On mixed polynomials of bidegree (n, 1)

Mohamed Elkadi, André Galligo

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Abstract

Specifying the bidegrees (n, m) of mixed polynomials $P(z, \bar{z})$ of the single complex variable z, with complex coefficients, allows to investigate interesting roots structures and counting; intermediate between complex and real algebra. Multivariate mixed polynomials appeared in recent papers dealing with Milnor fibrations, but in this paper we focus on the univariate case and $m = 1$, which is closely related to the important subject of harmonic maps. Here we adapt, to this setting, two algorithms of computer algebra: Vandermonde interpolation and a bissection-exclusion method for root isolation. Implemented in Maple, they are used to explore some interesting classes of examples.

1 Introduction

An expression $P(z, \bar{z}) = \sum_{k=0..n} \sum_{j=0..m} a_{k,j} z^k \bar{z}^j$ where z and $\bar{z}$ are complex conjugated, is called a (univariate) mixed polynomial of bidegree (n, m). We will assume $m \leq n$ and concentrate on the case where $m = 1$. Our aim is to study the roots in $\mathbb{C}$ of P. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and separating real and imaginary parts of P, i.e. writing $P = f(x, y) + ig(x, y)$ with $i^2 = -1$ and $z = x + iy$, we get a pair of real bivariate polynomials of degrees at most $n + m$. Conversely from a pair of bivariate polynomials $(f(x, y), g(x, y))$, letting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and $P = f + ig$, we get a univariate mixed polynomial. However, since the two representations are different, we can investigate interesting roots structures and develop algorithms, intermediate between complex and real algebra. This representation can be also used with several variables $(z_1, .., z_l)$; it received a renewed interest with the works in Algebraic Geometry of [25] on a new exotic sphere (à la Pham-Brieskorn); more recently Mutsuo Oka [21], thanks to mixed polynomials, answered a question of [18] on real generalizations of Milnor fibration theorem. Harmonic polynomials and rational maps are important special cases of mixed polynomials; they have been extensively studied and were applied to the study of gravitational lensing [14, 22]. Indeed after simplification, the roots finding problem for a mixed polynomial equation $P(z, \bar{z})$ of bidegree$(n, 1)$

reduces to the study of $\bar{z} = r(z)$, where r is a rational map; we will briefly recall some recent root counting formulas obtained in that field, [14, 24, 4].

Several techniques developed in Computer algebra seem useful to better investigate these objects. Specially in the case $m = 1$ where one expects properties similar to those of usual complex univariate polynomials. Unfortunately, the presentation of a univariate polynomial as a product via its roots is not valid in this context. Moreover, although P of bidegree $(n, 1)$ has $2n + 2$ coefficients, it may admit more than $2n + 2$ roots in $\mathbb{C}$. We will discuss and illustrate this behavior, directly related to bounding the number of zeros of harmonic maps. Beside the case $m = 1$, the results obtained so far on harmonic polynomials, see e.g. [32, 29, 16], concentrated on m near n. The study of the case $m = 2$ is still lagging behind.

In this paper, we adapt two basic algorithms in this new setting and use them to explore some interesting classes of examples, including random mixed polynomial of bi-degree $(n, 1)$ for rather large n. The first tool is a variant of Vandermonde matrix needed to interpolate $P(z, \bar{z})$, in such a way that we can prescribe some roots in $\mathbb{C}$ and then investigate the set of roots of P in $\mathbb{C}$. The second tool is a bisection-exclusion method which generalizes the classical one, see e.g. [31]. Together with a specific Newton process, it allows us to certify the set of complex roots we computed in each of our examples. We do not provide general complexity formulas but restrict ourselves to the case of mixed polynomials with simple roots (with an algorithm to check this property). Experiments, with the computer algebra system Maple, on mixed polynomials, of degrees $(n, 1)$, with given random distribution of coefficients allowed to observe interesting patterns.

The paper is organized as follows: In the next section 2, after some examples we present general properties of mixed polynomials and give an overview of results recently obtained on zeros counting of rational harmonic maps. In section 3, we construct generalized Vandermonde matrices and prove that they are generically invertible. In section 4, we present some investigation tools and together with examples; we investigate the effect of choosing the coefficients with several stochastic distributions. In section 5, we develop for the case $(n, 1)$, our bisection-exclusion method for locating the roots of P in $\mathbb{C}$, together with a Newton process and a test to check that a small disc contains only one root.

This paper is an amplification and a continuation of our presentation at the conference SNC'2014 [6].

We denote by $\bar{a}$ the complex conjugated of a complex number a, and by $\bar{P}$ the complex conjugated of a (mixed or usual) polynomial P, its coefficients are the complex conjugated of the coefficients of P.

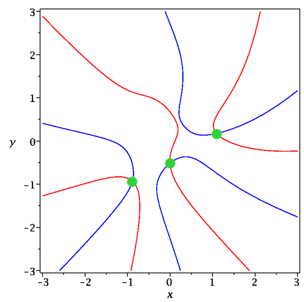

Figure 1: Example1

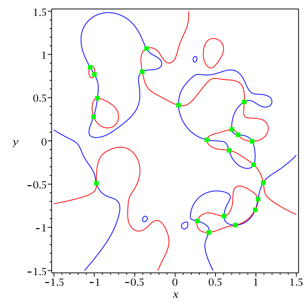

Figure 2: Example2

2 General properties

We begin with some examples of mixed polynomials and pictures of their roots.

Example 1 *A random mixed polynomial of bidegree* $(4, 1)$

$$P{:=}(4{-}3i)z^4\bar{z}{+}(3{+}7i)z^4{+}(8i)z^3\bar{z}{+}(7{+}9i)z^3{+}({-}6{-}9i)z^2\bar{z}$$

$$+(6{-}3i)z^2{+}({-}5{-}6i)z\bar{z}{+}(1{-}7i)z{+}({-}5{-}9i)\bar{z}{+}4{+}2i.$$

It has 3 roots in $\mathbb{C}$ *shown in green in Figure 1. Writing* $P = f(x, y)+ig(x, y)$*, the implicit curves defined by* $f = 0$ *and* $g = 0$ *are shown in red and blue.*

Example 2 *An example of a random polynomial of bidegree* $(17, 15)$ *with 19 roots, see Figure 2.*

Example 3 *Consider* $P = z\bar{z} + e$*, when* $e = -1$*, its roots form a circle; when* $e = 0$*, the only root is a point; while when* $e = 1$*,* P *has no root in* $\mathbb{C}$*.*

We briefly review some properties of univariate mixed polynomials inherited by their representations.

2.1 Factorization

The product of two mixed polynomials $P_3 = P_1P_2$ can be expressed by a set of algebraic conditions on their coefficients, identical to the set of conditions corresponding to "usual" bivariate polynomials with the same bidegrees. Therefore, the factorization properties and algorithms valid for bivariate polynomials, are also valid for univariate mixed polynomials.

