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Unit Versus Ad Valorem Taxes: The Private Ownership of Monopoly In General Equilibrium*

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Abstract

In an earlier paper [Blackorby and Murty; 2007] we showed that if a monopoly sector is imbedded in a general equilibrium framework and profits are taxed at one hundred percent, then unit (specific) taxation and ad valorem taxation are welfare-wise equivalent. In this paper, we consider private ownership of the monopoly sector. Given technical difficulties in making a direct general equilibrium comparison of unit and ad valorem taxation, we adopt a technique due to Guesnerie [1980] and Quinzii [1992] in a somewhat different context of increasing returns and non-convex economies to show that neither ad valorem taxation nor unit taxation Pareto dominates the other; although, generally, the two are not welfare-wise equivalent.

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Unit Versus Ad Valorem Taxes: The Private Ownership of Monopoly In General Equilibrium

by

Charles Blackorby and Sushama Murty

1. Introduction

It is well-known that, in a competitive environment, unit (or specific) taxation and ad valorem taxation are equivalent. Cournot [1838, 1960] realized that the two tax systems needed different treatment in the case of monopoly. Wicksell [1896, 1959] argued that ad valorem taxes dominate unit taxation in a monopoly; a complete demonstration of his claim was given by Suits and Musgrave [1955]. More specifically they demonstrated that, if the consumer price and quantity of the monopoly good remained unchanged, the government tax yield is higher with ad valorem taxes than under a regime of unit taxes. This follows because the profit-maximizing price of the monopolist is lower under ad valorem taxation than under unit taxation. Most recent work in this area has investigated forms of competition between pure monopoly and competition implicitly or explicitly accepting the above dominance argument. Delipalla and Keen [1992] examine different models of oligopoly with and without free entry to compare the two types of tax regimes while Lockwood [2004] shows, in a tax competition model, that tax competition is more intense with ad valorem taxes thus yielding a lower price in equilibrium.

Wicksell and Suits and Musgrave derived the above mentioned monopoly result in a partial equilibrium framework and claimed that ad valorem taxation was superior to unit taxation on welfare grounds. Recently, stronger and more explicit claims have been made: Skeath and Trandel [1994; p. 55] state that "in the monopoly case, given any unit excise tax, it is possible to find an ad valorem tax that Pareto dominates it."; Keen [1998; p. 9] states that "The conclusion—due to Skeath and Trandel—is thus strikingly unambiguous: with monopoly provision of a single good of fixed quality, consumers prefer ad valorem taxation because it leads to a lower price, firms prefer it because it leads to higher profits and government prefers it because it leads to higher revenue. There is no need to trade off the interests of these three groups: ad valorem taxation dominates specific."

It is this claimed welfare dominance of ad valorem taxes over unit taxation that we challenge in this paper. In the context of a general equilibrium model with a single monopoly sector, we show that the set of Pareto optima under unit taxation neither dominates the set of Pareto optima with ad valorem taxation nor does the set of ad valorem Pareto optima dominate the set of unit Pareto optima.

To be fair, none of the above authors claimed that they were talking about Pareto-efficient taxes. Nevertheless, if the economy is at a Pareto-inefficient equilibrium there will obviously be many tax regimes that dominate it. Focusing on Pareto-efficient taxes shows that at many Pareto optima, the optimal tax on the monopoly good is negative.[1] Of course, in the case of a subsidy to the monopolist, the intuition that Wicksell and Suits and Musgrave derived from the positive tax case is turned on its head: firstly, because the government has to obtain money from somewhere else in the system to subsidize the monopolist, and secondly, because, with a subsidy, the yield from an ad valorem tax is lower than from the unit tax that leads to the same profit maximizing output for the monopolist. Thus, in terms of the existing literature, unit subsidization dominates ad valorem subsidization. To deal with these issues we adopt a general equilibrium approach to the problem.

