $\tilde{u}_\alpha = R_{x_\alpha}^{\mu_\alpha} u_\alpha$. Then

$$\Delta_{g_\alpha}^2 \tilde{u}_\alpha + b_\alpha \mu_\alpha^2 \Delta_{g_\alpha} \tilde{u}_\alpha + c_\alpha \mu_\alpha^4 \tilde{u}_\alpha = \tilde{u}_\alpha^{2^\sharp - 1} \ , \tag{1.35}$$

where $g_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x)$. By (**??**), $\mu_\alpha \to 0$ as $\alpha \to +\infty$. It follows that $g_\alpha \to \xi$ in $C_{loc}^4(\mathbb{R}^n)$ as $\alpha \to +\infty$. Also there holds that $\tilde{u}_\alpha(0) = 1$ and that the $\tilde{u}_\alpha$'s are bounded in $\dot{H}^2$. Up to a subsequence, $\tilde{u}_\alpha \rightharpoonup \tilde{u}_\infty$ in H_{loc}^2 and

$$\Delta^2 \tilde{u}_\infty = \tilde{u}_\infty^{2^\sharp - 1} \ , \tag{1.36}$$

where $\Delta = -\mathrm{div}_\xi \nabla$ is the Euclidean Laplacian. Let $\tilde{\mathcal{S}}$ be given by

$$\tilde{\mathcal{S}} = \left\{ \lim_{\alpha \to +\infty} \frac{1}{\mu_\alpha} \exp_{x_\alpha}^{-1}(x_{i,\alpha}) \ , \ i \in I \right\} \ ,$$

where I consists of the indices i which are such that $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_\alpha)$ and $\mu_{i,\alpha} = o(\mu_\alpha)$. In what follows we let $K \subset\subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$ be compact, $x \in K$. By (**??**), (**??**) and (**??**), we have that

$$\begin{aligned} &\left| \tilde{u}_\alpha(x) - \mu_\alpha^{\frac{n-4}{2}} u_\infty(y_\alpha) - \mu_\alpha^{\frac{n-4}{2}} \sum_{i=1}^k B_\alpha^i(y_\alpha) \right|^{2^\sharp - 2} \\ &\le \left(\frac{D_\alpha(x_\alpha)}{D_\alpha(y_\alpha)} \right)^4 \big(1 + o(1)\big) + o(1) \ , \end{aligned} \tag{1.37}$$

where $y_\alpha = \exp_{x_\alpha}(\mu_\alpha x)$. It can be checked that

$$\mu_\alpha^{\frac{n-4}{2}} B_\alpha^i(y_\alpha) \to 0 \tag{1.38}$$

for all i, as $\alpha \to +\infty$. By (**??**), (**??**) and (**??**) we then get that

$$\tilde{u}_\alpha(x)^{2^\sharp - 2} \le \left(\frac{D_\alpha(x_\alpha)}{D_\alpha(y_\alpha)} \right)^4 \big(1 + o(1)\big) + o(1) \ . \tag{1.39}$$

Since $x \in K$, and $K \subset\subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$, there holds that $D_\alpha(x_\alpha) = O\left(D_\alpha(y_\alpha)\right)$. Hence, by (**??**), for any $K \subset\subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$, there exists $C > 0$ such that $|\tilde{u}_\alpha| \le C$ in K. By elliptic theory and (**??**) we then get that

$$\tilde{u}_\alpha \to \tilde{u}_\infty \tag{1.40}$$

in $C^4_{loc}(\mathbb{R}^n \backslash \tilde{\mathcal{S}})$ as $\alpha \to +\infty$. It holds that $0 \notin \tilde{\mathcal{S}}$ since, if not the case, we can write that $D_\alpha(x_\alpha) = o(\mu_\alpha)$ and we get a contradiction with (**??**). As a consequence, since $\tilde{u}_\alpha(0) = 1$, it follows that $\tilde{u}_\infty(0) = 1$ and thus that $\tilde{u}_\infty \not\equiv 0$. By (**??**) and Lin's classification [**?**] we then get that

$$\tilde{u}_\infty(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x - x_0|^2}{\sqrt{\lambda_n}}} \right)^{\frac{n-4}{2}} \tag{1.41}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that $\sqrt{\lambda_n}\lambda(\lambda - 1) + |x_0|^2 = 0$. Let $K \subset\subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$, $K \ne \emptyset$. By (**??**) and (**??**) there holds that $\tilde{u}_\alpha \to 0$ in $L^{2^\sharp}(K)$. By (**??**) we should get that $\int_K \tilde{u}_\infty^{2^\sharp} dx = 0$, a contradiction with (**??**). This ends the proof of Lemma **??**. $\square$

As already mentioned, it follows from (**??**) that there exists $C > 0$ such that

$$r_\alpha^{(n-4)/2} u_\alpha \le C \ . \tag{1.42}$$

Derivative companions to this estimate are as follows.

Lemma 1.4. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 5$, and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of* (**??**) *satisfying* (**??**). *There exists $C > 0$ such that, up to a subsequence,*

$$r_\alpha^{\frac{n-4}{2} + k} |\nabla^k u_\alpha| \le C \tag{1.43}$$

*in M for all α, and all $k=1,2,3$, where r_α is as in (**??**).*

Proof of Lemma **??**. Lemma **??** follows from Green's representation formula and (**??**). Let G_α be the Green's function of $\Delta_g^2+b_\alpha\Delta_g+c_\alpha$. By Green's representation formula,

$$u_\alpha(x)=\int_M G_\alpha(x,y)u_\alpha(y)^{2^\sharp-1}dv_g(y) \tag{1.44}$$

for all $x\in M$. Then, by (**??**),

$$\nabla^k u_\alpha(x)=\int_M \nabla_x^k G_\alpha(x,y)u_\alpha(y)^{2^\sharp-1}dv_g(y) \tag{1.45}$$

for all $x\in M$. There holds, see Grunau and Robert [**?**], that

$$\left|\nabla_y^k G_\alpha(x,y)\right|\le C d_g(x,y)^{4-n-k} \tag{1.46}$$

for all α, all $x,y\in M$ with $x\neq y$, and all $k\in\{0,1,2,3\}$. By (**??**), (**??**), (**??**) and Giraud's lemma, we then get that there exists $C>0$ such that

$$\begin{aligned}\left|\nabla^k u_\alpha(x)\right| &\le \int_M |\nabla_x^k G_\alpha(x,y)|u_\alpha(y)^{2^\sharp-1}dv_g(y)\\ &\le C\sum_{i=1}^N\int_M d_g(x,y)^{4-n-k}d_g(x_{i,\alpha},y)^{-(n+4)/2}dv_g(y)\\ &\le Cr_\alpha(x)^{-\frac{n-4}{2}-k}\ .\end{aligned}$$

This proves (**??**). □

Estimates such as in (**??**) and Lemma **??** are scale invariant estimates. When transposed to the Euclidean space, in the simple case of a single isolated blow-up point, we would get, for instance when $k=0$ and $k=2$, that $|x|^{(n-4)/2}|u(x)|\le C$ and $|x|^{n/2}|\Delta u(x)|\le C$. These two estimates are invariant with respect to the scaling $\lambda^{(n-4)/2}u(\lambda x)$ which leaves invariant both the equation $\Delta^2u=u^{2^\sharp-1}$ and the $\dot H^2$-norm. Scale invariant estimates, together with the Sobolev description (**??**), provide valuable informations on the u_α's. However they are still not strong enough to conclude to the theorem. Sharper estimates such as the ones in Proposition **??** are required. From now on we define the v_α's by

$$v_\alpha=\Delta_g u_\alpha+\frac{b_\alpha}{2}u_\alpha \tag{1.47}$$

for all α. By (**??**) there holds that

$$\Delta_g v_\alpha+\frac{b_\alpha}{2}v_\alpha=\tilde c_\alpha u_\alpha+u_\alpha^{2^\sharp-1}\ , \tag{1.48}$$

where $\tilde c_\alpha=\frac{b_\alpha^2}{4}-c_\alpha$. In particular, when $c_\alpha\le\frac{b_\alpha^2}{4}$, we get by the maximum principle that either $v_\alpha>0$ in M or $v_\alpha\equiv0$. When we also assume that $b_\alpha>0$, this implies that $v_\alpha>0$ when u_α is nontrivial. The following lemma is a key point toward the proof of Proposition **??**.

Lemma 1.5. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge5$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of positive real numbers satisfying that $c_\alpha-\frac{b_\alpha^2}{4}\le0$ for all α, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). Let u_∞ be such that, up to a subsequence, $u_\alpha\to u_\infty$ a.e. in M. There exists $C_1>0$ such that $v_\alpha\ge C_1u_\alpha^{2^\sharp/2}$*

*in M for all α, where the v_α's are as in (**??**). Assuming that either $u_\infty \not\equiv 0$ or $c-\frac{b^2}{4}<0$, where b and c are the limits of the sequences $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$, there also exists $C_2>0$ such that $v_\alpha \geq C_2 u_\alpha$ in M for all α.*

Proof of Lemma **??**. We use twice the basic remark that if Ω is an open subset of M, u, v are C^2-positive functions in Ω, and $x_0 \in \Omega$ is a point where $\frac{v}{u}$ achieves its supremum in Ω, then

$$\frac{\Delta_g v(x_0)}{v(x_0)} \geq \frac{\Delta_g u(x_0)}{u(x_0)} . \tag{1.49}$$

Indeed, $\nabla\left(\frac{v}{u}\right)=\frac{u\nabla v - v\nabla u}{u^2}$ so that $u(x_0)\nabla v(x_0)=v(x_0)\nabla u(x_0)$. Then,

$$\Delta_g\left(\frac{v}{u}\right)(x_0)=\frac{u(x_0)\Delta_g v(x_0)-v(x_0)\Delta_g u(x_0)}{u^2(x_0)}$$

and we get (**??**) by writing that $\Delta_g\left(\frac{v}{u}\right)(x_0)\geq 0$. In what follows we let $x_\alpha \in M$ be such that

$$\frac{u_\alpha(x_\alpha)^{2^\sharp/2}}{v_\alpha(x_\alpha)}=\max_{x\in M}\frac{u_\alpha^{2^\sharp/2}}{v_\alpha} . \tag{1.50}$$

Then, by (**??**),

$$\frac{\Delta_g u_\alpha(x_\alpha)^{2^\sharp/2}}{u_\alpha(x_\alpha)^{2^\sharp/2}} \geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} . \tag{1.51}$$

We compute

$$\Delta_g u_\alpha^{\frac{2^\sharp}{2}}=\frac{2^\sharp}{2}u_\alpha^{\frac{2^\sharp}{2}-1}\Delta_g u_\alpha-\frac{2^\sharp}{2}\left(\frac{2^\sharp}{2}-1\right)u_\alpha^{\frac{2^\sharp}{2}-2}|\nabla u_\alpha|^2 . \tag{1.52}$$

It follows from (**??**) and (**??**) that

$$\frac{2^\sharp}{2}\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} . \tag{1.53}$$

By (**??**) and (**??**) we then get that

$$\begin{aligned}\frac{2^\sharp}{2}\frac{v_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} &= \frac{2^\sharp}{2}\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)}+\frac{2^\sharp}{2}\frac{b_\alpha}{2}\\ &\geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}+\frac{2^\sharp}{2}\frac{b_\alpha}{2}\\ &= \frac{\Delta_g v_\alpha(x_\alpha)+\frac{b_\alpha}{2}v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}+\left(\frac{2^\sharp}{2}-1\right)\frac{b_\alpha}{2}\\ &= \frac{u_\alpha(x_\alpha)^{2^\sharp-1}}{v_\alpha(x_\alpha)}+\tilde{c}_\alpha\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}+\left(\frac{2^\sharp}{2}-1\right)\frac{b_\alpha}{2} ,\end{aligned}$$

where $\tilde{c}_\alpha=\frac{b_\alpha^2}{4}-c_\alpha$. By assumption, $b_\alpha>0$ and $\tilde{c}_\alpha \geq 0$. In particular, we get that

$$v_\alpha(x_\alpha) \geq \sqrt{\frac{2}{2^\sharp}}u_\alpha(x_\alpha)^{2^\sharp/2}$$

for all α. The estimate $v_\alpha \geq C u_\alpha^{2^\sharp/2}$ follows from the definition (**??**) of x_α. This proves the first part of Lemma **??**. In order to get the second part we let x_α be such that

$$\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}=\max_{x\in M}\frac{u_\alpha(x)}{v_\alpha(x)} . \tag{1.54}$$