2.2 Dimension

The real variety $V(P)$ defined in $\mathbb{C} = \mathbb{R}^2$ by $P = 0$, where P is an univariate mixed polynomial (non identically zero), can be either of dimension 1, 0 or -1 (i.e. $V(P)$ is empty).

In the first case, writing $P = f + ig$ as above, the bivariate polynomials $f(x, y)$ and $g(x, y)$ have a non constant gcd $h(x, y)$ which vanishes on a curve of $\mathbb{R}^2$. In other words this cannot happen if the gcd is constant, e.g. with probability 1 in a "random" case.

As we will see below, the third case cannot happen if the bidegree satisfy $n > m = 1$.

The most "common" case is the second one. If it is so, a natural question to ask is: what is the maximum number of roots for a given bidegree?

2.3 Topological degree

Let $P = f + ig$ as above. For each isolated root $z_j = (x_j, y_j)$, $j = 1..N$ of P we can attach the local topological degree of the map $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ at (x_j, y_j). Let us recall that the topological degree is defined as follows: since (f, g) is (locally) continuous and differentiable the image of a sufficiently small circle γ around (x_j, y_j) is a closed loop around $(f(x_j, y_j), g(x_j, y_j))$; the (signed) degree counts the number of turns (clockwise) of this loop. For a simple root, this degree is 1 or -1 according to the sign of the (non vanishing) jacobian determinant of (f, g) at that root.

In particular, near a simple root z_0 of P, the local equation of P can be written $\Phi := \bar{z} - \phi(z) = 0$; by a well known formula, the jacobian of Φ is equal to $|\phi'(z)|^2 - 1$; hence this jacobian is negative if and only if $|\phi'(z)| < 1$, in other words ϕ is locally a contraction map. We will return below to this condition when we consider attractive fixed points, see 2.5.2.

Now, a sufficiently big circle, containing all the roots, can be viewed either as a loop "around infinity" or as a loop around each root of P. So, one relates the degree "at infinity" to the sum of the local degrees at all roots of P. The degree at infinity for a “generic” univariate mixed polynomial of bidegree (n, m) is simply $n - m$ and this is always true when $m = 1$. This observation was turned into a precise theorem by Oka [21], who after factorizing the total degree part of P, $P_{m+n} = cz^p \bar{z}^q \prod_{j=1}^{s} (z + \gamma_j \bar{z})$, says that P is admissible iff for all j, $|\gamma_j| \neq 1$ and let $\epsilon(j) = 1$ if $|\gamma_j| < 1$ and $\epsilon(j) = -1$ if $|\gamma_j| > 1$. In that case, Oka proved that the sum of the local degrees is $p - q + \sum_j \epsilon(j)$. As a first consequence, if the sum of the local degrees is non zero, in particular if $m = 1, n > 1$, the zero set $V(P)$ of P cannot be empty. As a second consequence, in the "generic" case (for instance in the random case, as a claim with probability one) all roots of P are simple and the sum of the (signed) degrees is equal to $n - m$, this implies that the number of roots is $n - m + 2K$ where K is a non negative integer.

When $m = 1$, this result is known in the community of researchers on harmonic maps as the argument principle.

2.4 Resultants

Instead of calculating the resultant of the real representation $(f(x, y), g(x, y))$ of $P(z, \bar{z})$ to study the variety $V(P)$, we can use another resultant which respect the structure of $P(z, \bar{z})$.

Since $P(z, \overline{z}) = 0$ iff $\overline{P}(\overline{z}, z) = 0$, the complex roots of $P(z, \bar{z})$ can also consider as roots of the pair of "usual" polynomials $P(z, w)$ and $\bar{P}(w, z)$, such that $w = \bar{z}$.

The elimination of a variable in

$$\left\{ \begin{array}{l} P(z, w) = 0 \\ \overline{P}(w, z) = 0. \end{array} \right. \tag{1}$$

leads to a "biprojectif" resultant of degree $n^2 + m^2$, a consequence of multi-projective Bézout theorem, see [28]. In the discrete case, the number of solutions in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is $n^2 + m^2$.

Notice that the elimination of one variable in the system of polynomials equations $f(x, y) = 0$ and $g(x, y) = 0$ leads to $(n + m)^2$ solutions (x_j, y_j) in $\mathbb{P}^2(\mathbb{C})$. With this representation, the size of the Sylvester matrix is $2(n+m)$ whereas in the case of the resultant in (z, w) the size is $n + m$.

2.4.1 Real coefficients

Proposition 1 *Let $P(z, \bar{z}) = f(x, y) + ig(x, y)$ be a mixed polynomial of bidegree (n, m) with real coefficients; where f and g are real bivariate polynomials; let also $Y = y^2$. Then y is a factor of g, writing $g = y\hat{g}$, f and $\hat{g}$ are polynomials in (x, Y), that we denote by $\tilde{f}$ and $\tilde{g}$. Moreover, the number of solutions in $\mathbb{C}^2$ of $\tilde{f}(x, Y) = 0$, $\tilde{g}(x, Y) = 0$ is bounded by $\frac{n(n-1)}{2} + \frac{m(m-1)}{2}$.*

The proof is easy, see [6].

2.5 Rational harmonic map

Consider the case of mixed polynomials of bidegree $(n, 1)$: $P(z, \bar{z}) := \bar{z}q(z) - p(z)$ with $\deg(q) = n$, $\deg(p) \leq n$; after a translation on z also called Tchirnhausen transform, we can assume $\deg(p) \leq n-1$. We first characterize the case when dimension $V(P)$ equals one, then in the sequel of the paper, we will assume $n > 1$ and $\gcd(p, q) = 1$ which implies that the dimension of $V(P)$ is zero.

2.5.1 Dimension of $V(P)$

We already observed that the mixed equation of a circle of center $a \in \mathbb{C}$ and radius R is the mixed polynomial of bidegree $(1,1)$, $Q_{a,R} := z\bar{z} - \bar{a}z - a\bar{z} + |a|^2 - R^2 = 0$. In that case the imaginary part $g(x,y)$ of $Q_{a,R}$ is identically zero. Similarly, the mixed equation of a general line in $\mathbb{C} \setminus 0$ is the mixed polynomial of bidegree $(1,1)$, $L_a := az + \bar{a}\bar{z} - 1 = 0$.

Multiplying one of these two equations by a "usual" polynomial $p(z)$ of degree $n-1$, we get a mixed polynomial $P(z,\bar{z})$ of bidegree $(n,1)$ such that its zero set has real dimension one. Indeed its zero set $V(P)$ contains a circle or a line.