More specifically, we take a standard general equilibrium model in which a single monopoly sector has been imbedded. In particular we adapt the model of Guesnerie and Laffont [1978] (hereafter GL) to pose this question. We allow for private ownership of the monopoly firm and the competitive firm in the model. In addition, the government distributes its tax revenues by means of a demogrant which can be positive or negative.[2] The problem raised by private ownership is that the monopolist's profit under an ad valorem tax is not equal to its profit from an *equivalent* unit-tax for the same monopoly output level.[3] However, the sum of government revenue and monopoly profits does not change in the move to the equivalent unit-tax. This means that in the case of private ownership, for fixed profit shares and when the number of consumers is more than one, the incomes of the consumers change when moving from a unit-tax equilibrium to an equivalent ad valorem-tax equilibrium; hence, in general, a given unit-tax equilibrium is not an ad valorem-tax equilibrium of the same private ownership economy. Thus, there is no direct way to compare the set of unit-tax equilibria with the set of ad valorem-tax equilibria for a given private ownership economy. This remains true even in the special case where all consumers have quasi-linear preferences.

To see this suppose that all consumers have quasi-linear preferences that are linear in the monopoly good. Then, the demands for all competitive goods are independent of income and the demand for the monopoly good depends only upon aggregate income and not upon its distribution. Now, for a private ownership economy defined by a given allocation of profit shares, consider moving from a unit-tax equilibrium to an equivalent ad valorem one. Although the sum of government revenue and monopoly profits remains constant in this move, each consumer's income changes in two ways: first there is the direct change via this consumer's share in the monopoly profit and second there is the change in that consumer's demogrant. The latter implies if government revenue goes down,

[1] In fact, when personalized lump-sum transfers are permitted, the optimal tax on the monopoly good is always negative. See Guesnerie and Laffont [1978].

[2] A demogrant is a uniform lump-sum tranfer. For notational simplicity we consider the case of zero profit taxation; our result will hold however for any fixed level of profit taxation. In an earlier paper, Blackorby and Murty [2007], we studied the limiting case of 100 per cent profit taxation, which becomes a special case of the current model, when profit shares are equal across all consumers. In that paper we showed that the sets of unit-tax and of ad valorem-tax Pareto optima were the same.

[3] That is, the unit tax rate that can support the profit maximizing output of the monopolist under the ad valorem tax.

for example, then each consumer's income will decline by $1/H$ (H being the number of consumers) of this amount. In general these two change will not offset each other and every consumer's income changes when moving from the unit-tax equilibrium to the ad valorem one. These income changes do not affect the demand for competitive (the non monopoly) commodities and they do not change the aggregate demand for the monopoly good. Thus equilibrium prices and aggregate equilibrium quantities are the same under the two regimes. However, the consumption by each individual of the monopoly good is different in the two regimes (because its demand depends upon each consumer's income) and the hence the utilities experienced by the consumer in the two regimes are different. It is therefore impossible to make a direct comparison in the sense of Pareto of the unit-tax and ad valorem-tax equilibria. For example, if the original unit-tax equilibrium was Pareto optimal, there is no way of knowing directly if the resulting ad valorem-tax equilibrium is also a Pareto optimum—we only know that it is usually different.

In order to be able to make a comparison of the two tax regimes we proceed in an indirect manner which ultimately yields results. Consider the move from a unit-taxation to ad valorem taxation as the reverse is more or less the same. At every unit-tax equilibrium of a given private ownership economy, that is, for a given allocation of profit shares, there exist equivalent ad valorem tax rates which lead to same production decisions. However, as discussed above, under these ad valorem taxes, the given allocation of profit shares results in different distributions of consumer incomes and hence different consumption decisions. This lack of coordination between the production and consumption decisions (on account of maintaining the rigid income distribution rule) as we move from unit to equivalent ad valorem taxes motivates the use of a strategy followed by Guesnerie [1980] and explicated in Quinzii [1992] in another context: proving the existence of an efficient marginal cost pricing equilibrium in a non-convex economy with a given income distribution rule.

