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Trapped modes in thin and infinite ladder like domains: existence and asymptotic analysis

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Abstract: We are interested in a 2D propagation medium which is a localized perturbation of a reference homogeneous periodic medium. This reference medium is a "thick graph", namely a thin structure (the thinness being characterized by a small parameter $\varepsilon > 0$) whose limit (when ε tends to 0) is a periodic graph. The perturbation consists in changing only the geometry of the reference medium by modifying the thickness of one of the lines of the reference medium. The question we investigate is whether such a geometrical perturbation is able to produce localized eigenmodes. We have investigated this question when the propagation model is the scalar Helmholtz equation with Neumann boundary conditions . This amounts to solving an eigenvalue problem for the Laplace operator in an unbounded domain. We use a standard approach of analysis that consists in (1) find a formal limit of the eigenvalue problem when the small parameter tends to 0, here the formal limit is an eigenvalue problem for a second order differential operator along a graph; (2) proceed to an explicit calculation of the spectrum of the limit operator; (3) deduce the existence of eigenvalues as soon as the thickness of the ladder is small enough and construct an asymptotic expansion of the eigenvalues with respect to the small parameter. Following this approach, we prove the existence of localized modes provided that the geometrical perturbation consists in diminishing the width of one rung. Using the matched asymptotic expansion method, we obtain an asymptotic expansion at any order of the eigenvalues, which can be used for instance to compute a numerical approximations of these eigenvalues and associated eigenvectors.

Key-words: spectral theory, periodic media, quantum graphs, matched asymptotic expansion method

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Modes piégés dans des graphes "épais" infinis: existence and analyse asymptotique

Résumé : Nous consid/'erons la propagation des ondes dans un milieu qui est une perturbation locale d'un milieu périodique de référence. Ce milieu de référence est une graphe "épais", c'est à dire une structure fine (l'épaisseur de la structure étant caractérisée par un petit paramètre ε) dont la limite (quand ε tend vers 0) est un graphe périodique. La perturbation consiste à changer seulement la géométrie (et non les caractéristiques des matériaux) du milieu de référence en modifiant l'épaisseur d'un des barreaux du milieu de référence. Nous cherchons à savoir si une telle perturbation géométrique peut produire des modes piégés. Nous avons étudié cette question dans le cas de l'équation de Helmholtz avec des conditions de Neumann aux bords. Cette question se ramène à l'étude d'un problème aux valeurs propres pour l'opérateur laplacien dans un domaine non borné. Nous utilisons une démarche classique d'analyse qui consiste à (1) trouver la limite formelle du problème aux valeurs propres quand le petit paramètre tend vers 0, ici cette limite formelle est un problème aux valeurs propres pour un opérateur différentiel du second ordre défini sur un graphe; (2) effectuer un calcul explicite du spectre de cet opérateur limite ; (3) déduire l'existence de valeurs propres dès que lépaisseur du milieu est assez petite et construire un développement asymptotique des valeurs propres par rapport au petit paramètre. En suivant cette approche, nous prouvons l'existence de modes localisés dans le cas de l'échelle à condition que la perturbation géométrique consiste à diminuer la taille d'un des barreaux. Nous obtenons également un développement asymptotique à tout ordre des valeurs propres, qui peut être utilisé par exemple pour calculer une approximation numérique des valeurs propres et des vecteurs propres associés.

Mots-clés : théorie spectrale, milieux périodiques, graphes quantiques, développements asymptotiques raccordés

1   Introduction

Photonic crystals, also known as electromagnetic bandgap metamaterials, are 2D or 3D periodic media designed to control the light propagation. Indeed, the multiple scattering resulting from the periodicity of the material can give rise to destructive interferences at some range of frequencies. It follows that there might exist intervals of frequencies (called gaps) wherein the monochromatic waves cannot propagate. At the same time, for a given frequency located inside one of these gaps, a local perturbation of the crystal can produce defect mid-gap modes, that is to say solutions to the homogeneous time-harmonic wave equation that remains strongly localized in the vicinity of the perturbation. This localization phenomenon is of particular interest for a variety of promising applications in optics, for instance the design of highly efficient waveguides (cf. [17, 18]).

From a mathematical point of view, the presence of gaps is theoretically explained by the band-gap structure of the spectrum of the periodic partial differential operator associated with the wave propagation in such materials (Floquet-Bloch theory [9, 22]). In turn, the localization effect is directly linked to the possible presence of discrete spectrum appearing when perturbing the perfectly periodic operator. A thorough mathematical description of photonic crystals can be found [28]. However, without being exhaustive, let us remind the reader about a few important results on the topic. In the one dimensional case, it is well-known [7] that a periodic material has infinitely many gaps unless it is constant. By contrast, in 2d and 3d, a periodic medium might or might not have gaps. Nevertheless, several configurations where one on several gaps do exist can be found in [14, 15, 32, 34, 3] and references therein. In any case, except in dimension one, the number of gaps is expected to be finite. This statement, known as the Bethe Sommerfeld conjecture is fully demonstrated in [35, 36] for the periodic Schrödinger operator but is still partially open for Maxwell equations (see [44]). As for the localization effect, [12, 13, 1, 23, 27] exhibit situations where a compact (or lineic) perturbation of a periodic medium give rise to localized (resp. guided) modes. In these references, a local change in the material properties is required to ensure the localization phenomenon.

The aim of this paper is to complement the references mentioned above by proving the existence of localized midgap modes created by a geometrical perturbation of a particular periodic medium. We shall use a standard approach of analysis (used in [14, 34]) that consists in comparing the periodic medium with a reference one, for which theoretical results are available. In the present paper, we are interested in the Laplace operator with Neumann boundary condition in a ladder-like periodic waveguide. As the thickness of the rungs (proportional to a small parameter ε) tends to zeros, the domain shrinks to an (infinite) periodic graph. More precisely, the spectrum of the operator posed on the 2D domain tends to the spectrum of a self-adjoint operator posed on the limit graph ([40, 29, 41, 37]): this limit (or referent) operator consists of the second order derivative operator on each edge of the graph together with transmission conditions (called Kirchhoff conditions) at its vertices ([11, 8, 29]). As opposed to the initial operator, the spectrum of the limit operator can be explicitly determined using a finite difference scheme ([2, 10]). Based on this general methodology, we are able to prove that, for ε sufficiently small, the diminution of the thickness of one rung of the ladder gives rise to a localized mode. We point out that the analysis of quantum graphs has been a very active research aera for the last three decades and we refer the reader to the surveys [24, 25, 26] as well as the books [5, 38] for an overview and an exhaustive bibliography of this field.

2   Presentation of the problem

In the present work we study the propagation of waves in a ladder-like periodic medium (cf. figure 1). The homogeneous domain Ω_ε (we will call it ladder) consists of the infinite band of height L minus an infinite set of equispaced rectangular obstacles. The domain is 1-periodic in one space direction, corresponding to the variable x. The distance between two consecutive obstacles is equal to the distance from the obstacles to the boundary of the band and is denoted by ε.
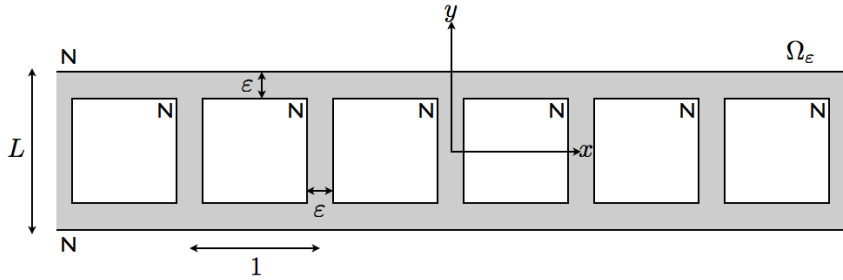

Figure 1: The unperturbed periodic ladder

The aim of the work is to find localized modes, that is solutions of the homogeneous scalar wave equation with Neumann boundary condition

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \partial_n u = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{1}$$

which are confined in the x-direction.

Without giving a strict mathematical formulation (this will be done in the following section) a localized mode can be understood as a solution of the wave equation (1), having the following form:

$$u(x,y,t) = v(x,y)\, e^{i\omega t}, \qquad v \in L_2(\Omega_\varepsilon), \tag{2}$$

where the factor $e^{i\omega t}$ shows the harmonic dependence on time whereas the function v (which does not depend on time) is in some sense confined (since it belongs to L_2). Injecting (2) into (1) leads to the following problem for the function v:

$$\begin{cases} -\Delta v = \omega^2 v & \text{in } \Omega_\varepsilon \ , \\ \partial_n u = 0 & \text{in } \partial\Omega_\varepsilon. \end{cases} \tag{3}$$

Problem (3) is an eigenvalue problem posed in the unbounded domain Ω_ε. It is well-known (cf. Theorem XIII.86 in [39], volume IV) that elliptic periodic operators in $2D$ domains have no eigenvalue. In order to create eigenvalues a perturbation has to be introduced. We introduce a local perturbation in this perfectly periodic domain by changing the distance between two consecutive obstacles from ε to $\mu\varepsilon$, $\mu > 0$ (see figure 2 in the case where $\mu \in (0,1)$). It corresponds to modify the width of the vertical rung of the ladder from $|x| < \varepsilon/2$ to $|x| < \mu\varepsilon/2$.

As we will see such a perturbation does not change the continuous spectrum of the underlying operator but it can introduce a non-empty discrete spectrum. Our aim is to find eigenvalues by playing with the values of μ and ε, ε being treated as a small parameter.
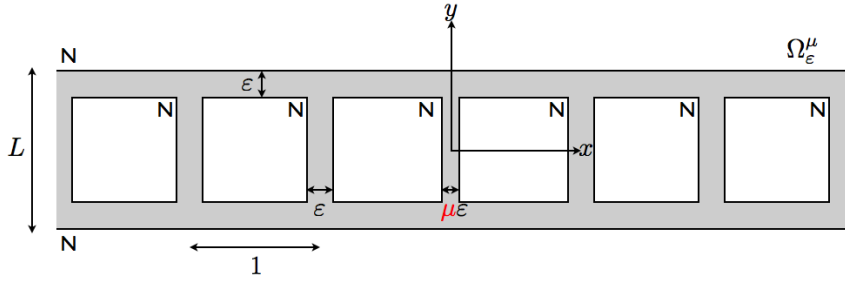
Figure 2: The Perturbed ladder

3   Mathematical formulation of the problem

This section describes a mathematical framework for the analysis of the spectral problem formulated above. We introduce the operator A_ε^μ, acting in the space $L_2(\Omega_\varepsilon^\mu)$, associated with the eigenvalue problem (3) in the perturbed domain:

$$A_\varepsilon^\mu u = -\Delta u, \qquad D(A_\varepsilon^\mu) = \left\{ u \in H_\Delta^1(\Omega_\varepsilon^\mu), \quad \partial_n u|_{\partial\Omega_\varepsilon^\mu} = 0 \right\}.$$

Here $H_\Delta^1(\Omega_\varepsilon^\mu) = \left\{ u \in H^1(\Omega_\varepsilon^\mu), \quad \Delta u \in L_2(\Omega_\varepsilon^\mu) \right\}$. The operator A_ε^μ is self-adjoint and positive. Our goal is to characterize its spectrum and, more precisely, to find sufficient conditions for existence of eigenvalues.

3.1   The essential spectrum of A_ε^μ

To determine the essential spectrum of the operator A_ε^μ, we consider the case $\mu = 1$, where the domain Ω_ε is perfectly periodic (figure 1). We will denote the corresponding operator A_ε. The Floquet-Bloch theory shows that the spectrum of periodic elliptic operators is reduced to its essential spectrum and has a band-gap structure [9, 39, 22]:

$$\sigma(A_\varepsilon) = \sigma_{ess}(A_\varepsilon) = \mathbb{R} \setminus \bigcup_{1 \leqslant n \leqslant N}]a_n, b_n[, \qquad (4)$$

where, in the previous formula, the union disappears if $N = 0$. For $N > 0$, the intervals $]a_n, b_n[$ are called spectral gaps. Their number N is conjectured to be finite (Bethe-Sommerfeld, 1933, [35, 36, 44]). The band-gap structure of the spectrum is a consequence of the following result given by the Floquet-Bloch theory:

$$\sigma(A_\varepsilon) = \bigcup_{\theta \in [-\pi, \pi]} \sigma\left(A_\varepsilon(\theta) \right). \qquad (5)$$

Here $A_\varepsilon(\theta)$ is the Laplace operator defined on the periodicity cell $\mathcal{C}_\varepsilon = \Omega_\varepsilon \cap \{x \in [-1/2, 1/2]\}$ (cf. figure 3) with θ-quasiperiodic boundary conditions on the lateral boundaries $\Gamma_\varepsilon^\pm = \partial\mathcal{C}_\varepsilon \cap \{x = \pm 1/2\}$ and homogeneous Neumann boundary conditions on the remaining part $\Gamma_\varepsilon = \partial\mathcal{C}_\varepsilon \setminus (\Gamma_\varepsilon^- \cup \Gamma_\varepsilon^+)$ of the boundary: for $\theta \in [-\pi, \pi]$,

$$A_\varepsilon(\theta) : L_2(\mathcal{C}_\varepsilon) \longrightarrow L_2(\mathcal{C}_\varepsilon), \qquad A_\varepsilon(\theta)u = -\Delta u,$$

$$D\left(A_\varepsilon(\theta) \right) = \left\{ u \in H_\Delta^1(\mathcal{C}_\varepsilon), \quad \partial_n u|_{\Gamma_\varepsilon} = 0, \quad u|_{\Gamma_\varepsilon^+} = e^{-i\theta} \, u|_{\Gamma_\varepsilon^-}, \quad \partial_x u|_{\Gamma_\varepsilon^+} = e^{-i\theta} \, \partial_x u|_{\Gamma_\varepsilon^-} \right\}.$$
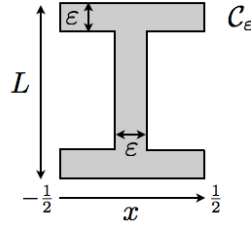
Figure 3: Periodicity cell

For each $\theta \in [-\pi, \pi]$ the operator $A_\varepsilon(\theta)$ is self-adjoint, positive and its resolvent is compact. Its spectrum is then a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda_1(\varepsilon, \theta) \leqslant \lambda_2(\varepsilon, \theta) \leqslant \cdots \leqslant \lambda_n(\varepsilon, \theta) \leqslant \ldots, \qquad \lim_{n \to \infty} \lambda_n(\varepsilon, \theta) = +\infty. \tag{6}$$

In (6) the eigenvalues are repeated with their multiplicity. The representative curves of the functions $\theta \mapsto \lambda_n(\varepsilon, \theta)$ are called dispersion curves and are known to be continuous and non-constant (cf. Theorem XIII.86, volume IV in [39]). The fact that the dispersion curves are non-constant implies that the operator A_ε has no eigenvalues. Finally, (5) can be rewritten as

$$\sigma(A_\varepsilon) = \bigcup_{n \in \mathbb{N}} \lambda_n\big(\varepsilon, [-\pi, \pi]\big),$$

which gives (4). The conjecture of Bethe-Sommerfeld means that for n large enough the intervals $\lambda_n\big(\varepsilon, [-\pi, \pi]\big)$ overlap.

Since $D(A_\varepsilon(-\theta)) = \overline{D(A_\varepsilon(\theta))}$ and the operators A_ε have real coefficients, the function $\lambda_n(\theta)$ are even. Thus, it is sufficient to consider $\theta \in [0, \pi]$ in (5). This will be used systematically in the rest of the paper.

As expected (this is related to Weyl's Theorem, see Ch.13 Vol. 4 in [39], Ch. 9 in [6], Theorem 1 in [12]), the essential spectrum is stable under a perturbation of the thickness of one rung of the ladder does.

Proposition 1. $\sigma_{ess}(A_\varepsilon^\mu) = \sigma_{ess}(A_\varepsilon)$.

This is a direct consequence of the following assertion.

Lemma 1. *Let* $\chi \in C^\infty(\Omega_\varepsilon)$ *be a function such that*

   (a) $\partial_n \chi|_{\partial \Omega_\varepsilon} = 0$,

   (b) $\exists\, M > 0$ *such that* $|x| > M \Rightarrow \chi(x, y) = 1$.

If $\{u_j\}_{j \in \mathbb{N}}$ *is a singular sequence for the operator* A_ε *corresponding to the value* λ, *then there exists a subsequence of* $\{\chi u_j\}_{j \in \mathbb{N}}$ *which is also a singular sequence for the operator* A_ε *corresponding to the value* λ.

Proof. By definition of a singular sequence, the sequence $\{u_j\}_{j \in \mathbb{N}}$ has the following properties:

   1. $u_j \in D(A_\varepsilon), \quad j \in \mathbb{N}$;

   2. $\inf\limits_{j} \|u_j\|_{L_2(\Omega_\varepsilon)} > 0$;

   3. $u_j \overset{w}{\longrightarrow} 0$ in $L_2(\Omega_\varepsilon)$;

   4. $A_\varepsilon u_j - \lambda u_j \longrightarrow 0$ in $L_2(\Omega_\varepsilon)$.

Let us show that there exists a subsequence of $\{\chi u_j\}_{j\in\mathbb{N}}$ which has the same properties. The property 1 is verified by the whole sequence χu_j thanks to property (a) . To prove property 2, it suffices to show that there exists a subsequence, still denoted u_j, such that

$$\|u_j\|_{L_2(K_\varepsilon)} \longrightarrow 0, \quad j \longrightarrow \infty, \quad \text{with } K_\varepsilon := \left\{ (x,y) \in \overline{\Omega_\varepsilon} \ / \ |x| \le M \right\}. \tag{7}$$

Indeed, (7) and property (b) imply $\inf\limits_{j} \|\chi u_j\|_{L_2(\Omega_\varepsilon)} \ge \inf\limits_{j} \|u_j\|_{L_2(\Omega_\varepsilon \cap \{|x|>M\})} > 0$. To prove (7), we write

$$\|\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} = (A_\varepsilon u_j - \lambda u_j, u_j)_{L_2(\Omega_\varepsilon)} + \lambda \|u_j\|^2_{L_2(\Omega_\varepsilon)}.$$

Then properties 3 and 4 imply that u_j is bounded in $H^1(\Omega_\varepsilon)$. By compactness, one can thus extract a subsequence that converges weakly $H^1(\Omega_\varepsilon)$ and strongly in $L^2(K_\varepsilon)$ to a limit which is necessarily 0 thanks to property 3, which proves (7). For the sequel, we work with the above subsequence.

The property 3 being obvious the only thing to show is the property 4 for the sequence $\{\chi u_j\}_{j\in\mathbb{N}}$. We have:

$$\|A_\varepsilon(\chi u_j) - \lambda(\chi u_j)\|^2_{L_2(\Omega_\varepsilon)} \le \|\chi(A_\varepsilon u_j - \lambda u_j)\|^2_{L_2(\Omega_\varepsilon)} + 2\|\nabla\chi \cdot \nabla u_j\|^2_{L_2(\Omega_\varepsilon)} + \|u_j \Delta\chi\|^2_{L_2(\Omega_\varepsilon)}.$$

The first and the last terms in the right-hand side tend to zero thanks to property 4 and (7)). Let us estimate the second term. Using first property (b), then properties (a) and (1) together with an integration by parts, we obtain

$$\|\nabla\chi \cdot \nabla u_j\|^2_{L_2(\Omega_\varepsilon)} = \int\limits_{K_\varepsilon} \nabla u_j \cdot \nabla\overline{u_j}\, |\nabla\chi|^2 \, dx = -\int\limits_{K_\varepsilon} u_j \left(|\nabla\chi|^2 \, \Delta\overline{u_j} + \nabla\big(|\nabla\chi|^2\big) \cdot \nabla\overline{u_j} \right)$$

which tends to 0 due to (7) since ∇u_j is bounded in $L^2(\Omega_\varepsilon)$ as well as Δu_j (by properties 3 and 4). $\qquad\square$

Proof of Proposition 1. It is sufficient to take a function χ in the previous lemma which does not depend on y, vanishes in a neighbourhood of the perturbed edge and such that $\nabla\chi$ vanishes in a neighbourhood of all vertical edges. Then, it follows from Lemma 1 that any singular sequence associated to λ of the operator A_ε provides the construction of a singular sequence of the operator A_ε^μ for the same λ and vice versa. $\qquad\square$

The essential spectrum of the operator $\sigma_{ess}(A_\varepsilon^\mu)$ having a band-gap structure, we will be interested in finding eigenvalues inside gaps (once the existence of gaps is established).

3.2   Towards the existence of eigenvalues: the method of study.

Our analysis consists of three main steps.

- First, we find a formal limit of the eigenvalue problem (3) when $\varepsilon \to 0$. To do so, we use the fact that, as ε goes to zero, the domains Ω_ε and Ω_ε^μ shrink to a graph $\mathcal{G}$. As a consequence, the formal limit problem will involve a self-adjoint operator $\mathcal{A}^\mu$ associated to a second order differential operator along the graph, the definition of which is strongly related to the fact that homogeneous Neumann boundary conditions are considered in the original problem. More precisely, at the limit $\varepsilon \to 0$, looking for an eigenvalue of A_ε^μ leads to search an eigenvalue of $\mathcal{A}^\mu$. This operator, that is well known from the works of [11, 8, 29] , will be described more rigorously in the next section.

- The second step is an explicit calculation of the spectrum of the limit operator. The essential spectrum is determined using the Floquet-Bloch theory (by solving a set of cell problems) while the discrete spectrum of the perturbed operator is found using a reduction to a finite difference equation (section 3). In particular, we shall show that the limit operator has infinitely eigenvalues as soon as $\mu < 1$ (and no eigenvalue when $\mu \geq 1$), that form a discrete subset of $\mathbb{R}^+$.

- Finally, when $\mu < 1$, we shall deduce the existence of an eigenvalue of A_ε^μ close to any such eigenvalue of $\mathcal{A}^\mu$ as soon as ϵ is small enough. The proof will be based on the construction of a quasi-mode (a kind approximation of the eigenfunction) and a criterion for the existence of eigenvalues of self-adjoint operators (see for instance Lemma 4 in [33]) and can be seen as a generalization of the well-known min-max principle for eigenvalues located below the lower bound of the essential spectrum.

An essential preliminary step is the decompostion of the operator A_ε^μ as the sum of two operators, namely its symmetric and antisymmetric parts. To do so, we introduce the following decomposition of $L_2(\Omega_\varepsilon^\mu)$:

$$L_2(\Omega_\varepsilon^\mu) = L_{2,s}(\Omega_\varepsilon^\mu) \oplus L_{2,a}(\Omega_\varepsilon^\mu),$$

where $L_{2,s}(\Omega_\varepsilon^\mu)$ and $L_{2,a}(\Omega_\varepsilon^\mu)$ are subspaces consisting of functions respectively symmetric and antisymmetric with respect to the axis $y = 0$:

$$L_{2,s}(\Omega_\varepsilon^\mu) = \left\{ u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = u(x,-y), \ \forall\, (x,y) \in \Omega_\varepsilon^\mu \right\},$$

$$L_{2,a}(\Omega_\varepsilon^\mu) = \left\{ u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = -u(x,-y), \ \forall\, (x,y) \in \Omega_\varepsilon^\mu \right\}.$$

The operator A_ε^μ is then decomposed into the orthogonal sum

$$A_\varepsilon^\mu = A_{\varepsilon,s}^\mu \oplus A_{\varepsilon,a}^\mu, \quad A_{\varepsilon,s}^\mu = A_\varepsilon^\mu\big|_{L_{2,s}(\Omega_\varepsilon^\mu)}, \qquad A_{\varepsilon,a}^\mu = A_\varepsilon^\mu\big|_{L_{2,a}(\Omega_\varepsilon^\mu)}.$$

Accordingly, the limit operator $\mathcal{A}^\mu$ is decomposed as:

$$\mathcal{A}^\mu = \mathcal{A}_s^\mu \oplus \mathcal{A}_a^\mu \tag{8}$$

The key point is that, as we shall see, contrary to the full operator $\mathcal{A}^\mu$ whose spectrum is $\mathbb{R}^+$, both operators $\mathcal{A}_s^\mu$ and $\mathcal{A}_a^\mu$ have spectral gaps (an infinity of them), each of them containing eigenvalues : these are isolated eigenvalues for $\mathcal{A}_s^\mu$ or $\mathcal{A}_a^\mu$, but embedded eigenvalues for $\mathcal{A}^\mu$. One deduces that the operators $A_{\varepsilon,s}^\mu$ and $A_{\varepsilon,a}^\mu$ have at least finitely many spectral gaps, the number of gaps tending to $+\infty$ when ε goes to 0: this is an important fact for applying the quasi-mode approach. Finally, using the method of matched asymptotic developments, we are able to compute the complete asymptotic expansions of the eigenvalues inside the gaps (section 4).