By (**??**) and (**??**),

$$\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \ge \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}$$

and we get with (**??**) that

$$\begin{aligned}\frac{v_\alpha(x_\alpha) - \frac{b_\alpha}{2}u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} &\ge \frac{\Delta_g v_\alpha(x_\alpha) + \frac{b_\alpha}{2}v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} - \frac{b_\alpha}{2}\\ &\ge \frac{\tilde{c}_\alpha u_\alpha(x_\alpha) + u_\alpha(x_\alpha)^{2^\sharp-1}}{v_\alpha(x_\alpha)} - \frac{b_\alpha}{2}\ .\end{aligned}$$

As a consequence,

$$\frac{v_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \ge \tilde{c}_\alpha \frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} + u_\alpha(x_\alpha)^{2^\sharp-2}\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}$$

and we get that

$$\frac{v_\alpha(x_\alpha)^2}{u_\alpha(x_\alpha)^2} \ge \tilde{c}_\alpha + u_\alpha(x_\alpha)^{2^\sharp-2}\ . \tag{1.55}$$

Assuming that $c-\frac{b^2}{4}<0$ there exists $\delta>0$ such that $c_\alpha \ge \delta$ for all α. Similarly, let us assume that $u_\infty \not\equiv 0$. If G_α stands for the Green's function of $\Delta_g^2 + b_\alpha\Delta_g + c_\alpha$, then

$$u_\alpha(x_\alpha) = \int_M G_\alpha(x_\alpha,\cdot)u_\alpha^{2^\sharp-1}dv_g \ge \int_{M\backslash\bigcup_i B_{x_i}(\delta')} G_\alpha(x_\alpha,\cdot)u_\alpha^{2^\sharp-1}dv_g\ .$$

By Lemma **??**, $u_\alpha \to u_\infty$ uniformly in compact subsets of $M\backslash\bigcup_{i=1}^k\{x_i\}$. Letting $\alpha\to+\infty$, and then $\delta'\to 0$, it follows that there exists $\delta>0$ such that $u_\alpha(x_\alpha)\ge\delta$ for all α. In particular, in both cases $c-\frac{b^2}{4}<0$ and $u_\infty\not\equiv 0$, we get with (**??**) that $v_\alpha(x_\alpha)\ge Cu_\alpha(x_\alpha)$ for some $C>0$ independent of α. By the definition of x_α in (**??**) it follows that $v_\alpha \ge Cu_\alpha$ in M for all α. This ends the proof of the lemma. □

At that point, given $\delta>0$, we define $\eta_\alpha(\delta)$ by

$$\eta_\alpha(\delta) = \max_{M\backslash\bigcup_i B_{x_{i,\alpha}}(\delta)} v_\alpha\ , \tag{1.56}$$

where v_α is as in (**??**). Then we prove the following first set of pointwise ε-sharp estimates on the u_α's and v_α's.

Lemma 1.6. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 5$, and $b,c>0$ be positive real numbers such that $c-\frac{b^2}{4}<0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha\to\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). Let $0<\varepsilon<\varepsilon_0$, where $\varepsilon_0>0$ is sufficiently small. There exist $R_\varepsilon>0$, $\delta_\varepsilon>0$, and $C_\varepsilon>0$ such that*

$$\begin{aligned} u_\alpha &\le C_\varepsilon\left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{4-n+(n-2)\varepsilon} + \eta_\alpha(\delta_\varepsilon)\right)\ ,\\ v_\alpha &\le C_\varepsilon\left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon) r_\alpha^{(2-n)\varepsilon}\right)\end{aligned} \tag{1.57}$$

*in $M\backslash\bigcup_{i=1}^k B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$ for all α, where $\mu_{i,\alpha}$ is as in (**??**), r_α is as in (**??**), μ_α is as in (**??**), v_α is as in (**??**), and η_α is as in (**??**).*

Proof of Lemma **??**. The first estimate we prove is the one on the v_α's from which we deduce then the estimate on the u_α's by using the Green's representation of u_α in terms of v_α. We establish the estimate on the v_α's for $0<\varepsilon<\frac{1}{2}$, and the estimate on the u_α's for $0<\varepsilon<\frac{1}{n-2}\min(2,n-4)$.

(1) *Proof of the estimate on the v_α's in* (**??**). We fix $0<\varepsilon<\frac{1}{2}$. Let G_1' be the Green's function of Δ_g+1 and let $\psi_{\alpha,\varepsilon}$ be given by

$$\psi_{\alpha,\varepsilon}(x)=\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon}\sum_i G_1'(x_{i,\alpha},x)^{1-\varepsilon}+\eta_\alpha(\delta_\varepsilon)\sum_i G_1'(x_{i,\alpha},x)^{\varepsilon}\ .$$

Given $R>0$ we let $\Omega_{\alpha,R}=\bigcup_i B_{x_{i,\alpha}}(R\mu_{i,\alpha})$, and let $x_\alpha\in M\backslash\Omega_{\alpha,R}$ be such that

$$\max_{M\backslash\Omega_{\alpha,R}}\frac{v_\alpha}{\psi_{\alpha,\varepsilon}}=\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)}\ .$$

First we claim that for $\delta_\varepsilon\ll 1$ and $R_\varepsilon\gg 1$ suitably chosen,

$$x_\alpha\in\partial\left(M\backslash\Omega_{\alpha,R}\right)\quad\text{or}\quad r_\alpha(x_\alpha)\ge\delta_\varepsilon\ . \tag{1.58}$$

We prove (**??**) by contradiction. We assume $x_\alpha\notin\partial\left(M\backslash\Omega_{\alpha,R}\right)$ and $r_\alpha(x_\alpha)<\delta_\varepsilon$. We have that

$$\frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}\ge\frac{\Delta_g\psi_{\alpha,\varepsilon}(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \tag{1.59}$$

and by direct computations, using standard properties of the Green's function G_1' such as its control by the distance to the pole, there also holds that since $0<\varepsilon<\frac{1}{2}$, there exist $C_0(\varepsilon),C_1(\varepsilon)>0$ such that

$$r_\alpha(x_\alpha)^2\frac{\Delta_g\psi_{\alpha,\varepsilon}(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)}\ge C_0(\varepsilon)-C_1(\varepsilon)r_\alpha(x_\alpha)^2\ . \tag{1.60}$$

By (**??**) and Lemma **??**,

$$\begin{aligned}r_\alpha(x_\alpha)^2\frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}&\le r_\alpha(x_\alpha)^2\frac{u_\alpha(x_\alpha)^{2^\sharp-1}}{v_\alpha(x_\alpha)}+r_\alpha(x_\alpha)^2\tilde{c}_\alpha\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}\\&\le Cr_\alpha(x_\alpha)^2u_\alpha(x_\alpha)^{\frac{2^\sharp-2}{2}}+Cr_\alpha(x_\alpha)^2\end{aligned} \tag{1.61}$$

for all α, where $\tilde{c}_\alpha=\frac{b_\alpha^2}{4}-c_\alpha$ and $C>0$ does not depend on α. By Lemma **??**,

$$\begin{aligned}\left(r_\alpha(x_\alpha)^2u_\alpha(x_\alpha)^{\frac{2^\sharp-2}{2}}\right)^2&\le Cr_\alpha(x_\alpha)^4+C\sum_i r_\alpha(x_\alpha)^4B_\alpha^i(x_\alpha)^{2^\sharp-2}+o(1)\\&\le C\delta_\varepsilon^4+\varepsilon_R\ ,\end{aligned} \tag{1.62}$$

where $\varepsilon_R\to 0$ as $R\to+\infty$. By (**??**)–(**??**) we get a contradiction by choosing $\delta_\varepsilon\ll 1$ and $R\gg 1$ sufficiently small. This proves (**??**). Now that we have (**??**), up to increasing R, we claim that thanks to Lemma **??**,

$$\max_{M\backslash\Omega_{\alpha,R}}\frac{v_\alpha}{\psi_{\alpha,\varepsilon}}\le C_\varepsilon\ . \tag{1.63}$$

Indeed, if $r_\alpha(x_\alpha)\ge\delta_\varepsilon$, then

$$\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)}\le\frac{v_\alpha(x_\alpha)}{\eta_\alpha(\delta_\varepsilon)}\left(\sum_i G_1'(x_{i,\alpha},x_\alpha)^{\varepsilon}\right)^{-1}\le C\ ,$$

while if $x_\alpha \in \partial\left(M\backslash\Omega_{\alpha,R}\right)$, we get that

$$\begin{aligned} v_\alpha(x_\alpha) &= v_\alpha\left(\exp_{x_{i,\alpha}}(\mu_{i,\alpha}z_\alpha)\right) \\ &= \mu_{i,\alpha}^{-\frac{n}{2}}\left(\Delta_{g_\alpha}(R^{\mu_{i,\alpha}}_{x_{i,\alpha}}u_\alpha) + \frac{b_\alpha\mu_{i,\alpha}^2}{2}(R^{\mu_{i,\alpha}}_{x_{i,\alpha}}u_\alpha)\right)(z_\alpha) \end{aligned}$$

for some i, where z_α is such that $x_\alpha = \exp_{x_{i,\alpha}}(\mu_{i,\alpha}z_\alpha)$, and $g_\alpha(x) = \left(\exp^\star_{x_{i,\alpha}} g\right)(\mu_{i,\alpha}x)$. Then, by Lemma **??**, and standard properties of G'_1,

$$\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \leq C\frac{\mu_{i,\alpha}^{-\frac{n}{2}}\mu_{i,\alpha}^{(n-2)(1-\varepsilon)}}{\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon}} \leq C\left(\frac{\mu_{i,\alpha}}{\mu_\alpha}\right)^{\frac{n-4}{2}-(n-2)\varepsilon} \leq C$$

up to choosing $R \gg 1$ such that $\partial B_0(R)\bigcap\mathcal{S}_{i,r} = \emptyset$, where $\mathcal{S}_{i,r}$ is as in (**??**). In particular, we get that (**??**) holds true. Noting that $G'_1(x_{i,\alpha},x) \leq Cr_\alpha(x)^{-(n-2)}$, this ends the proof of the estimate on the v_α's in (**??**).

