The evolutionary limit for models of populations interacting competitively via several resources

Nicolas Champagnat, Pierre-Emmanuel Jabin

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Nicolas Champagnat[1], Pierre-Emmanuel Jabin[1,2]

Abstract

We consider a integro-differential nonlinear model that describes the evolution of a population structured by a quantitative trait. The interactions between individuals occur by way of competition for resources whose concentrations depend on the current state of the population. Following the formalism of [16], we study a concentration phenomenon arising in the limit of strong selection and small mutations. We prove that the population density converges to a sum of Dirac masses characterized by the solution φ of a Hamilton-Jacobi equation which depends on resource concentrations that we fully characterize in terms of the function φ.

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1 Introduction

We are interested in the dynamics of a population subject to mutation and selection driven by competition for resources. Each individual in the population is characterized by a quantitative phenotypic trait $x \in \mathbb{R}$ (for example

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[1]TOSCA project-team, INRIA Sophia Antipolis – Méditerranée, 2004 rte des Lucioles, BP. 93, 06902 Sophia Antipolis Cedex, France,
E-mail: `Nicolas.Champagnat@sophia.inria.fr`
  [2]Laboratoire J.-A. Dieudonné, Université de Nice – Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France, E-mail: `jabin@unice.fr`

the size of individuals, their age at maturity, or their rate of intake of nutrients).

We study the following equation

$$\partial_t u_\varepsilon(t,x) = \frac{1}{\varepsilon}\left(\sum_{i=1}^k I_i^\varepsilon(t)\,\eta_i(x) - 1\right)u_\varepsilon(t,x) + M_\varepsilon(u_\varepsilon)(t,x), \qquad (1.1)$$

where M_ε is the mutation kernel

$$M_\varepsilon(f)(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} K(z)\left(f(x+\varepsilon z) - f(x)\right)dz, \qquad (1.2)$$

for a $K \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} zK(z)\,dz = 0$. Among many other ecological situations [15], this model is relevant for the evolution of bacteria in a chemostat [14, 16]. With this interpretation, $u^\varepsilon(t,x)$ represents the concentration of bacteria with trait x at time t, the function $\eta_i(x)$ represents the growth rate of the population of trait x due to the consumption of a resource whose concentration is I_i^ε, and the term -1 corresponds to the decrease of the bacteria concentration due to the constant flow out of the chemostat. Note that the growth and mutation terms in (1.1) are of different nature, as the total mutation rate $\int_{\mathbb{R}} K(z)$ of the population of trait x, which could either be produced by mutations during individuals' lives or occuring at births, does not depend on the resources concentrations I_i^ε. Note also that the resources consumption rates depend linearly on the resources concentrations. From a biological point of view, one might prefer a saturating functional response. Our method actually extends to all the models of [9, Thm. 1], including competitive Lotka-Volterra PDEs, but we chose here to keep the model simple to ease the presentation of the mathematical arguments. This model extends the one proposed in [16] to an arbitrary number of resources.

This equation has to be coupled with equations for the resources I_i, namely

$$I_i(t) = \frac{c_i}{1 + \int_{\mathbb{R}} \eta_i(x)\,u_\varepsilon(x)\,dx}, \qquad (1.3)$$

where $c_i > 0$. This corresponds to an assumption of fast resources dynamics with respect to the evolutionary dynamics. The resources concentrations are assumed to be at a (quasi-)equilibrium at each time t, which depends on the current concentrations u^ε.

The limit $\varepsilon \to 0$ corresponds to a simultaneous scaling of fast selection and small mutations. It was already considered in [16]. The following argument

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explains what limit behaviour for u^ε can be expected when $\varepsilon \to 0$. Defining φ_ε as

$$u_\varepsilon = e^{\varphi_\varepsilon/\varepsilon}, \quad \text{or } \varphi_\varepsilon = \varepsilon \log u_\varepsilon, \tag{1.4}$$

one gets the equation

$$\partial_t \varphi_\varepsilon = \sum_{i=1}^{k} I_i^\varepsilon(t)\, \eta_i(x) - 1 + H_\varepsilon(\varphi_\varepsilon), \tag{1.5}$$

where

$$H_\varepsilon(f) = \int_{\mathbb{R}} K(z)\, \left(e^{(f(x+\varepsilon z) - f(x))/\varepsilon} - 1 \right)\, dz. \tag{1.6}$$

At the limit $\varepsilon \to 0$ the Hamiltonian H_ε simply becomes

$$H(p) = \int_{\mathbb{R}} K(z)\, \left(e^{pz} - 1 \right)\, dz, \tag{1.7}$$

where p stands for $\partial_x f(x)$. So one expects Eq. (1.5) to lead to

$$\partial_t \varphi = \sum_{i=1}^{k} I_i(t)\, \eta_i(x) - 1 + H(\partial_x \varphi), \tag{1.8}$$

for some I_i which are unfortunately unknown since one cannot pass to the limit directly in (1.3). Therefore one needs to find a relation between the φ and the I_i at the limit. Under quite general assumptions on the parameters (see Lemma 3.1 below), the total population mass $\int_{\mathbb{R}} u_\varepsilon$ is uniformly bounded over time. This suggests that $\max_{x \in \mathbb{R}} \varphi(t, x) = 0$ should hold true for all $t \geq 0$. Together with (1.8), this gives a candidate for the limit dynamics as a solution to a Hamilton-Jacobi equation with Hamiltonian H and with unknown Lagrange multipliers I_i, subject to a maximum constraint. The limit population is then composed at time t of Dirac masses located at the maxima of $\varphi(t, \cdot)$.

This heuristics was justified in [16] in the case of a single resource (and when the resources evolve on the same time scale as the population), and the case of two resources was only partly solved. The mathematical study of the convergence to the Hamilton-Jacobi equation with maximum constraint and the study of the Hamilton-Jacobi equation itself have only be done in very specific cases [16, 2, 5, 26]. In fact the main problem in this proposed model is that the number of unknowns (the resources) may easily be larger than the

dimension of the constraint (formally equal to the number of points where $\varphi = 0$).

Our goal in this paper is to prove the convergence of φ_ε to a solution of (1.8), where we give a full characterization of the functions I_i. Those are no more considered as Lagrange multipliers for a set of constraints but are given by the solution φ itself. The new resulting model describes the evolution of a population as Dirac masses and is formally well posed.

The general problem of characterizing evolutionary dynamics as sums of Dirac masses under biologically relevant parameter scalings is a key tool in *adaptive dynamics*—a branch of biology studying the interplay between ecology and evolution [20, 22, 23, 13, 7]. The phenomenon of *evolutionary branching*, where evolution drives an (essentially) monotype population to subdivide into two (or more) distinct coexisting subpopulations, is particularly relevant in this framework [23, 18, 19]. When the population state can be approximated by Dirac masses, this simply corresponds to the transition from a population composed of a single Dirac mass to a population composed of two Dirac masses.

Several mathematical approaches have been explored to study this phenomenon. One approach consists in studying the stationary behaviour of an evolutionary model involving a scaling parameter for mutations, and then letting this parameter converge to 0. The stationary state has been proved to be composed of one or several Dirac masses for various models (for deterministic PDE models, see [4, 5, 12, 21, 17], for Fokker-Planck PDEs corresponding to stochastic population genetics models, see [3], for stochastic models, see [27], for game-theoretic models, see [11]). Closely related to these works are the notions of ESS (evolutionarily stable strategies) and CSS (convergence stable strategies) [23, 14], which allow one in some cases to characterize stable stationary states [4, 12, 21, 11].

The other main approach consists in studying a simultaneous scaling of mutation and selection, in order to obtain a limit dynamics where transitions from a single Dirac mass to two Dirac masses could occur. Here again, deterministic and stochastic approaches have been explored. The deterministic approach consists in applying the scaling of (1.1). The first formal results have been obtained in [16]. This was followed by several works on other models and on the corresponding Hamilton-Jacobi PDE [5, 26]. For models of the type we consider here, rigorous results (especially for the well posedness of the Hamilton-Jacobi eq. at the limit) mainly only exist in the case with just one resource, see [2] and [1] (one resource but multidimensional traits).