At this stage, it is worthwhile mentionning that the convergence of the spectrum of differential

operators in thin domains degenerating into a graph is not a new subject, particularly in the case of elliptic operators. In particular, for the Laplace operator with Neumann boundary conditions and in the case of compact domains, the convergence results (which are reduced to the convergence of eigenvalues) have been known since the works of Rubinstein-Schatzman [40] and Kuchment-Zheng [29]. Thanks to the Floquet Bloch theory, such results are easily transformed into analogous results for thin periodic domains (in this case, only continuous spectrum is involved). For general unbounded domains, a general (and somewhat abstract) theory has been developed by Post in [37] for the convergence of all spectral components. This theory can be applied to our problem. We have chosen here a more direct approach (based on quasi-modes), that will allow us to obtain a complete asymptotic expansion of the eigenvalues with respect to ε (See Section 5).

4 Spectral problem on the graph

4.1 The operator $\mathcal{A}^\mu$.

As $\varepsilon \to 0$, the domain Ω_ε tends to the periodic graph $\mathcal{G}$ represented on figure 4. Let us number the vertical edges of the graph $\mathcal{G}$ from left to right so that the set of the vertical edges is $\{e_j = \{j\} \times (-L/2, L/2)\}_{j \in \mathbb{Z}}$. The upper end of the edge e_j is denoted by M_j^+ and the lower one by M_j^-. The set of all the vertices of the graphs is then

$$\mathcal{M} = \{M_j^\pm\}_{j \in \mathbb{Z}}.$$

The horizontal edge joining the vertices $M_j^\pm$ and $M_{j+1}^\pm$ is denoted by $e_{j+\frac{1}{2}}^\pm = (j, j+1) \times \{\pm L/2\}$. The set of all the edges of the graph is

$$\mathcal{E} = \{e_j, e_{j+\frac{1}{2}}^\pm\}_{j \in \mathbb{Z}}$$

and we denote by $\mathcal{E}(M)$ the set of all the edges of the graph containing the vertex M.
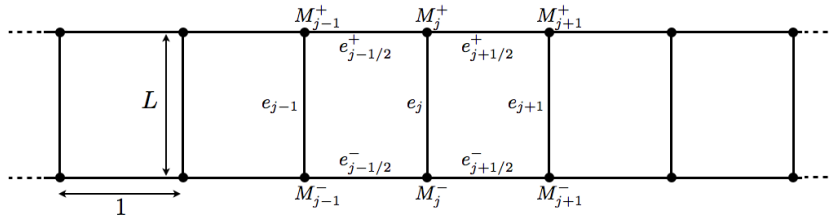
Figure 4: Limit graph $\mathcal{G}$

If u is a function defined on $\mathcal{G}$ we will use the following notation:

$$\mathbf{u}_j^\pm = u(M_j^\pm), \quad u_j(y) = u|_{e_j}, \quad u_{j+\frac{1}{2}}^\pm(x) = u|_{e_{j+\frac{1}{2}}^\pm}.$$

Let $w^\mu : \mathcal{E} \longrightarrow \mathbb{R}^+$ be a weight function which is equal to μ on the "perturbed edge" e_0, more precisely the limit of the perurbed rung $|x| < \mu\varepsilon/2$, and to 1 on the other edges:

$$w^\mu(e_0) = \mu, \quad w^\mu(e) = 1, \quad \forall\, e \in \mathcal{E}, \, e \neq e_0. \tag{9}$$

Let us now introduce the following functional spaces

$$L_2^\mu(\mathcal{G}) = \left\{ u \ / \ u \in L_2(e), \ \forall e \in \mathcal{E}; \quad \|u\|_{L_2^\mu(\mathcal{G})}^2 = \sum_{e \in \mathcal{E}} w^\mu(e) \|u\|_{L_2(e)}^2 < \infty \right\}, \tag{10}$$

$$H^1(\mathcal{G}) = \left\{ u \in L_2^\mu(\mathcal{G}) \ / \ u \in C(\mathcal{G}); \quad u \in H^1(e), \ \forall e \in \mathcal{E}; \quad \|u\|_{H^1(\mathcal{G})}^2 = \sum_{e \in \mathcal{E}} \|u\|_{H^1(e)}^2 < \infty \right\},$$

$$H^2(\mathcal{G}) = \left\{ u \in L_2^\mu(\mathcal{G}) \ / \ u \in C(\mathcal{G}); \quad u \in H^2(e), \ \forall e \in \mathcal{E}; \quad \|u\|_{H^2(\mathcal{G})}^2 = \sum_{e \in \mathcal{E}} \|u\|_{H^2(e)}^2 < \infty \right\}, \tag{11}$$

where $C(\mathcal{G})$ denotes the space of continuous functions on $\mathcal{G}$.:We define the limit operator $\mathcal{A}^\mu$ in $L_2^\mu(\mathcal{G})$ as follows: denoting u_e the restriction of u to e,

$$(\mathcal{A}^\mu u)_e = -u_e'', \qquad \forall e \in \mathcal{E}, \tag{12}$$

$$D(\mathcal{A}^\mu) = \left\{ u \in H^2(\mathcal{G}) \ / \ \sum_{e \in \mathcal{E}(M)} w^\mu(e) u_e'(M) = 0, \quad \forall M \in \mathcal{M} \right\}, \tag{13}$$

where $(u_e)'(M)$ stands for the derivative of the function u_e at the point M in the *outgoing direction*. The vertex relations in (13) are called Kirchhoff's conditions. Note that they all have an identical expression except at the vertices $M_0^\pm$. The following assertion as well as its proof can be found in [24], section 3.3.

Proposition 2 (Kuchment). *The operator $\mathcal{A}^\mu$ in the space $L_2^\mu(\mathcal{G})$ is self-adjoint. The corresponding closed sesqui-linear form has the following form:*

$$a^\mu[f,g] = (f',g')_{L_2^\mu(\mathcal{G})}, \quad \forall f,g \in D[a^\mu], \qquad D[a^\mu] = H^1(\mathcal{G}).$$

As for the ladder, we introduce the following decomposition of the space $L_2^\mu(\mathcal{G})$ into the spaces of symmetric and antisymmetric functions:

$$L_2^\mu(\mathcal{G}) = L_{2,s}^\mu(\mathcal{G}) \oplus L_{2,a}^\mu(\mathcal{G}),$$

$$L_{2,s}^\mu(\mathcal{G}) = \{ u \in L_2(\mathcal{G}) \ / \ u(x,y) = u(x,-y), \ \forall (x,y) \in \mathcal{G} \},$$

$$L_{2,a}^\mu(\mathcal{G}) = \{ u \in L_2(\mathcal{G}) \ / \ u(x,y) = -u(x,-y), \ \forall (x,y) \in \mathcal{G} \}.$$

Again, the operator $\mathcal{A}^\mu$ can be decomposed into the orthogonal sum

$$\mathcal{A}^\mu = \mathcal{A}_s^\mu \oplus \mathcal{A}_a^\mu,$$

with

$$\mathcal{A}_s^\mu = \mathcal{A}^\mu\big|_{L_{2,s}^\mu(\mathcal{G})}, \qquad \mathcal{A}_a^\mu = \mathcal{A}^\mu\big|_{L_{2,a}^\mu(\mathcal{G})},$$

which implies

$$\sigma(\mathcal{A}^\mu) = \sigma(\mathcal{A}_s^\mu) \cup \sigma(\mathcal{A}_a^\mu).$$

Thus, it is sufficient to study the spectrum of the operators $\mathcal{A}_s^\mu$ and $\mathcal{A}_a^\mu$ separately. The analysis of these two operators being analogous, we will present a detailed study of $\mathcal{A}_s^\mu$ (Section 4.2 and 4.3) and state the results for $\mathcal{A}_a^\mu$ (Section 4.4).

4.2 The essential spectrum of the operator $\mathcal{A}_s^\mu$

We shall study the spectrum of the operator $\mathcal{A}_s^\mu$ by a perturbation technique with respect to the case $\mu = 1$ which corresponds to the purely periodic case. The corresponding operator will be denoted by $\mathcal{A}_s$. Indeed, based on compact perturbation arguments (apply Proposition 1 by introducing a cut-off function which vanishes in a neighbourhood of the perturbed edge of the graph), we can prove the following proposition.

Proposition 3. *The essential spectra of $\mathcal{A}_s^\mu$ and $\mathcal{A}_s$ coincide:*

$$\sigma_{ess}(\mathcal{A}_s^\mu) = \sigma_{ess}(\mathcal{A}_s). \tag{14}$$

This reduces the study of the essential spectrum of $\mathcal{A}_s^\mu$ to the study of the spectrum of the purely periodic operator $\mathcal{A}_s$, which can be done through the Floquet-Bloch theory.

4.2.1 Description of the spectrum of $\mathcal{A}_s$ through Floquet-Bloch theory

As previously explained, the spectrum of the operator $\mathcal{A}_s$ can be studied using the Floquet-Bloch theory. On has then to study a set of problems posed on the periodicity cell of $\mathcal{G}$. Since we consider the subspace of symmetric functions with respect to the axis $y = 0$, this permits to reduce the problem to the lower half part of the periodicity cell $\mathcal{C}_- = \mathcal{G} \cap [-1/2, 1/2] \times [-L/2, 0]$ (see fig. 5).
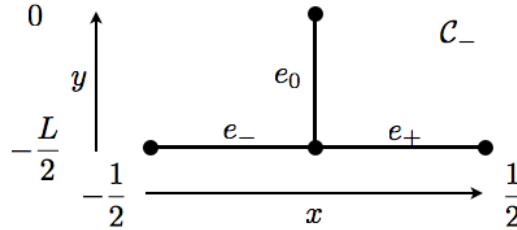
Figure 5: Periodicity cell

We introduce the spaces $L_2(\mathcal{C}_-)$ and $H^2(\mathcal{C}_-)$ analogously to (10), (11):

$$L_2(\mathcal{C}_-) = \{u \,/\, u_\ell := u|_{e_\ell} \in L_2(e_\ell),\, \ell \in \{0, +, -\}\},$$

$$H^2(\mathcal{C}_-) = \{u \in C(\mathcal{C}_-) / u_\ell \in H^2(e_\ell), \ell \in \{0, +, -\}\}.$$

We have then

$$\sigma(\mathcal{A}_s) = \bigcup_{\theta \in [0, \pi]} \sigma(\mathcal{A}_s(\theta)) \tag{15}$$

where $\mathcal{A}_s(\theta)$ is the following unbounded operator in $L_2(\mathcal{C}_-)$

$$\left[\mathcal{A}_s(\theta)u\right]_\ell = -u_\ell'', \ \ell \in \{0, +, -\}, \quad D(\mathcal{A}_s(\theta)) = \{u \in H^2(\mathcal{C}_-) \,/\, u \text{ satisfies (16) and } u_0'(0) = 0\}$$

$$(a) \quad u_+'(0) - u_-'(0) + u_0'(-L/2) = 0, \quad (b) \quad u_+(1/2) = e^{-i\theta}u_-(-1/2), \quad u_+'(1/2) = e^{-i\theta}u_-'(-1/2). \tag{16}$$

In the definition of $D(\mathcal{A}_s(\theta))$, the condition $u_0'(0) = 0$ corresponds to the symmetry with respect to $y = 0$, (16)-(a) is the Kirchhoff's condition with $\mu = 1$ and (16)-(b) are the θ-quasiperiodicity conditions. For each $\theta \in [0, \pi]$ the operator $\mathcal{A}_s(\theta)$ is self-adjoint and positive and its resolvent is compact due to the compactness of the embedding $H^1(\mathcal{C}_-) \subset L_2(\mathcal{C}_-)$. Consequently, its spectrum is a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda_{1,s}(\theta) \leqslant \lambda_{2,s}(\theta) \leqslant \cdots \leqslant \lambda_{n,s}(\theta) \leqslant \ldots, \qquad \lim_{n\to\infty} \lambda_{n,s}(\theta) = +\infty.$$

In the present case, the eigenvalues can be computed explicitly.

Proposition 4. *For $\theta \in [0, \pi]$, $\omega^2 \in \sigma(\mathcal{A}_s(\theta))$ if and only if ω is a solution of the equation*

$$2 \cos(\omega L/2)(\cos \omega - \cos \theta) = \sin \omega \sin(\omega L/2). \tag{17}$$

Proof. If $\omega^2 \neq 0$ is an eigenvalue of the operator $\mathcal{A}_s(\theta)$ then the corresponding eigenfunction $u = \{u_0, u_+, u_-\}$ is of the form

$$u_-(x) = a_- e^{i\omega x} + b_- e^{-i\omega x}, \qquad x \in [-1/2, 0], \tag{18}$$

$$u_+(x) = a_+ e^{i\omega x} + b_+ e^{-i\omega x}, \qquad x \in [0, 1/2], \tag{19}$$

$$u_0(y) = a_0 e^{i\omega y} + b_0 e^{-i\omega y}, \qquad y \in [-L/2, 0]. \tag{20}$$

Taking into account that $u \in D(\mathcal{A}_s(\theta))$, we arrive at the following linear system

$$a_- + b_- = a_+ + b_+ = a_0 e^{-i\omega L/2} + b_0 e^{i\omega L/2}, \tag{21}$$

$$a_0 = b_0, \tag{22}$$

$$b_- - a_- + a_+ - b_+ + a_0 e^{-i\omega L/2} - b_0 e^{i\omega L/2} = 0, \tag{23}$$

$$a_+ e^{i\omega/2} + b_+ e^{-i\omega/2} = e^{-i\theta}(a_- e^{-i\omega/2} + b_- e^{i\omega/2}), \tag{24}$$

$$a_+ e^{i\omega/2} - b_+ e^{-i\omega/2} = e^{-i\theta}(a_- e^{-i\omega/2} - b_- e^{i\omega/2}). \tag{25}$$

The relations (21) express the continuity of the eigenfunction at the vertex $(0, -L/2)$. The equation (22) comes from the condition $u_0'(0) = 0$. The relation (23) corresponds to (16)-(a) while (24) and (25) correspond to (16)-(b). Adding and substracting (24) and (25) lead to $a_- = a_+ e^{i(\theta+\omega)}$ and $b_- = b_+ e^{i(\theta-\omega)}$, which we can substitute into (21)–(23) to obtain the following system in (a_+, b_+, a_0)

$$M(\theta, \omega, L) \begin{pmatrix} a_+ \\ b_+ \\ a_0 \end{pmatrix} = 0 \quad \text{where} \quad M(\theta, \omega, L) := \begin{pmatrix} 1 - e^{i(\theta+\omega)} & 1 - e^{i(\theta-\omega)} & 0 \\ 1 & 1 & -2\cos(\frac{\omega L}{2}) \\ 1 - e^{i(\theta+\omega)} & -1 + e^{i(\theta-\omega)} & -2i\sin(\frac{\omega L}{2}) \end{pmatrix} \tag{26}$$

It is then easy to conclude since one obtains, after some computations omitted here

$$\det M(\theta, \omega, L) = 4e^{i\theta}\left(2\cos\left(\tfrac{\omega L}{2}\right)(\cos \omega - \cos \theta) - \sin \omega \sin\left(\tfrac{\omega L}{2}\right)\right).$$

For $\omega = 0$, the relations (18)–(20) are replaced by

$$u_-(x) = a_- + b_- x, \qquad x \in [-1/2, 0],$$
$$u_+(x) = a_+ + b_+ x, \qquad x \in [0, 1/2],$$
$$u_0(y) = a_0 + b_0 y, \qquad y \in [-L/2, 0].$$

Using the fact that $u \in D(\mathcal{A}_s(\theta))$ we have (instead of (21-25)):

$$a_- = a_+ = a_0, \qquad b_0 = 0, \qquad b_- = b_+, \qquad b_+ = b_-\, e^{-i\theta}, \qquad b_+ = a_-\, (e^{-i\theta} - 1).$$

One then easily sees that there exists a non-trivial solution if and only if $\theta = 0$ and that the corresponding eigenfunction is constant. Noticing that, for $\theta = 0$, $\omega = 0$ is solution of (17) allows us to conclude. $\qquad\square$

The reader will notice that when $L \in \mathbb{Q}$, the spectrum of $\mathcal{A}_s(\theta)$ has a particular structure: it is the image by the function $x \mapsto x^2$ of a periodic countable subset of $\mathbb{R}$. To see that, it suffices to remark that both functions at the left and right hand sides of (17) are periodic with a common period. As a consequence of (15), the spectrum of $\mathcal{A}_s(\theta)$ is the image by the function $x \mapsto x^2$ of a periodic subset of $\mathbb{R}$.

4.2.2   Characterization of the spectrum of $\mathcal{A}_s$

Using (15), Proposition 4 allows us to describe the structure of the spectrum of the operator $\mathcal{A}_s$. We first prove the existence of a countable infinity of gaps.

Proposition 5. *The following properties hold*

    1. $\sigma(\mathcal{A}_s) \supset \sigma_2 \cup \sigma_L$, where $\sigma_2 = \left\{ (\pi n)^2,\ n \in \mathbb{N} \right\}$ and $\sigma_L = \left\{ (2\pi n/L)^2,\ n \in \mathbb{N} \right\}$.

    2. The operator $\mathcal{A}_s$ has infinitely many gaps whose ends tend to infinity.

Proof.   1. For $\sin \omega = 0$ or $\sin(\omega L/2) = 0$, the equation (17) is satisfied for $\cos(\theta) = \cos(\omega)$ so that ω belongs to $\sigma(\mathcal{A}_s(\theta)) \subset \sigma(\mathcal{A}_s)$.

    2. Let $\omega_n = (2n+1)\pi/L$ (such that $\cos(\omega_n L/2) = 0$), let us distinguish two cases:

(a) $\sin(\omega_n) \neq 0$: the left hand side of equation (17) vanishes for all θ and, as $\sin(\omega_n L/2) \neq 0$, the right hand side does not. Then ω_n^2 does not belong to the spectrum of $\sigma(\mathcal{A}_s)$. Since $(2\pi n/L)^2$ and $(2\pi(n+1)/L)^2$ belong to $\sigma(\mathcal{A}_s)$ (in view of the point 1) there exists a gap which contains ω_n^2, strictly included in $((2\pi n/L)^2, (2\pi(n+1)/L)^2)$.

(b) $\sin(\omega_n) = 0$: (this case can occur only for special values of L, see remark 1), we know by point 1, that $\omega_n^2 \in \sigma(\mathcal{A}_s)$ and we are going to show that it exists $\delta > 0$ such that $((\omega_n - \delta)^2, \omega_n^2)$ and $(\omega_n^2, (\omega_n + \delta)^2)$ are in the resolvent set of $\mathcal{A}_s$. This will show the existence of two disjoint gaps of the form $(\omega_n^2 - l_n^-, \omega_n^2) \subset ((2\pi n/L)^2, \omega_n^2)$ and $(\omega_n^2, \omega_n^2 + l_n^+) \subset (\omega_n^2, (2\pi(n+1)/L)^2)$. Setting $\omega = \omega_n + z$ in relation (17) leads to

$$f_n(z) = 0 \quad \text{where} \quad f_n(z) := 2\sin(zL/2)(\cos \omega_n \cos z - \cos \theta) + \cos \omega_n \sin z \cos(zL/2)\,. \tag{27}$$

We have

$$f_n(z) = z(\cos \omega_n + L(\cos \omega_n - \cos \theta)) + o(z)$$

which cannot vanish for $0 < |z| < \delta$ for δ small enough, since $\cos \omega_n = \pm 1$. This implies that $\cos \omega_n + L(\cos \omega_n - \cos \theta) \neq 0$ for all θ.

The conclusion follows from the fact that the intervals $((2\pi n/L)^2, (2\pi(n+1)/L)^2)$ are disjoint, go to infinity with n, and contain one or two gaps. $\qquad\square$

Remark 1. *The case 2.(b) of the above proof can occur only for special values of L. Indeed, the reader will easily verify that the existence of ω such that $\sin(\omega) = \cos(\omega L/2) = 0$ is equivalent to the fact that*

$$L \in \mathbb{Q}_c := \left\{ q \in \mathbb{Q} \ / \ \exists \, (m,k) \in \mathbb{N} \times \mathbb{N}^* \ \text{such that} \ q = \frac{2m+1}{k} \ \text{(irreducible fraction)} \right\}. \qquad (28)$$

In fact, the condition (28) also influences the nature of the spectrum of $\mathcal{A}_s$. Indeed it can be shown that when L does not belong to $\mathbb{Q}_c$, the point spectrum of $\mathcal{A}_s$ is empty (i. e. the spectrum of $\mathcal{A}_s$ is purely continuous). When L belongs to $\mathbb{Q}_c$, it coincides with an infinity of eigenvalues of infinite multiplicity, associated with compactly supported eigenfunctions. It is worth noting that the presence of such eigenvalues is a specific feature of periodic graphs (see Section 5 in [25]).