(2) *Proof of the estimate on the u_α's in* (**??**). We fix $0 < \varepsilon < \frac{1}{n-2}\min(2, n-4)$. Let G'_2 be the Green's function of $\Delta_g + \frac{b}{4}$. Let $(x_\alpha)_\alpha$ be an arbitrary sequence of points such that $x_\alpha \in M\backslash\Omega_{\alpha,R}$ for all α, where $R > 0$ is to be chosen later on. There holds that

$$\begin{aligned} u_\alpha(x_\alpha) &= \int_M G'_2(x_\alpha,x)\left((\Delta_g+\frac{b}{4})u_\alpha\right)(x)dv_g(x) \\ &\leq \int_M G'_2(x_\alpha,x)v_\alpha(x)dv_g(x) \\ &\leq \int_{M\backslash\Omega_{\alpha,R_\varepsilon}} G'_2(x_\alpha,x)v_\alpha(x)dv_g(x) \\ &\quad + \sum_i\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} G'_2(x_\alpha,x)v_\alpha(x)dv_g(x) \end{aligned} \tag{1.64}$$

for all α, where R_ε is the radius obtained when proving the estimate on the v_α's in (**??**). We have that $G'_2(x_\alpha,x) \leq Cd_g(x_\alpha,x)^{2-n}$. Hence, by Giraud's lemma,

$$\begin{aligned} &\int_{M\backslash\Omega_{\alpha,R_\varepsilon}} G'_2(x_\alpha,x)r_\alpha(x)^{(2-n)(1-\varepsilon)}dv_g(x) \\ &\leq C\sum_i\int_M d_g(x_\alpha,x)^{2-n}d_g(x_{i,\alpha},x)^{2+(n-2)\varepsilon-n}dv_g(x) \\ &\leq C\sum_i d_g(x_{i,\alpha},x_\alpha)^{4-n+(n-2)\varepsilon} \end{aligned} \tag{1.65}$$

since $\varepsilon < \frac{n-4}{n-2}$. Now we fix $R > 0$ such that $R \geq 2R_\varepsilon$. Then $d_g(x_\alpha,x) \geq \frac{1}{2}d_g(x_{i,\alpha},x)$ for all $x \in B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$, and we get that

$$\begin{aligned} &\int_{B_{x_{i,\alpha}}(R_\varepsilon)} G'_2(x_\alpha,x)v_\alpha(x)dv_g(x) \\ &\leq Cd_g(x_{i,\alpha}x_\alpha)^{2-n}\int_{B_{x_{i,\alpha}}(R_\varepsilon)} v_\alpha dv_g \\ &\leq Cd_g(x_{i,\alpha}x_\alpha)^{2-n}\mathrm{Vol}_g\left(B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})\right)^{1/2}\|v_\alpha\|_{L^2(M)} \\ &\leq Cd_g(x_{i,\alpha}x_\alpha)^{2-n}\mu_{i,\alpha}^{n/2} \end{aligned} \tag{1.66}$$

since the v_α's are bounded in L^2. At last, still by Giraud's lemma,

$$\begin{aligned}&\int_{M\backslash\Omega_{\alpha,R_\varepsilon}} G_2'(x_\alpha,x) r_\alpha(x)^{(2-n)\varepsilon} dv_g(x)\\ &\le C\sum_i\int_M d_g(x_\alpha,x)^{2-n} d_g(x_{i,\alpha},x)^{(2-n)\varepsilon}dv_g(x)\\ &\le C\end{aligned} \tag{1.67}$$

since $0<\varepsilon<\frac{2}{n-2}$. Combining (**??**)–(**??**) with (**??**), thanks to the estimate on the v_α's in (**??**), we get that

$$\begin{aligned}u_\alpha(x_\alpha) \le{}& C\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon}\sum_i d_g(x_{i,\alpha},x_\alpha)^{4-n+(n-2)\varepsilon}\\ &+C\sum_i \mu_{i,\alpha}^{n/2} d_g(x_{i,\alpha}x_\alpha)^{2-n}+C\eta_\alpha(\delta_\varepsilon)\end{aligned} \tag{1.68}$$

There holds that

$$\begin{aligned}&\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} d_g(x_{i,\alpha},x_\alpha)^{4-n+(n-2)\varepsilon}+\mu_{i,\alpha}^{n/2}d_g(x_{i,\alpha}x_\alpha)^{2-n}\\ &=\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} d_g(x_{i,\alpha},x_\alpha)^{4-n+(n-2)\varepsilon}\\ &\quad\times\left(1+d_g(x_{i,\alpha},x_\alpha)^{-2-(n-2)\varepsilon}\mu_{i,\alpha}^{n/2}\mu_\alpha^{-\frac{n-4}{2}+(n-2)\varepsilon}\right)\end{aligned}$$

and

$$d_g(x_{i,\alpha},x_\alpha)^{-2-(n-2)\varepsilon}\mu_{i,\alpha}^{n/2}\mu_\alpha^{-\frac{n-4}{2}+(n-2)\varepsilon}\le C\left(\frac{\mu_{i,\alpha}}{\mu_\alpha}\right)^{\frac{n-4}{2}-(n-2)\varepsilon}\le C$$

since $d_g(x_{i,\alpha},x_\alpha)\ge R\mu_{i,\alpha}$. Coming back to (**??**) we get that the estimate on the u_α's in (**??**) holds true. This ends the proof of the lemma. □

Thanks to the estimates in Lemma **??** we can now prove Proposition **??**.

Proof of Proposition **??**. Consider the estimates: there exist $C>0$, $R>0$ and $\delta>0$ such that, up to a subsequence,

$$\left|\nabla^j u_\alpha\right|\le C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j}+\eta_\alpha(\delta)^{2^\sharp-1}\right) \tag{1.69}$$

in $M\backslash\Omega_{\alpha,R}$ for all $j=0,1,2,3$, and all α, where $\Omega_{\alpha,R}=\bigcup_i B_{x_{i,\alpha}}(R\mu_\alpha)$. We prove Proposition **??** by proving first these estimates, then by proving that we can replace $\eta_\alpha(\delta)^{2^\sharp-1}$ by $\|u_\infty\|_{L^\infty}$ in (**??**), and at last by proving that the estimates hold in the whole of M.

(1) *Proof of* (**??**). Let G_α be the Green's function of the fourth order operator $\Delta_g^2+b_\alpha\Delta_g+c_\alpha$. By Lemma **??**, given $0<\varepsilon\ll 1$, there exist $R_\varepsilon, C_\varepsilon,\delta_\varepsilon>0$ such that

$$u_\alpha\le C_\varepsilon\left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{4-n+(n-2)\varepsilon}+\eta_\alpha(\delta_\varepsilon)\right) \tag{1.70}$$

in $M\backslash\bigcup_{i=1}^k B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$. Since $\mu_{i,\alpha}\le\mu_\alpha$ by the definition of μ_α, there holds that $M\backslash\Omega_{\alpha,R}\subset M\backslash\bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_{i,\alpha})$. Let $(x_\alpha)_\alpha$ be a sequence in $M\backslash\Omega_{\alpha,R}$. We aim at proving that there exists $C,\delta>0$ such that, up to a subsequence,

$$\left|\nabla^j u_\alpha(x_\alpha)\right|\le C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha(x_\alpha)^{4-n-j}+\eta_\alpha(\delta)^{2^\sharp-1}\right) \tag{1.71}$$

for all α. Let $R = 2R_\varepsilon + 1$ and $\delta = \delta_\varepsilon$. We have that

$$u_\alpha(x) = \int_M G_\alpha(x,y) u_\alpha(y)^{2^\sharp - 1} dv_g(y) \tag{1.72}$$

for all α and $x \in M$. By combining (**??**) and (**??**) we then get that

$$\begin{aligned} \left|\nabla^j u_\alpha(x_\alpha)\right| &\le \int_M \left|\nabla_x^j G_\alpha(x_\alpha, x)\right| u_\alpha(x)^{2^\sharp-1} dv_g(x) \\ &\le C \int_M d_g(x_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\sharp-1} dv_g(x) \\ &\le \int_{M\setminus\Omega_{\alpha,R_\varepsilon}} d_g(x_\alpha,x)^{4-n-j} u_\alpha(x)^{2^\sharp-1} dv_g(x) \\ &\quad + \int_{\Omega_{\alpha,R_\varepsilon}} d_g(x_\alpha,x)^{4-n-j} u_\alpha(x)^{2^\sharp-1} dv_g(x) \end{aligned} \tag{1.73}$$

Let $k_n = (2^\sharp - 1)(n-2)$. By (**??**),

$$\begin{aligned} &\int_{M\setminus\Omega_{\alpha,R_\varepsilon}} d_g(x_\alpha,x)^{4-n-j} u_\alpha(x)^{2^\sharp-1} dv_g(x) \\ &\le C\mu_\alpha^{\frac{n+4}{2} - k_n\varepsilon} \sum_i A_\alpha^i + C\eta_\alpha(\delta_\varepsilon)^{2^\sharp-1} , \end{aligned} \tag{1.74}$$

where

$$A_\alpha^i = \int_{M\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x) .$$

Let $K_{i,\alpha} = \left\{x \text{ s.t. } d_g(x_\alpha,x) \le \frac{1}{2} d_g(x_{i,\alpha},x_\alpha)\right\}$. Then

$$\begin{aligned} A_\alpha^i &\le \int_{K_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x) \\ &\quad + \int_{K_{i,\alpha}^c\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x) \\ &= A_{1,\alpha}^i + A_{2,\alpha}^i . \end{aligned} \tag{1.75}$$

By the definition of $K_{i,\alpha}$, there holds that $d_g(x_\alpha,x) \le d_g(x_{i,\alpha},x)$ in $K_{i,\alpha}$. Hence, choosing $\varepsilon \ll 1$ sufficiently small such that $4 - k_n\varepsilon > 0$, we can write that

$$\begin{aligned} &\mu_\alpha^{4-k_n\varepsilon} \int_{K_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x) \\ &\le \int_{K_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_\alpha,x)^{4-n-j-\theta} d_g(x_{i,\alpha},x)^{\theta-n} \left(\frac{d_g(x_\alpha,x)}{d_g(x_{i,\alpha},x)}\right)^\theta dv_g(x) , \end{aligned}$$

where $0 < \theta \ll 1$ is chosen small, and by Giraud's lemma we get that

$$A_{1,\alpha}^i \le C\mu_\alpha^{-4+k_n\varepsilon} d_g(x_{i,\alpha},x_\alpha)^{4-n-j} . \tag{1.76}$$

In $K_{i,\alpha}^c$ there holds that $d_g(x_\alpha,x) \ge \frac{1}{2} d_g(x_{i,\alpha},x_\alpha)$, and we can directly write that

$$\begin{aligned} A_{2,\alpha}^i &\le C d_g(x_{i,\alpha},x_\alpha)^{4-n-j} \int_{K_{i,\alpha}^c\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x) \\ &\le C\mu_\alpha^{-4+k_n\varepsilon} d_g(x_{i,\alpha},x_\alpha)^{4-n-j} . \end{aligned} \tag{1.77}$$

At last, since $R \geq 2R_\varepsilon + 1$, we can write that $d_g(x_\alpha, x) \geq \frac{1}{2} d_g(x_{i,\alpha}, x_\alpha)$ for all α and all $x \in B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)$. Hence, by Hölder's inequality, for any i,

$$
\begin{aligned}
&\int_{B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} d_g(x_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\sharp - 1} dv_g(x) \\
&\leq C d_g(x_\alpha, x_{i,\alpha})^{4-n-j} \int_{B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} u_\alpha(x)^{2^\sharp - 1} dv_g(x) \\
&\leq C d_g(x_\alpha, x_{i,\alpha})^{4-n-j} \mathrm{Vol}_g\left(B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)\right)^{1 - \frac{2^\sharp - 1}{2^\sharp}} \|u_\alpha\|_{L^{2^\sharp}}^{2^\sharp - 1} \\
&\leq C \mu_\alpha^{\frac{n-4}{2}} r_\alpha(x_\alpha)^{4-n-j} .
\end{aligned}
\tag{1.78}
$$

Combining (**??**)–(**??**) we then get that (**??**) holds true. This proves (**??**).