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The stochastic approach is based on individual-based models, which are related to evolutionary PDE models as those in [12, 21] through a scaling of large population [8]. Using a simultaneous scaling of large population and rare mutations, a stochastic limit process was obtained in [6] in the case of a monotype population (i.e. when the limit process can only be composed of a single Dirac mass), and in [10] when the limit population can be composed of finitely many Dirac masses.

Finally note that the total population of individuals is typically very high, for bacteria for example. This is why even stochastic models will usually take some limit with infinite populations. Of course, this has some drawbacks. In particular the population of individuals around a precise trait may turn out to be low (even though the total population is large). As in the scaling under consideration, one has growth or decay of order $\exp(C/\varepsilon)$, it is in fact quite common that the population density of large sets of traits becomes much smaller than the density corresponding to a single individual, making the notion of density meaningless. One of the most important open problem would be to derive models that are both able of dealing with very large populations and still treat correctly the small subpopulations (keeping the stochastic effects or at least truncating the population with less than 1 individual).

There are already some attempts in this direction, mainly proposing models with truncation, see [25] and very recently [24]. In these works however the truncation is only made at some level $\exp(-C/\varepsilon)$.

In order to state our main result, we need some regularity and decay assumptions on the η_i, namely

$$\eta_i > 0, \quad \exists \bar{\eta} \in C_0(\mathbb{R}), \quad \forall x, \quad \sum_{i=1}^{k} \Big(|\eta_i(x)| + |\eta_i'(x)| + |\eta_i''(x)| \Big) \leq \bar{\eta}(x), \quad (1.9)$$

where $C_0(\mathbb{R})$ is the set of continuous function, tending to 0 as $x \to \pm\infty$.

Our method of characterization of the environmental variables I_i in (1.8) is based on the general principle that there exists a unique environment $I_1, \ldots, I_k$ corresponding to any given population state. Here, the population state at time t is given by the set $\{x \in \mathbb{R} : \varphi(t, x) = 0\}$. This principle was the basis of the work of Barles and Perthame [2] in the case of a single resource, where the previous correspondance is automatically satisfied under very general assumptions. In our case, we characterize I_i from $\omega := \{\varphi(t, \cdot) =$

0} by first constructing a *population measure* corresponding to the set ω, which has support in this set and which is a sort of metastable population equilibrium when there is no mutations (more specifically, we require it to be an ESS in the set of population measures with support in ω, see Prop. 1.1 below and [21]). Next, we construct the I_i from this population measure. The following assumptions characterize the situations where our approach can be applied, that is where uniqueness holds in the two steps previously described. The first assumption deals with the number of possible roots of the reproduction rate:

$$\exists 1 \leq \bar{k} \leq k, \ \forall I_1 \in [0, c_1], \ldots, I_k \in [0, c_k],$$

$$\text{the function } x \mapsto \sum_{i=1}^{k} I_i \, \eta_i(x) - 1 \text{ has at most } \bar{k} \text{ roots.} \quad (1.10)$$

We also require an invertibility condition on the matrix $\eta_i(x_j)$

$$\forall x_1 \ldots x_{\bar{k}} \text{ distinct, the } \bar{k} \text{ vectors of } \mathbb{R}^k$$

$$(\eta_i(x_1))_{i=1\ldots k}, \ldots, (\eta_i(x_{\bar{k}}))_{i=1\ldots k} \text{ are linearly independent} \quad (1.11)$$

Assumption (1.11) holds for generical choices of the functions η_i. Assumption (1.10) is more restrictive, specifically if $k = 1$, since it basically means that η_1 is monotone.

Note however that sometimes, one may have uniqueness of the environmental variables whereas the population measure is not unique (and (1.11) is violated). Most of our method would remain valid in such a case. This is the situation of a single resource, where almost nothing is required [2]. Nevertheless in more general situations, the conditions for which this kind of property holds are not currently identified. Even though (1.10) and (1.11) are likely more restrictive than would be strictly necessary, they have the advantage of being explicit and of enabling us to reconstruct the limit of the original u_ε. Moreover from the point of view of the biological interpretation, it is interesting to have a population equilibrium.

It may be hard to check (1.10) in specific models, but it is at least satisfied in large classes of parameters, for example if the derivatives $\eta_i^{(k)}$ are positive (or negative) for all $i = 1, \ldots, k$, in which case $\bar{k} = k$. For instance in the case $k = 2$, (1.10) and (1.11) are ensured by the convexity (or concavity) of η_1 and η_2.

This issue of uniqueness of the environmental variables is of course easier if the environment is larger. In particular, it would be considerably simplified by considering a model with (implicitly) infinitely many resources, like the Lotka-Volterra model

$$\partial_t u_\varepsilon(t,x) = \frac{1}{\varepsilon}\left(r(x) - \int_{\mathbb{R}} b(x-y)u_\varepsilon(t,y)dy\right) u_\varepsilon(t,x) + M_\varepsilon(u_\varepsilon)(t,x),$$

with an even function b. In this case, assumptions (1.10) and (1.11) should be replaced by the much simpler assumption that b as an operator is positive, i.e. that the Fourier transform of b is positive (see [9]). However, we chose to study here a model with explicit resources, as these are easier to interpret in terms of biological modeling.

Now, we may uniquely define the metastable measure associated with a set ω by

Proposition 1.1 *For any closed $\omega \subset \mathbb{R}$, there exists a unique finite nonnegative measure $\mu(\omega)$ satisfying*
i) $supp\,\mu \subset \omega$
ii) denoting $\bar{I}_i(\mu) = c_i/(1 + \int \eta_i(x)\,d\mu(x))$,

$$\sum_{i=1}^{k} \bar{I}_i(\mu)\,\eta_i(x) - 1 \leq 0 \ in\ \omega, \quad \sum_{i=1}^{k} \bar{I}_i(\mu)\,\eta_i(x) - 1 = 0 \ on\ supp\,\mu.$$

Now the limiting I_i in (1.8) are directly obtained by

$$I_i(t) = \bar{I}_i(\mu(\{x \in \mathbb{R} : \varphi(t,x) = 0\})), \tag{1.12}$$

i.e. φ is solution to the (closed) Hamilton-Jacobi equation

$$\partial_t \varphi = \sum_{i=1}^{k} \bar{I}_i(\mu(\{\varphi(t,\cdot) = 0\}))\,\eta_i(x) - 1 + H(\partial_x \varphi). \tag{1.13}$$

We prove

Theorem 1.1 *Assume $K \in C_c^\infty(\mathbb{R})$, $\int_{\mathbb{R}} zK(z)\,dz = 0$, (1.9), (1.10), (1.11), that the initial data $u_\varepsilon(t=0) > 0$ or $\varphi_\varepsilon(t=0)$ are C^2, satisfy*

$$\sup_\varepsilon \int_{\mathbb{R}} u_\varepsilon(t=0,x)\,dx < \infty, \quad \sup_\varepsilon \|\partial_x \varphi_\varepsilon(t=0,\cdot)\|_{L^\infty(\mathbb{R})} < \infty, \tag{1.14}$$

$$\inf_\varepsilon \inf_{x \in \mathbb{R}} \partial_{xx}\varphi(t=0,x) > -\infty, \tag{1.15}$$