Remark 2. *In the proof of the first point, gaps are located in the vicinity of the points λ satisfying $\cos(\lambda L/2) = 0$. These points are nothing else but the eigenvalues of the 1d Laplace operator defined on the vertical half edges $\{(x,y), x = j, -L/2 < y < 0\}$ with Dirichlet boundary condition at $y = -L/2$ and Neumann boundary condition at $y = 0$. The presence of gaps is therefore consistent with Theorem 5 of [25] dealing with gaps created by so-called graph decorations. Indeed, the vertical half edges can be seen as decorations of the infinite periodic graph $\mathcal{G}_0$ made of the set of the horizontal edges $e_{j+1/2}^+$.*

Next, we give a more precise description of the gap structure of $\sigma(\mathcal{A}_s)$ through a geometrical interpretation of equation (17). We first remark that as soon as $\omega \notin \{\pi\mathbb{Z}\} \cup \{2\pi\mathbb{Z}/L\}$, $\lambda = \omega^2$ belongs to $\sigma(\mathcal{A}_s)$ (i. e. ω is solution of (17)) if and only if

$$\exists \, \theta \in [0, \pi] \quad \text{such that} \quad \cos\theta \neq \cos\omega \quad \text{and} \quad \phi_L(\omega) = f(\theta, \omega), \qquad (29)$$

where the functions ϕ_L and f are defined by

$$\phi_L(\omega) := \frac{2}{\tan(\omega L/2)}, \quad f(\theta, \omega) := \frac{\sin\omega}{\cos\omega - \cos\theta}. \qquad (30)$$

In the following we reason in the (ω, y)-plane with y an additional auxiliary variable. We introduce the domain D

$$D = \left\{ \big(\omega, f(\theta, \omega)\big), (\omega, \theta) \in \mathbb{R} \times [0, \pi] \, \text{and} \, \cos\omega \neq \cos\theta \right\}. \qquad (31)$$

Lemma 2. *The domain D is the domain of the (ω, y)-plane, π-periodic with respect to ω, given by*

$$D = \bigcup_{n \in \mathbb{Z}} \left\{ D_0 + (n\pi, 0) \right\}, \quad D_0 := [0, \pi] \times \mathbb{R} \setminus \left\{ (\omega, y) \ / \ 0 < \omega < \pi, -\tan(\frac{\omega}{2})^{-1} < y < \tan(\frac{\omega}{2}) \right\}. \qquad (32)$$

Proof. The π-periodicity of the domain D with respect to ω follows from the identity $f(\theta, \omega + \pi) = f(\pi - \theta, \omega)$. To conclude, it suffices to remark that, for a given $\omega \in (0, \pi)$, if θ varies in the interval $[0, \omega)$ $\theta \mapsto f(\theta, \omega)$ is continuous and strictly decreasing from $-(\tan(\omega/2))^{-1}$ to $-\infty$ while, if θ varies in the interval $(\omega, \pi]$, $\theta \mapsto f(\theta, \omega)$ is continuous and strictly decreasing from $+\infty$ to $\tan(\omega/2)$. $\qquad \square$

Thanks to Proposition 5(1) and the characterization (29),

$$\sigma(\mathcal{A}_s) = \sigma_2 \cup \sigma_L \cup \left\{ \omega^2 \notin \sigma_L \ / \ (\omega, \phi_L(\omega)) \in D \right\}. \qquad (33)$$
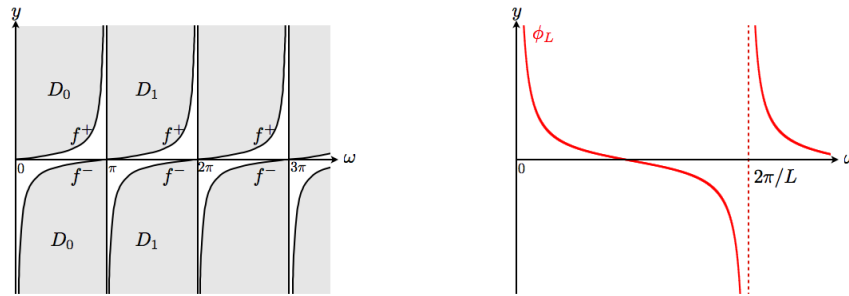
Figure 6: Representation of the D (grey part) and the curve $\mathcal{C}_L$ (for $L = 8$).

(a) $L = 8$　　　　　　　　(b) $L = 10\pi/7$
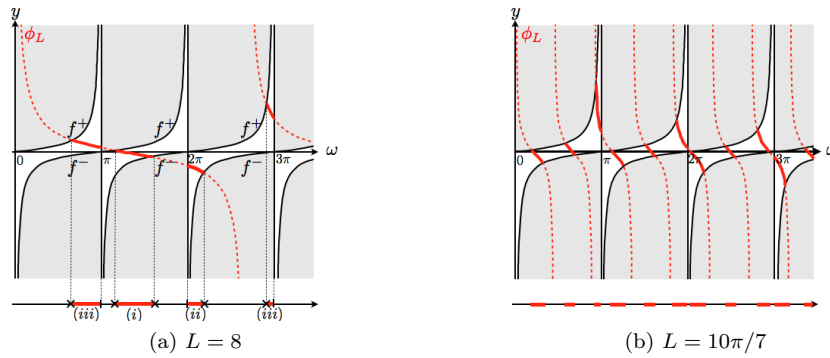
Figure 7: The images of the spectral gaps by $x \mapsto \sqrt{x}$. In the left picture, the three types of gaps are distinguished (according to the legend).
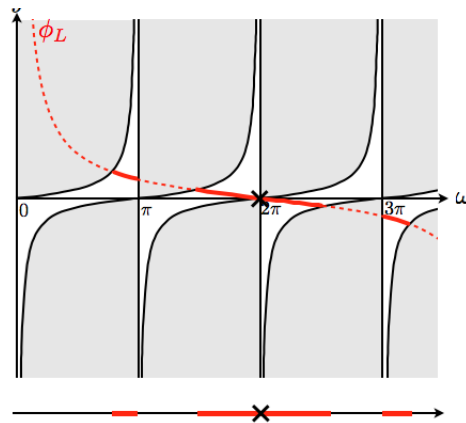
Figure 8: An example of eigenvalue of infinite multiplicity ($\omega = 2\pi$) obtained for $L = 1/2$. This eigenvalue separates a gap of type (ii) on the left from a gap of type (iii) on the right. This occurs for $L \in \mathbb{Q}_c$.

In other words, $\sigma(\mathcal{A}_s)$ is the union of σ_2, σ_L, and the image by the application $x \mapsto x^2$ of the projection on the line $y = 0$ of the intersection of the domain D with the curve $\mathcal{C}_L = \{(\omega, \phi_L(\omega)), \omega \in \mathbb{R}\}$. Thanks to this geometrical characterization, we shall be able to describe the structure of the gaps of the operator $\mathcal{A}_s$.

Let us introduce the π-periodic functions $f^{\pm} : \mathbb{R} \to \mathbb{R}^{\pm}$, such that, for any $\omega \in [0, \pi)$,

$$f^+(\omega) = \tan\frac{\omega}{2} \quad \text{and} \quad f^-(\omega) = -\left(\tan\frac{\omega}{2}\right)^{-1}.$$

The easy proof of the following result is left to the reader (see also Fig. 6, 7 and 8):

Proposition 6. *An interval* (ω_b^2, ω_t^2) *is a gap of the operator* $\mathcal{A}_s$ *if and only if* $[\omega_b, \omega_t] \cap \frac{2\pi\mathbb{Z}}{L} = \emptyset$ *and one of the following three possibilities holds:*

(i) *There exists* $n \in \mathbb{Z}$ *such that* $\pi n < \omega_b < \omega_t < \pi(n+1)$, *and,* $\phi_L(\omega_b) = f^+(\omega_b)$, $\phi_L(\omega_t) = f^-(\omega_t)$.

(ii) *There exists* $n \in \mathbb{Z}$ *such that* $\pi n = \omega_b < \omega_t < \pi(n+1)$, *and* $\phi_L(\omega_b) \leq 0$, $\phi_L(\omega_t) = f^-(\omega_t)$.

(iii) *There exists* $n \in \mathbb{Z}$ *such that* $\pi n < \omega_b < \omega_t = \pi(n+1)$, *and* $\phi_L(\omega_b) = f^+(\omega_b)$, $\phi_L(\omega_t) \geq 0$.

4.3 The discrete spectrum of $\mathcal{A}_s^{\mu}$

We are now interested in determining the discrete spectrum of $\mathcal{A}_s^{\mu}$. Suppose that ω^2 is not in the essential spectrum of $\mathcal{A}_s^{\mu}$, which implies in particular that $\omega \notin \pi\mathbb{Z}$ (see Prop. 5). Let u be a corresponding eigenfunction and let $\mathbf{u}_j = u(M_j^-) = u(M_j^+)$ (we consider symmetric functions). Since the eigenfunction u verifies the equation $-u'' + \omega^2 u = 0$ on each horizontal edge of the graph $\mathcal{G}$, one has

$$u_{j+\frac{1}{2}}(s) = \mathbf{u}_j \frac{\sin(\omega(1-s))}{\sin\omega} + \mathbf{u}_{j+1}\frac{\sin(\omega s)}{\sin\omega}, \qquad s := x - j \in [0,1], \qquad \forall j \in \mathbb{Z}. \qquad (34)$$

We first begin by excluding some particular cases:

Lemma 3. *If* $\cos\frac{\omega L}{2} = 0$, *then* ω^2 *is not in the discrete spectrum of* $\mathcal{A}_s^{\mu}$.

Proof. If $\cos\frac{\omega L}{2} = 0$ and $\omega \notin \pi\mathbb{Z}$, then ω^2 is an eigenvalue of infinite multiplicity (see Remark 1). Thus, it does not belong to the discrete spectrum of $\mathcal{A}_s^{\mu}$. Similarly, if $\cos\frac{\omega L}{2} = 0$ and $\omega \in \pi\mathbb{Z}$, then $\omega^2 \in \sigma_{ess}(\mathcal{A}_s^{\mu})$ (Prop. 5), which implies that it does not belong to the discrete spectrum of $\mathcal{A}_s^{\mu}$. $\square$

Thus we can assume that $\cos\frac{\omega L}{2} \neq 0$. In this case, on the vertical edges e_j, u_j is given by

$$u_j(y) = \mathbf{u}_j \frac{\cos(\omega y)}{\cos(\omega L/2)}, \qquad\qquad y \in [-L/2, L/2], \qquad \forall j \in \mathbb{Z}. \qquad (35)$$

According to (34)-(35), the function u is completely determined by the point values $\mathbf{u}_j$. Moreover, in order to ensure that $u \in L^2(\mathcal{G})$, the sequence $\mathbf{u}_j$ must be square integrable:

$$\sum_{j \in \mathbb{Z}} |\mathbf{u}_j|^2 < +\infty. \qquad (36)$$

It remains to express that u belongs to $D(\mathcal{A}_s^\mu)$ (see (13)), i.e. Kirchhoff's conditions are satisfied. Doing so, we obtain the following set of finite difference equations:

$$\mathbf{u}_{j+1} + 2\, g(\omega)\, \mathbf{u}_j + \mathbf{u}_{j-1} = 0, \qquad j \in \mathbb{Z}^*, \tag{37}$$

$$\mathbf{u}_1 + 2\, g_\mu(\omega)\, \mathbf{u}_0 + \mathbf{u}_{-1} = 0, \tag{38}$$

with

$$g(\omega) = -\cos\omega + \frac{\sin\omega}{\phi_L}, \tag{39}$$

$$g_\mu(\omega) = -\cos\omega + \mu\frac{\sin\omega}{\phi_L}. \tag{40}$$

Thus, we reduced the initial problem for a differential operator on the graph to a problem for a finite difference operator acting on sequences $\{\mathbf{u}_j\}_{j\in\mathbb{Z}}$. Looking for particular solutions of (37) for $j < 0$ and $j > 0$ under the form $\mathbf{u_j} = r^j$ leads to the characteristic equation

$$r^2 + 2\, g(\omega)\, r + 1 = 0. \tag{41}$$

At this point, we observe the following property

Lemma 4. *As soon as $\cos(\omega L/2) \neq 0$, one has the equivalence*

$$\omega^2 \in \sigma_{ess}(\mathcal{A}_s) \quad \Leftrightarrow \quad |g(\omega)| \leq 1.$$

Proof. Indeed, $|g(\omega)| \leq 1$ is equivalent to the existence of $\theta \in [0,\pi]$ such that

$$\cos(\theta) = g(\omega) = \cos\omega - \frac{1}{2}\sin\omega \tan(\omega L/2).$$

Since $\cos(\omega L/2) \neq 0$, this is equivalent to the characterization (17) of the essential spectrum. $\quad\square$

Since ω^2 does not belong to the essential spectrum of $\mathcal{A}_s^\mu$, $|g(\omega)| > 1$ and the discriminant $D(\omega)$ of (41) is strictly positive, which means that (41) has two distinct real solutions. Since the product of these solutions is equal to one, (41) has a unique solution $r(\omega) \in (-1, 1)$ given by

$$r(\omega) = -g(\omega) + sign(g(\omega))\sqrt{g^2(\omega) - 1}. \tag{42}$$

Joining (36) and (37), we deduce that there exists a constant $A \neq 0$

$$\mathbf{u}_j = A\, r(\omega)^{|j|}, \qquad j \in \mathbb{Z}. \tag{43}$$

It remains to enforce the Kirchhoff condition (38), which leads to

$$r(\omega) = -g^\mu(\omega). \tag{44}$$

Taking into account (39), (40) and (42), we arrive at the following relation:

$$sign(g(\omega))\sqrt{g^2(\omega) - 1} = (1 - \mu)(g(\omega) + \cos\omega).$$

Since $|g(\omega)| > 1$, $sign(g(\omega)) = sign(g(\omega) + \cos\omega)$ and we can rewrite the previous equality as

$$F(\omega) = \mu \quad \text{where } F(\omega) := 1 - \sqrt{\frac{g^2(\omega) - 1}{(g(\omega) + \cos\omega)^2}}. \tag{45}$$

For the rest of the analysis, it is useful to rewrite $F(\omega)$ (using (39)) as

$$F(\omega) = 1 - \sqrt{1 - \phi_L(\omega)\,(\phi_L(\omega) + \phi_2(\omega))},$$

where $\phi_2(\omega) = 2/\tan\omega$.

Remark 3. *Let (ω_b^2, ω_t^2) be a gap of the operator $\mathcal{A}_s^\mu$. Since $|g(\omega)| > 1$ in (ω_b, ω_t), F is well defined and continuous in (ω_b, ω_t). However, F might blow up (together with the function ϕ_2) as ω tends to ω_t or ω_b, i.e. at the extremities of the gap.*

Theorem 1. *For $\mu \geq 1$, the discrete spectrum of the operator $\mathcal{A}_s^\mu$ is empty. For $0 < \mu < 1$, let (ω_b^2, ω_t^2) be a gap of the operator $\mathcal{A}_s^\mu$:*

(a) *If (ω_b^2, ω_t^2) is a gap of type (i), then $\mathcal{A}_s^\mu$ has exactly two simple eigenvalues $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$ that satisfy $\omega_b < \omega_1 < \omega_2 < \omega_t$.*

(b) *If (ω_b^2, ω_t^2) is a gap of type (ii) or (iii), then $\mathcal{A}_s^\mu$ has exactly one simple eigenvalue $\lambda_1 = \omega_1^2$ such that $\omega_b < \omega_1 < \omega_t$.*

Proof. Assume that $\mu \geqslant 1$. If ω^2 belongs to the discrete sprectrum of $\mathcal{A}_s^\mu$, then ω^2 is in a gap of $\mathcal{A}_s^\mu$ and Equation (45) is satisfied. But this is impossible because $|g(\omega)| > 1$, which means in particular that $\mu = F(\omega) < 1$.

Then, we consider the case $0 < \mu < 1$. We investigate the variations of F for the different types of gaps described in Prop. 6:

- Gap of type (i): as a preliminary step, one can verify that $|g(\omega_b)| = |g(\omega_t)| = 1$ (using for instance the definition (39) of g together with the fact that $\phi_L(\omega_b) = f^+(\omega_b)$ and $\phi_L(\omega_t) = f^-(\omega_t)$, see Prop. 6), which implies that

$$F(\omega_b) = F(\omega_t) = 1. \tag{46}$$

Then, let us investigate the variations of the function $\phi_L \varphi$, with $\varphi = \phi_L + \phi_2$: first, since $\phi_L(\omega_b) = f^+(\omega_b) > 0$ and $\phi_L(\omega_t) = f^-(\omega_t) < 0$ (see Prop. 6), the strictly decaying function ϕ_L, which is continuous in the interval $[\omega_b, \omega_t]$, has exactly one zero in (ω_b, ω_t). We denote it by c. Besides, the fonction φ is continuous and strictly decaying in the interval $[\omega_b, \omega_t]$ (Prop. 6 ensuring the existence of $n \in \mathbb{Z}$ such that $[\omega_b, \omega_t] \subset (n\pi, (n+1)\pi)$, we deduce that ϕ_2 is continuous in $[\omega_b, \omega_t]$). Moreover, it satisfies $\varphi(\omega_b) > 0$ and $\varphi(\omega_t) < 0$. Indeed, a direct computation shows that

$$\forall n \in \mathbb{Z}, \quad \forall \omega \in (n\pi, (n+1)\pi), \quad -f^+(\omega) < \phi_2(\omega) < -f^-(\omega). \tag{47}$$

As a consequence, $\varphi(\omega_b) = f^+(\omega_b) + \phi_2(\omega_b) > 0$ and $\varphi(\omega_t) = f^-(\omega_t) + \phi_2(\omega_t) < 0$. As a result, φ has exactly one zero in (ω_b, ω_t). We denote it by d.

Noting that (46) implies that $\phi_L(\omega_b)\varphi(\omega_b) = \phi_L(\omega_t)\varphi(\omega_t) = 1$, we deduce that the function $\phi_L \varphi$, which is continuous on $[\omega_b, \omega_t]$, is strictly decaying from 1 to 0 in the interval $[\omega_b, \min(c, d)]$, is strictly increasing from 0 to 1 in the interval $[\max(c, d), \omega_t]$, and is negative in the interval $(\min(c, d), \max(c, d))$. It follows that F, which is therefore also continuous in $[\omega_b, \omega_t]$, is strictly decaying from 1 to 0 in the interval $[\omega_b, \min(c, d)]$, is negative in the interval $(\min(c, d), \max(c, d))$, and is strictly increasing from 0 to 1 in $[\max(c, d), \omega_t]$. As a result, for any $\mu \in (0, 1)$, Equation (45) has exactly two solutions in (ω_b, ω_t), the first one belonging to $(\omega_b, \min(c, d))$ and the second one to $(\max(c, d), \omega_t)$.

- Gap of type (ii): in this case, $\omega_b \in \mathbb{Z}\pi$ and the function F blows up in the neighborhood ω_b unless $\cos(\omega_b L/2) = 0$. More precisely, we can prove that

$$\lim_{\omega \to \omega_b^+} F(\omega) = \begin{cases} 1 - \sqrt{1 + 2L} < 0 & \text{if } \cos(\omega_b L/2) = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

By contrast, since $\omega_t \notin \mathbb{Z}\pi$ and as for the first kind of gap, we can prove that

$$F(\omega_t) = 1. \tag{48}$$

Then, here again, we investigate the variations of the function $\phi_L \varphi$, with $\varphi = \phi_L + \phi_2$. In view of Prop 6, the function ϕ_L is continuous, strictly decaying and negative in the interval $(\omega_b, \omega_t]$. Then, the function $\varphi = \phi_L + \phi_2$ is continuous in $(\omega_b, \omega_t]$, strictly decaying, and (thanks to (47)) satisfies

$$\lim_{\omega \to \omega_b^+} \varphi(\omega) = +\infty \quad \text{and} \quad \lim_{\omega \to \omega_t^-} \varphi(\omega) = f^-(\omega_t) + \phi_2(\omega_t) < 0.$$

As result, φ has still exactly one zero in (ω_b, ω_t). We denote it by d.

Noting that (48) implies that $\phi_L(\omega_t)\varphi(\omega_t) = 1$, we deduce that the function $\phi_L \varphi$, which is continuous in $(\omega_b, \omega_t]$, is negative in (ω_b, d), and stricly increasing from 0 to 1 in $[d, \omega_t]$. Thus, the function F, which is continuous in (ω_b, ω_t), is negative in (ω_b, d), and strictly increasing from 0 to 1 on $[d, \omega_t]$. Consequently, for any $\mu \in (0, 1)$, Equation (45) has exactly one solution (that belongs to (d, ω_t)). The proof for the gaps of type (iii) follows the same way.

$\square$

4.4   The spectrum of the operator $\mathcal{A}_a^\mu$.

We will now briefly describe the modifications of the previous considerations in the case of the operator $\mathcal{A}_a^\mu$. The operator corresponding to the periodic case $\mu = 1$ is denoted by $\mathcal{A}_a$. First, based on compact perturbation arguments, we can prove the following proposition, which is analogous to Prop. 3:

Proposition 7. *The essential spectra of $\mathcal{A}_a^\mu$ and $\mathcal{A}_a$ coincide:*

$$\sigma_{ess}(\mathcal{A}_a^\mu) = \sigma_{ess}(\mathcal{A}_a). \tag{49}$$

Besides, using the Floquet-Bloch Theory, we obtain the analogue of Proposition 4 in the anti-symmetric case (we refer the reader to Section 4.2 for the definition of $\mathcal{A}_a(\theta)$):

Proposition 8. *For $\theta \in [0, \pi]$, $\omega^2 \in \sigma(\mathcal{A}_a(\theta))$ if and only if $\omega \neq 0$ and ω is a solution of the equation*

$$2\sin(\omega L/2)\left(\cos \omega - \cos \theta\right) = -\sin \omega \cos(\omega L/2). \tag{50}$$

Thanks to the previous characterization, and similarly to the results of Proposition 5, we can describe the structure of the spectrum of $\mathcal{A}_a$:

Proposition 9. *The following properties hold:*

1. *$\sigma(\mathcal{A}_s) \supset \sigma_{2,a} \cup \sigma_{L,a}$, where $\sigma_{2,a} = \left\{(\pi n)^2,\ n \in \mathbb{N}^*\right\}$ and $\sigma_{L,a} = \left\{\left((2n+1)\pi/L\right)^2,\ n \in \mathbb{N}\right\}$.*

2. *The operator $\mathcal{A}_a$ has infinitely many gaps whose ends tend to infinity.*

Then, excluding here again the particular cases $\omega \in \{\pi\mathbb{Z}\} \cup \{2\pi\mathbb{Z}/L\}$, the computation of the discrete spectrum $\mathcal{A}_a$ leads to the set (37)-(38) of finite-difference equations substituting $g_a(\omega)$ and $g_a^\mu(\lambda)$ for $g(\omega)$ and $g^\mu(\omega)$:

$$g_a(\lambda) = -\cos \lambda + \frac{1}{2}\sin \lambda \tan\left(\lambda L/2 + \pi/2\right),$$

$$g_a^\mu(\lambda) = -\cos \lambda + \frac{\mu}{2}\sin \lambda \tan\left(\lambda L/2 + \pi/2\right).$$

The investigation of the characteristic equation (41) then provides the following characterization for the discrete spectrum of $\mathcal{A}_a$:

$$\omega^2 \in \sigma_d(\mathcal{A}_a^\mu) \quad \Leftrightarrow \quad \mu = 1 - \sqrt{\frac{g_a^2(\omega) - 1}{(g_a(\omega) + \cos\omega)^2}}. \tag{51}$$

Finally, as in the symmetric case (see Theo. 1), a detailed analysis of (51) allows us to prove the following result of the existence of eigenvalues:

Theorem 2. *For $\mu \geqslant 1$ the discrete sprectrum of $\mathcal{A}_a^\mu$ is empty. For $0 < \mu < 1$, there exists either one or two eigenvalues in each gap of $\mathcal{A}_a^\mu$.*

4.5 The spectrum of the operator $\mathcal{A}$.

As we have seen, both of the operators $\mathcal{A}_s$ and $\mathcal{A}_a$ have infinitely many gaps. However, it turns out that the gaps of one operator overlap with the spectral bands of the other one, so that the full operator $\mathcal{A}$ have no gap.

