Local distribution of the parts of unequal partitions in arithmetic progressions I

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To Kálmán Győry, Attila Pethő, János Pintz and András Sárközy,
on the occasion of their birthdays even if 260=70+70+60+60 is not an unequal partition.

Abstract. In [4], András Sárközy and the authors proved that for almost all unequal partitions of an integer n, the parts are evenly distributed in residue classes modulo d for $d = o(n^{1/2})$. In this paper, we study very precisely the local distribution in arithmetic progressions of the parts of unequal partitions. We obtain some asymptotic formulae for the number of unequal partitions of n with exactly N_r parts congruent to r mod d, $1 \leqslant r \leqslant d$. Our results show that $(N_1, \dots, N_d)$ behaves like a random Gaussian vector. This illustrates the fact that the distribution of the parts of unequal partitions in residue classes is much more uniform that in the case of unrestricted partitions.

1. Introduction

Recently András Sárközy and the authors obtained various statistical results on the distribution of the summands of partitions in residue classes. In [3], they proved that the parts of almost all partitions of n are well distributed in arithmetic progressions modulo d for $d < n^{1/2-\varepsilon}$. However, they also observed some limitation in this well distribution due to the fact that the parts with small moduli are more frequent (cf. [13]). This leads the authors to study precisely in [6] the distribution of the parts in residue classes. We obtained an asymptotic formula and found that if N_r denotes the number of parts congruent to r modulo d for a random partition of n then $(N_1, \dots, N_d)$ behaves like a random vector with a vectorial gamma distribution. In particular we showed that if $1 \leqslant a < b \leqslant d$ and $d \leqslant n^{1/8-\varepsilon}$ then the number of partitions of n such that $N_a \geqslant N_b$ is

$$(1\cdot1) \qquad (1+o(1))\frac{p(n)}{\Gamma(a/d)\Gamma(b/d)}\int_0^\infty x^{\frac{a}{d}-1}\mathrm{e}^{-x}\Big(\int_x^\infty y^{\frac{b}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x,$$

where Γ is the Euler gamma function.

Next we observed that for fixed d and large enough n this quantity is

$$(1\cdot2) \qquad \geqslant p(n)\big(\frac{1}{2}+\frac{b-a}{12d}\big).$$

Thus this result reflects well the fact that the parts in small moduli residue classes are more frequent. We also proved that for d fixed, the number of partitions of n such that $N_1 > N_2 > \cdots > N_d$ is $p(n)(c(d)+o(1))$ with an explicit $c(d) > \frac{1}{d!}$ given by some multiple truncated gamma integrals.

In this paper we investigate these questions for the unequal partitions. Since the parts are all distinct, we conjectured in [1] and in [4] that the preponderance of the "small" residue classes disappears, so that the distribution is more regular. In [4] we have already proved that the parts are well-distributed modulo d for $d = o(\sqrt{n})$. Let $d \in \mathbb{N}^*$, $\mathcal{D}$ a non-empty subset of $\{1, \dots, d\}$ and $\mathcal{D}^c = \{1, \dots, d\} \smallsetminus \mathcal{D}$ its complement. Let $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\}$ be a multiset of $|\mathcal{D}|$ non-negative integers. The aim of this paper is to study now $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$,

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the number of unequal partitions of n with exactly N_r parts congruent to r modulo d for all $r \in \mathcal{D}$. We adopt the convention $\Pi_d^*(0, \mathcal{R}_\mathcal{D}) = 1$ if $\mathcal{R}_\mathcal{D} = \{0, ..., 0\}$ and 0 otherwise.

Erdős and Lehner [8] proved that almost all of the $q(n)$ unequal partitions of n contain

$$(1 + o(1))\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}$$

parts. Furthermore, they also mentioned that the number of unequal partitions with at most $\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n} + y\sqrt[4]{n}$ parts is given by a Gaussian integral. Moreover, in [4] we stated that for $1 \leqslant r \leqslant d$, $d = o(\sqrt{n})$, for all but $o(q(n))$ unequal partitions of n the number of parts congruent to r mod d is

$$(1 + o(1))\frac{2\sqrt{3}\log 2\sqrt{n}}{\pi d}.$$

Therefore one can expect each N_r "mostly close" to

$$k_0 := \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d}. \tag{1·3}$$

More precisely we will suppose that for all $r \in \mathcal{D}$ we have

$$|N_r - k_0| \leqslant \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}, \tag{1·4}$$

where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. We can already note that the above cited result of Erdős and Lehner implies that the factor $n^{\frac{1}{4}}$ in the upper bound of (1·4) is very important. During the different proofs, the following two quantities will appear frequently:

$$R_\mathcal{D} = \sum_{r \in \mathcal{D}} rN_r, \tag{1·5}$$

$$Q_\mathcal{D} = d\sum_{r \in \mathcal{D}} \frac{N_r(N_r - 1)}{2}. \tag{1·6}$$

When $\mathcal{D} = \{1, \ldots, d\}$, we will simply write R and Q. Let

$$\delta := \begin{cases} g.c.d.(d, a_1, \ldots, a_k) & \text{if } \mathcal{D}^c = \{a_1, \ldots, a_k\} \\ d & \text{if } \mathcal{D}^c = \varnothing. \end{cases} \tag{1·7}$$