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and that $\varphi_\varepsilon(t = 0, \cdot)$ converges to a function φ^0 for the norm $\| \cdot \|_{W^{1,\infty}(\mathbb{R})}$.
Then up to the extraction of a subsequence in ε, φ_ε converges to some φ
uniformly on any compact subset of $[0, T] \times \mathbb{R}$ and in $W^{1,p}([0, T] \times K)$ for
any $T > 0$, $p < \infty$ and any compact K. In particular, φ is continuous.
The function I_i^ε converges to I_i in $L^p([0, T])$ for any $T > 0$, $p < \infty$, where
I_i is defined from φ as in (1.12), and I_i is approximately right-continuous
*for **all** $t \geq 0$. The function φ is a solution to (1.8) almost everywhere in*
t, x with initial condition $\varphi(t = 0, \cdot) = \varphi^0$. Moreover if one defines $\psi =$
$\varphi - \sum_{i=1}^k \int_0^t I_i(s) \, ds \, \eta_i(x)$, then ψ is a viscosity solution to

$$\partial_t \psi(t, x) = H\left(\partial_x \psi + \sum_{i=1}^k \int_0^t I_i(s) \, ds \, \eta_i(x) \right). \qquad (1.16)$$

We recall that a function f on $[0, +\infty)$ is approximately right-continuous at
$t \geq 0$ if t is a point of Lebesgue right-continuity of f, i.e.

$$\lim_{s \to 0} \frac{1}{s} \int_t^{t+s} |f(\theta) - f(t)| d\theta = 0.$$

Notice that, under the assumptions of Theorem 1.1, $\varphi_\varepsilon(t = 0, x) \to -\infty$
when $x \to \pm\infty$ since $\int_\mathbb{R} u_\varepsilon < \infty$ and φ_ε is uniformly Lipschitz. Be also
careful that we assume $\|\varphi_\varepsilon(t = 0, \cdot) - \varphi^0\|_{W^{1,\infty}(\mathbb{R})} \to 0$ even though $\varphi_\varepsilon(t = 0)$
(and thus φ^0) is not bounded.

From a practical point of view, computing the solution u_ε of Eq. (1.1) is often
too costly for small ε. This result allows to approximate the population
density u_ε for small ε by the simpler $\mu(\{\varphi(t, \cdot) = 0\})$, where φ may be
obtained by a discretization of (1.13), in the fashion of those done in [16].
Rigorous numerical analysis of this kind of Hamilton-Jacobi equations is
however still very preliminary.

On a more theoretical level, this theorem justifies in simple cases the classical
view of evolution (population at local equilibrium at every time). It also
indicates that for small ε, the behaviour of the population is determined
by a closed equation which means that evolution remains deterministic and
stable in the parameter ε.

In the proofs below, C denotes a numerical constant which may change from
line to line.

2 Proof of Prop. 1.1

2.1 Uniqueness

Assume that two measures μ_1 and μ_2 satisfy both points of Prop. 1.1. We first prove that they induce the same resources $\bar{I}_i$ and then conclude that they are equal.

1st step: Uniqueness of the $\bar{I}_i$. The argument here is essentially an adaptation of [21]. First note that

$$\int_{\mathbb{R}} \left(\sum_{i=1}^{k} \bar{I}_i(\mu_1)\, \eta_i(x) - 1 \right) d\mu_2 + \int_{\mathbb{R}} \left(\sum_{i=1}^{k} \bar{I}_i(\mu_2)\, \eta_i(x) - 1 \right) d\mu_1 \leq 0, \quad (2.1)$$

since μ_1 and μ_2 are non negative and by the point ii, $\sum_{i=1}^{k} \bar{I}_i(\mu_j)\, \eta_i(x) - 1$ is non positive on ω for $j = 1, 2$.
On the other hand since $\sum_{i=1}^{k} \bar{I}_i(\mu_j)\, \eta_i - 1$ vanishes on the support of μ_j, one has for instance

$$\int_{\mathbb{R}} \left(\sum_{i=1}^{k} \bar{I}_i(\mu_1)\, \eta_i - 1 \right) d\mu_2 = \int_{\mathbb{R}} \left(\sum_{i=1}^{k} (\bar{I}_i(\mu_1) - \bar{I}_i(\mu_2))\, \eta_i \right) d\mu_2$$

$$= \sum_{i=1}^{k} (\bar{I}_i(\mu_1) - \bar{I}_i(\mu_2)) \int_{\mathbb{R}} \eta_i \, d\mu_2$$

$$= \sum_{i=1}^{k} (\bar{I}_i(\mu_1) - \bar{I}_i(\mu_2))\, (c_i/\bar{I}_i(\mu_2) - 1),$$

by the definition of $\bar{I}_i(\mu_2)$.
Since one has

$$\sum_{i=1}^{k} c_i\, (\bar{I}_i(\mu_1) - \bar{I}_i(\mu_2))\, (1/\bar{I}_i(\mu_2) - 1/\bar{I}_i(\mu_1)) = \sum_{i=1}^{k} c_i\, \frac{(\bar{I}_i(\mu_1) - \bar{I}_i(\mu_2))^2}{\bar{I}_i(\mu_1)\, \bar{I}_i(\mu_2)} \geq 0.$$

one deduces from (2.1) that

$$\bar{I}_i(\mu_1) = \bar{I}_i(\mu_2), \quad i = 1 \ldots k. \tag{2.2}$$

2nd step: Uniqueness of μ. It is not possible to deduce that $\mu_1 = \mu_2$ directly from (2.2). This degeneracy (the possibility of having several equilibrium

measures, all corresponding to the same environment) is the reason why we require additional assumptions on the η_i.

First of all by Assumption (1.10), point i and thanks to (2.2), we know that μ_1 and μ_2 are both supported on a set consisting of at most $\bar{k}$ points $\{x_1, \ldots, x_{\bar{k}}\}$, which are the roots of $\sum_i \bar{I}_i(\mu_1)\eta_i(x) - 1 = \sum_i \bar{I}_i(\mu_2)\eta_i(x) - 1$ (possibly completed by arbitrary distinct points if this function has less than $\bar{k}$ roots). Therefore one may write

$$\mu_j = \sum_{l=1}^{\bar{k}} \alpha_l^j \, \delta_{x_l}.$$

Now (2.2) tells that $\int \eta_i d\mu_1 = \int \eta_i d\mu_2$ which means that

$$\sum_{l=1}^{\bar{k}} \alpha_l^1 \, \eta_i(x_l) = \sum_{l=1}^{\bar{k}} \alpha_l^2 \, \eta_i(x_l), \quad \forall i = 1 \ldots k.$$

To conclude it remains to use condition (1.11) and get that $\alpha_l^1 = \alpha_l^2$.

2.1.1 Existence

The basic idea to get existence is quite simple: Solve the equation[1]

$$\partial_t \nu = \left(\sum_{i=1}^{k} \bar{I}_i(\nu) \, \eta_i(\cdot) - 1 \right) \nu, \tag{2.3}$$

and obtain the equilibrium measure μ as the limit of $\nu(t)$ as $t \to +\infty$. This is done by considering the entropy

$$L(\nu) = \sum_{i=1}^{k} c_i \, \log \bar{I}_i(\nu) + \int d\nu$$

$$= \sum_{i=1}^{k} c_i \log c_i - \sum_{i=1}^{k} c_i \, \log \left(1 + \int \eta_i \, d\nu \right) + \int d\nu. \tag{2.4}$$

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[1]Existence and uniqueness are trivial for (2.3), for example by Cauchy-Lipschitz theorem in the set of finite positive measures equipped with the total variation norm.

As $-\log$ is convex and $\eta_i \geq 0$, then L itself is a convex function of ν. Moreover if $\nu(t)$ solves (2.3), one has

$$\frac{d}{dt}L(\nu(t)) = -\int \left(\sum_{i=1}^k \bar{I}_i(\nu)\,\eta_i(x) - 1\right)^2 d\nu. \qquad (2.5)$$

Therefore one expects the limit of ν and the equilibrium measure we are looking for to be the minimum of L.