The following property is attributed to Ph. Davis since it was first mentioned in his 1974 book [5]. We also gave a simple proof in [6].

Proposition 2 *The only possible curve contained in the zero set $V(P)$ of a mixed polynomial $P(z,\bar{z})$ of bidegree $(n,1)$ is either a circle or a line.*

Now, we will only consider the case where $V(P) = 0$ and assume $n > 1$.

2.5.2 Counting roots of $\bar{z} = r(z)$

We assume $n > 1$, $\deg(p) \leq n-1$, $\gcd(p,q) = 1$, and $\gcd(q,q') = 1$; $\deg(q) = n$. The roots of $P(z,\bar{z}) = \bar{z}q(z) - p(z)$ are the roots of $\bar{z} = r(z)$, with $r(z) := \frac{p(z)}{q(z)}$. We will also write $r(z) := \sum_{j=1}^{n} \frac{\mu_j}{z - z_j}$, where z_j denotes the (distinct) roots of $q(z)$ and $\mu_j \in \mathbb{C}$. Counting the roots of $\bar{z} = r(z)$ has been an active field of research due to its interpretation in gravitational lensing, see e.g. [22] and important progresses have been achieved.

Theorem 1 ([14]) *The number $N(r,n)$ of roots of $\bar{z} = r(z)$ is bounded by $5n-5$.*

Theorem 2 ([24]) *There exists a family of rational functions r_n, $n > 1$ such that $N(r_n, n) = 5n - 5$.*

Theorem 3 ([4]) *There exists a family of rational functions $r_{n,k}$, $n > 1, k = 0,..,2n-2$, such that $N(r_{n,k}, n) = n - 1 + 2k$.*

Let us briefly comment these results. We already observed that $N(r,n) \leq n^2 + 1$ and that $N(r,n) = n - 1 + 2K$, by the count of topological degrees (see section 2.3). Let z_0 be a (simple) root of $P = f + ig$, hence of $z = \overline{r(z)}$. Then a straightforward computation shows that the topological degree at z_0 of (f,g) is 1, (resp. -1), iff $|r'(z_0)| > 1$, (resp. $|r'(z_0)| < 1$). Moreover, z_0 is called sense preserving, (resp. reversing), and z_0 is a repelling, (resp. attractive), fixed point of the discrete dynamics $z_{l+1} := \overline{r(z_l)}$, $l \in \mathbb{N}$. Denoting by N_+ and N_- the numbers of attractive and repelling fixed points, we have $N_- = N_+ + (n-1)$, then $N(r,n) = N_+ + N_- = 2N_+ + n - 1$. Therefore, the first result reduces to prove that $N_+ \leq 2(n-1)$. The strategy, developed in [14], is to show that each of the N_+ attractive fixed point, also attracts at least $n+1$ critical points of the rational function $Q(z) := \overline{r(\overline{r(z)})}$, which has $2(n^2-1)$ critical points.

Rhies' examples [24] are invariant under rotations centered at the origin of angle $\frac{2\pi}{n}$, in particular this makes the number of roots easier to count. They have a physical interpretation since they correspond to a configuration of equal masses ($\mu_j = \mu > 0$) equally spaced on a circle centered at the origin and an additional small mass at the origin (which gives rise to a set of solutions very near to the origin).

The construction of generalizations of this configuration, in [4], proceeds for a fixed k, by induction on n, by adding a small enough mass which produces the expected effect but does not destroy the previous count, moving a little bit (almost infinitesimally) the previous roots.

The remaining question is: What happens far from these regular configurations and their small perturbations?

3 Vandermonde matrices

In this section, we consider the interpolation problem for finding the $N = (n+1)(m+1)$ coefficients of a univariate mixed polynomial $P(z, \bar{z})$ of bidegree (n, m), knowing its values at N points $w_j,\ j = 1..N$ of $\mathbb{C}$. Writing the corresponding linear constraints, we obtain a square (N, N) complex matrix which is a generalization of the classical Vandermonde matrix. Its determinant Δ is a (not identically zero) multivariate mixed polynomial. Unfortunately, the property that when the N points w_j are pairwise distinct then Δ does not vanish, which holds true for usual polynomials, is not true for mixed polynomials.

Since we are interested by characterizing P by its roots, we will consider variants of that problem. First, we normalize to 1 the constant coefficient (we could similarly have fixed the highest bidegree coefficient) to get rid of the trivial solution. Then for the simple roots problem, we force P to vanish on $N-1$ points $w_j,\ j = 1..N-1$. While for the case of simple and double roots problem, we force P to have a simple root at $N-1-2K$ points and a double root (with a specified direction) at K other pairs of points and directions (w_j, θ_j) $j = 1..K$, with $w_j \in \mathbb{C}$ and $\theta_j \in [0, \pi[$.

To ease the presentation, we consider separately the two cases and skip the study of the interpolation problem which is very similar to the first case. With the same approach, we can analyze mixed polynomials with real coefficients.

3.1 Simple roots

Given $N-1$ distinct points w_l, $l = 1..N-1$, or equivalently a point $W \in \mathbb{C}^{N-1}$, and sorting the pairs (k,j), $k = 1..N-1, j = 1..N-1$ lexicographically, we construct the $(N-1, N-1)$ square matrix A whose l-th row is formed by the evaluation at w_l of the monomial $z^k \bar{z}^j$. Let us denote by $\Delta(W)$ its determinant.

Proposition 3 $\Delta(W)$ *is a non identically zero mixed polynomial (in several variables).*

Proof: It is clear that $\Delta(W)$ is a mixed polynomial. We will show that it admits a higher derivative in $(z, \bar{z})$ non identically equal to zero.

Since $\Delta(W)$ is a determinant, its derivatives are linear function of its rows. For a fixed l, observe that the derivative, with respect to $(w_l, \bar{w}_l)$, of a row where the variables $(w_l, \bar{w}_l)$ do not appear, is just a zero row. While the highest order derivative, with respect to $(w_l, \bar{w}_l)$, of a row where the only appearing variables are $(w_l, \bar{w}_l)$, is a row with a non zero constant entry and all the other entries are zero.

We proceed by induction. We first consider the higher derivative $\Delta_1(W)$ of order n in w_1 and order m in $\bar{w}_1$ of Δ. By the previous observations, it is a determinant of a matrix similar to A but where the first row has been replaced by $n!m!$ times the unit row $(1, 0, ..., 0)$: a 1 followed by $N-2$ zeros. Hence $\Delta_1(W)$ does not depend on $(w_1, \bar{w}_1)$ and is equal to $n!m!$ times the determinant of the first principal $(N-2, N-2)$ sub matrix of A.

The argument can be repeated, and the proposition is proved by induction. •

When $n = m = 1$, three points w_1, w_2, w_3 in $\mathbb{C} \setminus 0$ determine a unique circle or a unique line (if they are aligned).