We remark that if $n \geqslant 1$ and $\Pi_d^*(n, \mathcal{R}_\mathcal{D}) \geqslant 1$ then n must satisfy:

$$n \equiv R_\mathcal{D} \, (\mathrm{mod}\, \delta). \tag{1·8}$$

Theorem 1.1. *Let $\varepsilon > 0$. The following two propositions hold.*

(i) Let $d \leqslant n^{1/4-\varepsilon}$, $\mathcal{D} = \{1, \ldots, d\}$ and $n \equiv R_\mathcal{D} \, (\mathrm{mod}\, d)$. Let $\mathcal{R} = \mathcal{R}_\mathcal{D} = \{N_1, \ldots, N_d\}$ be a multiset of integers satisfying (1·4)*. Then we have*

$$\Pi_d^*(n, \mathcal{R}_D) = (1 + o(1))q(n)\frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{d/2}$$
$$\times \exp\Big\{ - \frac{2\sqrt{3}\log^2 2}{\pi\big(1 - \frac{12(\log 2)^2}{\pi^2}\big)\sqrt{n}}\Big(\sum_{r=1}^{d}(N_r - k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r=1}^{d}(N_r - k_0)^2\Big\}.$$

(ii) We suppose now that $d \leqslant n^{1/6-\varepsilon}$ and $\mathcal{D} \subset \{1,\ldots,d\}$. Then under (1·8) and (1·4) we have

$$\Pi_d^*(n,\mathcal{R}_D) = q(n)\frac{\delta(1+o(1))}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2}$$
$$\times \exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(N_r-k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r\in\mathcal{D}}(N_r-k_0)^2\Big).$$

It is surprising to see that the range for the general $\mathcal{D}$ is only $d \leqslant n^{1/6-\varepsilon}$ and not $d \leqslant n^{1/4-\varepsilon}$ as in assertion (i). We are obliged to add this condition only in Section 8 of Part II [7] devoted to the terms $S_0(\lambda)$. It is perhaps possible with more care to have a general result valid for $d \leqslant n^{1/4-\varepsilon}$.

In this Theorem we already see that the distribution of the parts of unequal partitions in residue classes is more regular than in the context of unrestricted partitions. This distribution of $(N_r : r \in \mathcal{D})$ behaves like a vectorial Gaussian whose associated density depends only on the cardinality of $\mathcal{D}$ and not of its elements.

A first consequence is that the numbers N_r are almost surely very close to k_0.

Corollary 1.2. *Let $\mathcal{D} \subset \{1,\ldots,d\}$, $d^3|\mathcal{D}| \leqslant n^{1/2-3\varepsilon}$ with $\varepsilon \in]0,10^{-2}[$, $w(n)$ a non-decreasing function such that $\lim_{n\to\infty} w(n) = \infty$ and $w(n) = O(n^\varepsilon)$. Suppose that*

$$\frac{w(n)|\mathcal{D}|^{5/3}}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)}\Big) = o(1) \tag{1·9}$$

when $n \to \infty$. Then for almost all unequal partitions of n the number of summands $\equiv r \,(\mathrm{mod}\, d)$ are between $\lceil \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} - \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\rceil d$ and $\lfloor \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} + \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\rfloor d$ simultaneously for all $r \in \mathcal{D}$.

If $|\mathcal{D}| \approx d$ then (1·9) becomes $d^{3/2}w(n)(\log n)^{-1/2}\exp\Big(-\frac{\pi\log n}{2\sqrt{3}dw^2(n)}\Big) = o(1)$. We can apply Corollary 1.2 with $d = o\Big(\sqrt{\frac{\log n}{\log\log n}}\Big)$, $w(n) \leqslant \sqrt{\frac{\log n}{\log\log n}}$. Then we have proved that in almost all unequal partitions of n the number of summands $\equiv r \,(\mathrm{mod}\, d)$ are between $\lceil \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} - \frac{n^{1/4}\sqrt{\log n}}{d^2 w(n)}\rceil d$ and $\lfloor \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} + \frac{n^{1/4}\sqrt{\log n}}{d^2 w(n)}\rfloor d$ simultaneously for $r = 1,\ldots,d$.