Since the η_i are bounded, one finds

$$L(\nu) \geq -C + c\int d\nu,$$

for two numerical constants C and c. Consequently L is bounded from below on $M_+^1(\omega)$ the set of nonnegative Radon measures on ω. In addition, one may restrict to a bounded subset of $M_+^1(\omega)$ to compute the infimum of L. As any ball of $M_+^1(\omega)$ is compact for the weak-* topology (dual of continuous functions with compact support), L attains its infimum, or

$$M_0 = \{\nu \in M_+^1(\omega),\ L(\nu) \leq L(\nu')\ \forall \nu' \in M_+^1(\omega)\} \neq \emptyset.$$

Now take any $\mu \in M_0$ then take ν the solution to (2.3) with $\nu(t=0) = \mu$. $L(\nu)$ is non increasing and since it is already at a minimum initially, it is necessarily constant. By (2.5), this means that

$$\sum_{i=1}^k \bar{I}_i(\mu)\,\eta_i - 1 = 0 \text{ on } \operatorname{supp}\mu.$$

Hence μ is in fact a stationary solution to (2.3) and it satisfies point i and the second part of point ii of Prop. 1.1. Note by the way that the uniqueness argument in fact tells that there is a unique element in M_0.

It only remains to check the first part of point ii. By contradiction assume that there exists a point $x_0 \in \omega$ s.t.

$$\sum_{i=1}^k \bar{I}_i(\mu)\,\eta_i(x_0) - 1 > 0.$$

Let $\alpha > 0$ and define $\nu_\alpha = \mu + \alpha \delta_{x_0} \in M^1_+(\omega)$. Now compute

$$L(\nu_\alpha) = \int d\mu + \alpha - \sum_{i=1}^k c_i \log\left(1 + \int \eta_i d\mu + \alpha \eta_i(x_0)\right) + \sum_{i=1}^k c_i \log c_i$$

$$= \int d\mu + \alpha - \sum_{i=1}^k c_i \left(\log\left(1 + \int \eta_i d\mu\right) + \frac{\alpha \, \eta_i(x_0)}{1 + \int \eta_i d\mu} + O(\alpha^2)\right)$$

$$+ \sum_{i=1}^k c_i \log c_i$$

$$= L(\mu) - \alpha \left(\sum_{i=1}^k \bar{I}_i(\mu) \, \eta_i(x_0) - 1\right) + O(\alpha^2).$$

Thus $L(\nu_\alpha) < L(\mu)$ for α small enough which is impossible as μ is an absolute minimum of L.

Consequently the first part of ii is satisfied and the proof of Prop. 1.1 complete.

3   Proof of Theorem 1.1

3.1   A priori estimates for Eq. (1.5)

We denote by $BV_{\text{loc}}(\mathbb{R})$ the set of functions on $\mathbb{R}$ with bounded variation on any compact subset of $\mathbb{R}$, by $M^1(\omega)$ the set of signed Radon measures on the subset ω of $\mathbb{R}$ equipped with the total variation norm.

We show the following estimates on the solution to (1.5)

Lemma 3.1 *Let φ_ε be a solution to (1.5) with initial data φ_ε^0 such that $\int_{\mathbb{R}} e^{\varphi_\varepsilon^0(x)/\varepsilon} dx < \infty$, $\partial_x \varphi_\varepsilon^0 \in L^\infty(\mathbb{R})$ and $\partial_{xx}\varphi_\varepsilon^0$ uniformly bounded from below. Then for any $T > 0$*

$$\|\partial_t \varphi_\varepsilon\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x \varphi_\varepsilon\|_{L^\infty([0,T],BV_{\text{loc}}(\mathbb{R})\cap L^\infty(\mathbb{R}))} + \|\partial_{tx}\varphi_\varepsilon\|_{L^\infty([0,T],M^1)} \le C_T,$$
$$\forall\, t \le T,\, x \in \mathbb{R}, \quad \partial_{xx}\varphi_\varepsilon(t,x) \ge -C_T, \quad H_\varepsilon(\varphi_\varepsilon) \ge -C_T\varepsilon,$$

$$\forall\, t \le T,\, \int_{\mathbb{R}} u_\varepsilon(t,x)\, dx \le C_T, \quad \varphi_\varepsilon(t,x) \le C_T\varepsilon\, \log 1/\varepsilon.$$

where C_T only depends on the time T, $\int_{\mathbb{R}} e^{\varphi_\varepsilon^0(x)/\varepsilon} dx$, $\|\partial_x \varphi_\varepsilon^0\|_{L^\infty(\mathbb{R})}$ and the infimum of $\partial_{xx}\varphi_\varepsilon^0(x)$.

Proof. We start with the easy bound on the total mass.

Step 0: Bound on the total mass. First notice that because of (1.9), there exists $R > 0$ s.t.

$$\forall |x| > R, \quad \sum_{i=1}^{k} \eta_i(x) \leq 1/2.$$

Let ψ be a regular test function with support in $|x| > R$, taking values in $[0, 1]$ and equal to 1 on $|x| > R+1$. Using the fact that $I_i^\varepsilon(t) \leq 1$, we compute

$$\frac{d}{dt} \int_{\mathbb{R}} \psi(x) \, u_\varepsilon(t, x) \, dx \leq - \frac{1}{2\varepsilon} \int_{\mathbb{R}} \psi(x) \, u_\varepsilon(t, x) \, dx$$

$$+ \frac{1}{\varepsilon} \int_{\mathbb{R}^2} K(z)(\psi(x - \varepsilon z) - \psi(x)) \, u_\varepsilon(t, x) \, dz \, dx$$

$$\leq - \frac{1}{2\varepsilon} \int_{\mathbb{R}} \psi(x) \, u_\varepsilon(t, x) \, dx + C \int_{\mathbb{R}} u_\varepsilon(t, x) \, dx.$$

On the other hand as each $\eta_i > 0$, one has for some constant C

$$I_i^\varepsilon(t) = \frac{c_i}{1 + \int \eta_i \, u_\varepsilon \, dx} \leq \frac{C}{1 + \int (1 - \psi) u_\varepsilon \, dx}.$$

Therefore with the same kind of estimate

$$\frac{d}{dt} \int_{\mathbb{R}} (1 - \psi(x)) \, u_\varepsilon(t, x) \, dx \leq C \int_{\mathbb{R}} u_\varepsilon(t, x) \, dx$$

$$+ \frac{1}{\varepsilon} \left(\frac{C}{1 + \int (1 - \psi) u_\varepsilon \, dx} - 1 \right) \int_{\mathbb{R}} (1 - \psi(x)) \, u_\varepsilon(t, x) \, dx.$$

Summing the two

$$\frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon(t, x) \, dx \leq \frac{1}{\varepsilon} \left(\frac{C}{1 + \int (1 - \psi) u_\varepsilon \, dx} - 1 \right) \int_{\mathbb{R}} (1 - \psi(x)) \, u_\varepsilon(t, x) \, dx$$

$$- \frac{1}{2\varepsilon} \int_{\mathbb{R}} \psi(x) \, u_\varepsilon(t, x) \, dx + C \int_{\mathbb{R}} u_\varepsilon(t, x) \, dx.$$

Since the sum of the first two terms of the r.h.s. is negative if $\int u_\varepsilon$ is larger than a constant independent of ε, this shows that $\int u_\varepsilon(t, x) \, dx$ remains uniformly bounded on any finite time interval.