In our context, we proceed in the following manner. First, for each private ownership economy, that is, for each possible allocation of shares to the consumers, we construct the unit-tax utility possibility frontier—the set of all possible unit-tax Pareto optima given those fixed shares in the profits. Next we construct the outer envelope of these utility possibility frontiers. That is, for each feasible fixed level of utilities for persons 2 through H, we maximize, by choosing the allocation of private shares, the utility of consumer one. Picking a particular fixed set of shares, say $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_H)$, we then search along this unit-tax envelope to see if there is a point on it that is also supported as an equilibrium of $\bar{\theta}$ private-ownership ad valorem economy. Under some regularity conditions we show such a point (a vector of consumers' utilities), say $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_H)$, exists by a fixed-point argument. Since $\bar{u}$ lies on the unit envelope, there exists a share profile, say $\bar{\psi} = (\bar{\psi}_1, \ldots, \bar{\psi}_H)$, such that the Pareto frontier of the corresponding unit-tax economy is tangent to the unit envelope at $\bar{u}$. We show that under our regularity conditions, at $\bar{u}$, the consumer incomes and equilibrium prices and quantities in the ad valorem and unit economies are the same. However, we find that $\bar{\psi}$ is not equal to $\bar{\theta}$ and that $\bar{u}$ will never belong to the utility possibility set of the $\bar{\theta}$ ownership unit economy *unless* the shares in $\bar{\theta}$ were all equal to $1/H$ (and hence equivalent to one hundred per cent profit taxation problem we have already solved) or the optimal tax on the monopolist happened to be equal to zero. In this way, we obtain a point on the Pareto frontier of a $\bar{\theta}$ private-ownership economy with ad valorem taxes which is not present on the Pareto frontier of a $\bar{\theta}$ private-ownership economy with unit taxes, demonstrating that unit taxation does not dominate ad valorem when the monopoly is privately owned. The converse is proved in a similar way.

We begin with a description of the economy and describe both unit-tax and ad valorem-tax equilibria in these two economies. In Section 3, we construct the envelope of the unit-tax utility possibility frontiers and describe how to find an ad valorem-tax equilibrium along this unit-tax envelope. This leads to Theorem 1 which shows that when a point on the unit-tax envelope has both a unit-tax and an ad valorem-tax equilibrium representation, then the private ownership shares must be different in the two economies (unless the original shares were equal to $1/H$). In a similar manner, it is possible to show (by searching for a unit-tax equilibrium along the ad valorem-tax envelope) that, for a given allocation of shares, ad valorem taxation does not dominate unit taxation.[4] Taken together, these results substantiate the claim made above that neither tax system Pareto-dominates the other. Section 4 answers an ancillary question. How do the two second-best envelopes compare to each other and to the first-best utility-possibility frontier? We find that the unit-tax equilibrium allocations on the unit envelope backed by profiles with positive shares for all consumers are, in fact, first-best. The ad valorem representations of these allocations will also hence be first-best and will lie also on the ad valorem envelope provided the supporting share profiles are non-negative.[5] Section 5 concludes. All proofs are contained in Appendix A. In Appendix B, we show that the assumptions made in theorems in the main body of the paper can be justified from economic primitives.

2. Description of the Economy.

Consider an economy where $\mathcal{H}$ is the index set of consumers who are indexed by h. The cardinality of $\mathcal{H}$ is H. There are $N+1$ goods, of which the good indexed by 0 is the monopoly good. The remaining goods are produced by competitive firms.

The aggregate technology of the competitive sector is Y^c,[6] the technology of the monopolist is $Y^0 = \{(y_0, y^m) | y_0 \leq g(y^m)\}$, and the technology of the public sector for producing g units of a public good is $Y^g(g) = \{y^g \in \mathbf{R}^N_+ | \; F(y^g) \geq g\}$. For all $h \in \mathcal{H}$, the *net* consumption set is $X^h \subseteq \mathbf{R}^{N+1}$. The aggregate endowment is denoted by $(\omega_0, \omega) \in \mathbf{R}^{N+1}_{++}$. Suppose this is distributed among consumers as $\langle \omega_0^h, \omega^h \rangle$.[7] For all $h \in \mathcal{H}$, a *net* consumption bundle is denoted by (x_0^h, x^h) (so that the gross consumption is $(x_0^h + \omega_0^h, x^h + \omega^h)$), and u^h denotes the utility function defined over the *net* consumption

[4] Since the exercise is repetitive, we do not prove the analogue of Theorem 1 for this case.