This result implies Theorem 3.2 (resp. the estimate for $\Pi_1^*(n,\mathcal{R})$ implies Theorem 3.3) of Erdős and Lehner [8]. For the moduli in question the above Corollary improves [4].

This is another illustration of the fact that for almost all unequal partitions of n the parts are evenly distributed in residue classes.

When $|\mathcal{D}|$ is small, for example if $\mathcal{D} = \{a,b\}$, then (1·9) becomes

$$\frac{w(n)}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2^{7/3}\sqrt{3}w^2(n)}\Big) = o(1).$$

Thus this Corollary may be applied in all the range $d \leqslant n^{1/6-2\varepsilon}$.

A first application of this corollary is the following result which announces that for almost all unequal partitions, two residue classes don't have the same number of parts.

Corollary 1.3. *Let $d \leqslant n^{1/6-\varepsilon}$ with $\varepsilon \in]0,10^{-2}[$, and $1 \leqslant a < b \leqslant d$. The number of unequal partitions of n with the same number of summands in the residue classes a and b modulo d is $o(q(n))$.*

Another application is the following Corollary 1.4 which solves some conjecture of [1] or [4].

Corollary 1.4. *Let $d \leqslant n^{1/6-\varepsilon}$ with $\varepsilon \in]0, 10^{-2}[$, and $1 \leqslant a < b \leqslant d$. There are $\frac{q(n)}{2} + o(q(n))$ unequal partitions of n with more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$.*

This corollary shows that there are no dominant classes for the distribution of the parts of unequal partitions. We recall that it is not the case for the unrestricted partitions (see (1·1) and (1·2)). In the following last corollary we prove some other conjectures of [4] illustrating again the fact that all the residue classes are very equitably represented by the parts of the unequal partitions.

In particular we show that there exist $q(n)(\frac{1}{d!} + o(1))$ unequal partitions of n such that $N_1 > N_2 > \ldots > N_d$.

Corollary 1.5. *Let d be a fixed integer. The following two assertions are satisfied.*

(i) For any $1 \leqslant a \leqslant d$, the number of unequal partitions of n with more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$ for all $b \in \{1, \ldots, d\} \smallsetminus \{a\}$ is $q(n)(\frac{1}{d} + o(1))$.

(ii) Let σ be a permutation of the set $\{1, \ldots, d\}$. The number of unequal partitions of n such that there are more parts $\equiv \sigma(i) \pmod d$ than parts $\equiv \sigma(j) \pmod d$ for all $1 \leqslant i < j \leqslant d$ is $q(n)(\frac{1}{d!} + o(1))$.

Unfortunately, the proof of Theorem 1.1 is too long to be completely presented in this paper. Thus we will state in this paper only few steps for the case $\mathcal{D} = \{1, \ldots, d\}$ and the main parts of the proofs will be in [7]. In the next section we will study the generating function associated with $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$, next we introduce the saddle point method and state some preliminary formulae.

2. The generating function

Let $H_{\mathcal{D}}(z)$ be the associated generating function of $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$:

$$H_{\mathcal{D}}(z) = \sum_{n=0}^{\infty} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) z^n.$$

We will also need the generating function associated with the unequal partitions:

$$h(z) = \sum_{n=0}^{\infty} q(n) z^n = \prod_{j=1}^{\infty} (1 + z^j).$$

Let $\mathcal{D}^c = \{1, \ldots, d\} \smallsetminus \mathcal{D}$. We will prove the following lemma.

Lemma 2.1. *For $|z| < 1$, we have:*

$$\begin{aligned} H_{\mathcal{D}}(z) &= \Big\{ \prod_{\substack{r \in \mathcal{D}^c \\ j \in \mathbb{N}}} \Big(1 + z^{r+jd}\Big) \Big\} \frac{z^{\sum_{r\in\mathcal{D}} rN_r + d \sum_{r\in\mathcal{D}} \frac{N_r(N_r-1)}{2}}}{\prod_{r\in\mathcal{D}} \prod_{j=1}^{N_r} (1 - z^{jd})} \\ &= z^{R_{\mathcal{D}} + Q_{\mathcal{D}}} h(z) \prod_{r \in \mathcal{D}} \Big\{ \prod_{j=1}^{N_r} (1 - z^{jd})^{-1} \prod_{j=0}^{\infty} (1 + z^{r+jd})^{-1} \Big\}. \end{aligned}$$

First proof of Lemma 2.1 in the case $\mathcal{D} = \{1, \ldots, d\}$. According to Euler's theorem, for $|t| < 1$ and $|q| < 1$, we have

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{n(n-1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1 + tq^n), \tag{2·1}$$

for example, see [12] Theorem 348, p.280.