When $n = 2, m = 1$, five points w_1, w_2, w_3, w_4, w_5 in $\mathbb{C} \setminus 0$ on a circle centered at the origin of radius R, satisfy $w_l \bar{w}_l = R^2$, hence $w_l^2 \bar{w}_l = R^2 w_l$, for all $l = 1..5$. In other words, with the previous notations, the first and the fourth column of the determinant Δ are proportional; hence $\Delta = 0$. This indicates that, unlike in the usual polynomial case, the zero locus of Δ can be rather complicated. Here the "bad" points W correspond to a factorization of a mixed polynomial P of bidegree $(2,1)$ into a mixed polynomial of bidegree $(1,1)$ $(z\bar{z}/R^2 - 1)$ and a polynomial of bidegree $(1,0)$.

3.2 Simple and double roots

We consider the case when we impose a double root at w in the direction $u = e^{i\theta}$. Infinitesimally, this amounts to consider the limit situation where P vanishes at w and at $w + \epsilon u$ when ϵ tends to zero. We keep the first row $L_1 = (w^k \bar{w}^j)_{k,j}$ like in the previous matrix A, but we replace the following row $L_2 = [(w + \epsilon u)^k (\bar{w} + \epsilon \bar{u})^j)_{k,j}]$ by the limit L_2' of the linear combination

$(L_2 - L_1)/\epsilon$ when ϵ tends to zero. More precisely we have $L_2' = [(kw^{k-1}u\bar{w}^j + jw^k\bar{u}\bar{w}^{j-1})_{k,j}]$. Factoring out $\bar{u} = e^{-i\theta}$, we obtain

$$L'(w, \bar{w}, \theta) = e^{-i\theta}[kw^{k-1}e^{2i\theta}\bar{w}^j + jw^k\bar{w}^{j-1}]$$

for $k = 1..N-1, j = 1..N-1$.

We denote by B this second generalization of the Vandermonde matrix corresponding to the case where we force P to have a simple root at $N-1-2K$ points $w_{2K+l}, l = 1..N-2K-1$, and a double root at K other pairs of points and directions $(w_l, u_l = e^{i\theta_l})$ $l = 1..K$. B is also a square $(N-1, N-1)$ matrix, its first $2K$ rows $L_1, L_1', L_2, L_2', ...L_K'$ are modeled as described in the previous paragraph. Dividing out each L_l' by the corresponding $e^{-i\theta_l}$ we obtain a determinant that we denote by $\Delta(W, \Theta)$. To prove that it is not identically zero, it is sufficient to exhibit one of its higher derivative which is not identically zero. We follow roughly the same argument than in the previous subsection. Here we first perform the K differentiations with respect to $\theta_1, .., \theta_K$ and divide out by the factors $2ie^{2i\theta_l}, l = 1..K$; which amounts to get another determinant $\Delta'(W)$ where the K rows L_l' have been replaced by the K rows $(kw_l^{k-1}\bar{w_l}^j)_{k,j}$. Hence we got rid of the variables θ_l.

Now, we perform on Δ' the maximum higher differentiation with respect to w_1 and $\bar{w_1}$ i.e. $2n-1$ times with respect to w_1 and $2m$ times with respect to $\bar{w_1}$; we obtain a determinant which first row is $n!m!$ times the unit row and the other rows are unchanged. Again, we perform the maximum higher differentiation with respect to w_1 and $\bar{w_1}$ so, we obtain a constant times a sub-principal minor of Δ', of two orders less, which does not contain neither w_1 nor $\bar{w_1}$.

We can iterate the argument. So we proved by induction the following generalization of the previous proposition.

Proposition 4 $\Delta(W, \Theta)$ *is a non identically zero mixed polynomial.*

3.3 Real coefficients

We consider the case of mixed polynomials P with real coefficients.

We force a real mixed polynomial P of bidegree $(n, 1)$ to vanish at K distinct pairs of conjugated complex numbers $w_1, \bar{w_1}, ..., w_K, \bar{w_K}$ and $N-1-2K$ real numbers w_{2K+k}, with $N = 2n+1$. We also normalize the coefficient of $z^n\bar{z}$ to 1.

If the corresponding generalized Vandermonde matrix is invertible, the unique solution P will have real coefficients, the reason is that $P(w_l, \bar{w_l}) = 0$ implies $\bar{P}(\bar{w_l}, w_l) = 0$ hence, $\bar{P}$ satisfies the same equations $\bar{P}(w_l, \bar{w_l}) = 0$ for $l = 1..N-1$, then by unicity $\bar{P} = P$.

The first $2K$ rows of the generalized Vandermonde determinant are made of K pairs of conjugated rows, hence can be replaced by rows formed by their real and imaginary parts. It turns out that we can find adapted higher differentiations to generalize the argument of the section 3.

Requiring only $r < 2n+1$ such linear independent conditions, we expect to obtain an affine space of dimension $2n+1-r$.

Then, using (real) discriminant loci or quadrics defined from 2.5.2, we can imposer extra conditions to maximize the number of real roots. See section 4 for examples computed with this approach.

3.4 Exploration tools and examples

The techniques developed in the previous section are useful for finding examples and investigating roots sets.

3.4.1 A probabilist exploration tool with real coefficients

We write, a mixed polynomial $P := \bar{z}z^n + \sum a_j z^j + \bar{z}\sum b_j z^j$ then, $P = f(x,y) + ig(x,y)$. Since the $2n$ coefficients a_j and b_j are real, y is a factor of g. Moreover f and g/y are polynomials in (x,Y) with $Y = y^2$.

We call $R(Y)$ the discriminant with respect to x of f and g/y. It is a polynomial in Y; its coefficients are polynomials in the coefficients a_j and b_j. We also consider $F(x) = f(x,0)$, which is a real polynomial of degree $n+1$. The roots of P in $\mathbb{C}$ are: first, the real roots of F and second, the real pairs (x,y) with $Y = y^2$ such that Y is a non negative root of R; then, generically, x can be written as a polynomial in Y. defined by the system $(f, g/y)$ (e.g. thanks to a Groebner basis, or with minors of the Sylvester matrix).

We proceed as follows. We first require that $x = 0$ is a root of F if the degree of F is odd. We (repeatedly) choose randomly real values $x_1, x_2, .., x_{n-1}$; and real positive values $Y_1, Y_2, .., Y_{n-1}$, and we require that P vanishes at the $2(n-1)$ pairs of conjugated roots $(x_k + i\sqrt{Y}, x_k - i\sqrt{Y})$, $k = 1..n-1$. These constraints are independent with probability one. They define an affine subspace of dimension 2 that we parameterize with two coefficients, say a_1 and b_0.