Proposition 10.
$$\sigma(\mathcal{A}) = \mathbb{R}^+.$$

Proof. Let us suppose that there exists ω such that $\omega^2 \notin \sigma(\mathcal{A})$ (of course, the same is true fore some open neighborhood of ω). We first note that $\omega \notin \sigma_L \cup \sigma_{L,a}$, since these sets are either in the spectrum of $\mathcal{A}_a^\mu$ or in the spectrum $\mathcal{A}_a^\mu$ (Prop. 5 and Prop. 9). As a consequence, $\cos(\omega L/2) \neq 0$ and $\sin(\omega L/2) \neq 0$. Then, since $\omega^2 \notin \sigma(\mathcal{A})$, the characterizations (17)-(50) of the essential spectrum of $\mathcal{A}_s$ and $\mathcal{A}_a$ (divided respectively by $\cos(\omega L/2)$ and $\sin(\omega L/2)$) imply that

$$\left| -\cos\omega + \frac{1}{2}\sin\omega\tan(\omega L/2) \right| > 1 \quad \text{and} \quad \left| \cos\omega + \frac{\sin\omega}{2\tan(\omega L/2)} \right| > 1. \tag{52}$$

Introducing $a = \tan(\omega L/2)$, the system (52) can be rewritten as

$$\begin{cases} \dfrac{a^2}{4}\sin^2\omega - a\sin\omega\cos\omega + \cos^2\omega > 1, & (53) \\[3mm] \dfrac{1}{4a^2}\sin^2\omega + \dfrac{1}{a}\sin\omega\cos\omega + \cos^2\omega > 1. & (54) \end{cases}$$

Multiplying (54) by a^2 and taking the sum with (53) we obtain

$$\frac{1}{4}(1 + a^2)\sin^2\omega + (1 + a^2)\cos^2\omega > 1 + a^2,$$

which is impossible. $\qquad\square$

Let us then remark that the set of eigenvalues of $\mathcal{A}^\mu$, which is the union of the sets of eigenvalues of $\mathcal{A}_s^\mu$ and $\mathcal{A}_a^\mu$, is embedded in the essential spectrum of $\mathcal{A}^\mu$.

5 Existence of eigenvalues for the operator on the ladder

5.1 Main result and methodology

We return now to the case of the ladder. As it was mentioned before, instead of studying the full operator A_ε^μ we will study separately the operators $A_{\varepsilon,s}^\mu$, $A_{\varepsilon,a}^\mu$. Let us remind first the result, already proven for instance in [40, 29, 37] which states the convergence of the essential spectrum of the periodic operators $A_{\varepsilon,s}$ (resp. $A_{\varepsilon,a}$) to the essential spectrum of $\mathcal{A}_s$ (resp. $\mathcal{A}_a$).

Theorem 3 (Essential spectrum). *Let $\{(a_m, b_m), m \in \mathbb{N}^*\}$ be the gaps of the operator $\mathcal{A}_s$ (respectively $\mathcal{A}_a$) on the limit graph $\mathcal{G}$. Then, for each $m_0 \in \mathbb{N}^*$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}$ (respectively $A_{\varepsilon,a}$) has at least m_0 gaps $\{(a_{\varepsilon,m}, b_{\varepsilon,m}), 1 \leqslant m \leqslant m_0\}$ such that*

$$a_{\varepsilon,m} = a_m + O(\varepsilon), \quad b_{\varepsilon,m} = b_m + O(\varepsilon), \qquad \varepsilon \to 0, \qquad 1 \leqslant m \leqslant m_0.$$

In [37], O. Post proves the norm convergence of the resolvent of the laplacian with Neumann boundary conditions for a large class of thin domains shrinking to graphs. It consequently demonstrates the existence of eigenvalues of $A_{\varepsilon,s}$ (respectively $A_{\varepsilon,a}$) located in the gap of the essential spectrum and gives the rate of convergence of any eigenvalue $A_{\varepsilon,s}$ (respectively $A_{\varepsilon,a}$) toward an eigenvalue of $\mathcal{A}_s$ (respectively $\mathcal{A}_a$). The present paper provides a simple and constructive alternative proof of these known results and an asymptotic expansion of any eigenvalue with respect to ε at any order. Here is the main result of the paper.

Theorem 4 (Discrete spectrum). *Let (a, b) be a gap of the operator $\mathcal{A}_s^\mu$ (respectively $\mathcal{A}_a^\mu$) on the limit graph $\mathcal{G}$ and $\lambda^{(0)} \in (a, b)$ an eigenvalue of this operator. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}^\mu$ ($A_{\varepsilon,a}^\mu$) has an eigenvalue λ_ε inside a gap $(a_\varepsilon, b_\varepsilon)$ and for all $n \in \mathbb{N}$*

$$\lambda_\varepsilon = \sum_{k=0}^{n} \varepsilon^k \lambda^{(k)} + O(\varepsilon^{n+1}), \tag{55}$$

where the $\lambda^{(k)}$'s for $k \geq 1$ are defined by induction (see Section 5.5)

Remark 4. *Note that as every eigenvalue of the operators $\mathcal{A}_s^\mu$ (resp. $\mathcal{A}_a^\mu$) is simple (as established in Theorem 1), for ε small enough, λ_ε will be a simple eigenvalue of $A_{\varepsilon,s}^\mu$ (resp. $A_{\varepsilon,a}^\mu$), see [37].*

Let us mention that this asymptotic expansion can be used for instance to compute a numerical approximation of the eigenvalues of the operators defined on the ladder when $\varepsilon \to 0$ without mesh refinement. As we will see, the computation of the terms of the expansions only requires the solution of problems set on the limit graph with particular Kirchhoff conditions (see Section 5.4) and the solution of problems set in two different infinte normalized junctions (see Section 5.3). The associated numerical method will be detailed in a forthcoming paper.

We consider only in the following the proof for the eigenvalues of the operator $A_{\varepsilon,s}^\mu$, the case of the operator $A_{\varepsilon,a}^\mu$ being treated analogously.

Methodology of the proof The proof of Theorem 4 relies on the construction of a pseudo-mode, that is to say a symmetric function $u_{\varepsilon,n} \in H^1(\Omega_\varepsilon^\mu)$ such that for every symmetric function $v \in H^1(\Omega_\varepsilon^\mu)$

$$\left| \int_{\Omega_\varepsilon^\mu} (\nabla u_{\varepsilon,n} \nabla v - \lambda_{\varepsilon,n} u_\varepsilon v) \, d\Omega \right| \leqslant C \varepsilon^{n+1} \|u_\varepsilon\|_{H^1(\Omega_\varepsilon^\mu)} \|v\|_{H^1(\Omega_\varepsilon^\mu)}, \tag{56}$$

where

$$\lambda_{\varepsilon,n} = \sum_{k=0}^{n} \varepsilon^k \lambda^{(k)}. \tag{57}$$

By adapting the Lemma 4 for [33] (see Appendix A for a proof in a general case) if such a function exists, this provides an estimate of the distance from $\lambda_{\varepsilon,n}$ to the spectrum of $A^\mu_{\varepsilon,s}$

$$dist(\sigma(A^\mu_{\varepsilon,s}), \lambda_{\varepsilon,n}) \leqslant \widetilde{C}\varepsilon^{n+1} \tag{58}$$

with some constant $\widetilde{C}$ which does not depend on ε.

According to Theorem 3, for ε small enough there exists a constant C such that $\sigma_{ess}(A^\mu_{\varepsilon,s}) \cap [a + C\varepsilon, b - C\varepsilon] = \emptyset$. As a consequence, the intersection of the discrete spectrum $\sigma_d(A^\mu_{\varepsilon,s})$ and the interval $[\lambda_{\varepsilon,n} - \widetilde{C}\varepsilon^{n+1}, \lambda_{\varepsilon,n} + \widetilde{C}\varepsilon^{n+1}]$ is non empty. In other words, for ε sufficiently small, there exists at least one eigenvalue of $A^\mu_{\varepsilon,s}$ in a neighbourhood of $\lambda_{\varepsilon,n}$ of order ε^{n+1}.

The pseudo mode $u_{\varepsilon,n}$ and the associated expansion $\lambda_{\varepsilon,n}$ are constructed thanks to a formal asymptotic expansion of u^ε and λ^ε solution of the following eigenvalue problem

$$\text{Find } (u^\varepsilon, \lambda^\varepsilon), \ u^\varepsilon \in H^1(\Omega^\mu_\varepsilon), \ \|u_\varepsilon\|_{L^2(\Omega^\mu_\varepsilon)} = 1, \ \lambda^\varepsilon \in \mathbb{R}^+,$$

$$\left|\begin{array}{ll} \Delta u^\varepsilon + \lambda^\varepsilon u^\varepsilon = 0 & \text{in} \quad \Omega^\mu_\varepsilon, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on} \quad \partial\Omega^\mu_\varepsilon, \\ u^\varepsilon(x,y) = u^\varepsilon(x,-y) & \forall (x,y) \in \Omega^\mu_\varepsilon \end{array}\right. \tag{59}$$

This asymptotic expansion will be constructed by induction starting from an eigenvalue $\lambda^{(0)} \in (a,b)$ of the operator $\mathcal{A}^\mu_s$ and its associated eigenvector $u^{(0)}$.

Due to the multiscale nature of the problem, it is not possible to construct a simple asymptotic expansion of u^ε that would be valid in the whole domain Ω^μ_ε. We need to distinguish two asymptotic expansions of u^ε. The first one, describing the overall behaviour of u^ε far from the junctions, is expressed by means of the the longitudinal coordinate s ($s = x - j$ for the j-th horizontal thin slit and $s = y$ for the vertical thin slits) and is called the far field expansion. The second one is the near field expansion and is used to approximate u^ε in the neighborhood of each junction. Thus, it is expressed by means of the fast variables $((x-j)/\varepsilon, (y+L/2)/\varepsilon)$ near the j-th junction and is defined on a normalized unperturbed junction for $j \neq 0$ and a normalized perturbed junction for $j = 0$. Since both expansions are meant to be two approximations of the same function u^ε, they have to satisfy some matching conditions in some intermediate zones. This method is often called Matched Asymptotic Expansion. For complete and detailed descriptions of the method, we refer the reader to [43], [16] and [30] (cf. Part IV, which is dedicated to the eigenvalue problems). See also [4] for a recent application of the method to a spectral problem.

In Section 5.2, we give the ansatz for the far field and the near field expansions and derive the problems defined by induction satisfied by each far field and near field terms. We give also the matching conditions. We study in Section 5.3 and in Section 5.4 the well posedness of the problems satisfied by the near field terms and the far field terms respectively. We explain the algorithm of construction of each term in Section 5.5 and finally prove Theorem 4 by establishing (57) in Section 5.6.

Before entering the details, let us introduce some notations. The function u^ε being even with respect to y, it suffices to construct an asymptotic expansion of u^ε on the lower half part $\Omega^{\mu,-}_\varepsilon$ of Ω^μ_ε (comb shape domain):

$$\Omega^{\mu,-}_\varepsilon = \{(x,y) \in \Omega^\mu_\varepsilon \ \text{ s.t. } \ y < 0\}.$$
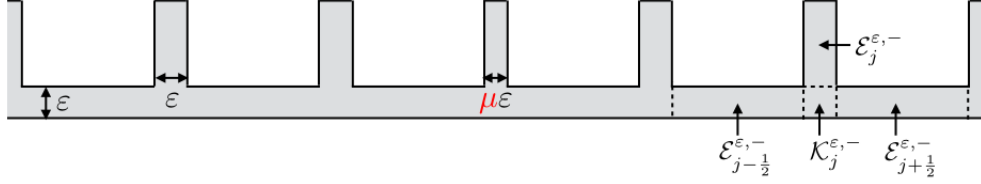
Figure 9: The domain $\Omega_\varepsilon^{\mu,-}$

As represented on Figure 9, we denote by $\mathcal{E}_{j+\frac{1}{2}}^{\varepsilon,-}$, $j \in \mathbb{Z}$, the horizontal edges of the domain $\Omega_\varepsilon^{\mu,-}$

$$\mathcal{E}_{j+\frac{1}{2}}^{\varepsilon,-} = \left(j + \varepsilon\mu_j/2, (j+1) - \varepsilon\mu_{j+1}/2\right) \times \left(-L/2, -L/2 + \varepsilon\right),$$

by $\mathcal{E}_j^{\varepsilon,-}$ its vertical edges

$$\mathcal{E}_j^{\varepsilon,-} = \left(j - \varepsilon\mu_j/2, j + \varepsilon\mu_j/2\right) \times \left(-L/2 + \varepsilon, 0\right),$$

and by $\mathcal{K}_j^{\varepsilon,-}$ the junctions

$$\mathcal{K}_j^{\varepsilon,-} = \left(j - \varepsilon\mu_j/2, j + \varepsilon\mu_j/2\right) \times \left(-L/2, -L/2 + \varepsilon\right)$$

where for all $j \in \mathbb{Z}$, $\mu_j = 1$ if $j \neq 0$ and $\mu_0 = \mu$ (with the notations of Section 4.1 $\mu_j = w^\mu(e_j)$ where the function w^μ is defined by (9) and the $\{e_j\}_{j \in \mathbb{Z}}$ are the corresponding vertical edges of the graph $\mathcal{G}$).

5.2   Asymptotic expansions : ansatz and equations

Until the end of the paper, $\lambda^{(0)} \in (a,b)$ is an eigenvalue of the operator $\mathcal{A}_s^\mu$ and $u^{(0)}$ is an associated eigenvector :

$$u^{(0)} \in D(\mathcal{A}_s^\mu), \quad \mathcal{A}_s^\mu \, u^{(0)} = \lambda^{(0)} \, u^{(0)} \tag{60}$$

i.e. denoting $\forall j \in \mathbb{Z}$ $u_{j+\frac{1}{2}}^{(0)} \equiv u^{(0)}\big|_{e_{j+1/2}^\pm}$ and $u_j^{(0)} \equiv u^{(0)}\big|_{e_j}$ (using the notations of Section 4.1)

$$\forall j \in \mathbb{Z}, \quad \begin{cases} \partial_s^2 u_{j+\frac{1}{2}}^{(0)}(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(k)}(s) = 0, \quad s = x - j \in [0,1], \\ \partial_y^2 u_j^{(0)}(y) + \lambda^{(0)} u_j^{(0)}(y) = 0, \quad y \in [-L/2, 0], \\ \partial_y u_j^{(0)}(0) = 0, \\ u_{j-\frac{1}{2}}^{(0)}(1) = u_j^{(0)}(-L/2) = u_{j+\frac{1}{2}}^{(0)}(0), \\ \partial_s u_{j+\frac{1}{2}}^{(0)}(0) - \partial_s u_{j-\frac{1}{2}}^{(0)}(1) + \mu_j \, \partial_y u_j^{(0)}(-L/2) = 0. \end{cases} \tag{61}$$

where, to ease the reading of the next sections, we denote ∂_s (resp. ∂_y) the derivative with respect to s (resp. to y). We choose $u^{(0)}$ as follows: $\forall j \in \mathbb{Z}$,

$$u_{j+\frac{1}{2}}^{(0)}(s) = r^{|j|} \frac{\sin\left(\sqrt{\lambda^{(0)}}(1-s)\right)}{\sin\sqrt{\lambda^{(0)}}} + r^{|j+1|} \frac{\sin\left(\sqrt{\lambda^{(0)}}s\right)}{\sin\sqrt{\lambda^{(0)}}}, \qquad s = x - j \in [0,1], \tag{62}$$

$$u_j^{(0)}(y) = r^{|j|} \frac{\cos\left(\sqrt{\lambda^{(0)}}y\right)}{\cos\left(\sqrt{\lambda^{(0)}}L/2\right)}, \qquad\qquad y \in [-L/2, L/2]. \tag{63}$$

Here r stands for the quantity $r(\sqrt{\lambda^{(0)}})$ defined in (42). Note that $u^{(0)}$ is exponentially decaying.

We propose an asymptotic expansion for λ^ε and u^ε solution of (59) constructed by induction starting from $\lambda^{(0)}$ and $u^{(0)}$. We first suppose a formal decomposition for the eigenvalue:

$$\lambda^\varepsilon = \sum_{k\in\mathbb{N}} \varepsilon^k \lambda^{(k)}. \tag{64}$$

In the sequel, we will impose by convention that $\lambda^{(-1)} = \lambda^{(-2)} = 0$.

Mimicking the approach of [20, 19, 42], we use the following ansatz (see Figure 10 for an schematic illustration of this ansatz):

Figure 10: Schematic representation of the asymptotic expansion (NF: near field, FF: far field).

- **Far field asymptotic expansion:** in the horizontal thin slits $\mathcal{E}^{\varepsilon,-}_{j+\frac{1}{2}}$ ($j \in \mathbb{Z}$), far from their extremities, we assume that the following expansion holds:

$$u^\varepsilon(x,y) \equiv u^\varepsilon_{j+\frac{1}{2}}(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k u^{(k)}_{j+\frac{1}{2}}(s), \quad s = x - j. \tag{65}$$

with $u^{(0)}_{j+\frac{1}{2}}$ given by (62). In the same way, in the vertical thin slits $\mathcal{E}^{\varepsilon,-}_j$ ($j \in \mathbb{Z}$) and far from their bottom extremity, we assume that

$$u^\varepsilon(x,y) \equiv u^\varepsilon_j(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k u^{(k)}_j(y). \tag{66}$$

with $u^{(0)}_j$ given by (63).

Far field terms $u^{(k)}_{j+\frac{1}{2}}$ (resp. $u^{(k)}_j$) are defined on the edge $e^-_{j+\frac{1}{2}}$ (resp. $e^-_j = e_j \cap \{y < 0\}$) of the half graph $\mathcal{G}^- = \mathcal{G} \cap \{y < 0\}$, the term $k = 0$ is given by (62-63). They are independant of ε. In what follows, for any k, we denote by $u^{(k)}$ (far field of order k) the set of far field functions $\{u^{(k)}_{j+\frac{1}{2}}, u^{(k)}_j\}_{j\in\mathbb{Z}}$. By convention, we will impose that $u^{(-1)} = u^{(-2)} = 0$ and we look for $u^{(k)} \in L^\mu_2(\mathcal{G}^-)$, $L^\mu_2(\mathcal{G}^-)$ being defined as in (10).

Substituting the far field expansions (65-66) and the expansion (64) of the eigenvalue into the eigenvalue problem (59), taking into account the symmetry of $u^{(k)}_j$, and separating formally the different powers of ε, we arrive at the following set of problems for the far field terms $u^{(k)}$, $k \in \mathbb{N}^*$:

$\forall j \in \mathbb{Z}$,

$$
\begin{cases}
\partial_s^2 u_{j+\frac{1}{2}}^{(k)}(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(k)}(s) = -\displaystyle\sum_{m=0}^{k-1} \lambda^{(k-m)} u_{j+\frac{1}{2}}^{(m)}(s), \quad s \in (0,1). \\[4mm]
\partial_y^2 u_j^{(k)}(y) + \lambda^{(0)} u_j^{(k)}(y) = -\displaystyle\sum_{m=0}^{k-1} \lambda^{(k-m)} u_j^{(m)}(y), \quad y \in (-L/2, 0), \qquad \partial_y u_j^{(k)}(0) = 0.
\end{cases}
$$

$$(67)$$

Far field terms $u^{(k)}$ are still not completely defined by (67). To link the functions $(u_{j+\frac{1}{2}}^{(k)}, u_j^{(k)})$ $(j \in \mathbb{Z})$, we need to prescribe transmission conditions at the vertices of the graph $\mathcal{G}$. These transmission conditions will directly result from the matching conditions.

- **Near field asymptotic expansion:** in the neighborhood of the junctions $\mathcal{K}_j^{\varepsilon,-}$ $(j \in \mathbb{Z})$, we seek an expansion of the form

$$
u^\varepsilon(x,y) \equiv U_j^\varepsilon(x,y) = \sum_{k \in \mathbb{N}} \varepsilon^k U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{y+L/2}{\varepsilon}\right) \tag{68}
$$

The near field terms $U_j^{(k)}(X,Y)$ do not depend on ε. For $j \neq 0$ (the unperturbed junctions), they are defined in the infinite rescaled unperturbed junction $\mathcal{J}$

$$
\mathcal{J} = \mathbb{R} \times (0,1) \, \bigcup \, (-1/2, 1/2) \times (1, +\infty).
$$

For $j = 0$, $U_0^{(k)}$ is defined in the rescaled perturbed junction

$$
\mathcal{J}_0 = \mathbb{R} \times (0,1) \, \bigcup \, (-\mu/2, \mu/2) \times [1+\infty). \tag{69}
$$

To be more concise, in the sequel, we shall say that the near fields terms are defined in the rescaled junction $\mathcal{J}_j$, where $\mathcal{J}_0$ is defined by (69) and $\mathcal{J}_j = \mathcal{J}$ for any $j \in \mathbb{Z}^*$ (see Figure 11). Note that each $\mathcal{J}_j$ is made of three unbounded branches. In what follows, for any k, we denote by $U^{(k)}$ (near field of order k) the set of near field functions $\{U_j^{(k)}\}_{j \in \mathbb{Z}}$. By convention, we will impose that $U^{(-1)} = U^{(-2)} = 0$.

As usual (see [19], [20]), we look for near field terms $U_j^{(k)}$ that belong to $H_{loc}^1(\mathcal{J}_j)$ and are not exponentially growing in the infinite branches

$$
B_{j,-} = \mathcal{J}_j \cap \{X < -\mu_j/2\}, \quad B_{j,+} = \mathcal{J}_j \cap \{X > \mu_j/2\} \quad \text{and} \quad B_{j,0} = \mathcal{J}_j \cap \{Y > 1\}. \tag{70}
$$

More precisely, this corresponds to impose that for all j and all k, $U_j^{(k)} \in \mathcal{V}_j$ where

$$
\mathcal{V}_j = \left\{ U \in H_{loc}^1(\mathcal{J}_j), w_j U \in H^1(\mathcal{J}_j) \right\}
$$

with

$$
w_j = \begin{vmatrix} 1 & \text{in } K_j = \mathcal{J}_j \setminus (B_{j,-} \cup B_{j,+} \cup B_{j,0}), \\ e^{-\sqrt{\pm(X \mp \mu_j/2)}} & \text{in } B_{j,\pm}, \\ e^{-\sqrt{Y-1}} & \text{in } B_{j,0}. \end{vmatrix}
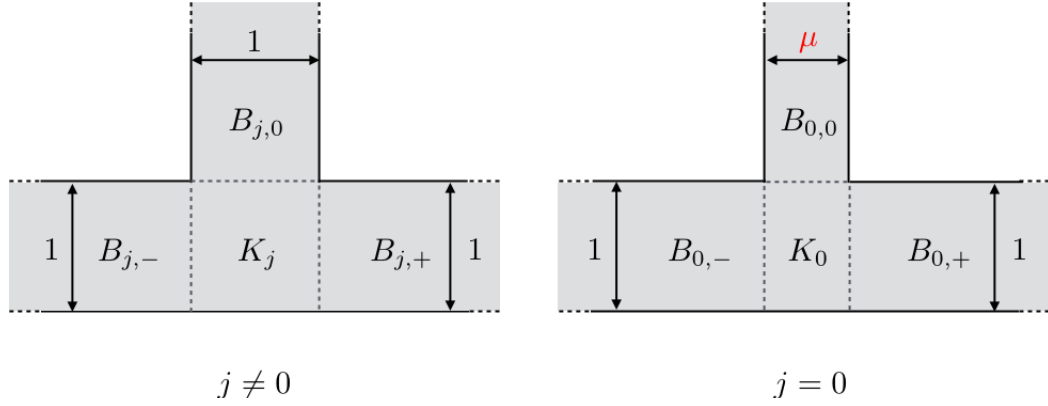$$

Figure 11: The jonctions $\mathcal{J}_j$ (perturbed for $j = 0$ and unperturbed for $j \neq 0$)

After injecting the near field expansion (68) and the eigenvalue expansion (64) into (59) and separating formally the different powers of ε, we find a set of problems for the near field functions $U^{(k)}$, $k \in \mathbb{N}$:

$$\forall j \in \mathbb{Z}, \qquad \begin{cases} \Delta U_j^{(k)} = -\sum_{m=0}^{k-2} \lambda^{(k-m-2)} U_j^{(m)} & \text{in } \mathcal{J}_j \\ \dfrac{\partial U_j^{(k)}}{\partial n} = 0 & \text{on } \partial \mathcal{J}_j. \end{cases} \tag{71}$$

As for the far field terms, near field terms $U_j^{(k)}$ are not completely defined by (71) (for instance, any constant function satisfies Problem (71) for $k \leq 1$). We need to prescribe their behavior at infinity (in the three infinite branches $B_{j,\delta}$, $\delta \in \{+, -, 0\}$), which here again results from the matching conditions. In the present case, the derivation of the matching conditions relies on a modal expansion of the near field terms in the three infinite branches $B_{j,-}$, $B_{j,+}$ and $B_{j,0}$ of $\mathcal{J}_j$.