For $z, w_r \in \mathbb{C}$, $|z| < 1$, and $|w_r| < |z|^{-r}$ $(1 \leqslant r \leqslant d)$ we have

$$\prod_{r=1}^{d}\prod_{k_r=0}^{\infty}(1+w_r z^{r+k_r d}) = \sum_{N_1=0}^{\infty}\cdots\sum_{N_d=0}^{\infty}\Big(\sum_{n=0}^{\infty}\Pi_d^*(n,\{N_1,\ldots,N_d\})z^n\Big)w_1^{N_1}\cdots w_d^{N_d}. \tag{2·2}$$

On the other hand, for $1 \leqslant r \leqslant d$, we write $w_r z^{r+k_r d} = (w_r z^r)(z^d)^{k_r}$ and we apply (2·1) with $t = w_r z^r$, $q = z^d$:

$$\begin{aligned}\prod_{r=1}^{d}\prod_{k_r=0}^{\infty}(1+w_r z^{r+k_r d}) &= \prod_{r=1}^{d}\prod_{k_r=0}^{\infty}(1+(w_r z^r)(z^d)^{k_r})\\ &= \prod_{r=1}^{d}\Big(1+\sum_{N_r=1}^{\infty}\frac{(w_r z^r)^{N_r}(z^d)^{N_r(N_r-1)/2}}{(1-z^d)(1-z^{2d})\cdots(1-z^{N_r d})}\Big)\\ &= \prod_{r=1}^{d}\sum_{N_r=0}^{\infty}\frac{w_r^{N_r}.z^{rN_r+dN_r(N_r-1)/2}}{\prod_{j=1}^{N_r}(1-z^{jd})}\\ &= \sum_{N_1=0}^{\infty}\cdots\sum_{N_d=0}^{\infty}\Big(\frac{z^{R+Q}}{\prod_{r=1}^{d}\prod_{j=1}^{N_r}(1-z^{jd})}\Big)w_1^{N_1}\cdots w_d^{N_d}.\end{aligned} \tag{2·3}$$

We finish the proof by comparing the coefficient of $w_1^{N_1}\cdots w_d^{N_d}$ in (2·2) and (2·3).

Second proof of Lemma 2.1. Let $n \equiv R_D \,(\mathrm{mod}\,\delta)$ and $n \geqslant 1$. To each unequal partition Π^* of n we can assign two integers a, b such that $n = a + b$ and a is the sum of all parts congruent to an r modulo d with $r \in \mathcal{D}$ and b is the sum of the other parts. Let $Q(a, \mathcal{R}_{\mathcal{D}}, \mathcal{D})$ denote the number of unequal partitions of a with exactly N_r parts congruent to r modulo d for $r \in \mathcal{D}$ and with no parts congruent to any $j \in \mathcal{D}^c$. We also consider $\tilde{Q}(b, \mathcal{D})$, the number of unequal partitions of the integer b with no parts congruent to any $r \in \mathcal{D}$. With these notations we have

$$\begin{aligned}H_{\mathcal{D}}(z) &= \sum_{a,b\in\mathbb{N}} Q(a,\mathcal{R}_{\mathcal{D}},\mathcal{D})\tilde{Q}(b,\mathcal{D})z^{a+b}\\ &= \sum_{a\in\mathbb{N}} Q(a,\mathcal{R}_{\mathcal{D}},\mathcal{D})z^a\sum_{b\in\mathbb{N}}\tilde{Q}(b,\mathcal{D})z^b\\ &= S_a S_b,\end{aligned}$$

say. For the sum S_b, we have

$$S_b = \prod_{\substack{r\in\mathcal{D}^c\\ j\in\mathbb{N}}}(1+z^{r+dj}). \tag{2·4}$$

Let λ be a partition counted in $Q(a, \mathcal{R}_{\mathcal{D}}, \mathcal{D})$ for some $a \geqslant 1$. This partition is of type:

$$a = \sum_{r\in\mathcal{D}}\sum_{i=1}^{N_r}(r+d\lambda_{i,r}),\ \text{with } 0 \leqslant \lambda_{1,r} < \ldots < \lambda_{N_r,r}\ (r\in\mathcal{D}).$$

This gives

$$a = \sum_{r\in\mathcal{D}} rN_r + d\sum_{r\in\mathcal{D}}\sum_{i=1}^{N_r}\lambda_{i,r}.$$

Inserting this in S_a we obtain

$$S_a = z^{\sum_{r\in\mathcal{D}} rN_r} \prod_{r\in\mathcal{D}} \sum_{0\leqslant\lambda_{1,r}<\ldots<\lambda_{N_r,r}} z^{d\sum_{i=1}^{N_r}\lambda_{i,r}}.$$