We denote by F_1 the evaluation of F (divided by x, if the degree of F is odd) and by R_1 the evaluation of R divided by $(Y - Y_1)..(Y - Y_{n-1})$.
We compute the discriminant A_1 of F_1, viewed as as a polynomial in x. Its graph is the image (by a change of coordinates) of a section of a "swallowtail", studied in Catastrophe theory, it delimits portions of the plane (a_1, b_0) where $F1$ admits a fixed number of real roots.

Then, we compute the discriminant of R_1, viewed as as a polynomial in Y. It admits generically two factors: a squared polynomial in (a_1, b_0), and a simple polynomial in (a_1, b_0), that we denote by A_2. The graphs of $A_2 = 0$ and $R_1 = 0$ delimit regions where R_1 admits a fixed number of non negative real roots. We further analyze these regions to choose the coefficients (a_1, b_0).

A variant of this method is to exploit the fact (recorded in section 2) that over $n-1$, the number of roots of P increases as twice the number of roots at which the topological degree is -1. So one can explore regions where the corresponding jacobian are negative. The evaluation of the jacobian at a point(x_0, y_0) is a quadratic form in the coefficients. So, we get regions delimited by conics in the parameter plane. Notice that when P has real coefficients, the sign of the jacobian is the same at two conjugated roots.

3.4.2 Example with $n = 5$

We wish to compute an example with maximal number of roots, different from Rhies'one, with less symmetries.

To simplify the computation and the presentation, we set to 1 the leading coefficient and perform a translation on z, $z := z - \alpha$, (similar to the Tchirnhausen transform) in order to fix one term in the expansion of P.

For $n = 5$, following Theorem 1, the maximal possible number of roots is $5n - 5 = 20$, while the number of coefficients of P is only 11.

We proceed as explained above: interpolating at the origin and at 4 pairs of conjugated roots, corresponding to $(x = 1, Y = 1), (x = -1, Y = 2), (x = -3, Y = 4), (x = 3, Y = 4)$, we end with two free parameters a_0, b_0. Then, we construct 4 conics defined by the jacobians of (f, g) at these 4 pairs. Then we can delimit a region where 3 of the 4 jacobians are negative, see Figure 3. We chose in that region, after two trials, the value $a_0 = 70, b_0 = 40$ which corresponds to a mixed polynomial with 8 pairs of conjugated roots and 4 real roots, hence the maximum number of roots. These roots are shown in Figure 4: the graphs of real and imaginary parts are colored in red and blue, the 8 attractive fixed points of $\bar{z} = r(z)$ are indicated by black solid boxes and the 12 repelling ones by green solid discs, while the 5 poles of r are indicated by brown diamonds. Observe the distributions of the intersections points (and their color) on the different ovals.

3.4.3 Physical configurations

We consider $\bar{z} = r(z)$, with $r := \frac{p}{q} = \sum \frac{\mu_j}{z - z_j}$, where $\gcd(q, q') = 1,\ \gcd(p, q) = 1$; in the physical interpretation μ_j are masses and should be real positive. Moreover, here we will assume here that p, q have real coefficients. This hypothesis implies some symmetries, in particular if $\mu_j > 0$ and $z_k = \bar{z_j}$ then $\mu_j = \mu_k$.

The family of examples studied by [24], and by [4], used this representation.

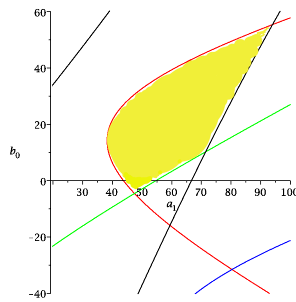

Figure 3: 4 conics

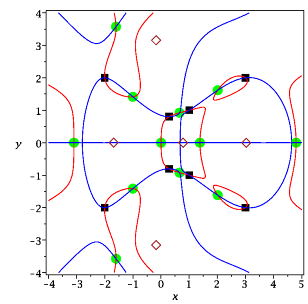

Figure 4: 20 roots

If we fix all the input data (except two) and express the constraints, we get nonlinear constraints which are difficult to deal with. So it is better to first convert the representation into a polynomial fraction and then, with linear constraints, apply the methodology of the previous subsection.

3.5 Random Mixed polynomials

In this section, we fix a distribution law of real numbers (such as the normal Gaussian or the uniform, with mean zero). Then we study the roots of a mixed polynomial $P(z, \bar{z})$, which coefficients are chosen using this law.

A natural question is to find the expectation of the number of roots w.r.t. these choices. We made some statistics on different kinds of distributions. Recently, a related question has been studied theoretically in [23], the authors provided a generalized integral formula.

3.5.1 Uniform distribution

For $n = 30, 31, 41, 50$, we computed the roots of 100 mixed polynomials of bidegree $(n, 1)$, which coefficients are integers uniformly distributed in $-10..10$ and collected the number of roots. From these statistics, we observed the same pattern shown in Figure 5 which corresponds to $n = 100$. For all these values of n, a typical distribution of the solutions roughly concentrate around the unit circle.

For this uniform distribution of real coefficients, the experimental average number of roots is about n. Moreover in about two third of the cases, we get $n - 1$ i.e. the minimum number of roots respecting the lower bound provided by the topological degree (see section 2). Notice that for the "physical" case, [22] showed that the minimum number of roots is indeed $n + 1$ and not $n - 1$.

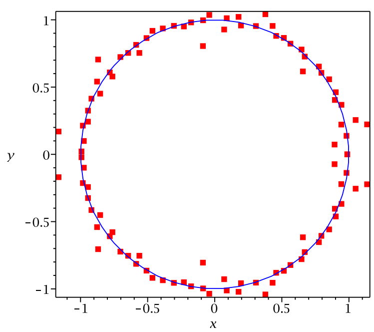

Figure 5: With uniform coefficients

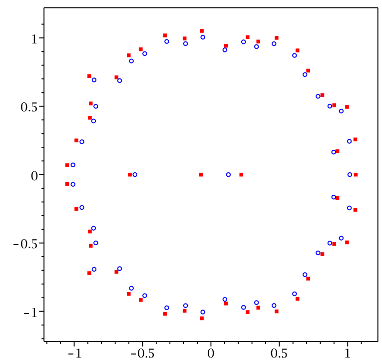

Figure 6: Physical configuration

3.5.2 Condensation

For "usual" univariate polynomials, when the coefficients are real and uniformly distributed, say in $-10..10$, the average number of roots is about $(2/\pi)Log(n)$, while when the size of the coefficients depends exponentially on the exponents then, as proved in [17], the number becomes a $O(n^s), 0 < s < 1$ and increase with the level of exponentiation; this phenomena was observed by Majundar and Scher in [17] who compared it to a condensation process. Therefore, one wonders if the same kind of behavior occurs for random mixed polynomials of bidegree $(n, 1)$. Our experiments indicate that it is not the case.