Step 1: Bound on $\partial_x \varphi_\varepsilon$. This is a classical bound for solutions to Hamilton-Jacobi equations. Here we still have to check that it remains true uniformly at the ε level. Compute

$$
\begin{aligned}
\partial_t \partial_x \varphi_\varepsilon = {} & \sum_{i=1}^k I_i^\varepsilon(t)\, \eta_i'(x) \\
& + \int K(z)\, e^{\frac{\varphi_\varepsilon(t,x+\varepsilon z)-\varphi_\varepsilon(t,x)}{\varepsilon}}\, \frac{\partial_x \varphi_\varepsilon(t,x+\varepsilon z)-\partial_x \varphi_\varepsilon(t,x)}{\varepsilon}\, dz.
\end{aligned}
\tag{3.1}
$$

We first observe that, as $I_i^\varepsilon \in [0,\ \max_i c_i]$ and $\sum_i |\eta_i'(x)| \le \bar\eta(x)$

$$
\begin{aligned}
|\partial_t \partial_x \varphi_\varepsilon| & \le \max_i c_i\ \bar\eta(x) + \frac{2}{\varepsilon} \int_{\mathbb{R}} K(z) e^{|z|\, \|\partial_x \varphi_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}} \|\partial_x \varphi_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\, dz \\
& \le \frac{C}{\varepsilon} e^{C\|\partial_x \varphi_\varepsilon(t,\cdot)\|_{L^\infty(\rho)}},
\end{aligned}
$$

since K has compact support. This entails

$$
\|\partial_x \varphi_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|\partial_x \varphi_\varepsilon^0\|_{L^\infty(\mathbb{R})} + \frac{C}{\varepsilon} \int_0^t e^{C\|\partial_x \varphi_\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R})}}\, ds,
$$

from which easily follows that $\partial_x \varphi_\varepsilon \in L^\infty([0,t_\varepsilon],\mathbb{R})$ for some $t_\varepsilon > 0$, which may (for the moment) depend on ε.

Now we use the classical maximum principle. Fix $t \in [0,T]$ such that $C_{\varepsilon,t} := \|\partial_x \varphi_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$. For any $x \in \mathbb{R}$ such that $\partial_x \varphi_\varepsilon(t,x) > \sup_y \partial_x \varphi_\varepsilon(t,y) - \alpha$, where the constant $\alpha > 0$ will be specified later, we have

$$
\partial_t \partial_x \varphi_\varepsilon(t,x) \le \max_i c_i\ \bar\eta(x) + \int_{\mathbb{R}} K(z) e^{|z|\, C_{t,\varepsilon}} \frac{\alpha}{\varepsilon}\, dz \le C\Big(1 + \frac{\alpha}{\varepsilon} e^{C\, C_{t,\varepsilon}}\Big).
$$

Therefore, choosing $\alpha = \varepsilon e^{-C\, C_{t,\varepsilon}}$, we obtain

$$
\frac{d}{dt} \sup_x \partial_x \varphi_\varepsilon(t,x) \le C,
$$

for a constant C independent of $t < t_\varepsilon$ and of ε. Using a similar argument for the minimum, we deduce that $t_\varepsilon > T$ and that $\partial_x \varphi_\varepsilon$ is bounded on $[0,T] \times \mathbb{R}$ by a constant depending only on T and $\|\partial_x \varphi_\varepsilon^0\|_{L^\infty(\mathbb{R})}$.

Step 2: First bound on $H_\varepsilon(\varphi_\varepsilon)$ and bounds on $\partial_t \varphi_\varepsilon$ and φ_ε. Simply note that

$$
\begin{aligned}
-\int_{\mathbb{R}} K(z)\, dz \le H_\varepsilon(\varphi_\varepsilon(t))(x) & = \int_{\mathbb{R}} K(z)\, e^{\frac{\varphi_\varepsilon(t,x+\varepsilon z)-\varphi_\varepsilon(t,x)}{\varepsilon}}\, dz - \int_{\mathbb{R}} K(z)\, dz \\
& \le \int K(z)\, e^{|z|\, \|\partial_x \varphi_\varepsilon\|_{L^\infty([0,T],\mathbb{R})}}\, dz \le C.
\end{aligned}
$$

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Consequently, directly from Eq. (1.5),

$$|\partial_t \varphi_\varepsilon| \leq \max_i c_i \, \bar\eta(x) + C,$$

hence ending the proof of the full Lipschitz bound on φ_ε.
To get the upper bound on φ_ε, simply note that because of the uniform Lipschitz bound on φ_ε

$$\varphi_\varepsilon(t,y) \geq \varphi_\varepsilon(t,x) - C_T \, |y - x|,$$

so

$$\int_{\mathbb{R}} u_\varepsilon(t,y) \, dy \geq \int_{\mathbb{R}} e^{\varphi_\varepsilon(t,x)/\varepsilon} \, e^{-C_T \, |y-x|/\varepsilon} \, dy \geq 2C_T^{-1} \, \varepsilon \, e^{\varphi_\varepsilon(t,x)/\varepsilon}.$$

Hence the bound on the total mass yields that $\varphi_\varepsilon \leq C_T \, \varepsilon \, \log 1/\varepsilon$.

Step 3: BV bound on $\partial_x \varphi_\varepsilon$. As for $\partial_x \varphi_\varepsilon$, we begin with a maximum (actually, minimum) principle. First from (1.5)

$$\partial_t \partial_{xx} \varphi_\varepsilon \geq -\bar\eta(x) + \int_{\mathbb{R}} K(z) \, e^{\frac{\varphi_\varepsilon(t,x+\varepsilon z) - \varphi_\varepsilon(t,x)}{\varepsilon}} \, \frac{\partial_{xx}\varphi_\varepsilon(t, x + \varepsilon z) - \partial_{xx}\varphi_\varepsilon(t, x)}{\varepsilon} \, dz$$
$$+ \int_{\mathbb{R}} K(z) \, e^{\frac{\varphi_\varepsilon(t,x+\varepsilon z) - \varphi_\varepsilon(t,x)}{\varepsilon}} \, \frac{(\partial_x\varphi_\varepsilon(t, x + \varepsilon z) - \partial_x\varphi_\varepsilon(t, x))^2}{\varepsilon} \, dz.$$

The last term is of course non negative and so with the same argument as before, we get

$$\frac{d}{dt} \inf_x \partial_{xx}\varphi_\varepsilon(t,x) \geq -C,$$

where C does not depend on ε. This proves the uniform lower bound on $\partial_{xx}\varphi_\varepsilon$. On the other hand, for any measurable subset A of $[x_1, x_2]$,

$$\int_{x_1}^{x_2} (\mathbb{I}_{x \in A} - \mathbb{I}_{x \notin A})\partial_{xx}\varphi_\varepsilon(t,x) \, dx = \int_{x_1}^{x_2} \partial_{xx}\varphi_\varepsilon(t,x) \, dx$$
$$- 2\int_{x_1}^{x_2} \partial_{xx}\varphi_\varepsilon(t,x) \, \mathbb{I}_{x \notin A} dx$$
$$\leq \partial_x\varphi_\varepsilon(t,x_2) - \partial_x\varphi_\varepsilon(t,x_1) + C \, |x_2 - x_1| \leq 2\|\partial_x\varphi_\varepsilon\|_{L^\infty([0,T],\mathbb{R})} + C \, |x_2 - x_1|.$$

This indeed shows that $\partial_{xx}\varphi_\varepsilon(t,\cdot)$ belongs to $M^1([x_1, x_2])$ with total variation norm less than $2\|\partial_x\varphi_\varepsilon\|_{L^\infty([0,T],\mathbb{R})} + C \, |x_2 - x_1|$. Thus, $\partial_{xx}\varphi_\varepsilon$ belongs to the space $L^\infty([0, T], M^1(\mathbb{R}))$, which entails $\partial_x\varphi_\varepsilon \in L^\infty([0,T], BV_{\text{loc}}(\mathbb{R}))$.

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Finally, it follows from (3.1) that

$$|\partial_{tx}\varphi_\varepsilon(t,x)| \leq \bar\eta(x)\max_i c_i + \int K(z)e^{|z|\,\|\partial_x\varphi_\varepsilon\|_{L^\infty}}\frac{|\partial_x\varphi_\varepsilon(t,x+\varepsilon z)-\partial_x\varphi_\varepsilon(t,x)|}{\varepsilon}\,dz$$

$$\leq c\left(\bar\eta(x) + \int K(z)\int_0^z |\partial_{xx}\varphi_\varepsilon(t,x+\varepsilon\theta)|\,d\theta dz\right).$$

Integrating, by Fubini

$$\int_{x_1}^{x_2}|\partial_{tx}\varphi_\varepsilon|\,dx \leq C\int_{x_1}^{x_2}\bar\eta(x)\,dx + C\int_{x_1-\varepsilon\rho}^{x_2+\varepsilon\rho}|\partial_{xx}\varphi_\varepsilon|\,dx,$$

where ρ is such that the support of K is included in the ball centered at 0 of radius ρ. This ends the proof of all the bounds on the derivatives of φ_ε.