The different sums in $\lambda_{i,r}$ are geometric:

$$\begin{aligned}\sum_{0\leqslant\lambda_{1,r}<\ldots<\lambda_{N_r,r}} z^{d\sum_{i=1}^{N_r}\lambda_{i,r}} &= \sum_{0\leqslant\lambda_{1,r}<\ldots<\lambda_{N_r-1,r}} z^{d\sum_{i=1}^{N_r-1}\lambda_{i,r}} \sum_{\lambda_{N_r,r}=1+\lambda_{N_r-1,r}}^{+\infty} z^{d\lambda_{N_r,r}}\\ &= \sum_{0\leqslant\lambda_{1,r}<\ldots<\lambda_{N_r-1,r}} z^{d\sum_{i=1}^{N_r-1}\lambda_{i,r}} \frac{z^{d(1+\lambda_{N_r-1,r})}}{1-z^d}.\end{aligned}$$

By iteration we obtain

$$S_a = \frac{z^{R_{\mathcal{D}}+Q_{\mathcal{D}}}}{\prod_{r\in\mathcal{D}}\prod_{j=1}^{N_r}(1-z^{jd})}. \tag{2·5}$$

Formulae (2·4) and (2·5) end the proof of Lemma 2.1. □

Remark. The main part of the second proof was to obtain (2·5). It is also possible to obtain this formula *via* the Euler identity like in the first proof given in the case $\mathcal{D} = \{1,\ldots,d\}$. Thus the first method can be also used for general $\mathcal{D}$.

We end this section with an elementary lemma on Q_D and R_D.

Lemma 2.2. *(i) If we suppose only*

$$|N_r - k_0| = o\Big(\frac{\sqrt{n}}{d}\Big) \qquad (r = 1,\ldots,d), \tag{2·6}$$

then we have:

$$R_{\mathcal{D}} = O(|\mathcal{D}|\sqrt{n}), \tag{2·7}$$

$$Q_{\mathcal{D}} = \frac{d|\mathcal{D}|k_0^2}{2} + d\sum_{r\in\mathcal{D}} k_0(N_r-k_0) + O\big(d\sum_{r\in\mathcal{D}}(N_r-k_0)^2\big) + O(|\mathcal{D}|\sqrt{n}). \tag{2·8}$$

(ii) Under (1·4) *we have*

$$\begin{aligned}Q_{\mathcal{D}} + R_{\mathcal{D}} = \frac{d|\mathcal{D}|k_0^2}{2} - \frac{d}{2}|\mathcal{D}|k_0 + d\sum_{r\in\mathcal{D}} k_0(N_r-k_0) + \frac{d}{2}\sum_{r\in\mathcal{D}}(N_r-k_0)^2 + k_0\sum_{r\in\mathcal{D}} r\\ + O\Big(\frac{n^{1/4}|\mathcal{D}|^{1/3}d^{2/3}\sqrt{\log n}}{w(n)}\Big).\end{aligned} \tag{2·9}$$

Remark. If we use (2·6) in the first error term of (2·8) we obtain (for $d = o(\sqrt{n})$)

$$Q_{\mathcal{D}} = \frac{d|\mathcal{D}|k_0^2}{2} + d\sum_{r\in\mathcal{D}} k_0(N_r-k_0) + o\big(|\mathcal{D}|\frac{n}{d}\big), \tag{2·10}$$

but we will need later a more precise estimate.

Proof. By (2·6), $N_r = O(d^{-1}\sqrt{n})$ for $r \in \mathcal{D}$, thus we have

$$R_{\mathcal{D}} = \sum_{r\in\mathcal{D}} rN_r = O(|\mathcal{D}|\sqrt{n}).$$

For $Q_{\mathcal{D}}$ we must be a little more precise :

$$\begin{aligned} Q_{\mathcal{D}} &= \frac{d}{2}\sum_{r\in\mathcal{D}}(N_r^2 - N_r) = \frac{d}{2}\sum_{r\in\mathcal{D}}(k_0 + N_r - k_0)^2 + O(|\mathcal{D}|\sqrt{n}) \\ &= \frac{d|\mathcal{D}|k_0^2}{2} + d\sum_{r\in\mathcal{D}} k_0(N_r - k_0) + \frac{d}{2}\sum_{r\in\mathcal{D}}(N_r - k_0)^2 + O(|\mathcal{D}|\sqrt{n}). \end{aligned}$$