It would be interesting to find random distributions of coefficients which increase the number of attractive fixed points of the corresponding discrete dynamics; hence the number of roots of P.

3.6 Equidistribution of poles

We also considered a random case which approximate equidistribution of poles with same mass: We took a Kac type polynomial q of degree n (with standard normal distribution of coefficients) such that its roots roughly concentrate near the circle centered at the origin of radius 1 and its derivative p and considered the mixed polynomial $P = n\bar{z}q - p$. We found about $n+1$ solutions near the unit circle. Figure 6 shows a typical set of solutions of P in red solid boxes together with the n roots of q in blue circles (3 more solutions, approximately at -1.7, $2 + 2.8i$ and $22.8i$ do not appear on the picture, very near to 3 roots of q). Notice the “almost pairing” between the red solid boxes and the blue circles almost aligned with the origin, the interpretation is that the non real solutions are near-by the non real roots of q, moreover the solutions seem a bit farther from the origin than the roots of q.

Compare with the patterns of the roots of q' described in [13] which, in the uniform random case, also form an "almost aligned pairing" with the roots of q but are contained inside the convex hull of the roots of q.

Inverse problem: Given a random distribution of masses, positioned at the roots of a polynomial $q(z)$. We can express the physical situation thanks to an additional polynomial $p(z)$. We observe the $n+1$ solutions Z_k of the equation $n\bar{z}q(z) - q'(z) = 0$. The problem of recovering the masses and $q(z)$ from the set $Z_k,\ k = 1..n+1$ can be addressed using linear algebra, relying on the Vandermonde matrices of section 3.

For higher values of n, all these explorations require efficient methods for finding (i.e. isolating in a small box) the root of a mixed polynomial. It is the subject of the next section.

4 Numerical computations

Our aim is to locate, up to a prescribed precision, the roots in $\mathbb{C}$ of the equation

$$P := \bar{z}B(z) - A(z) = 0 \quad (*)$$

where A and B are co-prime complex polynomials of degree n; we also let $Q := \frac{A}{B}$. Equivalently this amounts to locate the roots of $\bar{z} - Q(z) = 0$. Near a simple root of P, we can adapt Newton process to our setting.

Hypothesis: In all this section, we assume that all the roots of $(*)$ are simple.

4.1 Newton process

Newton process near a simple root is based on a linear approximation of an equation. Here, we set $w_{k+1} := \overline{z_{k+1}}$ and solve, in w_{k+1}, z_{k+1} the linear system formed by the pair of conjugated linear equations:

$$w_{k+1} - z_{k+1}Q'(z_k) = Q(z_k) - z_kQ'(z_k);$$

$$z_{k+1} - w_{k+1}\overline{Q'(z_k)} = \overline{Q(z_k)} - \overline{z_k}\overline{Q'(z_k)}.$$

Then, Newton iteration can be written:

$$z_{k+1} := z_k - \frac{(Q(z_k) - \overline{z_k})\overline{Q'(z_k)} + (\overline{Q(z_k)} - z_k)}{|Q'(z_k)|^2 - 1}.$$

Alternatively, considering linear approximations of $(*)$ and its conjugate, we get another formula:

$$z_{k+1} := z_k - \frac{(z_k\overline{B'(z_k)} - \overline{A'(z_k)})(\overline{z_k}B(z_k) - A(z_k)) - B(z_k)(z_k\overline{B'(z_k)} - \overline{A'(z_k)})}{|z_k\overline{B'(z_k)} - \overline{A'(z_k)}|^2 - |B(z_k)|^2}$$

As usual, if the root is simple the process converges quadratically in a small enough neighborhood of the root, which is difficult to determine efficiently. In theory, one should be able to adapt the bounds given by Smale and his coworkers, see [1]. But, since we focus on justifying the computations of our examples, it suffices to keep subdividing till we get convergence, up to a prescribed precision.

Near a simple root, the subdivision follows a bisection-exclusion scheme.

4.2 A bisection-exclusion algorithm

Bisection-exclusion algorithms are classical methods used to isolate roots of a polynomial or an analytic function, they were initiated by H. Weyl in [30] in 1924, then improved by P. Henrici, see some historical hints in [31], which provides a up to date presentation of the subject.

4.2.1 First reduction

If z_0 is a root of $(*)$, then $u_0 := \frac{1}{z_0}$ satisfies, after simplification, the equation

$$\bar{u}A_1(u) - B_1(u) = 0 \quad (**)$$

where A_1 and B_1 are the so called reciprocal polynomials of A and B (i.e. just invert the order in the list of the coefficients).

Since the two equations are quite similar, we can apply twice a root finding algorithm restricted to the unit disc. Moreover if the coefficients of the polynomials are real numbers (which will be frequent in our examples) we can further restrict our study to the closed upper half unit disc, and complete by conjugation.

4.2.2 Principle

We will update a domain of the complex plane, equal to the union of a set $\mathcal{S}$ of squares $S(z, c)$, z is the center of that square and c is its half length side. At the initialization, the domain includes the unit disc. A test function M is also chosen; given a square S, it returns a real number $M(S)$. $\mathcal{S}$ is inductively updated as follows: For each square S of $\mathcal{S}$, if $M(S) > 0$ then S is just removed from $\mathcal{S}$, else S is divided in 4 smaller squares which are added at the end of the list $\mathcal{S}$, while S is removed from $\mathcal{S}$.

An invariant of the loop is that the roots of P are always contained in the union of the squares of $\mathcal{S}$, while the length side of the squares decreases. The semantic of the exclusion test function M is that if $M(S) > 0$ then the square S does not contain any root.

4.2.3 Exclusion test functions

A first (natural) exclusion test function is

$$M_0(S(z_0, \frac{t}{\sqrt{2}})) = |\bar{z_0} - Q(z_0)| - t(1 + \sum_{k=1}^{\infty} \frac{|Q^{(k)}(z_0)|}{k!} t^{k-1})$$

obtained using the Taylor expansion of Q. Its positivity implies that for any z in the disc of radius $\frac{t}{\sqrt{2}}$ centered at z_0, $|\bar{z} - Q(z)| > 0$ holds. Since the square $S(z_0, t)$ is contained in that disk, there are no root in $S(z_0, t)$.