As a result, we can prove the following modal decomposition (a complete proof of this result can be found in [20] p. 316):

Lemma 5. *Assume that there exists a sequence of near field terms $U^{(k)}$ such that for all k and for all $j \in \mathbb{Z}$, $U_j^{(k)} \in \mathcal{V}_j$ is solution to (71). Then, for any $k \in \mathbb{N}$ and any $j \in \mathbb{Z}$ there exist a polynomial $p_{j,0}^{(k)} \in \mathbb{P}_{k+1}$ and a family $\left\{ p_{j,\ell,0}^{(k)} \in \mathbb{P}_{[k/2]}, \ \ell \neq 0 \right\}$ ([k/2] being the floor of $k/2$) such that*

$$U_j^{(k)}(X, Y) = p_{j,0}^{(k)}(Y) + \sum_{\ell=1}^{+\infty} p_{j,\ell,0}^{(k)}(Y)\, e^{-\ell\pi/\mu_j |Y|}\, f_\ell^j(X) \quad \text{in } B_{j,0}. \tag{72}$$

where $f_\ell^j(X) = \sqrt{2/\mu_j}\, \sin(\ell\pi X/\mu_j)$ if ℓ is odd and $\sqrt{2/\mu_j}\, \cos(\ell\pi X/\mu_j)$ if ℓ is even.

Similarly, there exist two polynomials $p_{j,\pm}^{(k)} \in \mathbb{P}_{k+1}$ and two families of polynomials $\left\{ p_{j,\ell,\pm}^{(k)} \in \mathbb{P}_{[k/2]}, \ \ell \neq 0 \right\}$ such that

$$U_j^{(k)}(X,Y) = p_{j,\pm}^{(k)}(X) + \sum_{\ell=1}^{+\infty} p_{j,\ell,\pm}^{(k)}(X) \, e^{-\ell\pi|X|} \, g_\ell(Y) \quad in \ B_{j,\pm}. \tag{73}$$

where $g_\ell(Y) = \sqrt{2}\cos(\ell\pi Y)$.

Remark 5. *It can be shown that the series in (72) (resp. (73)) converge in $H^1(B_{j,0})$ (resp. in $H^1(B_{j,\pm})$).*

Remark 6. *The proof of Lemma 5 is done by induction and using separation of variables techniques. For each k, and for each junctions $\mathcal{J}_j$, $U_j^{(k)}$ can be decomposed, in each band $B_{j,\delta}$, $\delta \in \{0,+,-\}$, thanks to an orthonormal basis of eigenvectors, indexed by $\ell \in \mathbb{N}$, of the transverse laplacian with Neumann boundary conditions. For instance, the first eigenvector (corresponding to $\ell = 0$) is the constant function 1. The coefficients of the decomposition are functions of the longitudinal variable (Y for the band $B_{j,0}$ and X for the bands $B_{j,\pm}$). According to (71), these functions are obtained solving second order ordinary differential equations with a source term depending on the previous terms $\lambda^{(m)}$ and $U_j^{(m)}$ for $m \leq k-2$. Thus we can show by induction that the first coefficient of the decomposition (corresponding to $\ell = 0$) is a polynomial $p_{j,\delta}^{(k)}$ of degree k. Except its affine part - the "ax+b"-part of the polynomial, this polynomial can be computed by induction thanks to the previous terms $\lambda^{(m)}$ and $p_{j,\delta}^{(m)}$ for $m \leq k - 2$. The affine part of $p_{j,\delta}^{(k)}$ will be determined thanks to the matching conditions (see (76) and (77) and Remark 7). In the same way, we can show by induction that the other coefficients of the decomposition (corresponding to $\ell \in \mathbb{N}^*$) are a decaying exponential times a polynomial $p_{j,\ell,\delta}^{(k)}$ of degree $[k/2]$. Except its constant part, each polynomial can be computed by induction thanks to the previous terms $\lambda^{(m)}$ and $p_{j,\ell,\delta}^{(m)}$ for $m \leq k - 2$. The constant part will be determined indirectly thanks to the matching conditions and the equations (see (96)).*

- **Matching conditions** To find the missing information (transmission conditions at the vertices of the graph for the far field terms and behavior at infinity for the near field terms), we shall write the so-called matching conditions that ensure that far field and near field expansions coincide in some intermediate areas. Indeed, far field and near field expansions are assumed to be both valid in some intermediate areas $\mathcal{M}_j^\varepsilon$, $j \in \mathbb{Z}$, localized at the left, right and above each junction K_j^ε,

$$\mathcal{M}_{j,m}^\varepsilon = \mathcal{M}_{j,-}^\varepsilon \cup \mathcal{M}_{j,+}^\varepsilon \cup \mathcal{M}_{j,0}^\varepsilon.$$

Then the matching zone $\mathcal{M}_{j,+}^\varepsilon$ corresponds to $x - j \to 0$ for the far field and to $X = (x - j)/\varepsilon \to +\infty$ for the near field. Typically, we can choose for instance (see Figure 12 for this example), for any integer j, the left and right intermediate areas $\mathcal{M}_{j,-}^\varepsilon$ and $\mathcal{M}_{j,+}^\varepsilon$ of the form

$$\mathcal{M}_{j,-}^\varepsilon = (0,\varepsilon) \times \left(j - 2\sqrt{\varepsilon}, j - \sqrt{\varepsilon}\right), \quad \mathcal{M}_{j,+}^\varepsilon = (0,\varepsilon) \times \left(j + \sqrt{\varepsilon}, j + 2\sqrt{\varepsilon}\right),$$

and the vertical intermediate areas of the form

$$\mathcal{M}_{j,0}^\varepsilon = \left(j - \frac{\mu_j\varepsilon}{2}, j + \frac{\mu_j\varepsilon}{2}\right) \times \left(-\frac{L}{2} + \sqrt{\varepsilon}, -\frac{L}{2} + 2\sqrt{\varepsilon}\right).$$
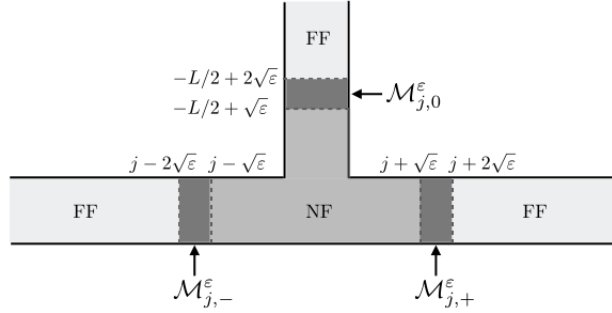
Figure 12: the matching area $\mathcal{M}_{j,m}^\varepsilon = \mathcal{M}_{j,-}^\varepsilon \cup \mathcal{M}_{j,+}^\varepsilon \cup \mathcal{M}_{j,0}^\varepsilon$ for $j \neq 0$ (NF: near field, FF: far field).

Assuming that the far field term $u_{j+\frac{1}{2}}^{(k)}$ is smooth (this assumption will be a posteriori verified, see Property 1 in Proposition 13), it can be decomposed into Taylor series in $\mathcal{M}_{j,+}^\varepsilon$ (i.e. in a neighborhood of the point $x = j$). Thus, the relation (65) can be (formally) rewritten as

$$u_{j+\frac{1}{2}}^\varepsilon(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k \sum_{\ell\in\mathbb{N}} \partial_s^\ell u_{j+\frac{1}{2}}^{(k)}(0)\frac{(x-j)^\ell}{\ell!}, \qquad j\in\mathbb{Z}. \tag{74}$$

Analogously, we can also substitute the near field terms in $\mathcal{M}_{j,+}^\varepsilon$ with their modal expansion (73) into the near field expansion (68), neglecting the exponentially decaying part

$$u_{j+\frac{1}{2}}^\varepsilon(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k p_{j,+}^{(k)}\left(\frac{x-j}{\varepsilon}\right) + O(\varepsilon^\infty), \qquad j\in\mathbb{Z}. \tag{75}$$

Then, writing $X = (x-j)/\varepsilon$ in (74) and (68), we can relate the two expansions :

$$\sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} \varepsilon^{k+l}\partial_s^\ell u_{j+\frac{1}{2}}^{(k)}(0)\frac{X^\ell}{\ell!} = \sum_{k\in\mathbb{N}} \varepsilon^k p_{j,+}^{(k)}(X), \qquad j\in\mathbb{Z}.$$

Identifying the terms of the order ε^k, we finally obtain the following expressions for the polynomials $p_{j,+}^{(k)}$:

$$p_{j,+}^{(k)}(X) = \sum_{l=0}^k \partial_s^\ell u_{j+\frac{1}{2}}^{(k-\ell)}(0)\frac{X^\ell}{\ell!}, \qquad k\in\mathbb{N}, \qquad j\in\mathbb{Z}. \tag{76}$$

Obviously, we can obtain two other sets of matching conditions by identifying near field and far field expansions in the matching areas $\mathcal{M}_{j,-}^\varepsilon$ and $\mathcal{M}_{j,0}^\varepsilon$:

$$p_{j,-}^{(k)}(X) = \sum_{\ell=0}^k \partial_s^\ell u_{j-\frac{1}{2}}^{(k-\ell)}(1)\frac{X^\ell}{\ell!} \quad\text{and}\quad p_{j,0}^{(k)}(Y) = \sum_{\ell=0}^k \partial_y^\ell u_j^{(k-\ell)}(-L/2)\frac{Y^\ell}{\ell!}, \qquad k\in\mathbb{N}, \qquad j\in\mathbb{Z}. \tag{77}$$

Remark 7. *At each step $k \in \mathbb{N}$, it suffices to match the affine part of $p_{j,\delta}^{(k)}$, for $\delta \in \{0, +, -\}$. Indeed, the terms of degree larger than 2 of $p_{j,\delta}^{(k)}$ automatically match as soon as the equations (67) and (71) and the matching conditions of lower order are satisfied.*

5.3 Conditions for the well-posedness of the near field problems

Let us suppose in this section that, for a fixed $k \geq 1$, (1) the terms $\lambda^{(m)}$ and the near field terms $U^{(m)}$ are known for $m \leq k - 1$; (2) the far field terms $u^{(m)}$ are known for $m \leq k$ and (3) the matching conditions are satisfied up to the order k. We want to give sufficient and necessary conditions on these previous terms ensuring the existence and the uniqueness of the near field term $U^{(k)}$ satisfying both the near field problem (71) and the matching conditions (76-77). It is given in the following proposition:

Proposition 11. *Let $j \in \mathbb{Z}$. Problem (71-76-77) has an unique solution $U_j^{(k)}$ in $\mathcal{V}_j$ if and only if the two following conditions hold*

1. *the far field term $u^{(k-1)}$ satisfies the generalized Kirchhoff conditions*

$$\partial_s u_{j+\frac{1}{2}}^{(k-1)}(0) - \partial_s u_{j-\frac{1}{2}}^{(k-1)}(1) + \mu_j \partial_y u_j^{(k-1)}\big(-L/2\big) = \Xi_j^{(k-2)}, \tag{78}$$

   where $\Xi_j^{(k-2)}$ is defined in (87);

2. *the far field term $u^{(k)}$ satisfies the jump conditions*

$$u_{j-\frac{1}{2}}^{(k)}(1) - u_j^{(k)}\big(-L/2\big) = \Delta_{j,-}^{(k-1)}, \qquad j \in \mathbb{Z}, \tag{79}$$

$$u_j^{(k)}\big(-L/2\big) - u_{j+\frac{1}{2}}^{(k)}(0) = \Delta_{j,+}^{(k-1)}, \qquad j \in \mathbb{Z}, \tag{80}$$

   where $\Delta_{j,-}^{(k-1)}$ and $\Delta_{j,+}^{(k-1)}$ are defined in (91) and (92).

The rest of this section is devoted to the proof of this proposition, which will be done in two steps. We first suppose that Problem (71-76-77) has a solution $U_j^{(k)}$ in $\mathcal{V}_j$ and we show that (78-79-80) have to be satisfied. In a second step, supposing that these conditions are fulfilled, we construct a solution to (71-76-77) in $\mathcal{V}_j$. This solution being defined uniquely, this will end the proof of the proposition.

Proof of Proposition 11

(1) Let us suppose that there exists a solution $U_j^{(k)} \in \mathcal{V}_j$. We introduce first the so-called Dirichlet to Neumann maps T and T_j (which are positive and symmetric operators (see [42])):

$$T : \begin{cases} H^{1/2}(0,1) \to H^{-1/2}(0,1) \\ \varphi \mapsto T\varphi := \sum_{\ell \geq 1} \ell\pi \, (\varphi, g_\ell)_{L^2} \, g_\ell \end{cases} \qquad T_j : \begin{cases} H^{1/2}\left(-\frac{\mu_j}{2}, \frac{\mu_j}{2}\right) \to H^{-1/2}\left(-\frac{\mu_j}{2}, \frac{\mu_j}{2}\right) \\ \varphi \mapsto T_j\varphi := \sum_{\ell \geq 1} \ell\pi/\mu_j \, (\varphi, f_\ell^j)_{L^2} \, f_\ell^j \end{cases} \qquad (j \in \mathbb{Z}). \tag{81}$$

where the functions f_ℓ^j and g_ℓ are defined in Lemma 5.

Because $U_j^{(k)} \in \mathcal{V}_j$ is solution to the near field equation (71) fulfilling the matching conditions (76-77), its restriction $V_j^{(k)}$ to the rescaled bounded domain $K_j = (-\mu_j/2, \mu_j/2) \times (0,1)$ satisfies

$$\begin{cases} \Delta V_j^{(k)} = \Phi_j^{(k-2)} \text{ in } K_j, \\ \pm \partial_X V_j^{(k)} + T V_j^{(k)} = g_{j,\pm}^{(k-1)} \text{ on } \Sigma_{j,\pm}, \\ \partial_Y V_j^{(k)} + T_j V_j^{(k)} = g_{j,0}^{(k-1)} \text{ on } \Sigma_{j,0}, \end{cases} \tag{82}$$

where $\Sigma_{j,\pm} = \partial K_j \cap \partial B_{j,\pm}$, $\Sigma_{j,0} = \partial K_j \cap \partial B_{j,0}$ and

$$\Phi_j^{(k-2)} = -\sum_{m=0}^{k-2} \lambda^{(k-m-2)} U_j^{(m)}, \tag{83a}$$

$$g_{j,+}^{(k-1)} = \sum_{\ell=0}^{k-1} \frac{(\mu_j/2)^\ell}{\ell!} \partial_s^{\ell+1} u_{j+\frac{1}{2}}^{(k-\ell-1)}(0) + \sum_{\ell=1}^{+\infty} \partial_X p_{j,\ell,+}^{(k)}(\mu_j/2) e^{-\ell\pi\mu_j/2} g_\ell, \tag{83b}$$

$$g_{j,-}^{(k-1)} = -\sum_{\ell=0}^{k-1} \frac{(-\mu_j/2)^\ell}{\ell!} \partial_s^{\ell+1} u_{j-\frac{1}{2}}^{(k-\ell-1)}(1) + \sum_{\ell=1}^{+\infty} \partial_X p_{j,\ell,-}^{(k)}(-\mu_j/2) e^{-\ell\pi\mu_j/2} g_\ell, \tag{83c}$$

$$g_{j,0}^{(k-1)} = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \partial_y^{\ell+1} u_j^{(k-\ell-1)}(-L/2) + \sum_{\ell=1}^{+\infty} \partial_Y p_{j,\ell,0}^{(k)}(1) e^{-\ell\pi/\mu_j} f_\ell^j. \tag{83d}$$

In view of Remark 6, the functions $\partial_X p_{j,\ell,\pm}^{(k)}$ and $\partial_Y p_{j,\ell,0}^{(k)}$ only depend on the terms λ_m, $U_j^{(m)}$ for $m \leq k-2$. Consequently, if the near fields terms $U^{(m)}$, the far fields terms $u^{(m)}$ and the eigenvalue $\lambda^{(m)}$ are defined up to order $m = k-1$, then the functions $g_{j,\delta}^{(k-1)}$ for $\delta \in \{0, +, -\}$ are entirely determined (which justifies its indexation) and they are in $H^{-1/2}(\Sigma_{j,\delta})$.

Let us remind the following well posedness result, obtained by adapting Lemma 2.3.2 in [42] to the geometry of our problem.

Lemma 6. *Let* $j \in \mathbb{Z}$, $\Phi \in L_2(K_j)$, $g_\pm \in H^{-1/2}(\Sigma_{j,\pm})$, $g_0 \in H^{-1/2}(\Sigma_{j,0})$. *There exists a unique modulo an additive constant solution* $V \in H^1(K_j)$ *to the problem*

$$\begin{cases} \Delta V = \Phi \ in \ K_j, \\ \partial_y V(X,0) = 0 \ on \ \Sigma_{j,N} \\ \pm\partial_X V + TV = g_\pm \ on \ \Sigma_{j,\pm}, \\ \partial_Y V + T_j V = g_0 \ on \ \Sigma_{j,0}. \end{cases} \tag{84}$$

if and only if the following compatibility condition is satisfied:

$$\langle g_+, 1 \rangle_{\Sigma_{j,+}} + \langle g_-, 1 \rangle_{\Sigma_{j,-}} + \langle g_0, 1 \rangle_{\Sigma_{j,0}} = \int_{K_j} \Phi. \tag{85}$$

where for $\delta \in \{+, -, 0\}$, $\langle g_\delta, f \rangle_{\Sigma_{j,\delta}}$ *denotes the duality brackets for* $g_\delta \in H^{-1/2}(\Sigma_{j,\delta})$ *and* $f \in H^{1/2}(\Sigma_{j,\delta})$.

As a consequence, Problem (82) has a solution in $H^1(K_j)$ only if the following compatibility condition is satisfied:

$$\left\langle g_{j,+}^{(k-1)}, 1 \right\rangle_{\Sigma_{j,+}} + \left\langle g_{j,-}^{(k-1)}, 1 \right\rangle_{\Sigma_{j,-}} + \left\langle g_{j,0}^{(k-1)}, 1 \right\rangle_{\Sigma_{j,0}} = \int_{K_j} \Phi_j^{(k-2)}. \tag{86}$$

The previous condition is equivalent to the non-homogeneous Kirchhoff's transmission conditions (78) for the far field function $u^{(k-1)}$

$$\partial_s u_{j+\frac{1}{2}}^{(k-1)}(0) - \partial_s u_{j-\frac{1}{2}}^{(k-1)}(1) + \mu_j \, \partial_y u_j^{(k-1)}\left(-\frac{L}{2}\right) = \Xi_j^{(k-2)},$$

with

$$\Xi_j^{(k-2)} = -\sum_{\ell=1}^{k-1} \left(\frac{(\mu_j/2)^\ell}{\ell!} \partial_s^{\ell+1} u_{j+\frac{1}{2}}^{(k-\ell-1)}(0) - \frac{(-\mu_j/2)^\ell}{\ell!} \partial_s^{\ell+1} u_{j-\frac{1}{2}}^{(k-\ell-1)}(1) + \frac{\mu_j}{\ell!} \partial_y^{\ell+1} u_j^{(k-\ell-1)}(-L/2) \right)$$

$$+ \int_{K_j} \Phi_j^{(k-2)}. \quad (87)$$

Moreover, because $U_j^{(k)}$ satisfies the matching conditions (76-77) then its restriction $V_j^{(k)}$ to K_j fulfills (note that for any integer $\ell \neq 0$, $\int_{\Sigma_{j,\pm}} f_\ell^j d\sigma = \int_{\Sigma_{j,0}} g_\ell d\sigma = 0$)

$$\int_{\Sigma_{j,+}} V_j^{(k)} = \sum_{\ell=0}^{k} \frac{(\mu_j/2)^\ell}{\ell!} \partial_s^\ell u_{j+\frac{1}{2}}^{(k-\ell)}(0), \qquad \int_{\Sigma_{j,-}} V_j^{(k)} = \sum_{\ell=0}^{k} \frac{(-\mu_j/2)^\ell}{\ell!} \partial_s^\ell u_{j-\frac{1}{2}}^{(k-\ell)}(1), \qquad (88)$$

$$\int_{\Sigma_{j,0}} V_j^{(k)} = \mu_j \sum_{\ell=0}^{k} \frac{1}{\ell!} \partial_y^\ell u_j^{(k-\ell)}(-L/2). \qquad (89)$$

Let us introduce the profile functions $\mathcal{N}_j^- \in H^1(K_j)$, $j \in \mathbb{Z}$ defined as the unique (modulo an additive constant) solution to the problem (84) with $\Phi = 0$, $g_+ = 0$, $g_- = 1$, $g_0 = -1/\mu_j$. Multiplying equation (82) by $\mathcal{N}_j^-$ and integrating by parts over the domain K_j gives

$$\int_{\Sigma_{j,+}} \left(\partial_X \mathcal{N}_j^- V_j^{(k)} - \mathcal{N}_j^- \partial_X V_j^{(k)} \right) - \int_{\Sigma_{j,-}} \left(\partial_X \mathcal{N}_j^- V_j^{(k)} - \mathcal{N}_j^- \partial_X V_j^{(k)} \right) + \int_{\Sigma_{j,0}} \left(\partial_Y \mathcal{N}_j^- V_j^{(k)} - \mathcal{N}_j^- \partial_Y V_j^{(k)} \right) = -\int_{K_j} \Phi_j^{(k-2)} \mathcal{N}_j^-.$$

Using the fact that $V_j^{(k)}$ satisfies the conditions (88-89) and that the DtN operators are symmetric, we obtain

$$\sum_{\ell=0}^{k} \frac{(-\mu_j/2)^\ell}{\ell!} \partial_s^\ell u_{j-\frac{1}{2}}^{(k-\ell)}(1) - \sum_{\ell=0}^{k} \frac{1}{\ell!} \partial_y^\ell u_j^{(k-\ell)}(-L/2) - \sum_{\delta \in \{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, \mathcal{N}_j^- \right\rangle_{\Sigma_{j,\delta}} = -\int_{K_j} \Phi_j^{(k-2)} \mathcal{N}_j^-$$

$$(90)$$

This implies the condition on the jumps of the function $u_j^{(k)}$

$$u_{j-\frac{1}{2}}^{(k)}(1) - u_j^{(k)}(-L/2) = \Delta_{j,-}^{(k-1)}, \qquad j \in \mathbb{Z},$$

with

$$\Delta_{j,-}^{(k-1)} = \sum_{\ell=1}^{k} \left(\frac{1}{\ell!} \partial_y^\ell u_j^{(k-\ell)}(-L/2) - \frac{(-\mu_j/2)^\ell}{\ell!} \partial_s^\ell u_{j-\frac{1}{2}}^{(k-\ell)}(1) \right) + \sum_{\delta \in \{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, \mathcal{N}_j^- \right\rangle_{\Sigma_{j,\delta}} - \int_{K_j} \Phi_j^{(k-2)} \mathcal{N}_j^-.$$

$$(91)$$

In a similar way, we introduce the profile functions $\mathcal{N}_j^+$ that are the unique (modulo an additive constant) solutions to the problems (84) with $\Phi = 0$, $g_+ = -1$, $g_- = 0$, $g_0 = 1/\mu_j$. Using a symmetry argument, we can see that $\mathcal{N}_j^+(X, Y) = -\mathcal{N}_j^-(-X, Y)$ for any integer $j \in \mathbb{Z}$. Then, similar computations yields the other set of jump conditions

$$u_j^{(k)}(-L/2) - u_{j+\frac{1}{2}}^{(k)}(0) = \Delta_{j,+}^{(k-1)}, \qquad j \in \mathbb{Z},$$

with

$$\Delta_{j,+}^{(k-1)} = \sum_{\ell=1}^{k} \left(\frac{(\mu_j/2)^\ell}{\ell!} \partial_s^\ell u_{j+\frac{1}{2}}^{(k-\ell)}(0) - \frac{1}{\ell!} \partial_y^\ell u_j^{(k-\ell)}\left(-\frac{L}{2} \right) \right) + \sum_{\delta \in \{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, \mathcal{N}_j^+ \right\rangle_{\Sigma_{j,\delta}} - \int_{K_j} \Phi_j^{(k-2)} \mathcal{N}_j^+.$$

$$(92)$$

Let us emphasize that these conditions remain the same if one adds any constant to the profile functions.