In the computation of the main term in the end of the proof of Theorem 1.1, we will need stronger condition for N_r (see (1·4)). If we use this condition then we obtain :

$$R_{\mathcal{D}} = k_0\sum_{r\in\mathcal{D}} r + O\Big(\frac{n^{1/4}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}\sum_{r\in\mathcal{D}} r\Big), \tag{2·11}$$

and
(2·12)

$$Q_{\mathcal{D}} = \frac{d|\mathcal{D}|k_0^2}{2} - \frac{d}{2}|\mathcal{D}|k_0 + d\sum_{r\in\mathcal{D}} k_0(N_r - k_0) + \frac{d}{2}\sum_{r\in\mathcal{D}}(N_r - k_0)^2 + O\Big(\frac{n^{1/4}|\mathcal{D}|^{1/3}d^{2/3}\sqrt{\log n}}{w(n)}\Big).$$

Next it remains to sum (2·11) and (2·12) to obtain (2·9).

3. The saddle point method in the case $\mathcal{D} = \{1, \dots, d\}$

For $0 < \varrho < 1$, it follows from Lemma 2.1 and the Cauchy formula that

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= \frac{1}{2i\pi}\int_{|v|=\varrho} v^{-n-1}H_{\mathcal{D}}(v)\,\mathrm{d}v \\ &= \frac{1}{2i\pi}\int_{|v|=\varrho} \frac{v^{-n-1+R_{\mathcal{D}}+Q_{\mathcal{D}}}}{\prod_{r\in\mathcal{D}}\prod_{j=1}^{N_r}(1 - v^{jd})}\,\mathrm{d}v. \end{aligned}$$

Let $x > 0$ to be specified later, $\varrho = \mathrm{e}^{-x}$, $z = x + iy$, $v = \exp(-z)$. We also define the following functions for $\Re w > 0$:

$$f(w) = \prod_{j=1}^{+\infty}(1 - \exp(-jw))^{-1}; \tag{3·1}$$

$$g_k(w) = \prod_{j=1}^{k}(1 - \exp(-jw))^{-1} = f(w)\prod_{j=k+1}^{+\infty}(1 - \exp(-jw)). \tag{3·2}$$

For $n \geqslant 1$ and $n \equiv R_{\mathcal{D}} \pmod d$, the integrand is periodic in y and we have

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d}\Big\{\prod_{r=1}^{d} g_{N_r}(d(x + iy))\Big\}\exp((n - \mathcal{R}_{\mathcal{D}} - Q_{\mathcal{D}})(x + iy))\,\mathrm{d}y. \tag{3·3}$$

4. A first upper bound in the case $\mathcal{D}=\{1,\dots,d\}$

We begin this section by a lemma which gives a first simple estimation of the function g_k.

Lemma 4.1. *Let $w=t+ib$ with $t>0$. We have*

$$g_k(w)=f(w)\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw)\big(\frac{1}{mw}-\frac{1}{2}\big)+O\big(|w|(kt)^{-1}+|w|^2t^{-2}k^{-1}\big)\Big).$$

Proof. By the definition of g_k we have:

$$g_k(w)=f(w)\exp\Big(-\sum_{\nu=k+1}^{\infty}\log\frac{1}{1-\exp(-\nu w)}\Big).$$

We develop the log, intervert the summations and compute the geometric sum (since $t>0$ all these manipulations are valid) :

$$\begin{aligned}g_k(w)&=f(w)\exp\Big(-\sum_{\nu=k+1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-\nu mw)\Big)\\&=f(w)\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\exp(-(k+1)mw)}{(1-\exp(-mw))}\Big)\\&=f(w)\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\exp(-kmw)}{(\exp(mw)-1)}\Big).\end{aligned}$$

We want to approximate $(\exp(mw)-1)^{-1}$ by $\frac{1}{mw}-\frac{1}{2}$. This is possible when m is "small" and we will prove that the contribution of the "large" m's is sufficiently small.

$$\begin{aligned}g_k(w)=f(w)\exp\Big(&-\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw)\big(\frac{1}{mw}-\frac{1}{2}\big)\\&+\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw)\big(\frac{1}{mw}-\frac{1}{2}-\frac{1}{\exp(mw)-1}\big)\Big).\end{aligned}$$

It remains to obtain an upper bound for the last sum in m. Let

$$T:=\Big|\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw)\Big(\frac{1}{mw}-\frac{1}{2}-\frac{1}{\exp(mw)-1}\Big)\Big|.$$