Conclusion. It only remains to show the sharp lower bound on $H_\varepsilon(\varphi_\varepsilon)$. Let us write

$$H_\varepsilon(\varphi_\varepsilon) \geq \int_{\mathbb{R}} K(z)\exp\left(\int_0^1 z\,\partial_x\varphi_\varepsilon(t,x+\theta z\,\varepsilon)\,d\theta\right)dz - \int_{\mathbb{R}}K(z)\,dz.$$

The BV bound on $\partial_x\varphi_\varepsilon$ shows that this function admits right and left limits at all $x \in \mathbb{R}$. Let us denote $\partial_x\varphi_\varepsilon(t,x^+)$ the limit on the right and $\partial_x\varphi_\varepsilon(t,x^-)$ the limit on the left. As $\partial_{xx}\varphi_\varepsilon$ is bounded from below, we know in addition that

$$\forall x, \quad \partial_x\varphi_\varepsilon(t,x^+) \geq \partial_x\varphi_\varepsilon(t,x^-).$$

By differentiating once more

$$\int_0^1 z\,\partial_x\varphi_\varepsilon(t,x+\theta z\,\varepsilon)\,d\theta \geq z\,\partial_x\varphi_\varepsilon(t,x^+)$$

$$+ \int_0^1 z\int_0^1 \theta\,z\,\varepsilon\,\partial_{xx}\varphi_\varepsilon(t,x+\theta'\theta z\varepsilon)\,d\theta'd\theta$$

$$\geq z\,\partial_x\varphi_\varepsilon(t,x^+) - C\,\varepsilon\,z^2,$$

again as $\partial_{xx}\varphi_\varepsilon$ is bounded from below. Finally

$$H_\varepsilon(\varphi_\varepsilon) \geq \int_{\mathbb{R}}K(z)\exp(z\,\partial_x\varphi_\varepsilon(t,x^+) - C\,\varepsilon\,z^2)\,dz - \int_{\mathbb{R}}K(z)\,dz$$

$$\geq H(\partial_x\varphi_\varepsilon(t,x^+)) - C\,\varepsilon,$$

where H is defined as in (1.7) and since K is compactly supported. Because we assumed that $\int_{\mathbb{R}}zK(z)\,dz = 0$, we have $H(p) \geq 0$ for any p, which ends the proof of Lemma 3.1.

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3.2 Passing to the limit in φ_ε

From the assumptions in Theorem 1.1, Lemma 3.1 gives uniform bounds on φ_ε.

Therefore up to an extraction in ε (still denoted with ε), there exists a function φ on $[0,T] \times \mathbb{R}$ such that $\partial_t \varphi \in L^\infty([0,T] \times \mathbb{R})$, $\partial_x \varphi \in L^\infty([0,T], BV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}))$, $\partial_{tx} \varphi \in L^\infty([0,T], M^1_{\mathrm{loc}}(\mathbb{R}))$ and $\partial_{xx} \varphi$ uniformly lower bounded on $[0,T] \times \mathbb{R}$, satisfying

$$\begin{aligned} \varphi_\varepsilon &\longrightarrow \varphi \quad \text{uniformly in } C(K) \text{ for any compact } K \text{ of } [0,T] \times \mathbb{R}, \\ \partial_x \varphi_\varepsilon &\longrightarrow \partial_x \varphi \quad \text{in any } L^p_{\mathrm{loc}}([0,T], \mathbb{R}), \ p < \infty. \end{aligned} \tag{3.2}$$

The first convergence follows from Arzéla-Ascoli theorem. For the second convergence, observe that $\|\partial_x \varphi_\varepsilon\|_{L^\infty([0,T],BV_{\mathrm{loc}}(\mathbb{R}))} + \|\partial_{tx} \varphi_\varepsilon\|_{L^\infty([0,T],M^1(\mathbb{R}))} \leq C_T$ implies that $\partial_x \varphi_\varepsilon$ is uniformly bounded in $L^\infty([0,T] \times \mathbb{R}) \cap BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$. The convergence in L^p_{loc} follows by compact embedding. We also have $\varphi \leq 0$ since otherwise the uniform bound on $\int_\mathbb{R} u_\varepsilon(t,x) \, dx$ would be contradicted. As the I_i^ε are bounded, it is possible to extract weak-* converging subsequences (still denoted with ε) to some $I_i(t)$.

Now, we write again

$$H_\varepsilon(\varphi_\varepsilon) = \int_\mathbb{R} K(z) \left(\exp\left(\int_0^1 z \, \partial_x \varphi_\varepsilon(t, x + \varepsilon \, z \, \theta) \, d\theta \right) - 1 \right) dz.$$

From the L^∞ bound on $\partial_x \varphi_\varepsilon$ and its strong convergence, one deduces that

$$H_\varepsilon(\varphi_\varepsilon) \longrightarrow H(\partial_x \varphi) \text{ in } L^1_{\mathrm{loc}}. \tag{3.3}$$

Therefore one may pass to the limit in (1.5) and obtain (1.8) (for the moment in the sense of distribution; the equality *a.e.* will follow from the convergence of I_i^ε in $L^p([0,T])$, proved below).

In addition by following [16] or [2], one may easily show that $\psi(t,x) = \varphi(t,x) - \sum_{i=1}^k \int_0^t I_i(s) \, ds \, \eta_i(x)$ is a viscosity solution to (1.16). We refer the reader to these references for this technical part.

It remains to obtain (1.12), the approximate right-continuity of I_i for *all* time t and the convergence of I_i^ε to I_i in $L^p([0,T])$ for $p < \infty$. This requires some sort of uniform continuity on the I_i^ε which is the object of the rest of the proof.

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3.3 Continuity in time for the I_i^ε

First of all note that, as suggested by the simulations of [16], there are examples where the I_i have jumps in time at the limit. So we will only be able to prove their right-continuity.

This regularity in time comes from the stability of the equilibrium defined through (1.12) and Prop. 1.1. Therefore let us define

$$\bar{I}_i(t) = \bar{I}_i(\mu(\{\varphi(t, .) = 0\})),$$

where $\bar{I}_i$ and μ are given by Prop. 1.1 and φ is the uniform limit of φ_ε as taken in the previous subsection.

Our first goal is the following result.

Lemma 3.2 *For any fixed s, there exist functions σ_s, $\tilde{\sigma} \in C(\mathbb{R}_+)$ with $\sigma_s(0) = \tilde{\sigma}(0) = 0$ s.t.*

$$\int_s^t |I_i^\varepsilon(r) - \bar{I}_i(s)|^2 \, dr \le (t - s) \, \sigma_s(t - s) + \tilde{\sigma}(\varepsilon).$$

Remark. Of course the whole point is that σ_s is uniform in ε. It is also crucial for the following that $\tilde{\sigma}$ does not depend on s.

3.3.1 Proof of Lemma 3.2

Step 0: φ has compact level sets.
Observe that $\varphi_\varepsilon(t = 0, x) \to -\infty$ when $x \to \pm\infty$ since $\int_{\mathbb{R}} u_\varepsilon(t = 0, x) \, dx < \infty$ and $\partial_x \varphi(t = 0)$ is bounded. Because of the uniform convergence of $\varphi_\varepsilon(t = 0)$ to φ^0 on $\mathbb{R}$, one deduces that $\varphi^0(x) \to -\infty$ when $x \to \pm\infty$.
Since $\partial_x \varphi \in L^\infty([0, T], \mathbb{R})$ and $I_i(t) \in [0, \max_i c_i]$, it follows from (1.8) that $\partial_t \varphi \in L^\infty([0, T], \mathbb{R})$ and thus $\varphi(t, x) \to -\infty$ when $x \to \pm\infty$ for all $t \ge 0$.
Therefore, the set

$$\Omega := \{(t, x) \in [0, T] \times \mathbb{R} : \varphi(t, x) \ge -1\}$$

is compact.