(2) Conversely, let us suppose now that the conditions (78-79-80) are satisfied. Let us remind that the compatibility condition of Problem of (82-83) is equivalent to the generalized Kirchhoff condition (78). Using Lemma 6, we know that it exists an unique modulo an additive constant solution $V_j^{(k)}$ of (82-83). Let us fix the constant by enforcing $V_j^{(k)}$ to satisfy the first average condition of (88). We construct first the extension $V_{j,+}^{(k)}$ of $V_j^{(k)}$ to the band $B_{j,+}$. The trace $\varphi_{j,+}^{(k)} \in H^{1/2}(\Sigma_{j,+})$ of the function $V_j^{(k)}$ on the boundary $\Sigma_{j,+}$ has the following modal decomposition

$$\varphi_{j,+}^{(k)} = \sum_{\ell \in \mathbb{N}} \varphi_{j,\ell,+}^{(k)} g_\ell, \quad \text{where} \quad \sum_{\ell \in \mathbb{N}} \ell \left(\varphi_{j,\ell,+}^{(k)} \right)^2 < +\infty. \tag{93}$$

In view of the first average condition of (88), we have in particular

$$\varphi_{j,0,+}^{(k)} = p_{j,+}^{(k)}(\mu_j/2). \tag{94}$$

The extension is constructed thanks to the modal expansion (73) in $B_{j,+}$:

$$V_{j,+}^{(k)}(X,Y) = p_{j,+}^{(k)}(X) + \sum_{\ell=1}^{+\infty} p_{j,\ell,+}^{(k)}(X) e^{-\ell\pi X} g_\ell(Y), \tag{95}$$

where, as explained in Remark 6, the coefficients of the polynomial $p_{j,\ell,+}^{(k)}$ of degree equal or larger than one are completely determined by the previous near-field and far-field terms. The constant part of the polynomial $p_{j,\ell,+}^{(k)}$ is chosen so that the trace of the extension $V_{j,+}^{(k)}$ on $\Sigma_{j,+}$ is $\varphi_{j,+}^{(k)}$. In other words

$$\forall \ell \in \mathbb{N}^*, \quad p_{j,\ell,+}^{(k)}\left(\frac{\mu_j}{2} \right) e^{-\frac{\ell\pi\mu_j}{2}} = \varphi_{j,\ell,+}^{(k)}. \tag{96}$$

Therefore, we have defined an extension $V_{j,+}^{(k)} \in H_{loc}^1(B_{j,+})$ of $V_j^{(k)}$ which satisfies, by construction, the near field problem (71) in $B_{j,+}$ and whose trace on $\Sigma_{j,+}$ coincides with the trace of $V_j^{(k)}$. It remains to prove that its normal trace on $\Sigma_{j,+}$ coincides with the one of $V_j^{(k)}$. We have

$$\partial_X V_{j,+}^{(k)}\Big|_{\Sigma_{j,+}} = \partial_X p_{j,+}^{(k)}(\mu_j/2) + \sum_{\ell \in \mathbb{N}^*} \partial_X p_{j,\ell,+}^{(k)}(\mu_j/2) e^{-\frac{\ell\pi\mu_j}{2}} g_\ell + \sum_{\ell \in \mathbb{N}^*} \ell\pi p_{j,\ell,+}^{(k)}(\mu_j/2) e^{-\frac{\ell\pi\mu_j}{2}} g_\ell.$$

The first two terms of the r.h.s. correspond to the function $g_{j,+}^{(k-1)}$ defined in (83b) and the last term corresponds to $-T\left(V_j^{(k)}|_{\Sigma_{j,+}} \right)$. Since $V_j^{(k)}$ satisfies Problem (82), we end up with

$$\partial_X V_{j,+}^{(k)}\Big|_{\Sigma_{j,+}} = \partial_X V_j^{(k)}\Big|_{\Sigma_{j,+}}. \tag{97}$$

To construct the extension $V_{j,0}^{(k)} \in H_{loc}^1(B_{j,0})$ of $V_j^{(k)}$, we need to show that if $V_j^{(k)}$ satisfies the first average condition of (88) and if $u^{(k)}$ satisfies the jump conditions (79) then $V_j^{(k)}$ satisfies the

average condition (89). Multiplying the first equation of Problem (82) with the profile function $\mathcal{N}_j^+$ and integrating by parts over the bounded domain K_j, we obtain

$$\frac{1}{\mu_j} \int_{\Sigma_{j,0}} V_j^{(k)} = \int_{\Sigma_{j,+}} V_j^{(k)} + \int_{K_j} \phi_j^{(k-2)} \mathcal{N}_j^+ - \sum_{\delta \in \{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, \mathcal{N}_j^+ \right\rangle_{\Sigma_{j,\delta}}$$

Since the jump condition (80) is satisfied, we have

$$\int_{K_j} \phi_j^{(k-2)} \mathcal{N}_j^+ - \sum_{\delta \in \{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, \mathcal{N}_j^+ \right\rangle_{\Sigma_{j,\delta}} = p_{j,0}^{(k)}(1) - p_{j,+}^{(k)}(\mu_j/2),$$

which yields to

$$\int_{\Sigma_{j,0}} V_j^{(k)} = \mu_j \, p_{j,0}^{(k)}(1). \tag{98}$$

Once this average condition is satisfied, the construction of the extension $V_{j,0}^{(k)}$ is strictly similar than the previous one. In the same way, we show that $V_j^{(k)}$ satisfies the second average condition of (88) and we can build an extension $V_{j,-}^{(k)} \in H^1_{loc}(B_{j,-})$ of $V_j^{(k)}$. $V_j^{(k)}$ and its extensions allow to construct uniquely a solution of Problem (71) satisfying the matching conditions (76-77).

5.4   Conditions for the well-posedness of the far field problems

Collecting the results of the previous sections (far field equations (67), Kirchhoff conditions (78), and non-homogeneous jump conditions (79-80)), we arrive at the following set of problems for the far field functions $u^{(k)}$, $k \geq 1$:

$$\forall j \in \mathbb{Z}, \quad \begin{cases} \partial_s^2 u_{j+\frac{1}{2}}^{(k)}(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(k)}(s) = -\lambda^{(k)} u_{j+\frac{1}{2}}^{(0)}(s) - f_{j+\frac{1}{2}}^{(k-1)}(s), \quad s \in (0,1), \\ \partial_y^2 u_j^{(k)}(y) + \lambda^{(0)} u_j^{(k)}(y) = -\lambda^{(k)} u_j^{(0)}(y) - f_j^{(k-1)}(y), \quad y \in (-L/2, 0), \\ \partial_y u_j^{(k)}(0) = 0, \\ \partial_s u_{j+\frac{1}{2}}^{(k)}(0) - \partial_s u_{j-\frac{1}{2}}^{(k)}(1) + \mu_j \partial_y u_j^{(k)}(-L/2) = \Xi_j^{(k-1)}, \\ u_{j-\frac{1}{2}}^{(k)}(1) - u_j^{(k)}(-L/2) = \Delta_{j,-}^{(k-1)}, \quad u_j^{(k)}(-L/2) - u_{j+\frac{1}{2}}^{(k)}(0) = \Delta_{j,+}^{(k-1)}. \end{cases} \tag{99}$$

where

$$f_{j+\frac{1}{2}}^{(k-1)} = \sum_{m=1}^{k-1} \lambda^{(k-m)} u_{j+\frac{1}{2}}^{(m)}, \qquad f_j^{(k-1)} = \sum_{m=1}^{k-1} \lambda^{(k-m)} u_j^{(m)}, \qquad j \in \mathbb{Z}. \tag{100}$$

The conditions for well-posedness of the far-field problems (99) are given in the following Proposition whose proof is analogous to the proof of Theorem 4.10 by [31] (see also Corollary 2.2 and Theorem 2.13).

Proposition 12. *Let* $k \geq 1$. *Assume that* $f^{(k-1)} \in L_2^\mu(\mathcal{G}^-)$, $\left\{ \Xi_j^{(k-1)} \right\}_{j \in \mathbb{Z}} \in l_2(\mathbb{Z})$ *and* $\left\{ \Delta_{j,\pm}^{(k-1)} \right\}_{j \in \mathbb{Z}} \in l_2(\mathbb{Z})$. *Problem* (99) *has a solution* $u^{(k)} \in L_2^\mu(\mathcal{G}^-)$ *if and only if*

$$\lambda^{(k)} = \left\| u^{(0)} \right\|_{L_2^\mu(\mathcal{G}^-)}^{-2} \left(2 \sum_{j \in \mathbb{Z}} \widetilde{\Xi}_j^{(k-1)} \boldsymbol{u}_j^{(0)} - \left(f^{(k-1)}, u^{(0)} \right)_{L_2^\mu(\mathcal{G}^-)} \right). \tag{101}$$

where

$$\widetilde{\Xi}_j^{(k-1)} = \Xi_j^{(k-1)} - \frac{\sqrt{\lambda^{(0)}}}{\sin\sqrt{\lambda^{(0)}}}\left(\Delta_{j+1,-}^{(k-1)} - \Delta_{j-1,+}^{(k-1)} + \cos\sqrt{\lambda^{(0)}}\left(\Delta_{j,+}^{(k-1)} - \Delta_{j,-}^{(k-1)}\right)\right). \qquad (102)$$

The solution $u^{(k)}$ of the previous Lemma is defined up to any eigenvector. The uniqueness is clearly restored by imposing the orthogonality condition

$$\left(u^{(k)}, u^{(0)}\right)_{L_2^\mu(\mathcal{G}^-)} = 0. \qquad (103)$$

5.5   Algorithm of construction of the complete asymptotic expansion

By repeating applications of Proposition 12 and Proposition 11 (successively), we are able to define a recursive procedure to construct all the terms of the different asymptotic expansions (far field expansion, near field expansion and eigenvalue expansion) up to any order. The construction is done by induction, starting from $\lambda^{(0)}$ and the (already defined) limit far field term $u^{(0)}$ (see (62-63)).

Order 0.   As mentioned above, the far field problem of order 0 (99) corresponds to an eigenvalue problem for the operator $\mathcal{A}_s^\mu$. $\lambda^{(0)}$ is a given eigenvalue of $\mathcal{A}_s^\mu$ and $u^{(0)}$ is its associated eigenvector satisfying $u_0^{(0)}(-L/2) = 1$.

Next, let us construct the limit near field terms $U_j^{(0)}$, $j \in \mathbb{Z}$. In view of Proposition 11, there exists a unique solution of (82) (for $k = 0$) satisfying the matching conditions (76-77) since by convention $u^{(-1)} = 0$ (and then it satisfies the Kirchhoff conditions (78)) and $u^{(0)}$ satisfies the jump conditions (79-80). It is then easy to see that for all j, $U_j^{(0)}$ is constant and is given by

$$U_j^{(0)} = u_{j+\frac{1}{2}}^{(0)}(0).$$

Order 1.   For $k = 1$, we have for all $j \in \mathbb{Z}$

$$f_{j+1/2}^{(0)} = f_j^{(0)} = 0\ , \quad \Phi_j^{(-1)} = 0\ , \quad g_{j,+}^{(0)} = \partial_s u_{j+\frac{1}{2}}^{(0)}(0)\ , \quad g_{j,-}^{(0)} = \partial_s u_{j-\frac{1}{2}}^{(0)}(1)\ , \quad g_{j,0}^{(0)} = \partial_y u_j^{(0)}(-L/2),$$

and

$$\Delta_{j,-}^{(0)} = +\partial_y u_j^{(0)}(-L/2) + \frac{\mu_j}{2}\partial_s u_{j-\frac{1}{2}}^{(0)}(1) + \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^0, \mathcal{N}_j^-\right\rangle_{\Sigma_{j,\delta}},$$

$$\Delta_{j,+}^{(0)} = -\partial_y u_j^{(0)}(-L/2) + \frac{\mu_j}{2}\partial_s u_{j+\frac{1}{2}}^{(0)}(0) + \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^0, \mathcal{N}_j^+\right\rangle_{\Sigma_{j,\delta}},$$

$$\Xi_j^{(0)} = \mu_j\,\lambda^{(0)}\,u_{j+\frac{1}{2}}^{(0)}(0).$$

We set, as suggested by (101) in Proposition 12, that

$$\lambda^{(1)} = 2\left\|u^{(0)}\right\|_{L_2^\mu(G)}^{-2} \sum_{j\in\mathbb{Z}}\widetilde{\Xi}_j^{(0)}u_j^{(0)}.$$

where $\widetilde{\Xi}_j^{(0)}$ is given by (102) and the previous formulas. Because $u^{(0)}$ is exponentially decaying, we have that $\{\Xi_j^{(0)}\}_{j \in \mathbb{Z}} \in l_2(\mathbb{Z})$ and $\Delta_{j,\pm}^{(0)} \in l_2(\mathbb{Z})$. Thanks to Proposition 12, the problem (99) (with $k = 1$) has an unique solution $u^{(1)}$ which satisfies the orthogonality condition (103). Finally, in view of Proposition 11, there exists a unique solution $U^{(1)}$ of (82) (with $k = 1$) satisfying the matching conditions (76-77) since by definition, $u^{(0)}$ satisfies the Kirchhoff conditions (78)) and $u^{(1)}$ satisfies the jump conditions (79-80).

Higher orders. The previous reasoning can be repeated for any $k \geq 2$.

- We first set the value of $\lambda^{(k)}$ thanks to the previous terms as suggested in (101) in Proposition 12.

- Then, thanks to Proposition 12, we define the far field term $u^{(k)} \in L_2^\mu(\mathcal{G}^-)$ as the unique solution to (99) that satisfies the orthogonality condition (103).

- Finally, for any $j \in \mathbb{Z}$, we define $U_j^{(k)}$ as the unique non exponentially increasing solution of (82) satisfying the matching conditions (76-77). Its existence is guaranteed by Proposition 11 since, by construction, $u^{(k-1)}$ satisfies the Kirchhoff conditions (78)) and $u^{(k)}$ satisfies the jump conditions (79-80).

In this overall procedure, the first difficulty to tackle is to show that the far-field terms are smooth enough so that their successive derivatives used for the matching conditions have sense (this difficulty is usual when dealing with the matching expansion method, see for instance [20]). In the present case, the second and the main difficulty consists in showing that we can apply Proposition 11. This is possible because the far field terms $u_j^{(k)}$ and the near field terms $U_j^{(k)}$ are exponentially decaying as $|j|$ tends to $+\infty$. In the following proposition, we establish these properties, which allows us to justify the construction of the far-field and the near field terms by induction at any order.

Proposition 13. *The far field and near field terms $\left\{ u^{(k)}, U^{(k)} \right\}_{k \in \mathbb{N}}$ are defined by induction as explained previously and they satisfy the following properties.*

1. *For any $k \in \mathbb{N}$ and any $j \in \mathbb{Z}$, $u_{j+\frac{1}{2}}^{(k)}$ belongs to $C^\infty([0,1])$ and $u_j^{(k)}$ belongs to $C^\infty\left(\left[-\frac{L}{2}, 0\right]\right)$*

2. *For any $k \in \mathbb{N}$, there exist polynomials $\left\{ a_\ell^{(k)}, b_\ell^{(k)}, c_\ell^{(k)}, d_\ell^{(k)} \right\}$ of degree $k - \ell$, $0 \leqslant \ell \leqslant k$, such that*

$$u_{j+\frac{1}{2}}^{(k)}(s) = r^j \sum_{\ell=0}^{k} s^\ell \left(a_\ell^{(k)}(j) \cos\left(\sqrt{\lambda_0}s\right) + b_\ell^{(k)}(j) \sin\left(\sqrt{\lambda_0}s\right) \right), \qquad s \in [0,1], \qquad j \in \mathbb{N},$$

(104)

$$u_j^{(k)}(y) = r^j \sum_{\ell=0}^{k} y^\ell \left(c_\ell^{(k)}(j) \cos\left(\sqrt{\lambda_0}y\right) + d_\ell^{(k)}(j) \sin\left(\sqrt{\lambda_0}y\right) \right), \qquad y \in \left[-\frac{L}{2}, 0\right], \qquad j \in \mathbb{N}^*.$$

(105)

and there exist functions $\mathcal{U}_\ell^{(k)} \in H_{loc}^1(\mathcal{J}_j)$ such that

$$U_j^{(k)} = r^j \sum_{\ell=0}^{k} j^\ell \mathcal{U}_\ell^{(k)}, \qquad \forall j \in \mathbb{N}^*.$$

(106)

Proof. The proof is naturally done by induction. First, it is clear that the functions $u^{(0)}$ and $U^{(0)}$ constructed above satisfy the properties 1–2.

Suppose that for some $n \geqslant 1$ functions $\left\{ u^{(k)}, U^{(k)} \right\}_{k=0}^{n}$ and real numbers $\left\{ \lambda^{(k)} \right\}_{k=0}^{n+1}$ are well defined and constructed as explained above and suppose that $\left\{ u^{(k)}, U^{(k)} \right\}_{k=0}^{n}$ have the properties 1-2. We can then construct $\lambda^{(n+1)}$, then $u^{(n+1)}$ and finally $U^{(n+1)}$ as explained above (Propositions 12 and 11 apply). Let us show that the functions $u^{(n+1)}$, $U^{(n+1)}$ satisfy the properties 1–2 as well.

1. The smoothness of the functions $\left\{ u_j^{(n+1)}, u_{j+\frac{1}{2}}^{(n+1)} \right\}_{j \in \mathbb{Z}}$ follows from the smoothness of the functions $\left\{ f_j^{(n+1)}, f_{j+\frac{1}{2}}^{(n+1)} \right\}_{j \in \mathbb{Z}}$, in (99) (which only depends on the previous far-field terms $\left\{ u^{(k)} \right\}_{k=0}^{n}$).

2. Since the far field terms $\{ u^{(k)} \}_{k=0}^{n}$ satisfy the properties (104-105) and the near field terms $\{ U^{(k)} \}_{k=0}^{n}$ the property (106), there exists a family of polynomials $\left\{ \widetilde{a}_\ell^{(n)}, \widetilde{b}_\ell^{(n)}, \widetilde{c}_\ell^{(n)}, \widetilde{d}_\ell^{(n)} \right\}$ of degree $n - \ell$, $0 \leqslant \ell \leqslant n$, and three polynomials $\{ q_\delta, \delta \in \{+, -, 0\} \}$, of degree n, such that

$$F_{j+\frac{1}{2}}(s) = r^j \sum_{\ell=0}^{n} s^\ell \left(\widetilde{a}_\ell^{(n)}(j) \cos \left(\sqrt{\lambda_0} s \right) + \widetilde{b}_\ell^{(n)}(j) \sin \left(\sqrt{\lambda_0} s \right) \right), \qquad s \in [0, 1], \qquad j \in \mathbb{N},$$

(107)

$$F_j(y) = r^j \sum_{\ell=0}^{n} y^\ell \left(\widetilde{c}_\ell^{(n)}(j) \cos \left(\sqrt{\lambda_0} y \right) + \widetilde{d}_\ell^{(n)}(j) \sin \left(\sqrt{\lambda_0} y \right) \right), \qquad y \in \left[-\frac{L}{2}, 0 \right], \qquad j \in \mathbb{N}^*,$$

(108)

$$\Delta_{j,+}^{(n)} = r^j \, q_+(j), \qquad \Delta_{j,-}^{(n)} = r^j \, q_-(j), \qquad \Xi_j^{(n)} = r^j \, q_0(j), \qquad j \in \mathbb{N}^*, \tag{109}$$

where, for any integer j, $F_{j+\frac{1}{2}} = -\lambda^{(n+1)} u_{j+\frac{1}{2}}^{(0)} - f_{j+\frac{1}{2}}^{(n)}$ and $F_j = -\lambda^{(n+1)} u_j^{(0)} - f_j^{(n)}$. By definition, $u^{(n+1)}$ is solution of (99) and then using Appendix B.1, we can show that it satisfies (104-105).

Finally, to prove that U^{n+1} satisfy (106), let us note that the right-hand sides $\Phi_j^{(n-1)}$, $g_{j,\pm}^{(n)}$ and $g_{j,0}^{(n)}$ of problem (82) (for $k = n + 1$) and the right hand side of the average conditions (88-89) have the form r^j multiplied by polynomials in j of degree at most $n+1$ for $j \geqslant 1$. The problem (82-88-89), that we denote $P_j^{(n+1)}$, can be written, by linearity,

$$P_j^{(n+1)} = r^j \sum_{\ell=0}^{n+1} j^\ell \mathcal{P}_\ell^{(n+1)}, \qquad j \in \mathbb{N}^*. \tag{110}$$

where $\mathcal{P}_\ell^{(n+1)}$, for $0 \leqslant \ell \leqslant n + 1$ is the problem obtained from (82), (88-89) by keeping in $\Phi_j^{(n-1)}$, $g_{j,\pm}^{(n)}$, $g_{j,0}^{(n)}$ ($j \geqslant 1$) and the right-hand side of (88-89) only the coefficients in the terms containing $r^j j^\ell$. The problems $\{ \mathcal{P}_\ell^{(n+1)} \}_{\ell=0}^{n+1}$ do not depend on j. It is not so obvious to see that each problem $\mathcal{P}_\ell^{(n+1)}$ is well posed. But, applying the relation (110) to $1 \leqslant j \leqslant n + 2$ we see that each problem $\mathcal{P}_\ell^{(n+1)}$ is a linear combination of the problems $\{ P_j^{(n+1)} \}_{j=0}^{n+1}$ and, consequently, it has a unique solution in K that can be continued to a

function in $\mathcal{U}_\ell^{(n+1)} \in H_{loc}^1(\mathcal{J})$. Now from (110) applied to any $j \in \mathbb{N}^*$ the relation (106) follows.

$\square$

5.6 Construction of a pseudo-mode

The existence of the terms of the asymptotic expansion being proved, we can now construct a pseudo-mode.