We have

$$\begin{aligned}T&\leqslant\sum_{m\leqslant|w|^{-1}}\frac{1}{m}\exp(-kmt)\Big|\frac{1}{mw}-\frac{1}{2}-\frac{1}{\exp(mw)-1}\Big|\\&\quad+\sum_{m>|w|^{-1}}\frac{1}{|w|^{-1}}\exp(-kmt)\Big(\frac{1}{m|w|}+\frac{1}{2}+\frac{1}{|\exp(mw)-1|}\Big)\\&\leqslant T_1+T_2,\end{aligned}$$

say. Since for $m|w|\leqslant 1$, $(\exp(mw)-1)^{-1}=\frac{1}{mw}-\frac{1}{2}+O(m|w|)$ we have for T_1:

$$T_1\ll\sum_{m\leqslant|w|^{-1}}\frac{1}{m}\exp(-kmt)m|w|\ll\frac{|w|}{\mathrm{e}^{kt}-1}.\tag{4·1}$$

For T_2 we use the fact that $|\exp(mw)-1| \geqslant \exp(mt)-1 \geqslant mt \geqslant |w|^{-1}t$:

$$\begin{aligned} T_2 &\leqslant |w| \sum_{m>|w|^{-1}} \exp(-kmt)\Big(O(1)+\frac{1}{\exp(mt)-1}\Big) \\ &\leqslant O(|w|)\sum_{m=1}^{\infty}\exp(-kmt)+|w|\sum_{m>|w|^{-1}}\exp(-kmt)\frac{1}{|w|^{-1}t}. \end{aligned} \tag{4·2}$$

By (4·1) and (4·2) we obtain

$$T \leqslant \frac{1}{e^{kt}-1}\Big(O(|w|)+\frac{|w|^2}{t}\Big) \leqslant \frac{1}{k}\Big(O\big(\frac{|w|}{t}\big)+\frac{|w|^2}{t^2}\Big).$$

□

For the sake of orientation, let us consider the case $b=0$ in order to obtain a trivial upper bound for the modulus of the last complex integral. (We will concentrate on the main term in the exponent). For $t>0$, by Lemma 4.1 we obtain (with $\mathcal{R}=\mathcal{R}_{\mathcal{D}}, R=R_{\mathcal{D}}, Q=Q_{\mathcal{D}}$):
For $n \equiv R \,(\mathrm{mod}\, d)$,

$$\begin{aligned} \Pi_d^*(n,\mathcal{R}) &\leqslant \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d}\Big|\prod_{r=1}^{d} g_{N_r}(d(x+iy))\Big|\exp((n-R-Q)x)\,\mathrm{d}y \\ &\leqslant \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d}\Big\{\prod_{r=1}^{d} g_{N_r}(dx)\Big\}\exp((n-R-Q)x)\,\mathrm{d}y \\ &= \Big\{\prod_{r=1}^{d} g_{N_r}(dx)\Big\}\exp((n-R-Q)x). \end{aligned}$$

Assume provisorily that

$$d \leqslant n^{\frac{1}{2}-\varepsilon} \tag{4·3}$$

with some fixed positive ε and consider $g_k(t)$ for

$$|k-k_0| = o\Big(\frac{\sqrt{n}}{d}\Big), \tag{4·4}$$

this choice of k is motivated by the statistical result of Erdős and Lehner [8] on the number of the parts of a generic unequal partition (cf. the introduction).
Let

$$x_0 := \frac{\pi}{2\sqrt{3n}}, \qquad t := dx_0. \tag{4·5}$$

Then

$$k_0 t = k_0 d x_0 = \log 2. \tag{4·6}$$

During the computation of the main term S_0 we will use the following constant

$$C_2 := \sum_{m=1}^{\infty}\frac{1}{m^2 2^m}.$$

Lemma 4.2. *We have*
$$C_2 = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}.$$

This formula is probably well known but we however propose here a simple proof of it.
Proof. Since for $0 < x < 1$,
$$\Big(\sum_{m=1}^{\infty} \frac{x^m}{m^2}\Big)' = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m} = \frac{1}{x}\log\frac{1}{1-x},$$
we have
$$\begin{aligned} C_2 - 0 &= \int_0^{1/2} \frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x \\ &= -(\log 2)^2 - \int_0^{1/2} (\log x)\frac{\mathrm{d}x}{1-x}. \end{aligned}$$
In the integral we write $u = 1-x$:
$$\begin{aligned} C_2 &= -(\log 2)^2 + \int_{1/2}^{1} \frac{1}{u}\log\frac{1}{1-u}\,\mathrm{d}u \\ &= -(\log 2)^2 + \int_0^{1} \frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x - \int_0^{1/2} \frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x \\ &= -(\log 2)^2 + \frac{\pi^2}{6} - C_2. \end{aligned}$$
□

Now we can give a first estimate of $g_k(dx_0)$:

Lemma 4.3. *Under* (4·3) *and* (4·4) *we have:*
$$g_k(dx_0) = \exp\Big(\frac{\pi^2}{12dx_0} + \frac{(\log 2)^2}{2dx_0} + o\Big(\frac{\sqrt{n}}{d}\Big)\Big).$$