(2) *Proof that* (**??**) *holds true with* $\|u_\infty\|_{L^\infty}$ *instead of* $\eta_\alpha(\delta)^{2^\sharp - 1}$. If $u_\infty \not\equiv 0$ there is nothing to do since, by Lemma **??**, $\eta_\alpha(\delta) \leq C$. Now we prove that

$$
\eta_\alpha(\delta) \leq C \mu_\alpha^{(n-4)/2} \tag{1.79}
$$

in case $u_\infty \equiv 0$. This is sufficient to conclude to the validity of (**??**) with $\|u_\infty\|_{L^\infty}$ instead of $\eta_\alpha(\delta)^{2^\sharp - 1}$. We assume in what follows that $u_\infty \equiv 0$ and we define $\Omega_\alpha(\delta) = \bigcup_i B_{x_{i,\alpha}}(\delta)$. By (**??**),

$$
\max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha \leq C \mu_\alpha^{\frac{n-4}{2}} + C \eta_\alpha(\delta)^{2^\sharp - 1} , \tag{1.80}
$$

while, by (**??**), we can write that

$$
\max_{M \backslash \Omega_\alpha(\delta)} v_\alpha \leq C \max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha + C \|v_\alpha\|_{L^1} . \tag{1.81}
$$

Let G_3' be the Green's function of $\Delta_g + b$. There exists $\Lambda > 0$ such that $G_3' \geq \Lambda$ in M and since $v_\alpha \leq (\Delta_g + b) u_\alpha$ for $\alpha \gg 1$, we get with Green's representation formula that

$$
\Lambda \|v_\alpha\|_{L^1} \leq \int_M G_3'(x,y) v_\alpha(y) dv_g(y) \leq u_\alpha(x)
$$

for all α and all x. Therefore, thanks to (**??**),

$$
\eta_\alpha(\delta) \leq C \max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha . \tag{1.82}
$$

By Lemma **??**, there holds that $\max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha \to 0$ as $\alpha \to +\infty$. In particular $\eta_\alpha(\delta) \to 0$ as $\alpha \to +\infty$, and since by (**??**) and (**??**),

$$
\max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha \leq C \mu_\alpha^{\frac{n-4}{2}} + C \eta_\alpha(\delta)^{2^\sharp - 2} \max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha ,
$$

we get that

$$
\max_{M \backslash \Omega_\alpha(\delta/2)} u_\alpha \leq C \mu_\alpha^{\frac{n-4}{2}} . \tag{1.83}
$$

The existence of $C > 0$ such that (**??**) holds true follows from (**??**) and (**??**).

(3) *Proof that the estimates are global in* M. According to the preceding discussion, the estimates (**??**) hold in $M \backslash \Omega_{\alpha,R}$ for some $R > 0$. We are left with the proof that they also hold in $\Omega_{\alpha,R}$. By Lemmas **??** and **??**,

$$
r_\alpha^{\frac{n-4}{2} + j} |\nabla^j u_\alpha| \leq C
$$

in M for all $j = 0,1,2,3$. Noting that $r_\alpha^{-\frac{n-2}{4}-j} \le C r_\alpha^{4-n-j}$ in $\Omega_{\alpha,R}$, this ends the proof of the proposition. □

2. Proof of Theorem ??

We prove Theorem **??** by contradiction. We assume that (M,g) is conformally flat of dimension $n \ge 5$. We let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, $c - \frac{b^2}{4} < 0$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). The Pohozaev identity for fourth order equations can be written as follows: for any smooth bounded domain $\Omega \subset \mathbb{R}^n$, and any $u \in C^4(\overline{\Omega})$,

$$
\begin{aligned}
&\int_\Omega \left(x^k \partial_k u\right) \Delta^2 u dx + \frac{n-4}{2}\int_\Omega u \Delta^2 u dx \\
&= \frac{n-4}{2}\int_{\partial\Omega}\left(-u\frac{\partial \Delta u}{\partial \nu} + \frac{\partial u}{\partial \nu}\Delta u\right) d\sigma \\
&+ \int_{\partial\Omega}\left(\frac{1}{2}(x,\nu)(\Delta u)^2 - (x,\nabla u)\frac{\partial \Delta u}{\partial \nu} + \frac{\partial (x,\nabla u)}{\partial \nu}\Delta u\right) d\sigma \ ,
\end{aligned}
\tag{2.1}
$$

where ν is the outward unit normal to $\partial\Omega$ and $d\sigma$ is the Euclidean volume element on $\partial\Omega$. A preliminary lemma we prove is concerned with the Pohozaev identity, applied to the u_α's, in balls of radii $\sqrt{\mu_\alpha}$, where μ_α is as in (**??**). Without loss of generality, up to passing to a subsequence, we can suppose that $\mu_\alpha = \mu_{1,\alpha}$ for all α. Then we let $x_\alpha = x_{1,\alpha}$ for all α. We say x_α is the blow-up point associated with μ_α. The meaning of $\sqrt{\mu_\alpha}$ in this section is that it is precisely the distance up to which a bubble singularity like in (**??**), with $x_{i,\alpha} = x_\alpha$ and $\mu_{i,\alpha} = \mu_\alpha$, interact in the L^∞-topology. Namely, for such a B_α,

$$
\lim_{\alpha \to +\infty} \max_{M \setminus B_{x_\alpha}(R\sqrt{\mu_\alpha})} B_\alpha = \varepsilon_R \ ,
$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. In particular, $\max_{\partial B_{x_\alpha}(\delta_\alpha)} B_\alpha \to 0$ as $\alpha \to +\infty$ for any sequence $(\delta_\alpha)_\alpha$ of positive real numbers such that $\frac{\delta_\alpha}{\sqrt{\mu_\alpha}} \to +\infty$.

Lemma 2.1. *Let (M,g) be a smooth compact conformally flat Riemannian manifold of dimension $n \ge 5$, and $b,c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). There exists $\delta > 0$ and $K(u_\infty) \ge 0$ such that $K(u_\infty) > 0$ if $u_\infty \not\equiv 0$, and*

$$
\begin{aligned}
&\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} \left(A_g - b_\alpha g\right)(\nabla u_\alpha, \nabla u_\alpha) dv_g \\
&= o\left(\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g\right) - \left(K(u_\infty) + o(1)\right)\mu_\alpha^{\frac{n-4}{2}} \ .
\end{aligned}
\tag{2.2}
$$

*for all α, where A_g is as in (**??**), μ_α is as in (**??**), and x_α is the blow-up point associated with μ_α.*

Proof of Lemma **??**. Let $\overline{u}_\alpha$ be defined in bounded subsets of $\mathbb{R}^n$ by

$$
\overline{u}_\alpha(x) = u_\alpha\left(\exp_{x_\alpha}(\sqrt{\mu_\alpha} x)\right) \ . \tag{2.3}
$$

By Hebey, Robert and Wen [?], there exist $\delta > 0$, $A > 0$, and a biharmonic function $\hat{\varphi} \in C^4(B_0(2\delta))$ such that, up to a subsequence,

$$\overline{u}_\alpha(x) \to \frac{A}{|x|^{n-4}} + \hat{\varphi}(x) \tag{2.4}$$

in $C^3_{loc}(B_0(2\delta)\backslash\{0\})$ as $\alpha \to +\infty$, with the property that $\hat{\varphi}$ is nonnegative, and even positive in $B_0(2\delta)$ if $u_\infty \not\equiv 0$. Moreover, there also holds that for any α,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g = o(1) \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \,, \tag{2.5}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. Now we let x_∞ be the limit of the x_α's and let $\delta_0 > 0$ and $\hat{g}$ be such that $\hat{g}$ is flat in $B_{x_\infty}(4\delta_0)$. We write that $g = \varphi^{4/(n-4)}\hat{g}$ with $\varphi(x_\infty) = 1$, and let $\hat{u}_\alpha = u_\alpha\varphi$. Define

$$B_\alpha = \frac{4b_\alpha}{n-4}\varphi^{\frac{8-n}{n-4}}\hat{g} + \varphi^{\frac{12-n}{n-4}}A_g$$

and

$$\begin{aligned} h_\alpha = & \, b_\alpha\varphi^{\frac{2}{n-4}}\Delta_{\hat{g}}\varphi^{\frac{2}{n-4}} - \frac{n-2}{4(n-1)}b_\alpha\varphi^{\frac{8}{n-4}}S_g + c_\alpha\varphi^{\frac{8}{n-4}} \\ & - \frac{n-4}{2}Q_g\varphi^{\frac{8}{n-4}} + \varphi^{\frac{n+4}{n-4}}\mathrm{div}_g(A_g d\varphi^{-1}) \,, \end{aligned}$$

where Q_g is the Q-curvature of g and A_g is as in (**??**). By conformal invariance of the geometric Paneitz operator in the left hand side of (**??**), there holds that

$$\begin{aligned} & \Delta^2\hat{u}_\alpha + b_\alpha\varphi^{\frac{4}{n-4}}\Delta\hat{u}_\alpha - B_\alpha(\nabla\varphi, \nabla\hat{u}_\alpha) + h_\alpha\hat{u}_\alpha \\ & \quad + \varphi^{\frac{n+4}{n-4}}\mathrm{div}_g(\varphi^{-1}A_g d\hat{u}_\alpha) = \hat{u}_\alpha^{2^\sharp-1} \end{aligned} \tag{2.6}$$

in $B_{x_\infty}(4\delta)$, where A_g is as in (**??**), B_α, and h_α are as above, and $\Delta = \Delta_{\hat{g}}$ is the Euclidean Laplacian. As a remark, (**??**) can be rewritten as

$$\Delta^2\hat{u}_\alpha + \varphi^{\frac{4}{n-4}}\mathrm{div}_\xi\left((A_g - b_\alpha g)\, d\hat{u}_\alpha\right) + \cdots = \hat{u}^{2^\sharp-1} \,,$$

where the dots represent lower order terms. Now we let $\delta > 0$ be sufficiently small. We regard $\hat{u}_\alpha$ as a function in the Euclidean space and assimilate x_α to 0 thanks to the exponential map $\exp_{x_\alpha}$ with respect to g. With an abusive use of notations, we still denote by φ the function $\varphi \circ \exp_{x_\alpha}$, by A_g the tensor field $(\exp_{x_\alpha})^\star A_g$, and by $\hat{g}$ the metric $(\exp_{x_\alpha})^\star\hat{g}$. Applying the Pohozaev identity (**??**) to the $\hat{u}_\alpha$'s in $B_0(\delta\sqrt{\mu_\alpha})$ we get that

$$\begin{aligned} & \int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k\partial_k\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx + \frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha\Delta^2\hat{u}_\alpha dx \\ & = \frac{n-4}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(-\hat{u}_\alpha\frac{\partial\Delta\hat{u}_\alpha}{\partial\nu} + \frac{\partial\hat{u}_\alpha}{\partial\nu}\Delta\hat{u}_\alpha\right)d\sigma \\ & + \int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(\frac{1}{2}(x,\nu)(\Delta\hat{u}_\alpha)^2 - (x,\nabla\hat{u}_\alpha)\frac{\partial\Delta\hat{u}_\alpha}{\partial\nu} + \frac{\partial(x,\nabla\hat{u}_\alpha)}{\partial\nu}\Delta\hat{u}_\alpha\right)d\sigma \,. \end{aligned} \tag{2.7}$$

Integrating by parts, using (**??**), we can also write that

$$\begin{aligned}&\int_{B_0(\delta\sqrt{\mu_\alpha})}\left(x^k\partial_k\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx+\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha\Delta^2\hat{u}_\alpha dx\\&=b_\alpha\int_{B_0(\delta\sqrt{\mu_\alpha})}\varphi^{\frac{4}{n-4}}|\nabla\hat{u}_\alpha|^2dx-\int_{B_0(\delta\sqrt{\mu_\alpha})}\varphi^{\frac{8}{n-4}}A_g(\nabla\hat{u}_\alpha,\nabla\hat{u}_\alpha)dx\\&\quad+o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx\right)+O\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right)\\&\quad+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2(1+\hat{u}_\alpha^{2^\sharp-2})dx\right)+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx\right),\end{aligned}\tag{2.8}$$

where, in this equation, as already mentioned, we regard φ and A_g as defined in the Euclidean space. The proof of (**??**) involves only straightforward computations. By (**??**),

$$\begin{aligned}&\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2(1+\hat{u}_\alpha^{2^\sharp-2})dx=o\left(\mu_\alpha^{\frac{n-4}{2}}\right),\text{ and}\\&\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx=o\left(\mu_\alpha^{\frac{n-4}{2}}\right)\end{aligned}\tag{2.9}$$

while, by (**??**),

$$\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx=o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx\right).\tag{2.10}$$