[5] A similar argument can also be made for the relationship between the ad valorem-tax envelope and the first-best frontier.

[6] Aggregate profit maximization in this sector is consistent with individual profit maximization by many different firms, as we assume away production externalities.

[7] Any H dimensional vector of variables pertaining to all H consumers such as $(u_1, \ldots, u_H)$ is denoted by $\langle u^h \rangle$.

set. The production bundle of the competitive sector is denoted by y^c, of the public sector by y^g, and of the monopolist by (y_0, y^m), where $y^m \in \mathbf{R}^N_+$ is its vector of input demands.

The economy is summarized by $E = (\langle \omega_0^h, \omega^h \rangle, \langle X^h, u^h \rangle, Y^0, Y^c, Y^g)$. An allocation in this economy is denoted by $z = \big(\langle x_0^h, x^h \rangle, y_0, y^m, y^c, y^g\big)$. A private ownership economy is one where the consumers own shares in the profits of both the competitive and monopoly firms. A profile of consumer shares in aggregate profits is given by $\langle \theta_h \rangle \in \Delta_{H-1}$.[8] The consumer price of the monopoly good is $q_0 \in \mathbf{R}_{++}$, $q \in \mathbf{R}^N_{++}$ is the vector of consumer prices of the competitively supplied goods. The wealth of consumer h is given by w_h. The producer price of the monopoly good is $p_0 \in \mathbf{R}_{++}$, $p \in \mathbf{R}^N_+$ is the vector of producer prices of the competitively supplied goods. The individual and aggregate consumer demands for the monopoly good are given by

$$x_0(q_0, q, \langle w_h \rangle) = \sum_h x_0^h(q_0, q, w_h), \tag{2.1}$$

and the individual and aggregate consumer demand vectors for the competitively supplied commodities are given by

$$x(q_0, q, \langle w_h \rangle) = \sum_h x^h(q_0, q, w_h). \tag{2.2}$$

The indirect utility function of consumer h is denoted by $V^h(q_0, q, w^h)$.[9] We assume that the monopolist is naive, in the sense that it does not take into account the effect of its decision on consumer incomes.[10] Its cost and input demand functions are denoted by $C(y_0, p)$ and $y^m(y_0, p)$, respectively. The aggregate competitive profit and supply functions are denoted by $\Pi^c(p)$ and $y^c(p)$, respectively. We use the following general assumptions on preferences and technologies in our analysis.

Assumption 1: For all $h \in \mathcal{H}$, the gross consumption set is $X^h + \{(\omega_0^h, \omega)\} = \mathbf{R}^{N+1}_+$, the utility function u_h is increasing, strictly quasi-concave, and twice continuously differentiable in the interior of its domain X^h. This, in turn, implies that the indirect utility function V^h is twice continuously differentiable.[11] We also assume that the demand functions $(x_0^h(), x^h())$ are twice continuously differentiable on the interior of their domain.

Assumption 2: The technologies Y^0, Y^c, and $Y^g(g)$ are closed, convex, satisfy free disposability, and contain the origin. The public good production function F is strictly concave and twice continuously differentiable on the interior of its domain.

8 Δ_{H-1} is the $H-1$-dimensional unit simplex. Assuming that consumers have the same shares of monopoly and competitive sectors' profits makes the following analysis simpler without any loss of generality.

9 There is also a public good g but, as it remains constant throughout the analysis, it is suppressed in the utility function.

10 Likewise we assume that consumers are naive; they do not anticipate changes in theirs incomes due to change in the profits of the monopolist.