Step 1: One basic property of $\{\varphi = 0\}$.
Let us start with the following crucial observation

$$\forall s, \ \exists \tau_s \in C(\mathbb{R}_+) \text{ with } \tau_s(0) = 0, \text{ s.t. } \forall t \ge s,$$
$$\forall x \in \{\varphi(t, .) = 0\}, \ \exists y \in \{\varphi(s, .) = 0\} \text{ with } |y - x| \le \tau_s(t - s). \quad (3.4)$$

This is a sort of semi-continuity for $\{\varphi = 0\}$. It is proved very simply by contradiction. If it were not true, then

$$\exists s, \ \exists \tau_0 > 0, \ \exists t_n \to s, \ t_n \geq s, \ \exists y_n \in \{\varphi(t_n, .) = 0\},$$
$$d(y_n, \{\varphi(s, .) = 0\}) \geq \tau_0,$$

where $d(y, \omega) = \inf_{x \in \omega} |x - y|$ is the usual distance.

Since all the y_n belong to the compact set Ω of Step 0, we can extract a converging subsequence $y_n \to y$. As φ is continuous, $\varphi(s, y) = 0$ or $y \in \{\varphi(s, .) = 0\}$. On the other hand one would also have $d(y, \{\varphi(s, .) = 0\}) \geq \tau_0$ which is contradictory.

Step 2: The functional.
Denote

$$\mu_s = \mu(\{\varphi(s, .) = 0\}),$$

as given by Prop. 1.1. We look at the evolution of

$$F_\varepsilon(t) = \int_\mathbb{R} \log u_\varepsilon(t, x) \, d\mu_s(x) = \frac{1}{\varepsilon} \int_\mathbb{R} \varphi_\varepsilon(t, x) \, d\mu_s(x),$$

for $t \geq s$. Compute

$$\frac{d}{dt} F_\varepsilon(t) = \frac{1}{\varepsilon} \int_\mathbb{R} \left(\sum_{i=1}^k I_i^\varepsilon(t) \, \eta_i(x) - 1 \right) d\mu_s(x) + \frac{1}{\varepsilon} \int_\mathbb{R} H_\varepsilon(\varphi_\varepsilon(t)) \, d\mu_s.$$

Now write

$$\frac{1}{\varepsilon} \int_\mathbb{R} \left(\sum_{i=1}^k I_i^\varepsilon(t) \, \eta_i(x) - 1 \right) d\mu_s(x) = \frac{d}{dt} \int_\mathbb{R} u_\varepsilon(t, x) \, dx$$
$$- \frac{1}{\varepsilon} \int_\mathbb{R} \left(\sum_{i=1}^k I_i^\varepsilon(t) \, \eta_i(x) - 1 \right) (u_\varepsilon(t, x) \, dx - d\mu_s(x)).$$

As $\sum_{i=1}^k \bar{I}_i(s) \, \eta_i(x) - 1$ vanishes on the support of μ_s,

$$\frac{1}{\varepsilon} \int_\mathbb{R} \left(\sum_{i=1}^k I_i^\varepsilon(t) \, \eta_i(x) - 1 \right) d\mu_s(x) = \frac{d}{dt} \int_\mathbb{R} u_\varepsilon(t, x) - \frac{A(t)}{\varepsilon}$$
$$- \frac{1}{\varepsilon} \int_\mathbb{R} \left(\sum_{i=1}^k (I_i^\varepsilon(t) - \bar{I}_i(s)) \, \eta_i(x) \right) (u_\varepsilon(t, x) \, dx - d\mu_s(x)),$$

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with

$$A(t) = \int_{\mathbb{R}} \left(\sum_{i=1}^{k} \bar{I}_i(s)\, \eta_i(x) - 1 \right) u_\varepsilon(t, x)\, dx.$$

Notice that

$$\int_{\mathbb{R}} \left(\sum_{i=1}^{k} (I_i^\varepsilon(t) - \bar{I}_i(s))\, \eta_i(x) \right) (u_\varepsilon(t, x)\, dx - d\mu_s(x)) = -\sum_{i=1}^{k} c_i \frac{(I_i^\varepsilon(t) - \bar{I}_i(s))^2}{I_i^\varepsilon(t)\, \bar{I}_i(s)}.$$

So we deduce

$$\frac{1}{\varepsilon} \int_s^t \sum_{i=1}^{k} c_i \frac{(I_i^\varepsilon(r) - \bar{I}_i(s))^2}{I_i^\varepsilon(r)\, \bar{I}_i(s)} dr = \int_{\mathbb{R}} \log \frac{u_\varepsilon(t, x)}{u_\varepsilon(s, x)}\, d\mu_s - \int_{\mathbb{R}} (u_\varepsilon(t, x) - u_\varepsilon(s, x))\, dx$$

$$+ \int_s^t \frac{A(r)}{\varepsilon}\, dr - \frac{1}{\varepsilon} \int_s^t \int_{\mathbb{R}} H_\varepsilon(\varphi_\varepsilon(r))\, d\mu_s.$$

$$(3.5)$$

Step 3: Easy bounds.
Lemma 3.1 tells that

$$-H_\varepsilon(\varphi_\varepsilon) \leq C_T\, \varepsilon.$$

The total mass stays bounded in time so

$$-\int_{\mathbb{R}} (u_\varepsilon(t, x) - u_\varepsilon(s, x))\, dx \leq \int_{\mathbb{R}} (u_\varepsilon(t, x) + u_\varepsilon(s, x))\, dx \leq C.$$

And furthermore

$$\int_{\mathbb{R}} \log \frac{u_\varepsilon(t, x)}{u_\varepsilon(s, x)}\, d\mu_s = \frac{1}{\varepsilon} \int_{\mathbb{R}} (\varphi_\varepsilon(t, x) - \varphi_\varepsilon(s, x))\, d\mu_s$$

$$\leq \frac{1}{\varepsilon} \int_{\mathbb{R}} (\varphi(t, x) - \varphi(s, x))\, d\mu_s + \frac{2}{\varepsilon} \|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)},$$

where the last bound comes from the fact that, by Prop. 1.1, μ_s is supported on $\{\varphi(s, .) = 0\} \subset \Omega$, where Ω is defined in Step 0. Since in addition we know that $\varphi \leq 0$,

$$\int_{\mathbb{R}} \log \frac{u_\varepsilon(t, x)}{u_\varepsilon(s, x)}\, d\mu_s \leq \frac{2}{\varepsilon} \|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)}.$$

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Consequently we deduce from (3.5) the bound

$$\frac{1}{\varepsilon} \int_s^t \sum_{i=1}^k c_i \frac{(I_i^\varepsilon(r) - \bar{I}_i(s))^2}{I_i^\varepsilon(r)\,\bar{I}_i(s)} dr \leq C + \frac{2}{\varepsilon} \|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)} + \int_s^t \frac{A(r)}{\varepsilon} dr. \quad (3.6)$$

Step 4: Control on A and the measure of $\{x,\ \varphi_\varepsilon \sim 0\}$.
For some α_ε to be chosen later, decompose

$$\int_s^t A(r)\,dr = \int_s^t \int_\mathbb{R} \left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1 \right) u_\varepsilon(r,x) \mathbb{I}_{\varphi_\varepsilon(r,x) \leq -\alpha_\varepsilon}\,dx\,dr$$

$$+ \int_s^t \int_\mathbb{R} \left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1 \right) u_\varepsilon(r,x) \mathbb{I}_{\varphi_\varepsilon(r,x) \geq -\alpha_\varepsilon}\,dx\,dr.$$

For the first part, note again that by (1.9), there exists R s.t.

$$\forall |x| > R, \ \sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) \leq 1/2.$$