However, we prefer to rely on the finite Taylor expansions of the polynomials A and B as follows. We denote, as usual, by $A^{(k)}$ and $B^{(k)}$ the k-th derivative of A and B, and to ease the reading of the formulas we also denote by $A_0, A_0', A_0^{(k)}$ the values $A(z_0), A'(z_0), A^{(k)}(z_0)$ and similarly for B. We collect the terms of the Taylor expansions of $(*)$ at z_0 with $z = z_0 + te$, $t \in \mathbb{R}^+$, $|e| = 1$; we get:

$$P := \bar{z}B(z) - A(z) = (\bar{z_0}B_0 - A_0) + t(\sum_{k=0}^{n} \frac{B_0^{(k)}}{k!} t^k \bar{e} e^k)$$

$$+t(\sum_{k=1}^{n} \frac{\bar{z_0}B_0^{(k)} - A_0^{(k)}}{k!} t^{k-1} e^k).$$

Now, we define $M(S(z_0, t/\sqrt{2}))$ as

$$M := |\bar{z_0}B_0 - A_0| - t(\sum_{k=0}^{n} \frac{|B_0^{(k)}|}{k!} t^k) - t(\sum_{k=1}^{n} \frac{|\bar{z_0}B_0^{(k)} - A_0^{(k)}|}{k!} t^{k-1}).$$

Clearly, if $M(S(z_0, \frac{t}{\sqrt{2}})) > 0$ then P has no root in the square $S(z_0, \frac{t}{\sqrt{2}})$.

We also define another test function using the Taylor expansion of a linear combination of P and $\bar{P}$, which eliminates the term in $\overline{z - z_0}$ in a similar way than in the computation of Newton formula, expanding

$$\overline{B_0}P(z) - (\bar{z_0}B_0' - A_0')\overline{P(z)}.$$

Then, we obtain another test function M_1, which is equal to:

$$M_1 := |\overline{B_0}(\bar{z_0}B_0 - A_0) - (\bar{z_0}B_0' - A_0')(z_0\overline{B_0} - \overline{A_0})| - |z - z_0|||B_0|^2 - |z_0\overline{B_0'} - \overline{A_0'}||$$

$$- \sum_{j=2}^{n} \frac{|z - z_0|^j}{j!} |\overline{B_0}(\bar{z_0}B_0^{(j)} - A_0^{(j)}) - (\bar{z_0}B_0' - A_0')(z_0\overline{B_0^{(j)}} - \overline{A_0^{(j)}})|$$

$$- \sum_{k=1}^{n-1} \frac{|z - z_0|^{k+1}}{k!} |\overline{B_0}B_0^{(k)} - (\bar{z_0}B_0' - A_0')\overline{B_0^{(k)}}|.$$

4.2.4 Algorithm

We now describe the algorithm we implemented in Maple.

A precision 2^{-N} and a number of digits for the floating point computations are fixed; they depend on the degree n and some a priori knowledge (or assumption) on the geometry. As a pre-processing, a small integer m_0 is also fixed, it is used to subdivide the unit square into 2^{m_0+2} small squares of side length 2^{-m_0}, we initialize the set $\mathcal{S}$ to be a subset of these squares such that their union contains the unit disk. We order this set and call $\mathcal{T}$ the obtained list. The initial a priori division of the unit square allows to spare unnecessary evaluations of the function test, when the side length is not small enough. To give an idea of the parameters, for a degree $n = 100$ and a generic position of the roots, a typical choice we made for m_0 is 8, for Digits it is 30 and for N it is 40.

The algorithm consists of several loops which update the list $\mathcal{T}$, it calls 3 different test functions M, M_1 and T.

First, we update $\mathcal{T}$, recursively removing and testing with M its first square $S(z_0, 2^{-m})$, if the result is non positive we append at the end of $\mathcal{T}$ four new smaller squares obtained by dividing $S(z_0, 2^{-m})$ into four squares. We stop either when $\mathcal{T}$ is empty or when all the side length of its square are smaller than $2^{-N/2}$.

Examples show that the drawback of using M is that the size of the squares diminish but their number remain important and they form clusters of squares. The test function M_1 is a bit more complicated but is more efficient for diminishing the number of squares in the clusters

Second, we update $\mathcal{T}$, recursively removing and testing with M_1. We also delimit clusters of squares. We stop either when $\mathcal{T}$ is empty or when all the sizes of the clusters are smaller than $2^{-N/2}$.

Third, for each cluster of squares, we apply 10 times a Newton iteration, if it converges we get a candidate root and we apply a local injectivity test function (see next subsection), and subdivide further if needed till we reach the length side 2^{-N}.

Finally we return either FAIL or a list of approximations of simple roots.

4.2.5 Illustration

Let us give an illustration where A and B are random polynomials with real coefficients and degree $n = 50$. We chose $N = 20$ and start with about 250 squares. During the computation, the number of elements in $\mathcal{T}$ decreases and increases several times and eventually stabilizes. In Figures 7 and 8 we show two states of subdivision: after 3000 and 10000 calls of the function M, the side length were about 2^{-7} and 2^{-9} (a red solid-box correspond to the center of a square). Notice the appearance of clusters of small squares.

Then, to compare the effects of the two functions M and M_1 on a cluster

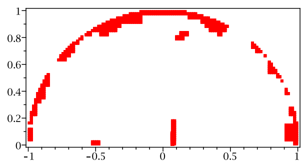

Figure 7: step1

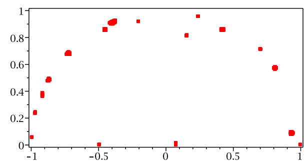

Figure 8: step2

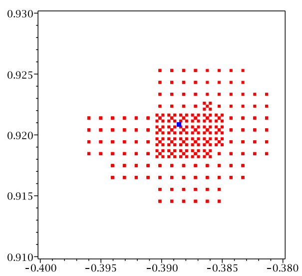

Figure 9: step3 with M

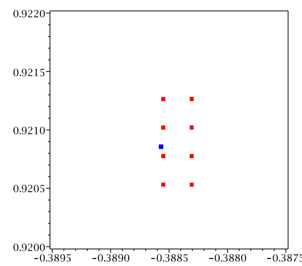

Figure 10: step3 with M_1

of squares, we performed two different computations. First, we performed a loop of 2000 calls of the function M, Figure 9 shows a zoom of a cluster near a root (represented by a blue solid-box), the side length is about 2^{-10}. Second, we performed a loop of 2000 calls of the function M_1, Figure 10 shows a zoom of a cluster near the same root, the side length is about 2^{-12}. Observe that the scales are different in the two pictures; at this stage of the computation, the function M_1 is more efficient since it produces smaller clusters with less squares.

4.3 A local result

Here, we prove that, in a sufficiently small neighborhood of a simple root of $P = \bar{z} - Q(z) = 0$, the test function M_1 behaves more efficiently than the test function M.