11 See Blackorby and Diewert [1979].

Assumption 3: The profit function of the competitive sector, Π^c, is assumed to be differentially strongly convex and the cost function $C(y_0, p)$ of the monopolist is assumed to be differentially strongly concave in prices and increasing and convex in output.[12] The competitive supply $y^c(p)$ is given by Hotelling's Lemma as $\nabla_p \Pi^c(p)$ and the input demands of the monopolist are given by $y^m(y_0, p) = \nabla_p C(y_0, p)$. The marginal cost $\nabla_{y_0} C(y_0, p)$ is positive on the interior of the domain of C.

2.1. A Unit-Tax Private-Ownership Equilibrium.

The monopolist's optimization problem, when facing a unit tax $t_0 \in \mathbf{R}$ and when the vector of unit taxes on the competitive goods is $t \in \mathbf{R}^N$, is

$$P_0^u\left(p, t_0, t, \langle w_h \rangle\right) := \text{argmax}_{p_0^u} \left\{ p_0^u \cdot x_0\left(p_0^u + t_0, p + t, \langle w_h \rangle\right) - C\left(x_0(p_0^u + t_0, p + t, \langle w_h \rangle), p\right) \right\}. \tag{2.3}$$

As discussed in detail in GL the profit function of the monopolist (the function over which it optimizes) is not in general concave. Following them we assume that the solution to monopolist's profit maximization problem is locally unique and smooth. Under assumptions 1, 2, and 3, the first-order condition for this problem is

$$\nabla_{q_0} x_0\left(p_0^u + t_0, p + t, \langle w_h \rangle\right) \ \left[p_0^u - \nabla_{y_0} C(y_0, p)\right] + x_0(p_0^u + t_0, p + t, \langle w_h \rangle) = 0 \tag{2.4}$$

which implicitly defines the solution $p_0^u = P_0^u(p, t_0, t, \langle w_h \rangle)$.

Assumption 4: P_0^u is single-valued and twice continuously differentiable function such that

$$\nabla_{t_0} P_0^u(p, t_0, t, \langle w_h \rangle) \neq -1. \tag{2.5}$$

As discussed in GL, $\nabla_{t_0} P_0^u \neq -1$ implies that the monopolist cannot undo all changes by the tax authority of t_0. Since consumer demands are homogeneous of degree zero in consumer prices and incomes, $\nabla_{q_0} x_0$ is homogeneous of degree minus one in these variables. Also, the cost function C is homogeneous of degree one in p. Hence, it follows that the left side of (2.4) is homogeneous of degree zero in p_0^u, p, t_0, t, and $\langle w_h \rangle$. This implies that the function $P_0^u(p, t_0, t, \langle w_h \rangle)$ is homogeneous of degree one in p, t_0, t, and $\langle w_h \rangle$.

A unit-tax equilibrium in private-ownership economy with shares $\langle \theta_h \rangle \in \Delta_{H-1}$ is given by[13]

$$-x\left(q_0, q, \langle w_h \rangle\right) + y^c(p) - y^m(y_0^u, p) - y^g \geq 0, \tag{2.6}$$

$$-x_0\left(q_0, q, \langle w_h \rangle\right) + y_0^u \geq 0, \tag{2.7}$$

$$p_0^u - P_0^u\left(p, t_0, t, \langle w_h \rangle\right) = 0, \tag{2.8}$$

$$w_h = \theta_h \left[p_0^u y_0^u - C(y_0^u, p) + \Pi^c(p)\right] + \frac{1}{H}\left[t^T(y^c - y^m - y^g) + t_0 y_0^u - p y^g\right], \ \forall h \in \mathcal{H} \tag{2.9}$$

and

$$F(y^g) - g \geq 0, \ p_0^u \geq 0, \ p \geq 0_N, \ q_0 = p_0^u + t_0 \geq 0, q = p + t \geq 0_N. \tag{2.10}$$

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[12] See Avriel, Diewert, Schaible, and Zang [1988].