Independently, we can also write with the change of variables $x=\sqrt{\mu_\alpha}y$ and (**??**) that if R_α stands for the right hand side in (**??**), then

$$\begin{aligned}\mu_\alpha^{-\frac{n-4}{2}}R_\alpha\to&\frac{n-4}{2}\int_{\partial B_0(\delta)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(\delta)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\end{aligned}\tag{2.11}$$

as $\alpha\to+\infty$, where

$$\tilde{u}(x)=\frac{A}{|x|^{n-4}}+\hat{\varphi}(x)\tag{2.12}$$

is given by (**??**) (so that $\Delta^2\hat{\varphi}=0$). Coming back to the Pohozaev identity (**??**), taking $\Omega=B_0(\delta)\backslash B_0(r)$, and since $\Delta^2\tilde{u}=0$ in Ω, it comes that

$$\begin{aligned}&\frac{n-4}{2}\int_{\partial B_0(\delta)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(\delta)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&=\frac{n-4}{2}\int_{\partial B_0(r)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(r)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\end{aligned}\tag{2.13}$$

for all $r > 0$. Combining (**??**), (**??**), and (**??**), letting $r \to 0$, we then get that

$$\mu_\alpha^{-\frac{n-4}{2}} R_\alpha \to K(u_\infty) \tag{2.14}$$

as $\alpha \to +\infty$, where $K(u_\infty) = (n-2)(n-4)^2 \omega_{n-1} A \hat{\varphi}(0)$. We have that $A > 0$ and we know that $\hat{\varphi}(0) > 0$ if $u_\infty \not\equiv 0$. It follows that $K(u_\infty) > 0$ if $u_\infty \not\equiv 0$. By combining (**??**)–(**??**), and (**??**), we can write that

$$\begin{aligned} & b_\alpha \int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx - \int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{8}{n-4}} A_g(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx \\ & = o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|^2 dx \right) + \big(K(u_\infty) + o(1) \big) \mu_\alpha^{\frac{n-4}{2}} , \end{aligned} \tag{2.15}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. The norm of $\nabla \hat{u}_\alpha$ in the first term of (**??**) is with respect to the Euclidean metric $\hat{g} = \xi$. Noting that $|\nabla u|^2_{\hat{g}} = \varphi^{4/(n-4)} |\nabla u|^2_g$, it follows from (**??**) that

$$\begin{aligned} & \int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{8}{n-4}} \big(A_g - b_\alpha g \big)(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx \\ & = o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|^2 dx \right) - \big(K(u_\infty) + o(1) \big) \mu_\alpha^{\frac{n-4}{2}} \end{aligned}$$

an equation from which we easily get with (**??**) that

$$\begin{aligned} & \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} \big(A_g - b_\alpha g \big)(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ & = o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \right) - \big(K(u_\infty) + o(1) \big) \mu_\alpha^{\frac{n-4}{2}} . \end{aligned} \tag{2.16}$$

This ends the proof of the lemma. □

Thanks to Lemma **??**, and to the estimates in Section **??**, we can prove Theorem **??**.

Proof of Theorem **??**. By Lemma **??** it suffices to prove that when $n = 5, 6, 7$,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g = o(\mu_\alpha^{\frac{n-4}{2}}) , \tag{2.17}$$

where $\delta > 0$ is as in Lemma **??**. For $\alpha \gg 1$ sufficiently large, we write that

$$\begin{aligned} \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \le & \sum_{i=1}^k \int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 dv_g \\ & + \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha}) \setminus \bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 dv_g , \end{aligned} \tag{2.18}$$

where $R > 0$ is as in Proposition **??**. By the embedding $H^2 \subset H^{1,2^\star}$, where $2^\star = \frac{2n}{n-2}$, the functions $|\nabla u_\alpha|$ are bounded in $L^{2^\star}$. Using Hölder's inequalities it follows that for any i,

$$\int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 dv_g \le C \mu_\alpha^{n(1-\frac{2}{2^\star})} = C\mu_\alpha^2 . \tag{2.19}$$

There holds that $\mu_\alpha^2 = o(\mu_\alpha^{\frac{n-4}{2}})$ when $n = 5, 6, 7$. Independently, thanks to Proposition **??**, we can write that

$$\begin{aligned}&\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 dv_g \\ &\le C\mu_\alpha^{\frac{n}{2}} + C\mu_\alpha^{n-4}\sum_{i=1}^k \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha},\cdot)^{6-2n} dv_g\ .\end{aligned} \tag{2.20}$$

There holds that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha},\cdot)^{6-2n} dv_g \le CS_\alpha \tag{2.21}$$

for all α, where $S_\alpha = 1$ when $n = 5$, $S_\alpha = \ln\frac{1}{\mu_\alpha}$ when $n = 6$, and $S_\alpha = \frac{1}{\mu_\alpha}$ when $n \ge 7$. Combining (**??**)–(**??**) we get (**??**). This ends the proof of Theorem **??**. □

3. Trace estimates

We prove trace estimates in this section. Such estimates are required to prove Theorem **??**. As in Section **??** we do not need to assume here that g is conformally flat. We let (M, g) be a compact Riemannian manifold and let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, where $c - \frac{b^2}{4} < 0$. We let also $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). We aim at proving that if A is a smooth $(2, 0)$-tensor field, the the integral of $A(\nabla u_\alpha, \nabla u_\alpha)$ around the maximum blow-up point x_α behaves like the trace of A at x_∞ times μ_α^2, where $x_\alpha \to x_\infty$ as $\alpha \to +\infty$. In what follows we define $\tilde I_1$ and $\tilde I_2$ to be the subsets of $\{1, \dots, k\}$ given by

$$\begin{aligned}\tilde I_1 &= \Big\{i = 1, \dots, k \text{ s.t. } d_g(x_{i,\alpha}, x_\alpha) = o(1)\Big\}\ , \text{ and} \\ \tilde I_2 &= \Big\{i = 1, \dots, k \text{ s.t. } d_g(x_{i,\alpha}, x_\alpha) = o(\sqrt{\mu_\alpha})\Big\}\ ,\end{aligned} \tag{3.1}$$

where the $x_{i,\alpha}$'s and k are given by the decomposition (**??**), μ_α is as in (**??**), and x_α is the blow-up point associated with μ_α. Namely, assuming that, up to a subsequence, $\mu_\alpha = \mu_{i_0,\alpha}$ for some i_0 and all α, then $x_\alpha = x_{i_0,\alpha}$.

Proposition 3.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 7$, and $b, c > 0$ be positive real numbers such that $c - \frac{b^2}{4} < 0$. Let also $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). Let A be a smooth $(2, 0)$-tensor field. Let $\delta > 0$ be such that $d_g(x_{i,\alpha}, x_\alpha) \ge 2\delta\sqrt{\mu_\alpha}$ for all α and all $i \notin \tilde I_2$, where μ_α is as in (**??**), x_α is the blow-up point associated with μ_α, and $\tilde I_2$ is as in (**??**). Then there exists $\beta > 0$ such that, up to a subsequence,*

$$\lim_{\alpha\to+\infty} \frac{1}{\mu_\alpha^2}\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \beta\, Tr_g(A)(x_\infty)\ , \tag{3.2}$$

where x_∞ is the limit of the x_α's. Similarly, if $u_\infty \equiv 0$, and $\delta > 0$ is such that $d_g(x_{i,\alpha}, x_\alpha) \ge 2\delta$ for all α and all $i \notin \tilde I_1$, where x_α is the blow-up point associated *with μ_α, and $\tilde I_1$ is as in (??), then*

$$\lim_{\alpha\to+\infty}\frac{1}{\mu_\alpha^2}\int_{B_{x_\alpha}(\delta)}A(\nabla u_\alpha,\nabla u_\alpha)dv_g=\beta\, Tr_g(A)(x_\infty)\tag{3.3}$$

for some $\beta>0$, where, here again, x_∞ is the limit of the x_α's.

Proof of Proposition **??**. Let I' be the subset of $\{1,\dots,k\}$ consisting of the i's such that $\mu_\alpha=O(\mu_{i,\alpha})$ and, for i given, let $\hat I_i$ be the subset of $\{1,\dots,k\}$ consisting of the j's such that $d_g(x_{i,\alpha},x_{j,\alpha})=O(\mu_{i,\alpha})$. Given $R>0$ we define $A_{i,\alpha,R}$ to be the annuli type sets

$$A_{i,\alpha,R}=B_{x_{i,\alpha}(R\mu_{i,\alpha})}\backslash\bigcup_{j\in\hat I_i}B_{x_{j,\alpha}}\left(\frac{1}{R}\mu_{i,\alpha}\right).$$

We claim that for any sequences $(\Omega_\alpha)_\alpha$ of domains in M,

$$\begin{aligned}&\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla u_\alpha|^2dv_g\le 2\int_{\Omega_\alpha}|\nabla u_\infty|^2dv_g+o\left(\mathrm{Vol}_g(\Omega_\alpha)^{\frac{2}{n}}\right)+\varepsilon_R\mu_\alpha^2\,,\\&\int_{\Omega_\alpha\backslash\bigcup_{i=1}^kB_{x_{i,\alpha}}(R\mu_\alpha)}|\nabla u_\alpha|^2dv_g\le O\left(\mathrm{Vol}_g(\Omega_\alpha)\right)+\varepsilon_R\mu_\alpha^2\end{aligned}\tag{3.4}$$

for all α, where $\varepsilon_R\to0$ as $R\to+\infty$. First we prove (**??**), then we prove (**??**) and at last we prove (**??**).

(1) *Proof of the first estimate in* (**??**). We use the Sobolev decomposition (**??**). Thanks to (**??**), by the Sobolev embedding $H^2\subset H^{1,2^\star}$, where $2^\star=\frac{2n}{n-2}$, by Höder's inequality, and since $n\ge7$ so that there holds $\mu_\alpha^{n-4}=o(\mu_\alpha^2)$, we can write that

$$\begin{aligned}\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla u_\alpha|^2dv_g&\le2\int_{\Omega_\alpha}|\nabla u_\infty|^2dv_g\\&+\sum_{j=1}^k\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla B_\alpha^j|^2dv_g+o\left(\mathrm{Vol}_g(\Omega_\alpha)^{\frac{2}{n}}\right)\,.\end{aligned}\tag{3.5}$$

Independently, for any j,

$$\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla B_\alpha^j|^2dv_g\le C\mu_{j,\alpha}^2\int_{\mathbb{R}^n\backslash\frac{1}{\mu_{j,\alpha}}K_\alpha}\frac{|x|^2}{(1+|x|^2)^{n-2}}dx+o(\mu_\alpha^2)\,,\tag{3.6}$$

where $K_\alpha=\exp_{x_{j,\alpha}}^{-1}\left(\bigcup_{i\in I'}A_{i,\alpha,R}\right)$. In case $j\notin I'$, then $\mu_{j,\alpha}=o(\mu_\alpha^2)$ and

$$\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla B_\alpha^j|^2dv_g=o(\mu_\alpha^2)\,.\tag{3.7}$$

In case $j\in I'$, then

$$\begin{aligned}&\int_{\Omega_\alpha\backslash\bigcup_{i\in I'}A_{i,\alpha,R}}|\nabla B_\alpha^j|^2dv_g\\&\le C\mu_{j,\alpha}^2\int_{\mathbb{R}^n\backslash\frac{1}{\mu_{j,\alpha}}\exp_{x_{j,\alpha}}^{-1}(A_{j,\alpha,R})}\frac{|x|^2}{(1+|x|^2)^{n-2}}dx+o(\mu_\alpha^2)\\&\le C\mu_{j,\alpha}^2\int_{\mathbb{R}^n\backslash K_R}\frac{|x|^2}{(1+|x|^2)^{n-2}}dx+o(\mu_\alpha^2)\end{aligned}\tag{3.8}$$

where $K_R = B_0(R)\backslash\bigcup_{i\in\hat{I}_j} B_{y_i}(\frac{2}{R})$ and y_i is the limit of the $\frac{1}{\mu_{j,\alpha}}\exp^{-1}_{x_{j,\alpha}}(x_{i,\alpha})$'s. The first estimate in (**??**) clearly follows from (**??**)–(**??**).