Therefore we may simply dominate

$$\int_s^t \int_\mathbb{R} \left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1 \right) u_\varepsilon(r,x) \mathbb{I}_{\varphi_\varepsilon(r,x) \leq -\alpha_\varepsilon}\,dx\,dr \leq C\,(t-s)\,e^{-\alpha_\varepsilon/\varepsilon}.$$

Concerning the second part, we constrain $1/2 \geq \alpha_\varepsilon \geq \|\varphi - \varphi_\varepsilon\|_{L^\infty(\Omega)}$ and may therefore bound

$$\int_s^t \left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1 \right) u_\varepsilon(r,x) \mathbb{I}_{\varphi_\varepsilon(r,x) \geq -\alpha_\varepsilon}\,dr$$

$$\leq \int_s^t \left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1 \right)^+ u_\varepsilon(r,x) \mathbb{I}_{\varphi(r,x) \geq -2\alpha_\varepsilon}\,dr,$$

where $(a)^+$ denotes the positive part of $a \in \mathbb{R}$. Now $\sum_{i=1}^k \bar{I}_i(s)\,\eta_i(x) - 1$ is nonpositive on $\{\varphi(s,.) = 0\}$ and so

$$\left(\sum_{i=1}^k \bar{I}_i(s)\,\eta_i - 1 \right)^+ \mathbb{I}_{\varphi(r,.)=0} \leq C \sup_{x \in \{\varphi(r,.)=0\}} \inf_{y \in \{\varphi(s,.)=0\}} |y - x| \leq C\,\tau_s(t-s),$$

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by Step 1 as the η_i are uniformly Lipschitz. For two sets O_1 and O_2, define in general
$$\delta(O_1, O_2) = \sup_{x \in O_1} \inf_{y \in O_2} |x - y|.$$

By the same argument, one gets
$$\left(\sum_{i=1}^{k} \bar{I}_i(s)\, \eta_i - 1 \right)^{+} \mathbb{1}_{\varphi(r,.) \geq -2\alpha_\varepsilon} \leq C\, \tau_s(t - s)$$
$$+ C\, \delta(\{\varphi(r,.) \geq -2\alpha_\varepsilon\}, \{\varphi(r,.) = 0\}).$$

Inequality (3.5) now becomes
$$\int_s^t \sum_{i=1}^{k} c_i \frac{(I_i^\varepsilon(r) - \bar{I}_i(s))^2}{I_i^\varepsilon(r)\, \bar{I}_i(s)} dr \leq C\, \varepsilon + 2\, \|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)} + C\,(t - s)\, e^{-\alpha_\varepsilon/\varepsilon}$$
$$+ C \int_s^t \tau_s(r - s)\, dr + C \int_s^t \delta(\{\varphi(r,.) \geq -2\alpha_\varepsilon\}, \{\varphi(r,.) = 0\}).$$
$$(3.7)$$

Conclusion. Eq. (3.5) indeed gives Lemma 3.2 if one defines
$$\sigma_s(t - s) = \frac{1}{t - s} \int_s^t \tau_s(r - s)\, dr,$$
$$\tilde{\sigma}(\varepsilon) = C\varepsilon + 2\|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)} + C\, T\, e^{-\alpha_\varepsilon/\varepsilon}$$
$$+ C \int_0^T \delta(\{\varphi(r,.) \geq -2\alpha_\varepsilon\}, \{\varphi(r,.) = 0\})\, dr.$$

Of course σ_s is continuous and, as τ_s is continuous and $\tau_s(0) = 0$, then trivially $\sigma_s(0) = 0$. Since $\{\varphi(r,.) \geq -2\alpha_\varepsilon\}$ and $\{\varphi(r,.) = 0\}$ are subsets of Ω, $\tilde{\sigma}(\varepsilon)$ is bounded for $\varepsilon \leq 1$, and thus, in order to complete the proof of Lemma 3.2, we only have to check that $\tilde{\sigma}(\varepsilon) \to 0$ when $\varepsilon \to 0$ for a convenient choice of α_ε. If we take $\alpha_\varepsilon \geq \|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega)}$ converging to 0 slowly enough to have $\alpha_\varepsilon/\varepsilon \to +\infty$, we only have to prove that
$$C \int_0^T \delta(\{\varphi(r,.) \geq -2\alpha_\varepsilon\}, \{\varphi(r,.) = 0\})\, dr \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

By dominated convergence it is enough that for any r
$$\delta(\{\varphi(r,.) \geq -2\alpha_\varepsilon\}, \{\varphi(r,.) = 0\}) \longrightarrow 0.$$

Just as in Step 1 this is a direct consequence of the continuity of φ.

3.4 Compactness of the I_i^ε and the proof of (1.12)

First notice that, simply passing to the limit in Lemma 3.2,

Lemma 3.3 *There exists $\sigma_s \in C(\mathbb{R}_+)$ with $\sigma_s(0) = 0$ s.t. $\forall i$*

$$\int_s^t |I_i(r) - \bar{I}_i(s)|^2 \, dr \leq (t - s)\, \sigma_s(t - s).$$

This means that at any point of Lebesgue continuity of I_i, one has $I_i = \bar{I}_i$. We recall that *a.e.* point is a Lebesgue point for I_i. As the I_i were defined only almost everywhere anyhow (they are weak-* limits), we may identify I_i and $\bar{I}_i$. This proves (1.12) and that $\bar{I}_i$ is approximately continuous on the right for any time t (and not only *a.e.* t).

Now let us prove the compactness in L^1_{loc} of each I_i^ε. We apply the usual criterion and hence wish to control

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i^\varepsilon(s)| \, dt \, ds.$$

Decompose

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i^\varepsilon(s)| \, dt \, ds \leq \int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i(s)| \, dt \, ds$$

$$+ \int_0^T \frac{1}{h} \int_s^{s+h} |I_i(t) - I_i^\varepsilon(s)| \, dt \, ds + \int_0^T \frac{1}{h} \int_s^{s+h} |I_i(t) - I_i(s)| \, dt \, ds.$$

The first and third terms are bounded directly from Lemmas 3.2 and 3.3: for example,

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i(s)| \, dt \, ds \leq \int_0^T \left(\frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i(s)|^2 \, dt \right)^{1/2} ds$$

$$\leq \int_0^T (\sigma_s(h) + \tilde{\sigma}(\varepsilon)/h)^{1/2} \, ds.$$

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The second term can be handled the same way after swapping the order of integration

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(s) - I_i(t)| \, dt \, ds = \int_0^{T+h} \frac{1}{h} \int_{\max(0,t-h)}^t |I_i^\varepsilon(s) - I_i(t)| \, ds \, dt$$

$$\leq \int_0^{T+h} \left(\frac{1}{h} \int_{\max(0,t-h)}^t |I_i^\varepsilon(s) - I_i(t)|^2 \, ds \right)^{1/2} dt$$

$$\leq \int_0^{T+h} (\sigma_t(h) + \tilde{\sigma}(\varepsilon)/h)^{1/2} \, dt.$$

So finally we bound

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i^\varepsilon(s)| \, dt \, ds \leq 3 \int_0^{T+h} (\sigma_s(h) + \tilde{\sigma}(\varepsilon)/h)^{1/2} \, ds$$

$$\leq 3 \int_0^{T+h} \sqrt{\sigma_s(h)} \, ds + 3(T+h)\sqrt{\tilde{\sigma}(\varepsilon)/h}.$$

Since of course the functions $\sigma_s(\cdot)$ can be chosen uniformly bounded in Lemma 3.2, again by dominated convergence, this shows that $\forall \tau > 0$, $\exists h$, $\exists \varepsilon_0(h)$ s.t. $\forall \varepsilon < \varepsilon_0(h)$

$$\int_0^T \frac{1}{h} \int_s^{s+h} |I_i^\varepsilon(t) - I_i^\varepsilon(s)| \, dt \, ds \leq \tau.$$

This is enough to get compactness of the I_i^ε in L^1_{loc} and then in any L^p_{loc} with $p < \infty$, which concludes the proof of Theorem 1.1.

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