[13] Lemmas B1 and B2 in the appendix demonstrate that the set of unit-tax equilibria is generically a $2N-1$-dimensional manifold.

2.2. An Ad Valorem-Tax Private-Ownership Equilibrium.

The monopolist's profit maximization problem, when confronted with ad valorem taxes (τ_0, τ) is[14]

$$\begin{aligned} &P_0^a(p, \tau_0, \tau, \langle R_h \rangle) := \\ \operatorname{argmax}_{p_0^a} &\Big\{ p_0^a x_0 \Big(p_0^a(1+\tau_0), p^T(I_N + \boldsymbol{\tau}), \langle R_h \rangle \Big) \\ &- C\Big(x_0(p_0^a(1+\tau_0), p^T(I_N + \boldsymbol{\tau}), \langle R_h \rangle), p \Big) \Big\}, \end{aligned} \tag{2.11}$$

Assumption 5: P_0^a is single valued and twice continuously differentiable function such that $(1+\tau_0)\nabla_{\tau_0} P_0^a \neq -P_0^a$. This assumption reflects that the monopolist can not fully undo the effect of the tax set by the government and is in fact implied by Assumption 4.

A monopoly ad-valorem tax equilibrium in a private ownership economy with shares $\langle \theta_h \rangle \in \Delta_{H-1}$ satisfies

$$-x\left(q, q_0, \langle R_h \rangle\right) + y^c(p) - y^m(p, y_0^a) - y^g \geq 0, \tag{2.12}$$

$$-x_0\left(q_0, q, \langle R_h \rangle\right) + y_0^a \geq 0, \tag{2.13}$$

$$p_0^a = P_0^a\left(p, \tau_0, \tau, \langle R_h \rangle\right), \tag{2.14}$$

$$R_h = \theta_h \left[p_0^a y_0^a - C(y_0^a, p) + \Pi^c(p) \right] + \frac{1}{H} \Big[\tau_0 p_0^a y_0^a + p^T \boldsymbol{\tau} \left[y^c - y^g - y^m \right] - p^T y^g \Big] \ \ \forall h \in \mathcal{H}, \tag{2.15}$$

$$F(y^g) - g \geq 0, \tag{2.16}$$

and

$$p_0^a \geq 0, \ p \geq 0_N, \ q_0 = p_0^a(1+\tau_0) \geq 0, \ q = (I_N + \boldsymbol{\tau})p \geq 0_N. \tag{2.17}$$

As in the unit-tax case, it can be shown that the function P_0^a is homogeneous of degree one in its arguments.

3. Unit Versus Ad Valorem Taxes In Private Ownership Economies.

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[14] Symbols in bold face such as $\boldsymbol{\tau}$ and $\mathbf{p}$ stand for diagonal matrices with diagonal elements being the elements of vectors τ and p, respectively.

3.1. An Ad Valorem-Tax Private-Ownership Equilibrium on the Envelope of Unit-Tax Utility Possibility Frontiers

For each possible profile of profit shares, $\langle\theta_h\rangle \in \Delta_{H-1}$, we obtain a unit-tax Pareto frontier by solving the following problem for all utility profiles $(u_2, \ldots, u_H)$ for which solution exists,

$$\begin{aligned}
\mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle) := & \max_{p_0^u, p, t_0, t, \langle w_h\rangle} V^1(p_0^u + t_0, p + t, w_1) \\
& \text{subject to} \\
& V^h(p_0^u + t_0, p + t, w_h) \geq u_h, \text{ for } h = 2, \ldots, H, \\
& -x\left(p_0^u + t_0, p + t, \langle w_h\rangle\right) + y^c(p) - y^m(y_0^u, p) - y^g \geq 0, \\
& -x_0\left(p_0^u + t_0, p + t, \langle w_h\rangle\right) + y_0^u \geq 0, \\
& p_0^u - P_0^u\left(p, t_0, t, \langle w_h\rangle\right) = 0, \\
& w_h = \theta_h\left[p_0^u y_0^u - C(y_0^u, p) + \Pi^c(p)\right] + \frac{1}{H}\left[t^T(y^c - y^m - y^g) + t_0 y_0 - p y^g\right] \forall h \in \mathcal{H}, \\
& \text{and} \\
& F(y^g) - g \geq 0.
\end{aligned} \tag{3.1}$$