(2) *Proof of the second estimate in* (**??**). Here we use Proposition **??**. By (**??**) we can write that

$$\begin{aligned}&\int_{\Omega_\alpha\backslash\bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_\alpha)}|\nabla u_\alpha|^2 dv_g\\&\le C\int_{\Omega_\alpha\backslash\bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_\alpha)}\left(1+\mu_\alpha^{n-4}r_\alpha^{6-2n}\right)dv_g\\&\le C\mathrm{Vol}_g(\Omega_\alpha)+C\mu_\alpha^{n-4}\sum_{j=1}^k\int_{\Omega_\alpha\backslash B_{x_{j,\alpha}}(R\mu_\alpha)}d_g(x_{j,\alpha},x)^{6-2n}dv_g\ ,\end{aligned}\tag{3.9}$$

and there holds that

$$\int_{\Omega_\alpha\backslash B_{x_{j,\alpha}}(R\mu_\alpha)}d_g(x_{j,\alpha},x)^{6-2n}dv_g\le C_1+C_2\mu_\alpha^{6-n}\int_{\mathbb{R}^n\backslash B_0(R)}|x|^{6-2n}dx\ .\tag{3.10}$$

Since $n\ge 7$, the second estimate in (**??**) follows from (**??**) and (**??**). This proves (**??**).

(3) *Proof of* (**??**). Let $K_\alpha^1=\bigcup_{i\in I'}A_{i,\alpha,R}$ and $K_\alpha^2=\bigcup_{i=1}^k B_{x_{i,\alpha}}(R\mu_\alpha)$. By the structure equation, (**??**), $A_{i,\alpha,R}\bigcap A_{j,\alpha,R}=\emptyset$ for all $i\ne j$ in I'. We start by writing that

$$\begin{aligned}&\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\\&=\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\bigcap K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g+\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\backslash K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\\&=\sum_{i\in J}\int_{A_{i,\alpha,R}}A(\nabla u_\alpha,\nabla u_\alpha)dv_g+\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\backslash K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\ ,\end{aligned}\tag{3.11}$$

where $J=I'\bigcap\tilde{I}_2$ and $\tilde{I}_2$ is as in (**??**), and that

$$\begin{aligned}&\left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\backslash K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\right|\\&\le\left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\backslash K_\alpha^2}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\right|+\left|\int_{K_\alpha^2\backslash K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\right|\ .\end{aligned}\tag{3.12}$$

By (**??**),

$$\begin{aligned}&\left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\backslash K_\alpha^2}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\right|\le o(\mu_\alpha^2)+C\varepsilon_R\mu_\alpha^2\ ,\\&\left|\int_{K_\alpha^2\backslash K_\alpha^1}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\right|\le o(\mu_\alpha^2)+C\varepsilon_R\mu_\alpha^2\ ,\end{aligned}\tag{3.13}$$

where $\varepsilon_R\to 0$ as $R\to+\infty$. Combining (**??**)–(**??**), it follows that

$$\begin{aligned}\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}A(\nabla u_\alpha,\nabla u_\alpha)dv_g=&\sum_{i\in J}\int_{A_{i,\alpha,R}}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\\&+o(\mu_\alpha^2)+\varepsilon_R(\alpha)\mu_\alpha^2\ ,\end{aligned}\tag{3.14}$$

where $\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\varepsilon_R(\alpha)=0$. We fix $i\in J$ and define g_α to be the metric in Euclidean space given by $g_\alpha(x)=\left(\exp^\star_{x_{i,\alpha}}g\right)(\mu_{i,\alpha}x)$. For $j\in\hat{I}_i$, we let also $a_{j,\alpha}$ be the point in $\mathbb{R}^n$ given by $a_{j,\alpha}=\mu_{i,\alpha}^{-1}\exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha})$. Since $j\in\hat{I}_i$ there holds that, up to a subsequence, $a_{j,\alpha}\to a_j$ in $\mathbb{R}^n$. We define $\tilde{u}_{i,\alpha}$ to be the function defined in the Euclidean space by $\tilde{u}_\alpha=R^{\mu_{i,\alpha}}_{x_{i,\alpha}}u_\alpha$, where the R^μ_x action is as in (**??**). In other words,

$$\tilde{u}_\alpha(x)=\mu_{i,\alpha}^{\frac{n-4}{2}}u_\alpha\left(\exp_{x_{i,\alpha}}(\mu_{i,\alpha}x)\right)\ .$$

Then

$$\int_{A_{i,\alpha,R}}A(\nabla u_\alpha,\nabla u_\alpha)dv_g=\mu_{i,\alpha}^2\int_{B_0(R)\backslash W_\alpha}A_\alpha(\nabla\tilde{u}_\alpha,\nabla\tilde{u}_\alpha)dv_{g_\alpha}\ , \tag{3.15}$$

where $A_\alpha(x)=\left(\exp^\star_{x_{i,\alpha}}A\right)(\mu_{i,\alpha}x)$, $W_\alpha=\bigcup_{j\in\hat{I}_i}\tilde{B}_{a_{j,\alpha}}(\frac{1}{R})$, and $\tilde{B}_{a_{j,\alpha}}(\frac{1}{R})$ is the ball of center $a_{j,\alpha}$ and radius $1/R$ with respect to g_α. Since $g_\alpha\to\xi$ in the C^4-topology, where ξ is the Euclidean metric, it follows from Lemma **??** and (**??**) that

$$\begin{aligned}&\int_{A_{i,\alpha,R}}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\\&=\mu_{i,\alpha}^2\int_{B_0(R)\backslash\bigcup_{i\in\hat{I}_i}B_{a_j}(\frac{1}{R})}\hat{A}_0(\nabla B,\nabla B)dx+o(\mu_{i,\alpha}^2)\\&=\mu_{i,\alpha}^2\int_{\mathbb{R}^n}\hat{A}_0(\nabla B,\nabla B)dx+o(\mu_{i,\alpha}^2)+\varepsilon_R\mu_\alpha^2\ ,\end{aligned} \tag{3.16}$$

where $\hat{A}_0=\left(\exp^\star_{x_\infty}A\right)(0)$, $a_{j,\alpha}\to a_j$ as $\alpha\to+\infty$, $\varepsilon_R\to 0$ as $R\to+\infty$, and B is as in (**??**). Since B is radially symmetrical, we get from (**??**) that

$$\begin{aligned}&\int_{A_{i,\alpha,R}}A(\nabla u_\alpha,\nabla u_\alpha)dv_g\\&=\mu_{i,\alpha}^2\frac{1}{n}\mathrm{Tr}_g(A)(x_\infty)\int_{\mathbb{R}^n}|\nabla B|^2dx+o(\mu_{i,\alpha}^2)+\varepsilon_R\mu_\alpha^2\ ,\end{aligned} \tag{3.17}$$

and (**??**) follows from (**??**) and (**??**) with $\beta=\left(\sum_{i\in J}\mu_i\right)\int_{\mathbb{R}^n}|\nabla B|^2dx$, where μ_i is the limit of $\frac{\mu_{i,\alpha}}{\mu_\alpha}$ as $\alpha\to+\infty$. Assuming $\mu_\alpha=\mu_{1,\alpha}$ for all α, there holds that $1\in J$ and $\beta>0$. This ends the proof of (**??**).

(4) *Proof of* (**??**). We take advantage of $u_\infty\equiv 0$. By (**??**) in Proposition **??**, and since $n\geq 7$, we get that

$$\begin{aligned}\int_{B_{x_\alpha}(\delta)\backslash K_\alpha^2}|\nabla u_\alpha|^2dv_g&\leq C\mu_\alpha^{n-4}\int_{B_{x_\alpha}(\delta)\backslash K_\alpha^2}r_\alpha(x)^{6-2n}dv_g(x)\\&\leq C\mu_\alpha^{n-4}\sum_{i=1}^k\int_{B_{x_\alpha}(\delta)\backslash B_{x_{i,\alpha}}(R\mu_\alpha)}d_g(x_{i,\alpha},x_\alpha)^{6-2n}dv_g\\&\leq C\mu_\alpha^{n-4}\sum_{i=1}^k\left(1+\mu_\alpha^{6-n}\int_{\mathbb{R}^n\backslash B_0(R)}|x|^{6-2n}dx\right)\\&=o(\mu_\alpha^2)+\varepsilon_R\mu_\alpha^2\ ,\end{aligned} \tag{3.18}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Then, we can write that

$$\begin{aligned}&\int_{B_{x_\alpha}(\delta)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ &= \sum_{i \in J'} \int_{A_{i,\alpha,R}} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + \int_{B_{x_\alpha}(\delta)\backslash K_\alpha^1} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \ ,\end{aligned} \tag{3.19}$$

where $J' = I' \bigcap \tilde{I}_1$ and $\tilde{I}_1$ is as in (**??**), while

$$\begin{aligned}&\left|\int_{B_{x_\alpha}(\delta)\backslash K_\alpha^1} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| \\ &\le \left|\int_{B_{x_\alpha}(\delta)\backslash K_\alpha^2} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| + \left|\int_{K_\alpha^2\backslash K_\alpha^1} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| \ .\end{aligned} \tag{3.20}$$

By (**??**) and (**??**) we then get from (**??**) and (**??**) that

$$\int_{B_{x_\alpha}(\delta)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \sum_{i \in J'} \int_{A_{i,\alpha,R}} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + o(\mu_\alpha^2) + \varepsilon_R \mu_\alpha^2 \ , \tag{3.21}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$, and (**??**) follow from (**??**) and (**??**). This ends the proof of the proposition. □

4. Proof of Theorem ?? when $n \ge 6$

We prove Theorem **??** by contradiction. We assume that (M,g) is conformally flat of dimension $n \ge 6$. We let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). We split the proof in the two cases $n = 6, 7$ and $n \ge 8$.