Without lost of generality, we can assume that the root is the origin, so $Q(0) = 0$, we also set $a := Q'(0)$. Since the root is simple, we have $|a| \neq 1$. Then, in a small neighborhood V of 0, $Q(z) = az + O(|z|^2)$, and $\overline{Q(z)} = \bar{a}\bar{z} + O(|z|^2)$. Let us specialize the functions M and M_1 for a square contained in V, using these notations, so here $A = az + O(|z|^2)$, $B = 1$ and we consider $|z_0| = O(t)$.

$$M(z_0, t) = |\bar{z}_0 - az_0| - t(1 + |a|) + O(t^2).$$

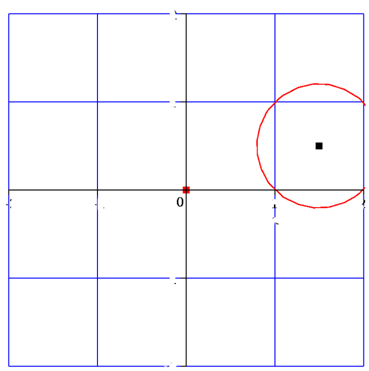

Figure 11: Adjacent squares

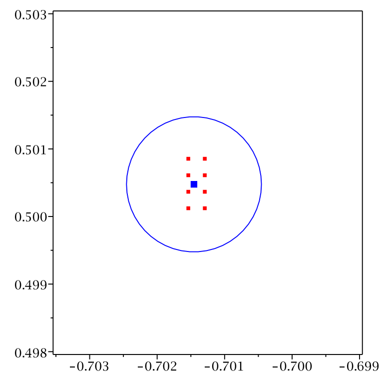

Figure 12: Injectivity disc

Recall that for any argument of z_0, the only inequality we can certify is $|\bar{z_0} - az_0| \geq |1 - |a|^2||z_0|$.

$$M_1(z_0, t) = |1 - |a|^2||z_0| - t|1 - |a|^2| + O(t^2) = |1 - |a|^2|(|z_0| - t) + O(t^2).$$

We see that when t tends to zero, while we cannot find a lower bound for M, for M_1 if $|z_0| > (1 + b)t$ with $b > 0$ then $(|z_0| - t)$ remain strictly positive and $\frac{M_1(z_0,t)}{t} > b + O(1)$. We have indeed the following proposition.

Proposition 5 *After a sufficient number of subdivisions using the exclusion function M_1, it remains at most 4 squares around each simple root of P.*

Proof: If the root is the origin or a point $2^{-l} + 2^{-k}i$ of the subdivision grid, then it is clear that the four adjacent squares cannot be excluded. So, we consider the next adjacent squares as indicated in Figure 11. To simplify the presentation, assume that the root is the origin and let $2t$ be the diagonal length of a square. Let also z_0 be the center of a next adjacent square, then $|z_0|^2 \geq \frac{13}{8}t^2$, hence $|z_0| > (1 + 0.25)t$. Applying the reasoning of the previous paragraph, this implies that for t sufficiently small, the function M_1 excludes all the near-by squares around the four squares adjacent to the root. •

4.4 Injectivity Criteria

The calls to test functions M and M_1 in the bisection process, provide a family of clusters of squares. We would like to check if each of these cluster contains one and only one root. We proceed as follows. Given such a cluster, we run several times a Newton process starting from the center of one of the squares, till we get a convergence to a value; let call it z_0. Then we consider a disc centered at z_0 containing all the squares of the cluster, and compute the following criteria based on an exclusion function T, if $T > 0$ then there are no other root in that disc. See an illustration in Figure 12.

For that purpose, we first write the equality $\bar{z_0} - Q(z_0) = 0$, then consider the Taylor expansion of $\bar{z} - Q(z)$ near z_0.

We get $\overline{z - z_0} - (z - z_0) \sum_{k=1}^{\infty} \frac{Q^{(k)}(z_0)}{k!} (z - z_0)^{k-1}$. A first local injectivity test function $T_0(t)$ is:

$$T_0 := ||Q'(z_0)| - 1| - \sum_{k=2}^{\infty} \frac{|Q^{(k)}(z_0)|}{k!} t^{k-1}.$$

If $T_0(t) > 0$ then there is no other root than z_0 in the disk of radius t centered at z_0. The test works pretty well (up to a prescribed precision) in the examples, the draw back is that the sum is infinite.

We prefer to similarly construct another test function $T(t)$, in terms of A and B, as follows:

$$T := ||B_0|^2 - |\bar{z_0} B_0' - A_0'|^2| - \sum_{k=2}^{n} \frac{t^{k-1}}{k!} |(z_0 \overline{B_0^{(k)}} - \overline{A_0^{(k)}})| - \sum_{k=2}^{n} \frac{t^k}{k!} |B_0^{(k)}|.$$

4.4.1 Illustration (continued)

In the previous illustrative example with $n = 50$, when m reached $m = 13$ in the algorithm, there were 17 clusters of square in the half unit disc. For each of them the Newton process converges and the injectivity criteria can be called. After 1000 more subdivisions it was satisfied. Hence, up to the precision of the computations, we could conclude that they were 17 roots in the half unit disc.

For the cluster around $z_0 = -.3885684 + .920856 38i$, shown in Figure 10, all the squares were contained in a disc of radius 10^{-10}, after 1000 more subdivisions, there remained only four squares in the clusters which is contained in a disc of radius 2^{-15}, then the injectivity criteria was satisfied.

In order to get rid of the clause "up to the precision of the computations", one should use interval arithmetic, as done e.g. in [26].

5 Conclusion

In this paper we presented univariate mixed polynomials from a Computer algebra view point. After some general results, we concentrated on the case where the bidegree is $(n, 1)$ and the coefficients are real numbers, aiming generalizations of behaviors of "usual" univariate polynomials; the complex plane $\mathbb{C}$ included in $\mathbb{C}^2$ is viewed as a substitute of the real axis $\mathbb{R}$ included in $\mathbb{C} = \mathbb{R}^2$. The equation $\bar{z}q(z) - p(z) = 0$ is (generically) equivalent to the equation $\bar{z} = \frac{p(z)}{q(z)}$. This second equation has been extensively studied for its application in gravitational lensing, and important results were obtained, that we briefly reviewed. We started its study from a computational point of view, and adapted to this setting, some efficient roots isolation algorithms. Relying on resultants and generalized Vandermonde matrices, we also constructed exploration tools and described some significant examples. Little is known on roots of mixed equations of bidegrees (n, m) with small $m > 1$. We plan to generalize the exposed methods to address the study of the equation $\bar{z}^2 = r(z)$ and beyond.

Also, the polynomial solutions of the bi-Laplacian Δ^2, called bi-harmonic functions, are of great interest since they have important applications, see e.g. [20]; they are real (and imaginary) part of mixed polynomials of the form $\bar{z}Q_1(z) + Q_2(z) + zQ_3(\bar{z}) + Q_4(\bar{z})$,

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