The envelope for the Pareto manifolds of all possible private ownership economies with unit taxes is obtained by solving the following problem for all utility profiles $(u_2, \ldots, u_H)$ for which solutions exist:

$$\begin{aligned}
\hat{\mathcal{U}}^u(u_2, \ldots, u_H) := & \max_{\langle\theta_h\rangle} \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle) \\
& \text{subject to} \\
& \sum_h \theta_h = 1, \\
& \theta_h \in [0, 1], h \in \mathcal{H}.
\end{aligned} \tag{3.2}$$

Suppose the solution to this problem is given by

$$\langle \overset{*}{\theta}{}_h^u \rangle = \langle \overset{*}{\theta}{}_h^u(u_2, ..., u_h) \rangle. \tag{3.3}$$

That is, for given utility levels, $(u_2, \ldots, u_H)$, $\langle \overset{*}{\theta}{}_h^u \rangle$ is the vector of shares that maximizes the utility of consumer 1.

We now generate an algorithm that picks out the ad valorem tax-equilibria that lie on the envelope of all unit-tax utility possibility frontiers corresponding to different private ownership economies.

Let A^u be the set of all allocations corresponding to the utility profiles on the unit envelope $\hat{\mathcal{U}}^u(u_2, ..., u_h)$. Define mappings $\langle x_0^h(z), x^h(z)\rangle, y_0(z), y^m(z), y^c(z)$, and $y^g(z)$, which, for every $z = \left(\langle x_0^h, x^h\rangle, y_0, y^m, y^c, y^g\right) \in A^u$, assign $\langle x_0^h(z), x^h(z)\rangle = \langle x_0^h, x^h\rangle$, $y^c(z) = y^c, y^g(z) = y^g, y_0(z) = y_0$, and $y^m(z) = y^m$.

Let $\rho^u : A^u \to \mathbf{R}^H$ with image $\rho^u(z) = \langle u^h(x_0^h(z), x^h(z))\rangle$ be a utility map of the allocations in A^u. That is, for every $z \in A^u$, the set of utility levels enjoyed by consumers at that allocation is $\rho^u(z)$. With some abuse of notation, let $\theta_h^u(z) = \overset{*}{\theta}{}_h^u(\rho^u(z))$ for all $h \in \mathcal{H}$ be the solution of the problem (3.2) at the allocation z.

To apply the strategy outlined in the introduction, which is based on a fixed point argument, we need to restrict all prices and taxes to a compact and convex set. A natural way to do so is to adopt a price normalization rule, which the equilibrium system allows as it is homogeneous of degree zero in the variables.[15] Let b be such a normalization rule such that the set

$$\mathcal{S}_b^u := \{(p_0, t_0, p, t) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}_+^N \times \mathbf{R}^N |\ \ b(p_0, t_0, p, t, \langle w_h\rangle) = 0 \text{ for any } \langle w_h\rangle\ \} \quad (3.4)$$

is compact. Let $\langle \bar{p}_0^u, \bar{t}_0, \bar{p}, \bar{t}\rangle$ and $(\underline{p}_0^u, \underline{t}_0, \underline{p}, \underline{t})$ be the vectors of maximum and minimum values attained by p_0^u, t_0, p, and t in $\mathcal{S}_b^u$. For example, $\bar{p}_0^u$ solves

$$\max\ \{p_0^u \in \mathbf{R}_+ |\ \exists t_0, p, t, \langle w_h\rangle \text{ such that } b(p_0^u, p, t_0, t, \langle w_h\rangle) = 0\}. \quad (3.5)$$

Define the mapping $\psi^u : A^u \to \mathcal{S}_b^u$ as $\psi^u(z) = (\psi_p^u(z), \psi_w^u(z))$, where $\psi_p^u(z) = (p_0^u(z), t_0(z), p(z), t(z))$ is the vector of unit taxes and producer prices associated with allocation z (a unit tax equilibrium), while $\psi_w^u(z) = \langle w^h(z)\rangle$ is the profile of consumer incomes associated with allocation z.