First we assume $n = 6, 7$. By Theorem **??** we know that $u_\infty \equiv 0$ in (**??**). Let $\mathcal{S} = \{x_1, \dots, x_N\}$ be the geometric blow-up set consisting of the limits of the $x_{i,\alpha}$'s, where the $x_{i,\alpha}$'s are as in Section **??**. Let x_α and μ_α be as in Sections **??** and **??**, μ_α being as in (**??**). We may assume $x_\alpha = x_{1,\alpha}$ for all α. Given $x_i \in \mathcal{S}$, since g is conformally flat, there exists (up to the assimilation of x_i with 0) a smooth positive function $\varphi > 0$ in a neighborhood U of x_i such that $\varphi^{4/(n-4)}\xi = g$ in $U = B_0(\delta_0)$, where ξ is the Euclidean metric. We may also assume $U \cap \mathcal{S} = \{x_i\}$. We define $\hat{u}_\alpha = \varphi u_\alpha$ and apply the Pohozaev identity (**??**) to $\eta_\delta \hat{u}_\alpha$ in $B_0(\delta)$ for $\delta \in (0, \delta_0)$, where $\eta_\delta(x) = \eta(\frac{2}{\delta}x)$ and η is such that $\eta \equiv 1$ in $B_0(1)$ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B_0(4/3)$. By Hebey, Robert and Wen [**?**], there holds that

$$\int_{B_0(\delta)\backslash B_0(\delta/2)} |\nabla^k \hat{u}_\alpha|^2 dx = o(1) \int_M |\nabla u_\alpha|^2 dx \tag{4.1}$$

for all $k = 0, 1, 2$, where $o(1) \to 0$ as $\alpha \to +\infty$, and there also holds since $u_\infty \equiv 0$ that

$$\frac{\int_{\mathcal{B}_\delta} |\nabla u_\alpha|^2 dv_g}{\int_M |\nabla u_\alpha|^2 dv_g} \to 1 \tag{4.2}$$

as $\alpha \to +\infty$, where $\mathcal{B}_\delta = \bigcup_{i=1}^N B_{x_i}(\delta)$. These estimates may be proved directly from Proposition **??**. By (**??**) and (**??**), following the computations in Hebey, Robert and Wen [?], we get from the Pohozaev identity that

$$\left|\int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} \left(A_g - b_\alpha g\right)(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx\right| \le C\left(\varepsilon_\delta + o(1)\right) \int_M |\nabla u_\alpha|^2 dv_g \ , \tag{4.3}$$

where $C > 0$ is independent of α and δ, A_g is as in (**??**), and ε_δ can be made independent of α and such that $\varepsilon_\delta \to 0$ as $\delta \to 0$. When $b \notin \mathcal{S}_w$, $A_g - b_\alpha g$ has a sign for $\alpha \gg 1$ sufficiently large. In particular, coming back to M, summing over $i = 1, \ldots, N$, it follows from (**??**), (**??**) and (**??**) that

$$\int_M |\nabla u_\alpha|^2 dv_g \le C\varepsilon_\delta \int_M |\nabla u_\alpha|^2 dv_g + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \ , \tag{4.4}$$

and we get a contradiction since $\varepsilon_\delta \to 0$ as $\delta \to 0$. This proves Theorem **??** when $n = 6$. When $n = 7$, we consider (**??**) around x_α, namely for $i = 1$. By (**??**), (**??**), and Proposition **??**,

$$\int_M |\nabla u_\alpha|^2 dv_g = O(\mu_\alpha^2) \ . \tag{4.5}$$

By (**??**) in Proposition **??**, (**??**), and (**??**), we then get by letting $\alpha \to +\infty$ and $\delta \to 0$ in (**??**) that $\frac{1}{n}\mathrm{Tr}_g(A_g)(x_\infty) = b$, where x_∞ is the limit of the x_α's. This proves Theorem **??** when $n = 7$.

Now we assume $n \ge 8$. Let x_α and μ_α be as above. By (**??**) and Proposition **??**,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g = O(\mu_\alpha^2) \tag{4.6}$$

for all $\delta > 0$. Applying Lemma **??** and Proposition **??**, it follows that

$$n\left(\frac{1}{n}\mathrm{Tr}_g(A_g)(x_\infty) - b + o(1)\right)\mu_\alpha^2 = -\frac{1}{\beta}\left(K(u_\infty) + o(1)\right)\mu_\alpha^{\frac{n-4}{2}} \ , \tag{4.7}$$

where $K(u_\infty)$ is as in Lemma **??**, $\beta > 0$ is as in Proposition **??**, and $x_\alpha \to x_\infty$ as $\alpha \to +\infty$. Assuming that $n \ge 9$ and $b \ne \frac{1}{n}\mathrm{Tr}_g(A_g)$ in M, the contradiction directly follows from (**??**) since, in that case, $\mu_\alpha^{(n-4)/2} = o(\mu_\alpha^2)$. This proves Theorem **??** when $n \ge 9$. In case $n = 8$ we have that $\mu_\alpha^{(n-4)/2} = \mu_\alpha^2$, and if we assume that $b < \frac{1}{n}\mathrm{Tr}_g(A_g)$ in M, then, again, we directly get a contradiction thanks to (**??**) using the signs of the two terms in (**??**). This ends the proof of Theorem **??**.

5. Proof of Theorem ?? when $n = 5$

We prove Theorem **??** in the 5-dimensional case by contradiction. We assume that (M, g) is conformally flat of dimension $n = 5$. We let $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ be converging sequences of real numbers with limits b and c as $\alpha \to \infty$, and $(u_\alpha)_\alpha$ be a bounded sequence in H^2 of positive nontrivial solutions of (**??**) satisfying (**??**). By Theorem **??** we know that $u_\infty \equiv 0$. We let $\mathcal{S}$ be the geometric blow-up set consisting of the limits of the $x_{i,\alpha}$'s as $\alpha \to +\infty$: $\mathcal{S} = \{x_1, \ldots, x_N\}$, where $N \le k$. In the case of clusters, $N < k$. We prove in what follows that there exist $\lambda_1, \ldots, \lambda_N \ge 0$ such that $\sum_{i=1}^N \lambda_i = 1$ and such that

$$\lambda_i^2 \mu_{x_i}(x_i) + \sum_{j \ne i} \lambda_i \lambda_j G(x_i, x_j) = 0 \tag{5.1}$$

for all $i = 1, \dots, N$, where G is the Green's function of $\Delta_g^2 + b\Delta_g + c$ and μ_x is its regular part as in (**??**). When $c < b^2/4$, which is assumed here, G is given by

$$G(x,y) = \int_M G_1(x,z)G_2(z,y)dv_g(z) \ ,$$

where G_1 (respectively G_2) is the Green's function of the second order Schrödinger operator $\Delta_g + d_1$ (respectively $\Delta_g + d_2$), and d_1, d_2 are as in (**??**) with b and c in place of b_α and c_α. Hence, $G > 0$ and Theorem **??** when $n = 5$ follows from (**??**). Note that (**??**) reduces to $\lambda_i^2 \mu_{x_i}(x_i) = 0$ in case $N = 1$, so that the positivity of the mass is required, in particular in the case of clusters.

We prove (**??**) in the sequel. By Theorem **??** and Proposition **??**, splitting M into the two subsets $\{r_\alpha \le R\mu_\alpha\}$ and $\{r_\alpha \ge R\mu_\alpha\}$, we easily get that there exists $C > 0$ such that, up to a subsequence, $\int_M u_\alpha^{2^\sharp-1}dv_g \le C\mu_\alpha^{1/2}$ for all α. By Lemma **??** we then easily get that there exists $c > 0$ such that, up to a subsequence,

$$\int_M u_\alpha^{2^\sharp-1}dv_g = \big(c + o(1)\big)\mu_\alpha^{\frac{1}{2}} \ . \tag{5.2}$$

Again by Theorem **??** and Proposition **??**, thanks also to (**??**), we get that for any compact subset Ω of $M\backslash\mathcal{S}$,

$$\frac{\int_\Omega u_\alpha^{2^\sharp-1}dv_g}{\int_M u_\alpha^{2^\sharp-1}dv_g} = o(1) \ . \tag{5.3}$$

In what follows we let $\delta_0 = \inf_{i\neq j} d_g(x_i, x_j)$. For $i = 1, \dots, N$, and $\delta \in (0, \delta_0)$, we define

$$\lambda_i = \lim_{\alpha\to+\infty} \frac{\int_{B_{x_i}(\delta)} u_\alpha^{2^\sharp-1}dv_g}{\int_M u_\alpha^{2^\sharp-1}dv_g} \ . \tag{5.4}$$

It follows from (**??**) that λ_i does not depend on δ and that $\sum_i \lambda_i = 1$. Let $\tilde{u}_\alpha$ be given by

$$\tilde{u}_\alpha = \frac{u_\alpha}{\int_M u_\alpha^{2^\sharp-1}dv_g} \ . \tag{5.5}$$

By (**??**) and (**??**) there holds that

$$\Delta_g^2\tilde{u}_\alpha + b_\alpha\Delta_g\tilde{u}_\alpha + c_\alpha\tilde{u}_\alpha = \tilde{\mu}_\alpha^4\tilde{u}_\alpha^{2^\sharp-1} \ ,$$

where $\tilde{\mu}_\alpha = O(\mu_\alpha)$. By Proposition **??** and (**??**) there also holds that for any compact subset $\Omega \subset M\backslash\mathcal{S}$ there exists $C_\Omega > 0$ such that $\tilde{u}_\alpha \le C_\Omega$ in Ω. Then, by standard elliptic theory, there exists $\tilde{u} \in C^4(M\backslash\mathcal{S})$ such that $\tilde{u}_\alpha \to \tilde{u}$ in $C^4_{loc}(M\backslash\mathcal{S})$ as $\alpha \to +\infty$. By Green's representation formula and the estimates in (**??**), we get that $\tilde{u}$ expresses as the sum of the $\lambda_i G_{x_i}$'s, where $G_{x_i} = G(x_i, \cdot)$. Summarizing, up to a subsequence,

$$\tilde{u}_\alpha \to \sum_{i=1}^N \lambda_i G_{x_i} \tag{5.6}$$

in $C^4_{loc}(M\backslash\mathcal{S})$ as $\alpha \to +\infty$, where the λ_i's are as in (**??**) and $\tilde{u}_\alpha$ is given by (**??**).

Now we fix $i \in \big\{1, \dots, N\big\}$. Since g is conformally flat, there exists (up to the assimilation of x_i with 0) a smooth positive function $\varphi > 0$ in a neighborhood U of x_i such that $\varphi^{4/(n-4)}\xi = g$ in $U = B_0(\delta_0)$, where ξ is the Euclidean metric. We may also assume $U \cap \mathcal{S} = \{x_i\}$. Define $\hat{u}_\alpha = \varphi u_\alpha$. Basic Riemannian estimates, going back to the equation for geodesics, yield

$$d_g(0,x) = |x|\varphi(0)^{\frac{2}{n-4}} \left(1 + \frac{1}{n-4}\left(\frac{\nabla\varphi(0)}{\varphi(0)}, x\right) + O(|x|^2)\right) , \tag{5.7}$$

where $(\cdot,\cdot)$ is the Euclidean scalar product. It follows from (**??**), (**??**) and (**??**) that

$$\lim_{\alpha\to+\infty} \frac{\hat{u}_\alpha}{\int_M u_\alpha^{2^\sharp-1} dv_g} = H_i \tag{5.8}$$

in $C^4_{loc}(U\backslash\{0\})$ as $\alpha \to +\infty$, where

$$H_i(x) = \frac{\lambda_i \varphi(x_i)^{-1}}{6\omega_4|x|} + \beta_i(x) \tag{5.9}$$

in $U\backslash\{0\}$, $\beta_i \in C^{0,\theta}(U)$ for $0 < \theta < 1$, β_i is smooth outside 0, and

$$\beta_i(0) = \left(\lambda_i \mu_{x_i}(x_i) + \sum_{j\neq i} \lambda_j G(x_i,x_j)\right) \varphi(x_i) . \tag{5.10}$$

By standard elliptic theory, following arguments as in Druet, Hebey and Vétois [**?**], there also holds that

$$\lim_{r\to0} \sup_{|x|=r} \sum_{k=1}^{3} |x|^k |\nabla^k \beta_i(x)| = 0 . \tag{5.11}$$