Let us first introduce a cut off function $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \leqslant \chi(x) \leqslant 1, \quad \forall x \in \mathbb{R}, \qquad \chi(x) = \left\{ \begin{array}{ll} 0, & x \leqslant 1, \\ 1, & x \geqslant 2. \end{array} \right.$$

Using this cut-off function, we construct a pseudo-mode that coincides with the truncated (at order n) far field expansion far from the junctions and with the truncated near field expansion close to them (see Figure 13):

$$u_{\varepsilon,n}(x,y) = u_{\varepsilon,n}^{\mathrm{FF}}(x,y) + u_{\varepsilon,n}^{\mathrm{NF}}(x,y), \quad \text{in } \Omega_\varepsilon^{\mu,-}. \tag{111}$$

where
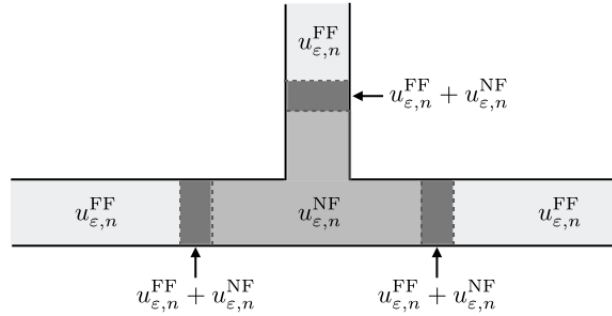
Figure 13: Schematic representation of the global approximation $u_{\varepsilon,n} = u_{\varepsilon,n}^{\mathrm{FF}} + u_{\varepsilon,n}^{\mathrm{NF}}$

$$u_{\varepsilon,n}^{\mathrm{FF}}(x,y) = \sum_{j\in\mathbb{Z}} \left[\left(\sum_{k=0}^n \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j) \right) \chi\left(\frac{x-j}{\varepsilon^\alpha}\right) \chi\left(\frac{j+1-x}{\varepsilon^\alpha}\right) + \left(\sum_{k=0}^n \varepsilon^k u_j^{(k)}(y) \right) \chi\left(\frac{2y+L}{2\varepsilon^\alpha}\right) \right],$$

$$\tag{112}$$

$$u_{\varepsilon,n}^{\mathrm{NF}}(x,y) = \sum_{j\in\mathbb{Z}} \left(\sum_{k=0}^n \varepsilon^k U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \left(1 - \chi\left(\frac{x-j}{\varepsilon^\alpha}\right) \right) \left(1 - \chi\left(\frac{j-x}{\varepsilon^\alpha}\right) \right) \left(1 - \chi\left(\frac{2y+L}{2\varepsilon^\alpha}\right) \right).$$

$$\tag{113}$$

Finally, we can show the following estimate (whose proof is postponed in the appendix B.2) which, as explained in Section 5.1, enables to prove Theorem 4.

Proposition 14. *For any $n \in \mathbb{N}$, $u_{\varepsilon,n} \in H^1(\Omega_\varepsilon^{\mu,-})$. Moreover, let $\lambda_{\varepsilon,n} = \sum\limits_{k=0}^{n} \varepsilon^k \lambda^{(k)}$, there exists a constant $C_n > 0$ such that*

$$\left| \int\limits_{\Omega_\varepsilon^{\mu,-}} (\nabla u_{\varepsilon,n} \nabla v - \lambda_{\varepsilon,n} u_{\varepsilon,n} v) \, d\Omega \right| \leqslant C_n \varepsilon^{n+1} \|u_{\varepsilon,n}\|_{H^1(\Omega_\varepsilon^{\mu,-})} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}, \qquad \forall v \in H^1(\Omega_\varepsilon^{\mu,-}),$$

for ε small enough.

A   Quasi-modes method

In this section we adapt the proof of Lemma 4 of [33] to show the following Proposition 15. This proposition is used to show the existence of eigenvalues for the operator $A_{\varepsilon,s}^\mu$ ($A_{\varepsilon,as}^\mu$), as explained at the beginning of Section 5.1.

Let H be a Hilbert space, A a self-adjoint positive definite operator:

$$\exists \alpha > 0 : \quad (Au, u) \geqslant \alpha \|u\|^2, \qquad \forall u \in D(A).$$

Let a be the closed positive definite sesquilinear form which corresponds to the operator A:

$$D[a] = D(A^{1/2}), \qquad a[u, u] \geqslant \alpha \|u\|^2, \qquad \forall u \in D[a].$$

We denote by $|\cdot|_a$ the norm in the space $D[a]$ corresponding to the scalar product

$$\langle u, v \rangle_a = a[u, v], \qquad \forall u, v \in D[a].$$

Proposition 15. *Suppose that $\lambda > \alpha$. If there exists $u \in D[a]$ such that*

$$|a[u, v] - \lambda(u, v)| \leqslant \varepsilon |u|_a |v|_a, \qquad \forall v \in D[a], \qquad \varepsilon < (\lambda + 1)^{-1}, \tag{114}$$

then

$$dist(\sigma(A), \lambda) \leqslant C\varepsilon, \qquad C = \lambda + 1.$$

To show this result, let us define the operator $B : D[a] \to D[a]$ as follows:

$$a[Bf, v] = (f, v), \qquad \forall v \in D[a].$$

We notice that this implies

$$Bf \in D(A), \qquad ABf = f, \qquad \forall f \in D[a]. \tag{115}$$

Let us prove now the following assertion.

Lemma 7.

$$\lambda \in \sigma(A) \quad <=> \quad \frac{1}{\lambda} \in \sigma(B), \qquad \forall \lambda > 0.$$

Proof. **Case of eigenvalues:**

1. Let λ be an eigenvalue of the operator A: $\; Af = \lambda f$. We have

$$(f, v) = \lambda^{-1}(Af, v) = a[\lambda^{-1}f, v], \qquad \forall v \in D[a].$$

By definition of the operator B this implies $Bf = \lambda^{-1}f$.
2. Let λ^{-1} be an eigenvalue of the operator B: there exists $f \in D[a]$ such that $Bf = \lambda^{-1}f$. From (115) it follows that $f \in D(A)$ and $Af = \lambda ABf = \lambda f$.

Case of continuous spectrum:

1. Suppose that $\lambda \in \sigma_c(A)$. Then, there exist a singular sequence $\{u_n\}_{n\in\mathbb{N}} \subset D(A)$ such that

   a) $\inf\limits_{n} \|u_n\| > 0$,

   b) $u_n \xrightarrow{w} 0$ in H,

   c) $\|(A - \lambda)u_n\| \longrightarrow 0$.

Let us show that $\{u_n\}$ is also a singular sequence for the operator B in $D[a]$ equipped with the norm $|\cdot|_a$.

   a) $\inf\limits_{n} |u_n|_a > 0$ is obviously verified.

   b) $u_n \xrightarrow{w} 0$ in $D[a]$. Indeed, using the properties (b), (c) of the sequence $\{u_n\}$, we obtain

$$a[u_n, v] = (Au_n, v) = ((A - \lambda)u_n, v) + \lambda(u_n, v) \longrightarrow 0, \qquad \forall v \in D[a].$$

   c) $|(B - \lambda^{-1})u_n|_a \longrightarrow 0$. Indeed,

$$\left|Bu_n - \lambda^{-1}u_n\right|_a = a[(B - \lambda^{-1})u_n, (B - \lambda^{-1})u_n] = (A(B - \lambda^{-1})u_n, (B - \lambda^{-1})u_n)$$
$$= \lambda^{-1}(\lambda u_n - Au_n, (B - \lambda^{-1})u_n) \leqslant C\|Au_n - \lambda u_n\| \longrightarrow 0.$$

2. Suppose that $\lambda^{-1} \in \sigma_c(B)$. Then, there exists a singular sequence $\{f_n\}_{n\in\mathbb{N}} \subset D[a]$ such that

   a) $\inf\limits_{n} |f_n|_a > 0$,

   b) $f_n \xrightarrow{w} 0$ in $D[a]$,

   c) $|(B - \lambda^{-1})f_n|_a \longrightarrow 0$.

Let us show that there exists a singular sequence for the operator A. We set $u_n = Bf_n \in D(A)$.

   a) $\inf\limits_{n} \|u_n\| > 0$. Indeed, from the properties (a) and (c) of the sequence $\{f_n\}$, we have

$$|u_n|_a^2 = (A^{1/2}Bf_n, A^{1/2}u_n) = (ABf_n, u_n) = (f_n, u_n),$$

   and therefore

$$\inf\limits_{n} \|u_n\| \geqslant \inf\limits_{n} \frac{|u_n|_a^2}{\|f_n\|} > 0.$$

b) $u_n \xrightarrow{w} 0$ in H. Indeed, the sequence $\{u_n\}$ being bounded, we can extract a subsequence which converges weakly to some element $h \in H$. We keep the same notation $\{u_n\}$ for the subsequence:

$$(u_n, w) \longrightarrow (h, w), \quad \forall w \in H. \tag{116}$$

For $w \in D[a]$ we have:

$$(u_n, w) = a[Bf_n, w] \longrightarrow 0, \qquad \forall w \in D[a], \tag{117}$$

where we used the property (b) of the sequence $\{f_n\}$. Thus, the relations (116) and (117) imply that

$$(h, w) = 0, \qquad \forall w \in D[a].$$

Since $D[a]$ is dense in H we conclude that $h = 0$.

c) $\|(A - \lambda)u_n\| \longrightarrow 0$. Indeed, we have

$$\|(A - \lambda)u_n\| = \|f_n - \lambda Bf_n\| \leqslant \alpha^{-1/2}|f_n - \lambda Bf_n|_a \longrightarrow 0.$$

$\square$

Proof of Proposition 15. Suppose that $u \in D[a]$ is such that (114) is verified. We have:

$$|a[u, v] - \lambda(u, v)| = |a[u, v] - \lambda(ABu, v)| = |a[u, v] - \lambda[Bu, v]| = |a[u - \lambda Bu, v]| \leqslant \varepsilon |u|_a |v|_a,$$

and hence,

$$\left|(B - \lambda^{-1})u\right|_a \leqslant \frac{\varepsilon}{\lambda}|u|_a.$$

The last relation implies that

$$dist(\sigma(B), \lambda^{-1}) \leqslant \frac{\varepsilon}{\lambda},$$

and for the operator A we obtain

$$dist(\sigma(A), \lambda) \leqslant \frac{\lambda \varepsilon}{1 - \varepsilon} < (\lambda + 1)\varepsilon,$$

where we took into account the relation $\varepsilon(1 + \lambda) < 1$.

B Technical results associated with the asymptotic expansion

B.1 Exponential decaying of the far-field terms

Let us consider the far field term u^{n+1} solution of (99) (with $k = n + 1$) where the right hand sides satisfy (107-108-109). Let us show that $u^{(n+1)}$ satisfy (104-105).

First of all, we notice that the first two lines of (99) together with the assumptions (107), (108) for the right-hand sides imply the forms (104), (105) of the solutions $u_{j+\frac{1}{2}}^{(n+1)}$, $u_j^{(n+1)}$ with

some coefficients $\left\{a_\ell^{(n+1)}(j), b_\ell^{(n+1)}(j)\ c_\ell^{(n+1)}(j), d_\ell^{(n+1)}(j),\ 0 \leqslant \ell \leqslant n+1\right\}$. In order to determine the dependence of these coefficients on j we inject the relations (104) and (107) into the first line of (99). We get

$$2(\ell+1)\sqrt{\lambda_0}\,b_{\ell+1}^{(n+1)}(j) + (\ell+1)(\ell+2)\,a_{\ell+2}^{(n+1)}(j) = \widetilde{a}_\ell^{(n)}(j), \qquad 0 \leqslant \ell \leqslant n,$$
$$-2(\ell+1)\sqrt{\lambda_0}\,a_{\ell+1}^{(n+1)}(j) + (\ell+1)(\ell+2)\,b_{\ell+2}^{(n+1)}(j) = \widetilde{b}_\ell^{(n)}(j), \qquad 0 \leqslant \ell \leqslant n,$$

where $a_{\ell+2}^{(n+1)}(j) = b_{\ell+2}^{(n+1)}(j) = 0$ which implies that

$$b_{n+1}^{(n+1)}(j) = \frac{\widetilde{a}_n^{(n)}(j)}{2(n+1)\sqrt{\lambda_0}}, \qquad a_{n+1}^{(n+1)}(j) = -\frac{\widetilde{b}_n^{(n)}(j)}{2(n+1)\sqrt{\lambda_0}}, \tag{118}$$

and the other coefficients, except $a_0^{(n+1)}(j)$ and $b_0^{(n+1)}(j)$, are determined by induction. An induction in ℓ shows that the coefficients $a_\ell^{(n+1)}(j)$, $b_\ell^{(n+1)}(j)$ are polynomials with respect to j of degree $n+1-\ell$ for $1 \leqslant \ell \leqslant n+1$. Indeed, taking into account the assumptions for the coefficients $\left\{a_\ell^{(n)}(j), b_\ell^{(n)}(j)\right\}_{\ell=0}^n$, from (118) we see that the coefficients $a_{n+1}^{(n+1)}(j)$, $b_{n+1}^{(n+1)}(j)$ are constants in j. If we suppose that $a_\ell^{(n+1)}(j)$, $b_\ell^{(n+1)}(j)$ are polynomials (with respect to j) of degree $n+1-\ell$ for all ℓ such that $m \leqslant \ell \leqslant n+1$ for some $1 < m \leqslant n+1$ then the relations (118) imply that $a_{m-1}^{(n+1)}(j)$, $b_{m-1}^{(n+1)}(j)$ are polynomials in j of degree $n+1-m+1$. Thus, it remains only to determine the behaviour of the coefficients $a_0^{(n+1)}(j)$, $b_0^{(n+1)}(j)$ with respect to j for $j \geqslant 0$.

Repeating the same argument applied to the coefficients $\left\{c_\ell^{(n+1)}(j), d_\ell^{(n+1)}(j)\right\}_{\ell=0}^{n+1}$, using the second line of (99), we get that the coefficients $c_\ell^{(n+1)}(j)$, $d_\ell^{(n+1)}(j)$ are polynomials with respect to j (for $j \geqslant 1$) of degree $n+1-\ell$ for $1 \leqslant \ell \leqslant n+1$. Besides, from the third line of (99), we obtain that

$$c_1^{(n+1)}(j) + \sqrt{\lambda_0}\,d_0^{(n+1)}(j) = 0$$

and we deduce that the coefficient $d_0^{(n+1)}(j)$ is, as $c_1^{(n+1)}(j)$ a polynomial with respect to j of degree n.

Let us now establish the dependence on j of the coefficients $a_0^{(n+1)}(j)$, $b_0^{(n+1)}(j)$, $c_0^{(n+1)}(j)$ using the three remaining equations of (99). We get for $j \geqslant 1$:

$$r^{j-1}\left(a_0^{(n+1)}(j-1)\cos\sqrt{\lambda_0}\ +\ b_0^{(n+1)}(j-1)\sin\sqrt{\lambda_0}\right) - r^j\,c_0^{(n+1)}(j)\cos(\frac{\sqrt{\lambda_0}L}{2}) = r^j\,p_1^{(n+1)}(j), \tag{119}$$

$$r^j\left(c_0^{(n+1)}(j)\cos(\frac{\sqrt{\lambda_0}L}{2}) - a_0^{(n+1)}(j)\right) = r^j\,p_3^{(n+1)}(j), \tag{120}$$

$$r^{j-1}\left(a_0^{(n+1)}(j-1)\sin\sqrt{\lambda_0} - b_0^{(n+1)}(j-1)\cos\sqrt{\lambda_0}\right) + r^j\left(b_0^{(n+1)}(j) + c_0^{(n+1)}(j)\sin(\frac{\sqrt{\lambda_0}L}{2})\right) = r^j\,p_3^{(n+1)}(j), \tag{121}$$

where $\left\{ p_i^{(n+1)} \right\}_{i=1}^{3}$ are polynomials in j of degree n. The system (119)–(121) reduces then to the following one:

$$c_0^{(n+1)}(j) = \frac{1}{\cos\left(\frac{\sqrt{\lambda_0}L}{2}\right)} \left(a_0^{(n+1)}(j) + p_2^{(n+1)}(j) \right), \qquad j \geqslant 1, \tag{122}$$

$$b_0^{(n+1)}(j) = \frac{1}{\sin\sqrt{\lambda_0}} \left(ra_0^{(n+1)}(j+1) - a_0^{(n+1)}(j)\cos\sqrt{\lambda_0} + r\left(p_1^{(n+1)}(j+1) + p_2^{(n+1)}(j+1) \right) \right), \qquad j \geqslant 0, \tag{123}$$

$$r^2\, a_0^{(n+1)}(j+2) + 2\, r\, g(\sqrt{\lambda_0})\, a_0^{(n+1)}(j+1) + a_0^{(n+1)}(j) = p^{(n+1)}(j), \qquad j \geqslant 0, \tag{124}$$

where the function g is defined in (39) and $p^{(n+1)}(j)$ is a polynomial in j of degree n which can be written

$$p^{(n+1)}(j) = \sum_{m=0}^{n} \rho_m j^m. \tag{125}$$

Let us study the finite difference equation (124). First, recalling that r and r^{-1} are solutions of the equation (41), we find the general (real) solution of the corresponding homogeneous equation:

$$\widetilde{a}_0^{(n+1)} = C + Dr^{-2j}, \qquad C, D \in \mathbb{R}, \qquad j \geqslant 0.$$

where $D = 0$ because $u^{(n+1)} \in L_2^{\mu}(G)$. Finally, the right-hand side of the equation (124) being a polynomial of degree n, we look for a solution which is a polynomial of degree $n+1$.

$$a_0^{(n+1)}(j) = \sum_{m=0}^{n+1} \alpha_m\, j^m.$$

Together with (125) it gives the following system for the coefficients $\{\gamma_m\}_{m=0}^{n+1}$:

$$\sum_{i=m}^{n+1} C_i^m r^2 \alpha_i + 2g(\sqrt{\lambda^{(0)}})\, r\alpha_m + \sum_{i=m}^{n+1} (-1)^{i-m} C_i^m r^2 \alpha_i = \rho_m, \qquad 0 \leqslant m \leqslant n+1,$$

where we have imposed $\rho_{n+1} = 0$. Since r is a solution of (41) the equation for $m = n+1$ is automatically verified and the other equations take the form

$$\sum_{i=m+1}^{n+1} C_i^m \left(\left(2^{i-m} - 1 \right) r^2 - 1 \right) \alpha_i = \rho_m, \qquad 0 \leqslant m \leqslant n, \tag{126}$$

which is a system with an upper-diagonal matrix with non-zero elements ($|r| < 1$) for the coefficients $\{\alpha_m\}_{m=1}^{n+1}$. The coefficient α_0 cannot be determined which corresponds to the general solution $\widetilde{a}_0^{(k)}(j) = C$, $j \geqslant 0$. Hence, $a_0^{(n+1)}(j)$ is a polynomial in j of degree $n+1$ for $j \geqslant 0$ as well as the coefficients $b_0^{(n+1)}(j)$ for $j \geqslant 0$ and $\widetilde{c}_0^{(n+1)}(j)$ for $j \geqslant 1$ (see (122), (123)). This finishes the proof.

B.2 Proof of Proposition 14

Here and in what follows we denote by C_n all the constants that does not depend on ε. We will now estimate the value

$$\mathcal{I}_{\varepsilon,n}(v) = \int\limits_{\Omega_\varepsilon^{\mu,-}} \left(\nabla u_{\varepsilon,n} \nabla v - \lambda_{\varepsilon,n} u_{\varepsilon,n} v \right) d\Omega \tag{127}$$

for any $v \in H^1(\Omega_\varepsilon^{\mu,-})$. After injecting (111)–(113) in (127) the value $\mathcal{I}_{\varepsilon,n}(v)$ can be decomposed into three main contributions

$$\mathcal{I}_{\varepsilon,n}(v) = \mathcal{I}_{\varepsilon,n}^{\mathrm{FF}}(v) + \mathcal{I}_{\varepsilon,n}^{\mathrm{NF}}(v) + \mathcal{I}_{\varepsilon,n}^{\mathrm{M}}(v), \tag{128}$$

corresponding to the far field consistency error

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{FF}}(v) = -\sum_{k=n+1}^{2n} \varepsilon^k \sum_{p=k-n}^{n} \lambda^{(k-p)} \sum_{j\in\mathbb{Z}} \Big[\int_{\Omega_\varepsilon^{\mu,-}} u_{j+\frac{1}{2}}^{(p)}(x-j) \chi\left(\frac{x-j}{\varepsilon^\alpha}\right) \chi\left(\frac{j+1-x}{\varepsilon^\alpha}\right) v(x,y) d\Omega,$$
$$- \int_{\Omega_\varepsilon^{\mu,-}} u_j^{(p)}(y) \chi\left(\frac{2y+L}{2\varepsilon^\alpha}\right) v(x,y) d\Omega \Big], \quad (129)$$

the near field consistency error

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{NF}}(v) = -\sum_{j\in\mathbb{Z}} \sum_{k=n-1}^{n} \varepsilon^k \sum_{p=0}^{k} \lambda^{(k-p)} \int_{\Omega_\varepsilon^{\mu,-}} U_j^{(p)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \left(1-\chi\left(\frac{x-j}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\frac{j-x}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\frac{2y+L}{2\varepsilon^\alpha}\right)\right) v(x,y) d\Omega$$

$$-\sum_{j\in\mathbb{Z}} \sum_{k=n+1}^{2n} \varepsilon^k \sum_{p=k-n}^{n} \lambda^{(k-p)} \int_{\Omega_\varepsilon^{\mu,-}} U_j^{(p)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \left(1-\chi\left(\frac{x-j}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\frac{j-x}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\frac{2y+L}{2\varepsilon^\alpha}\right)\right) v(x,y) d\Omega,$$

$$\tag{130}$$

and the matching error

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{M}}(v) = \mathcal{I}_\varepsilon^{M,+}(v) + \mathcal{I}_\varepsilon^{M,-}(v) + \mathcal{I}_\varepsilon^{M,0}(v), \tag{131}$$

where

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{M},+}(v) = -\varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k \left(u_{j+\frac{1}{2}}^{(k)}\right)'(x-j) - \sum_{k=0}^{n} \varepsilon^k \partial_x U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{x-j}{\varepsilon^\alpha}\right) v(x,y) d\Omega,$$

$$+ \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j) - \sum_{k=0}^{n} \varepsilon^k U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{x-j}{\varepsilon^\alpha}\right) \partial_x v(x,y) d\Omega, \quad (132)$$

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{M},-}(v) = \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k \left(u_{j-\frac{1}{2}}^{(k)}\right)'(x-j) - \sum_{k=0}^{n} \varepsilon^k \partial_x U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{j-x}{\varepsilon^\alpha}\right) v(x,y) d\Omega$$

$$- \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k u_{j-\frac{1}{2}}^{(k)}(x-j) - \sum_{k=0}^{n} \varepsilon^k U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{j-x}{\varepsilon^\alpha}\right) \partial_x v(x,y) d\Omega, \quad (133)$$

and

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{M},0}(v) = -\varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k \left(u_j^{(k)}\right)'(y) - \sum_{k=0}^{n} \varepsilon^k \partial_y U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{2y+L}{2\varepsilon^\alpha}\right) v(x,y) d\Omega$$

$$+ \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k u_j^{(k)}(y) - \sum_{k=0}^{n} \varepsilon^k U_j^{(k)}\left(\frac{x-j}{\varepsilon}, \frac{2y+L}{2\varepsilon}\right) \right) \chi'\left(\frac{2y+L}{2\varepsilon^\alpha}\right) \partial_y v(x,y) d\Omega. \quad (134)$$

The far field and near field consistency errors measure how the truncated expansions fails to satisfy the initial eigenvalue problem. The matching error measures the mismatch between near field and far field expansions in the matching areas.

To find an estimation of the near field consistency error and the matching error, we will use the following result, whose proof may be found in [21] (Lemma 3.10).