Proof.
As $n \to \infty$, by Lemma 4.1, we have
$$g_k(t) = f(t)\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}\exp(-k_0mt)\exp(-(k-k_0)mt)\big(\frac{1}{mt}-\frac{1}{2}\big) + O(k^{-1})\Big).$$
Next we use the fact that k is close to k_0:
$$\begin{aligned} g_k(t) = f(t)\exp\Big(&-\sum_{m=1}^{\infty}\frac{1}{m}\exp(-k_0mt)\big(\frac{1}{mt}-\frac{1}{2}\big) \\ &\times(1+O(|k-k_0|mt\exp(|k-k_0|mt)) + O(k_0^{-1})\Big). \end{aligned}$$
We develop the different terms, for $n \to \infty$,
$$\begin{aligned} g_k(t) = f(t)\exp\Big(&-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^2}(\mathrm{e}^{-k_0t})^m + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\mathrm{e}^{-k_0t})^m + \\ &+ O(|k-k_0|)\Big(\sum_{m=1}^{\infty}\frac{1}{m}\exp(-k_0mt/2) + t\sum_{m=1}^{\infty}\exp(-k_0mt/2)\Big) \\ &+ O(k_0^{-1})\Big). \end{aligned}$$

For $t = dx_0$ we obtain

$$g_k(dx_0) = f(dx_0)\exp\Big(-\frac{1}{dx_0}\sum_{m=1}^{\infty}\frac{2^{-m}}{m^2} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{2^{-m}}{m} + O(|k-k_0|)\Big(\sum_{m=1}^{\infty}\frac{2^{-m/2}}{m} + dx_0\sum_{m=1}^{\infty}2^{-m/2}\Big) + O(k_0^{-1})\Big)$$
$$= f(dx_0)\exp\Big(-\frac{1}{dx_0}\sum_{m=1}^{\infty}\frac{2^{-m}}{m^2} + o\Big(\frac{\sqrt{n}}{d}\Big)\Big)$$
$$= \exp\Big(\frac{\pi^2}{6dx_0} - \frac{1}{dx_0}\sum_{m=1}^{\infty}\frac{2^{-m}}{m^2} + o\Big(\frac{\sqrt{n}}{d}\Big)\Big).$$

This ends the proof of Lemma 4.3. □

If

$$(4{\cdot}7)\qquad |N_r - k_0| = o\Big(\frac{\sqrt{n}}{d}\Big) \qquad (r = 1, \dots, d)$$

as $n \to \infty$ then we have

$$(4{\cdot}8)\qquad \prod_{r=1}^{d} g_{N_r}(dx_0) = \exp\Big(\frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n})\Big)$$

and

$$\Pi_d^*(n,\mathcal{R}) \leqslant \exp\Big((n-R-Q)x_0 + \frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n})\Big).$$

Lemma 2.2, (1·3), (4·3), (4·6), and (4·7) yield that

$$(4{\cdot}9)\qquad R + Q = o(n) + \frac{d^2k_0^2}{2} + o(n) = \frac{1}{2}\Big(\frac{\log 2}{x_0}\Big)^2 + o(n).$$

Finally,

$$\Pi_d^*(d,\mathcal{R}) \leqslant \exp\Big(nx_0 + \frac{\pi^2}{12x_0} + o(\sqrt{n})\Big) = \exp\Big(\frac{\pi\sqrt{n}}{\sqrt{3}} + o(\sqrt{n})\Big).$$

This is trivial since

$$(4{\cdot}10)\qquad q(n) = (1+o(1))\frac{1}{4.3^{1/4}n^{3/4}}\exp\Big(\frac{\pi\sqrt{n}}{\sqrt{3}}\Big).$$

But we shall see in Part II [7] that the estimates can be improved.

We will use the classical splitting of the integral (3·3)

$$(4{\cdot}11)\qquad \Pi_d^*(n,\mathcal{R}_D) = S_0 + S_1 + S_2,$$

with

$$S_0 = \frac{d}{2\pi}\int_{|y|\leqslant y_1}\Big\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Big\}\exp((n-\mathcal{R}_{\mathcal{D}}-Q_{\mathcal{D}})(x_0+iy))\,\mathrm{d}y,$$

$$S_1 = \frac{d}{2\pi}\int_{y_1\leqslant|y|\leqslant y_2}\dots, \qquad S_2 = \frac{d}{2\pi}\int_{y_2\leqslant|y|\leqslant\pi/d}\dots$$

with $y_1 = n^{-\frac{3}{4}+\frac{\varepsilon}{3}}$ and $y_2 = 3\pi x_0$.

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