In order to prove (**??**) in our context we first note that by (**??**), β_i satisfies an equation like

$$\Delta^2\beta_i + A^{kl}\partial^2_{kl}\beta_i + B^k\partial_k\beta_i + D\beta_i = f_i \tag{5.12}$$

in $U\backslash\{0\}$, where the coefficients A^{kl}, B^k and D are smooth, and where f_i is such that $|f_i(x)| \leq C|x|^{-3}$ in $U\backslash\{0\}$. First, keeping in mind that we aim at proving (**??**), we claim that there exists $C > 0$ such that

$$\sum_{k=1}^{3} |x|^k |\nabla^k \beta_i(x)| \leq C \tag{5.13}$$

in $U\backslash\{0\}$. We argue by contradiction. Suppose that there exists $(x_m)_m$ in $U\backslash\{0\}$ such that $\sum_{k=1}^3 |x_m|^k |\nabla^k \beta_i(x_m)| \to +\infty$ as $m \to +\infty$. Since β_i is smooth in $U\backslash\{0\}$, there holds that $x_m \to 0$ as $m \to +\infty$. Let $\beta_{i,m}(x) = \beta_i(|x_m|x)$. By (**??**), thanks to standard elliptic theory, there exists $\beta \in C^4(\mathbb{R}^n\backslash\{0\})$ such that $\beta_{i,m} \to \beta$ in $C^3_{loc}(\mathbb{R}^n\backslash\{0\})$ as $m \to +\infty$ and $\Delta^2\beta = 0$ in $\mathbb{R}^n\backslash\{0\}$. We have that $|\beta| \leq C$ in $\mathbb{R}^n\backslash\{0\}$ since $\beta_i \in C^{0,\theta}(U)$. Then

$$\sum_{k=1}^{3} |x_m|^k \left|\nabla^k \beta_i(x_m)\right| = \sum_{k=1}^{3} \left|\nabla^k \beta_{i,m}\left(\frac{x_m}{|x_m|}\right)\right| \to \sum_{k=1}^{3} \left|\nabla^k \beta(y)\right| , \tag{5.14}$$

where y is the limit of the points $\frac{x_m}{|x_m|}$ as $m \to +\infty$. A contradiction, and this proves (**??**). Now we prove (**??**). Here again we argue by contradiction. We assume there exists $(x_m)_m$ in $U\backslash\{0\}$ such that

$$\sum_{k=1}^{3} |x_m|^k |\nabla^k \beta_i(x_m)| \geq C \tag{5.15}$$

for all m and some $C > 0$, and such that $x_m \to 0$ as $m \to +\infty$. We define $\beta_{i,m}$ as above. Then we get the existence of $\beta \in C^4(\mathbb{R}^n\backslash\{0\}$ such that $\beta_{i,m} \to \beta$ in $C^3_{loc}(\mathbb{R}^n\backslash\{0\})$ as $m \to +\infty$ and $\Delta^2\beta = 0$ in $\mathbb{R}^n\backslash\{0\}$. By (**??**), there holds that $\Delta^2\beta = 0$ in $\mathbb{R}^n$ in the sense of distributions and not only outside 0. Then β is smooth and, necessarily, see Adimurthi, Robert and Struwe [**?**], we get that $\beta \equiv C^{st}$ is a constant. Coming back to (**??**), we get a contradiction with (**??**). This proves (**??**).

From now on, given $\delta \in (0, \delta_0)$, we define

$$\begin{aligned} A_\delta = &-\frac{1}{2}\int_{\partial B_0(\delta)}\left(H_i\frac{\partial \Delta H_i}{\partial\nu} - \frac{\partial H_i}{\partial \nu}\Delta H_i\right)d\sigma + \frac{1}{2}\int_{\partial B_0(\delta)}(x,\nu)(\Delta H_i)^2 d\sigma \\ &-\int_{\partial B_0(\delta)}(x,\nabla H_i)\frac{\partial\Delta H_i}{\partial\nu}d\sigma + \int_{\partial B_0(\delta)}\frac{\partial(x,\nabla H_i)}{\partial\nu}\Delta H_i d\sigma\ , \end{aligned} \tag{5.16}$$

where ν is the unit outward normal to $\partial B_0(\delta)$ and H_i is as in (**??**)-(**??**). By (**??**) and (**??**),

$$\lim_{\delta\to 0} A_\delta = \frac{\lambda_i\varphi(x_i)^{-1}}{2}\beta_i(0)\ . \tag{5.17}$$

Independently, applying the Pohozaev identity (**??**) to $\hat{u}_\alpha$ in $B_0(\delta)$, we get by (**??**) and (**??**) that

$$\int_{B_0(\delta)}\left(x^k\partial_k\hat{u}_\alpha + \frac{1}{2}\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx = \left(cA_\delta + o(1)\right)\mu_\alpha\ . \tag{5.18}$$

By Proposition **??**, for any $k \in \{0,1,2\}$,

$$\int_{B_0(\delta)}\hat{u}_\alpha|\nabla^k\hat{u}_\alpha|dx \le \varepsilon_\delta(\alpha)\mu_\alpha \ \text{ and } \ \int_{B_0(\delta)}|\nabla\hat{u}_\alpha|^2dx \le \varepsilon_\delta(\alpha)\mu_\alpha\ , \tag{5.19}$$

where $\lim_{\delta\to 0}\limsup_{\alpha\to+\infty}\varepsilon_\delta(\alpha) = 0$. By (**??**) and (**??**), integrating by parts, we get that

$$\left|\int_{B_0(\delta)}\left(x^k\partial_k\hat{u}_\alpha + \frac{1}{2}\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx\right| \le \varepsilon_\delta(\alpha)\mu_\alpha\ , \tag{5.20}$$

where $\varepsilon_\delta(\alpha)$ is as above. Combining (**??**) and (**??**) it follows that $A_\delta \to 0$ as $\delta \to 0$. Coming back to (**??**) and (**??**), this proves (**??**). As already mentioned, this also proves Theorem **??** when $n = 5$.

Theorem **??** has an interpretation in terms of phase stability of solitons for the fourth order Schrödinger equation

$$i\frac{\partial u}{\partial t} + \Delta_g^2 u + \varepsilon\Delta_g u = |u|^{2^\sharp-2}u\ , \tag{5.21}$$

where $\varepsilon > 0$. Equations like (**??**) have been introduced by Karpman [**?**] and Karpman and Shagalov [**?**] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Among other possible references they have been investigated since then (local well-posedness, global well-posedness, scattering) by Fibich, Ilan, and Papanicolaou [**?**], Guo and Wang [**?**], Hao, Hsiao, and Wang [**?**, **?**], Pausader [**?**, **?**, **?**], Pausader and Shao [**?**], and Segata [**?**]. Solitons for (**??**) can be written as $ue^{-i\omega t}$, where $u : M \to \mathbb{R}$ satisfies (**??**) with $b = \varepsilon$ and $c = \omega$. We assume here that $\omega > 0$. If (**??**) with $b = \varepsilon$ and $c = \omega$ is stable, then phase stability holds true for (**??**) in the sense that for any sequence $u_\alpha e^{-i\omega_\alpha t}$ of solitons, with $\|u_\alpha\|_{H^2} \le \Lambda$

for some $\Lambda > 0$, if $\omega_\alpha \to \omega$ in $\mathbb{R}$, then, up to a subsequence, $u_\alpha \to u$ in C^4 and the sequence of solitons converges to another soliton. In other words, if (**??**) is table, then the sole convergence of the phase suffices to guarantee convergence of the solitons. A corollary to Theorem **??** is that phase stability holds true for (**??**) when the scalar curvature of the background space is positive, $\varepsilon > 0$ is sufficiently small, and $\omega \in (0, \varepsilon)$, up to the addition of extra assumptions when $n = 5$ in order to apply Theorem **??**.

6. Proof of Theorem ??

First we prove that $\mu_x(x) \geq 0$ for all x. Let P_0 be the geometric Paneitz operator as in the left hand side of (**??**), and $P_g = \Delta_g^2 + b\Delta_g + c$. Let also G_0 be the Green's function of P_0 and G be the Green's function of P_g. We fix $x \in M$, and let $\tau_x : M\backslash\{x\} \to \mathbb{R}$ be the function such that

$$G(x, \cdot) = G_0(x, \cdot) + \tau_x(\cdot) \tag{6.1}$$

in $M\backslash\{x\}$. When $n = 5$, τ_x extends continuously in M. Moreover, we have that $P_g\tau_x = -P_gG_0(x, \cdot) = (P_0 - P_g)G_0(x, \cdot)$ in $M\backslash\{x\}$. Noting that

$$(P_0 - P_g)G(x, \cdot) = O(d_g(x, \cdot)^{-3}) \ ,$$

we actually have that $\tau_x \in H_4^p(M) \cap C^{0,\theta}(M)$ for all $p \in (1, 5/3)$ and all $\theta \in (0, 1)$, where H_4^p is the Sobolev space of functions in L^p with four derivatives in L^p. In particular,

$$\tau_x(y) = \int_M G(y, \cdot)\left(P_0 - P_g\right)G_0(x, \cdot)dv_g$$

for all $y \in M$, and noting that $H_4^p \subset H_2^{\frac{5p}{5-2p}}$ and $\frac{5p}{5-2p} > 2$ for p close to $5/3$, we get that

$$\begin{aligned}\tau_x(x) &= \int_M G_0(x, \cdot)\left(P_0 - P_g\right)G_0(x, \cdot)dv_g + \int_M \tau_x P_g \tau_x dv_g \\ &= \int_M \big(A_g - bg\big)\left(\nabla G_0\left(x, \cdot\right), \nabla G_0(x, \cdot)\right)dv_g \\ &\quad + \int_M \left(\frac{1}{2}Q_g - c\right)G_0(x, \cdot)^2 dv_g + \int_M \left((\Delta_g\tau_x)^2 + b|\nabla\tau_x|^2 + c\tau_x^2\right)dv_g \ .\end{aligned} \tag{6.2}$$

By assumption, $bg \leq A_g$ and $c \leq \frac{1}{2}Q_g$. Hence $\tau_x \geq 0$ in M. Now we use the fact that g is conformally flat. In particular, there exists $\varphi > 0$ such that $g = \varphi^4\tilde{g}$ and $\tilde{g}$ is flat around x. The Green functions G_0 and $\tilde{G}_0$ of P_0 and $\tilde{P}_0$, where $\tilde{P}_0$ is the geometric Paneitz operator with respect to $\tilde{g}$, are related by

$$G_0(x, y) = \frac{\tilde{G}_0(x, y)}{\varphi(x)\varphi(y)} \tag{6.3}$$

for all $x \neq y$. Independently,

$$\tilde{G}_0(x, y) = \frac{1}{6\omega_4 d_{\tilde{g}}(x, y)} + A + \alpha_x(y) \ , \tag{6.4}$$

where α_x is continuous and such that $\alpha_x(x) = 0$. Combining (**??**) and (**??**), thanks to (**??**), we get that

$$G_0(x, y) = \frac{1}{6\omega_4 d_g(x, y)} + A + \tilde{\alpha}_x(y) \ ,$$

where $\tilde{\alpha}_x$ is such that $\tilde{\alpha}_x(x) = 0$. Coming back to (**??**), thanks to (**??**), we then get that

$$\mu_x(x) = A + \tau_x(x) \; . \tag{6.5}$$

By Humbert and Raulot [**?**], assuming the Yamabe invariant is positive, P_0 is coercive, and G_0 is positive, we have that $A > 0$ with equality if and only if (M, g) is conformally diffeomorphic to the unit sphere. Since $\tau_x(x) \geq 0$, and x is arbitrary, we proved that $\mu_x(x) \geq 0$ for all x, and that if $\mu_x(x) = 0$ for some x, then (M, g) is conformally diffeomorphic to the unit sphere.

We assume now that $\mu_x(x) = 0$ for some x. Then, by (**??**), $\tau_x(x) = 0$ and $A = 0$. In particular (M, g) is conformally diffeomorphic to the unit sphere and by (**??**), since $b, c > 0$, $bg \leq A_g$ and $c \leq \frac{1}{2} Q_g$, we get that $\tau_x \equiv 0$ and that

$$\frac{1}{2} Q_g \equiv c \text{ in } M \;\text{ and }\; (A_g - bg)\left(\nabla G(x, \cdot), \nabla G(x, \cdot)\right) \equiv 0 \text{ in } M \backslash \{x\} \; . \tag{6.6}$$

By first equation in (**??**), Q_g is constant, and since g is conformal to the round metric we get, see for instance Hebey and Robert [**?**] for the classification of all constant metrics, that g has constant sectional curvature. In particular, (M, g) is isometric to the 5-sphere with a constant multiple of the round metric. Then we also get that $A_g \equiv kg$ for some constant k and it follows from the second equation in (**??**) that necessarily $k = b$. In particular, $c \equiv \frac{1}{2} Q_g$ and $A_g \equiv bg$ in M. This ends the proof of Theorem **??**.

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