Lemma 8. *For any $\alpha > 0$, there exist $\varepsilon_0 > 0$ and two constants $C_1(\alpha)$ and $C_2(\alpha)$, which do not depend on ε, such that*

$$\|v\|_{L_1(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} \leqslant C_1(\alpha)\varepsilon^{\alpha+1/2} \|v\|_{H^1(]0,1[\times]0,\varepsilon[)}, \qquad \forall v \in H^1(]0,1[\times]0,\varepsilon[), \quad \forall \varepsilon < \varepsilon_0, \quad (135)$$

and

$$\|v\|_{L_2(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} \leqslant C_2(\alpha)\varepsilon^{\alpha/2} \|v\|_{H^1(]0,1[\times]0,\varepsilon[)}, \qquad \forall v \in H^1(]0,1[\times]0,\varepsilon[), \quad \forall \varepsilon < \varepsilon_0. \quad (136)$$

B.2.1 Estimation of the far field consistency error

Lemma 9. *There exist $\varepsilon_0 > 0$ and a constant $C_n > 0$ such that, for any $\varepsilon < \varepsilon_0$,*

$$\mathcal{I}_{\varepsilon,n}^{FF}(v) \leq C_n \, \varepsilon^{n+3/2} \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)} \tag{137}$$

Proof. Thanks to the Cauchy Schwarz and the discrete Cauchy Schwarz inequalities, we have

$$\left| \sum_{k=n+1}^{2n} \varepsilon^k \sum_{p=k-n}^{n} \lambda^{(k-p)} \sum_{j\in\mathbb{Z}} \int_{\Omega_\varepsilon^{\mu,-}} u_{j+\frac{1}{2}}^{(p)}(x-j)\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right) v(x,y)d\Omega, \right|$$

$$\leqslant C_n \sum_{k=n+1}^{2n} \varepsilon^k \sum_{p=k-n}^{n} \lambda^{(k-p)} \sum_{j\in\mathbb{Z}} \left(\int_0^1 \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| u_{j+\frac{1}{2}}^{(p)}(s) \right|^2 ds dy \right)^{1/2} \left(\int_j^{j+1} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} |v(x,y)|^2 dx dy \right)^{1/2},$$

$$\leqslant C_n \sum_{k=n+1}^{2n} \varepsilon^{k+\frac{1}{2}} \sum_{p=k-n}^{n} \lambda^{(k-p)} \left\| u^{(p)} \right\|_{L_2(G)} \|v\|_{L_2(\Omega_\varepsilon^{\mu,-})},$$

$$\leqslant C_n \varepsilon^{n+\frac{3}{2}} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}.$$

The second term of $\mathcal{I}_{\varepsilon,n}^{\mathrm{FF}}$ can be estimated in the same way. $\qquad\square$

B.2.2 Estimation of the near field consistency error

Lemma 10. *There exist $\varepsilon_0 > 0$ and a constant $C_n > 0$ such that, for any $\varepsilon < \varepsilon_0$,*

$$\mathcal{I}_{\varepsilon,n}^{NF}(v) \leq C_n \varepsilon^{n\alpha+1/2} \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)} \tag{138}$$

Proof. We divide $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF}}(v)$ into four main contributions (corresponding to different parts of the domain C_ε^μ) that will be estimated separately.

$$\left| \mathcal{I}_{\varepsilon,n}^{\mathrm{NF}}(v) \right| \leqslant \left(\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},0}(v) + \mathcal{I}_{\varepsilon,n}^{\mathrm{NF},1}(v) + \mathcal{I}_{\varepsilon,n}^{\mathrm{NF},2}(v) + \mathcal{I}_{\varepsilon,n}^{\mathrm{NF},3}(v) \right),$$

where,

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},0}(v) = \sum_{j\in\mathbb{Z}} \int_{Q_\varepsilon^j} \left| S_\varepsilon^j(x,y)v(x,y)\right| d\Omega, \qquad \mathcal{I}_{\varepsilon,n}^{\mathrm{NF},1}(v) = \sum_{j\in\mathbb{Z}} \int_{j+\varepsilon}^{j+\varepsilon^\alpha} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| S_\varepsilon^j(x,y)v(x,y)\right| d\Omega,$$

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},2}(v) = \sum_{j\in\mathbb{Z}} \int_{j-\varepsilon^\alpha}^{j-\varepsilon} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| S_\varepsilon^j(x,y)v(x,y)\right| d\Omega, \qquad \mathcal{I}_{\varepsilon,n}^{\mathrm{NF},3}(v) = \sum_{j\in\mathbb{Z}} \int_{j-\frac{\mu_j\varepsilon}{2}}^{j+\frac{\mu_j\varepsilon}{2}} \int_{-\frac{L}{2}+2\varepsilon}^{-\frac{L}{2}+\varepsilon^\alpha} \left| S_\varepsilon^j(x,y)v(x,y)\right| d\Omega,$$

with $Q_\varepsilon^j = \left[j-\mu_j\varepsilon, j+\mu_j\varepsilon\right] \times \left[-\frac{L}{2}, -\frac{L}{2}+2\varepsilon\right]$ and

$$S_\varepsilon^j(x,y) = \sum_{k=n-1}^{n} \varepsilon^k \sum_{p=0}^{k} \lambda^{(k-p)} U_j^{(p)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) + \sum_{k=n+1}^{2n} \varepsilon^k \sum_{p=k-n}^{n} \lambda^{(k-p)} U_j^{(p)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right),$$

We can estimate $\mathcal{I}_{\varepsilon,n}^{(\mathrm{NF},0)}(v)$ using the Cauchy-Schwarz inequality, the estimation (136), and taking into account the exponential decay of $U_j^{(k)}$ as j tends to $\pm\infty$ (Property 2 in Proposition 13):

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},0}(v) \leqslant \sum_{j\in\mathbb{Z}} \left(\sum_{k=n-1}^{n} \varepsilon^{k+1} \sum_{p=0}^{k} \left|\lambda^{(k-p)}\right| \left\|U_j^{(p)}\right\|_{L_2(K_j)} + \sum_{k=n+1}^{2n} \varepsilon^{k+1} \sum_{p=k-n}^{n} \left|\lambda^{(k-p)}\right| \left\|U_j^{(p)}\right\|_{L_2(K_j)} \right) \|v\|_{L_2(Q_\varepsilon^j)}$$

$$\leqslant C_n \varepsilon^n \sum_{p=0}^{n} \left(\sum_{j\in\mathbb{Z}} \left\|U_j^{(p)}\right\|_{L_2(K_j)}^2 \right)^{1/2} \left(\sum_{j\in\mathbb{Z}} \|v\|_{L_2(Q_\varepsilon^j)}^2 \right)^{1/2} \leqslant C_n \varepsilon^{n+\frac{1}{2}} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}. \quad (139)$$

In order to estimate the term $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},1}(v)$, $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},2}(v)$ and $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},3}(v)$, we use the fact that the near fields terms $U_j^{(p)}$ have a polynomial growth in the infinite branches $B_{j,\delta}$, $\delta \in \{+,-,0\}$ (see (72-73) and (76-77)): consequently, for $(x,y) \in (j+\varepsilon, j+\varepsilon^\alpha) \times \left(-\frac{L}{2}, -\frac{L}{2}+\varepsilon\right)$,

$$\left|S_\varepsilon^j(x,y)\right| \leqslant C_n(j) \left(\sum_{k=n-1}^{n} \varepsilon^{k+1} \sum_{p=0}^{k} \varepsilon^{p(\alpha-1)} + \sum_{k=n+1}^{2n} \varepsilon^{k+1} \sum_{p=k-n}^{n} \varepsilon^{p(\alpha-1)} \right) \leqslant C_n(j)\, \varepsilon^{(n-1)\alpha},$$

which, thanks to Property 2 of Proposition 13 and (135), implies

$$\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},1}(v) \leqslant C_n \varepsilon^{n\alpha+\frac{1}{2}} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}. \tag{140}$$

The terms $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},2}(v)$ and $\mathcal{I}_{\varepsilon,n}^{\mathrm{NF},3}(v)$ can be estimated in the same way. $\qquad\square$

B.2.3 Estimation of the matching error

Lemma 11. *There exist $\varepsilon_0 > 0$ and a constant $C_n > 0$ such that, for any $\varepsilon < \varepsilon_0$,*

$$I_{\varepsilon,n}^M(v) \leq C_n \varepsilon^{n\alpha+1/2} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})} \tag{141}$$

Proof. We only do the proof for $I_\varepsilon^{\mathrm{M},+}(v)$, the estimations of $I_{\varepsilon,n}^{\mathrm{M},-}(v)$ and $I_\varepsilon^{\mathrm{M},0}(v)$ being analogous. Let us now consider the first line $I_{\varepsilon,n}^{\mathrm{M},+,1}(v)$ of $I_{\varepsilon,n}^{\mathrm{M},+}(v)$. In view of the modal expansion (72), we

get

$$\left| I_{\varepsilon,n}^{\mathrm{M},+,1}(v) \right| \leqslant C_n\, \varepsilon^{-\alpha} \left(\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{P}_\varepsilon^j(x)v(x,y) \right| dxdy + \sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{E}_\varepsilon^j(x,y)v(x,y) \right| dxdy \right),$$

where

$$\mathcal{P}_\varepsilon^j(x) = \sum_{k=0}^{n} \varepsilon^k \left(u_{j+\frac{1}{2}}^{(k)} \right)'(x-j) - \sum_{k=0}^{n} \varepsilon^{k-1} \left(p_{j,+}^{(k)} \right)' \left(\tfrac{x-j}{\varepsilon} \right),$$

$$\mathcal{E}_\varepsilon^j(x,y) = \sum_{k=0}^{n} \varepsilon^k \partial_x \left(\sum_{p\in\mathbb{N}^*} p_{j,p,+}^{(k)} \left(\tfrac{x-j}{\varepsilon} \right) e^{-\frac{p\pi(x-j)}{\varepsilon}} f_p \left(\tfrac{2y+L}{2\varepsilon} \right) \right).$$

In view of the matching conditions (76), we can rewrite $\mathcal{P}_\varepsilon^j$ as

$$\mathcal{P}_\varepsilon^j(x) = \sum_{k=0}^{n} \varepsilon^k \left(\left(u_{j+\frac{1}{2}}^{(k)} \right)'(x-j) - \sum_{l=0}^{n-k-1} \frac{d^l \left(u_{j+\frac{1}{2}}^{(k)} \right)'}{ds^l} \Bigg|_{s=0} \frac{(x-j)^l}{l!} \right).$$

and therefore, for $x \in [j+\varepsilon^\alpha,\, j+2\varepsilon^\alpha]$,

$$\left| \mathcal{P}_\varepsilon^j(x) \right| \leqslant \sum_{k=0}^{n} \varepsilon^k \, (2\varepsilon^\alpha)^{n-k} \left\| \frac{d^{n-k+1} u_{j+\frac{1}{2}}^{(k)}}{ds^{n-k+1}} \right\|_{L_\infty([0,1])} \leqslant C_n(j)\, \varepsilon^{\alpha n}.$$

Then, using (135) and Property 2 of Proposition 13, we get

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{P}_\varepsilon^j(x)v(x,y) \right| dxdy \leqslant C_n\, \varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}.$$

Besides, noting that $\mathcal{E}_\varepsilon^j$ is exponentially decaying ($\alpha < 1$), for any positive integer N, we have

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{E}_\varepsilon^j(x,y)v(x,y) \right| dxdy \leqslant C_n(N)\varepsilon^N \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}, \qquad \forall n \in \mathbb{N}. \tag{142}$$

Thus,

$$\left| I_{\varepsilon,n}^{\mathrm{M},+,1}(v) \right| \leqslant C_n\, \varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}. \tag{143}$$

Similarly, we can bound the second line $I_{\varepsilon,n}^{\mathrm{M},+,2}(v)$ of $I_{\varepsilon,n}^{\mathrm{M},+}(v)$:

$$\left| I_{\varepsilon,n}^{\mathrm{M},+,2}(v) \right| = \left| \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\Omega_\varepsilon^{\mu,-}} \left(\sum_{k=0}^{n} \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j) - \sum_{k=0}^{n} \varepsilon^k U_j^{(k)} \left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon} \right) \right) \chi' \left(\tfrac{x-j}{\varepsilon^\alpha} \right) \partial_x v(x,y)d\Omega, \right|$$

$$\leqslant C_n\, \varepsilon^{-\alpha} \left(\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{T}_\varepsilon^j(x)\partial_x v(x,y) \right| dxdy + \sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| \mathcal{J}_\varepsilon^j(x,y)\partial_x v(x,y) \right| dxdy \right),$$

where

$$\mathcal{T}_\varepsilon^j(x) = \sum_{k=0}^n \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j) - \sum_{k=0}^n \varepsilon^k p_{j,0}^{(k)}\left(\frac{x-j}{\varepsilon}\right), \quad \mathcal{J}_\varepsilon^j(x,y) = \sum_{k=0}^n \sum_{p\in\mathbb{N}^*} \varepsilon^k p_{j,p,0}^{(k)}\left(\frac{x-j}{\varepsilon}\right) e^{-\frac{p\pi(x-j)}{\varepsilon}} f_p\left(\frac{2y+L}{2\varepsilon}\right).$$

Using the matching conditions (76), the term $\mathcal{T}_\varepsilon^j$ can be rewritten as

$$\mathcal{T}_\varepsilon^j(x) = \sum_{k=0}^n \varepsilon^k \left(u_{j+\frac{1}{2}}^{(k)}(x-j) - \sum_{l=0}^{n-k} \left.\frac{d^l u_{j+\frac{1}{2}}^{(k)}}{ds^l}\right|_{s=0} \frac{(x-j)^l}{l!}\right),$$

which implies

$$\left|\mathcal{T}_\varepsilon^j(x)\right| \leqslant \sum_{k=0}^n \varepsilon^k \,(2\varepsilon^\alpha)^{n-k+1} \left\|\frac{d^{n-k+1} u_{j+\frac{1}{2}}^{(k)}}{ds^{n-k+1}}\right\|_{L_\infty([0,1])} \leqslant C_n(j)\,\varepsilon^{\alpha(n+1)}, \qquad x\in[j+\varepsilon^\alpha,\, j+2\varepsilon^\alpha].$$

Thus, using Cauchy-Schwartz inequality, we get:

$$\sum_{j\in\mathbb{N}} \int_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{T}_\varepsilon^j(x)\partial_x v(x,y)\right| dxdy \leqslant C_n\,\varepsilon^{\alpha(n+1)+\frac{1}{2}}\,\|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}.$$

Analogously to (142), the second term is exponentially decaying and can be estimated as follows:

$$\sum_{j\in\mathbb{N}} \int_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{J}_\varepsilon^j(x,y)v(x,y)\right| dxdy \leqslant C_n(N)\varepsilon^N, \qquad \forall n\in\mathbb{N}.$$

Finally,

$$\left|I_{\varepsilon,n}^{\mathrm{M},+,2}(v)\right| \leqslant C_n\,\varepsilon^{\alpha n+\frac{1}{2}}\|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}. \tag{144}$$

Collecting (143) and (144) yields the desired inequality:

$$\left|I_{\varepsilon,n}^{\mathrm{M},+}(v)\right| \leqslant C_n\,\varepsilon^{\alpha n+\frac{1}{2}}\|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}. \tag{145}$$

$\square$

B.2.4   Conclusion

Collecting the results of the lemmas 9, 10 and 11, we can conclude that for ε small enough and for any $\alpha\in(0,1)$, there exists a constant $C_n(\alpha)>0$ such that

$$\left|\mathcal{I}_{\varepsilon,n}(v)\right| \leqslant C_n(\alpha)\varepsilon^{\alpha n+\frac{1}{2}}\|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}, \qquad \forall v\in H^1(\Omega_\varepsilon^{\mu,-}),$$

Moreover, for ε small enough we have the following estimate:

$$\|u_{\varepsilon,n}\|_{L_2(\Omega_\varepsilon^{\mu,-})}^2 \geqslant C_n \sum_{j\in\mathbb{Z}} \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|u_{j+\frac{1}{2}}^{(0)}(s,y)\right|^2 dsdy + \int_{-\frac{\mu_j\varepsilon}{2}}^{\frac{\mu_j\varepsilon}{2}} \int_{-\frac{L}{4}}^{0} \left|u_j^{(0)}(x,y)\right|^2 dxdy \right) \geqslant C_n\varepsilon. \tag{146}$$

where C_n is a constant that does not depend on ε.

This ends the proof of Proposition 14.

References

[1] H. Ammari and F. Santosa. Guided waves in a photonic bandgap structure with a line defect. *SIAM J. Appl. Math.*, 64(6):2018–2033 (electronic), 2004.

[2] Y. Avishai and J. M. Luck. Quantum percolation and ballistic conductance on a lattice of wires. *Physical Review B*, 45(3):1074, 1992.

[3] F. L. Bakharev, S. A. Nazarov, and K. M. Ruotsalainen. A gap in the spectrum of the Neumann-Laplacian on a periodic waveguide. *Appl. Anal.*, 92(9):1889–1915, 2013.

[4] A. Bendali, A. Huard, A. Tizaoui, S. Tordeux, and J.P. Vila. Asymptotic expansions of the eigenvalues of a 2-D boundary-value problem relative to two cavities linked by a hole of small size. *C. R. Math. Acad. Sci. Paris*, 347(19-20):1147–1152, 2009.

[5] Gregory Berkolaiko and Peter Kuchment. *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[6] M. Sh. Birman and M. Z. Solomjak. *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller.

[7] G. Borg. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.*, 78:1–96, 1946.

[8] Robert Carlson. Adjoint and self-adjoint differential operators on graphs. *Electronic Journal of Differential Equations*, 6(1998):1–10, 1998.

[9] M. S. P. Eastham. *The spectral theory of periodic differential equations*. Edinburgh : Scottish Academic Press, Edinburgh-London, distributed by chatto and windus edition, 1973.

[10] Pavel Exner. Lattice kronig-penney models. *Physical review letters*, 74(18):3503, 1995.

[11] Pavel Exner. Contact interactions on graph superlattices. *Journal of Physics A: Mathematical and General*, 29(1):87, 1996.

[12] A. Figotin and A. Klein. Localized classical waves created by defects. *J. Statist. Phys.*, 86(1-2):165–177, 1997.

[13] A. Figotin and A. Klein. Midgap defect modes in dielectric and acoustic media. *SIAM J. Appl. Math.*, 58(6):1748–1773 (electronic), 1998.

[14] A. Figotin and P. Kuchment. Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model. *SIAM J. Appl. Math.*, 56(1):68–88, 1996.

[15] A. Figotin and P. Kuchment. Band-gap structure of spectra of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals. *SIAM J. Appl. Math.*, 56(6):1561–1620, 1996.

[16] A. M. Il′in. *Matching of asymptotic expansions of solutions of boundary value problems*, volume 102 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by V. Minachin [V. V. Minakhin].

[17] J. D. Joannopoulos, R. D. Meade, and J. N. Winn. *Photonic Crystal - Molding the Flow of Light*. Princeton Univeristy Press, 1995.

[18] S.G. Johnson and J. D. Joannopoulos. *Photonic Crystal - The road from theory to practice.* Kluwer Acad. Publ., 2002.

[19] P. Joly and A. Semin. Construction and analysis of improved Kirchoff conditions for acoustic wave propagation in a junction of thin slots. In *Paris-Sud Working Group on Modelling and Scientific Computing 2007–2008*, volume 25 of *ESAIM Proc.*, pages 44–67. EDP Sci., Les Ulis, 2008.

[20] P. Joly and S. Tordeux. Matching of asymptotic expansions for wave propagation in media with thin slots. I. The asymptotic expansion. *Multiscale Model. Simul.*, 5(1):304–336 (electronic), 2006.

[21] P. Joly and S. Tordeux. Matching of asymptotic expansions for waves propagation in media with thin slots. II. The error estimates. *M2AN Math. Model. Numer. Anal.*, 42(2):193–221, 2008.

[22] P. Kuchment. *Floquet theory for partial differential equations*, volume 60 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1993.

[23] P. Kuchment and B. Ong. On guided waves in photonic crystal waveguides. In *Waves in periodic and random media (South Hadley, MA, 2002)*, volume 339 of *Contemp. Math.*, pages 105–115. Amer. Math. Soc., Providence, RI, 2003.

[24] Peter Kuchment. Quantum graphs. I. Some basic structures. *Waves Random Media*, 14(1):S107–S128, 2004. Special section on quantum graphs.

[25] Peter Kuchment. Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J. Phys. A*, 38(22):4887–4900, 2005.

[26] Peter Kuchment. Quantum graphs: an introduction and a brief survey. In *Analysis on graphs and its applications*, volume 77 of *Proc. Sympos. Pure Math.*, pages 291–312. Amer. Math. Soc., Providence, RI, 2008.

[27] Peter Kuchment and Beng-Seong Ong. On guided electromagnetic waves in photonic crystal waveguides. In *Operator theory and its applications*, volume 231 of *Amer. Math. Soc. Transl. Ser. 2*, pages 99–108. Amer. Math. Soc., Providence, RI, 2010.

[28] Peter Kuchment and Yehuda Pinchover. Integral representations and Liouville theorems for solutions of periodic elliptic equations. *J. Funct. Anal.*, 181(2):402–446, 2001.

[29] Peter Kuchment and Hongbiao Zeng. Convergence of spectra of mesoscopic systems collapsing onto a graph. *J. Math. Anal. Appl.*, 258(2):671–700, 2001.

[30] V. Maz'ya, S. Nazarov, and B. Plamenevskij. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. I*, volume 111 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2000. Translated from the German by Georg Heinig and Christian Posthoff.

[31] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[32] S. A Nazarov. An example of multiple gaps in the spectrum of a periodic waveguide. *Mat. Sb.*, 201(4):99–124, 2010.

[33] S. A. Nazarov. Opening of a gap in the continuous spectrum of a periodically perturbed waveguide. *Mat. Zametki*, 87(5):764–786, 2010.

[34] S. A. Nazarov. The asymptotic analysis of gaps in the spectrum of a waveguide perturbed with a periodic family of small voids. *J. Math. Sci. (N. Y.)*, 186(2):247–301, 2012. Problems in mathematical analysis. No. 66.

[35] L. Parnovski. Bethe-Sommerfeld conjecture. *Ann. Henri Poincaré*, 9(3):457–508, 2008.

[36] L. Parnovski and A. V. Sobolev. Bethe-Sommerfeld conjecture for periodic operators with strong perturbations. *Invent. Math.*, 181(3):467–540, 2010.

[37] Olaf Post. Spectral convergence of quasi-one-dimensional spaces. *Ann. Henri Poincaré*, 7(5):933–973, 2006.

[38] Olaf Post. *Spectral analysis on graph-like spaces*, volume 2039 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[39] M. Reed and B. Simon. *Methods of modern mathematical physics v. I-IV*. Academic Press, New York, 1972-1978.

[40] Jacob Rubinstein and Michelle Schatzman. Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplacian spectrum. *Arch. Ration. Mech. Anal.*, 160(4):271–308, 2001.

[41] Yoshimi Saito. The limiting equation for Neumann Laplacians on shrinking domains. *Electron. J. Differential Equations*, pages No. 31, 25 pp. (electronic), 2000.

[42] A. Semin. *Propagation d'ondes dans des jonctions de fentes minces*. PhD thesis, Université Paris-Sud, 2010.

[43] M. Van Dyke. *Perturbation methods in fluid mechanics*. Applied Mathematics and Mechanics, Vol. 8. Academic Press, New York, 1964.

[44] M. Vorobets. On the Bethe-Sommerfeld conjecture for certain periodic Maxwell operators. *J. Math. Anal. Appl.*, 377(1):370–383, 2011.

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