For every $z \in A^u$, define $q_0(z) = p_0^u(z) + t_0(z)$ and $q(z) = p(z) + t(z)$. Since $\mathcal{S}_b^u$ is compact, there exist $(\bar{q}_0, \bar{q}\rangle$ and $(\underline{q}_0, \underline{q})$ which denote the vector of maximum and minimum possible consumer prices that can be attained under the adopted price normalization rule.

For all $z \in A^u$ we can separate $q(z)$ and $q_0(z)$ into ad valorem taxes and producer prices defined by functions $(\tau_0(z), \tau(z))$ and $(p_0^a(z), p(z))$, which ensure that $(y_0(z), y(z))$ and $(p_0^a(z), p(z))$ are the profit maximizing outputs and prices in the monopoly and the competitive sector when the ad valorem taxes are $(\tau_0(z), \tau(z))$, that is, $(\tau_0(z), \tau(z))$ and $(p_0^a(z), p(z))$ solve

$$\begin{aligned} q_0(z) &= p_0^a(z)(1 + \tau_0(z)) \text{ and} \\ p_0^a(z) = P_0^a &\left(\tau_0(z), p(z), t(z), \langle w^h(z)\rangle\right) > 0 \\ q(z) &= (\boldsymbol{\tau}(z) + I_N)p(z). \end{aligned} \quad (3.6)$$

This implies, from arguments such as Suits and Musgrave, that for every $z \in A^u$, if[16]

$$\frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))} > -1 \quad (3.7)$$

[15] The rationale for this including a discussion of valid normalization rules and a proof that their choice does not affect the solution (Lemma B3) is in the appendix.

[16] See also Blackorby and Murty.

then

$$\tau_0(z) = \frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))}, \tag{3.8}$$

$$\tau(z) = \mathbf{p}(z)^{-1} t(z), \tag{3.9}$$

and using (3.8), we have[17]

$$p_0^a(z) = \frac{q_0(z)}{1+\tau_0(z)} = \frac{q_0(z)\ \nabla_{y_0} C(y_0(z), p(z))}{\nabla_{y_0} C(y_0(z), p(z))\ +\ t_0(z)} > 0. \tag{3.10}$$

Given (i) an appropriate normalization rule and (ii) the fact that for a monopolist $p_0^u(z) \geq \nabla_{y_0} C(y_0(z), p(z))$, we have for every $z \in A^u$

$$p(z) \in [\underline{p}, \bar{p}],\ p_0^u(z) \in [\underline{p}_0^u, \bar{p}_0^u],\ t(z) \in [\underline{t}, \bar{t}],\ t_0(z) \in [\underline{t}_0, \bar{t}_0],\ q_0(z) \in [\underline{q}_0, \bar{q}_0],\ q(z) \in [\underline{q}, \bar{q}], \tag{3.11}$$

and[18]

$$\nabla_{y_0} C(y_0(z), p(z)) \in [\underline{p}_0^u, \bar{p}_0^u,]. \tag{3.12}$$

Further, assuming that $\tau_0(z)$ and $p_0^a(z)$ are continuous functions, (3.8)–(3.12) imply that there exist compact intervals $[\underline{\tau}_0, \bar{\tau}_0]$ and $[\underline{p}_0^a, \bar{p}_0^a]$ such that for every $z \in A^u$, we have

$$\tau_0(z) \in [\underline{\tau}_0, \bar{\tau}_0], \tag{3.13}$$

$$\tau(z) \in [\underline{\tau}, \bar{\tau}] \tag{3.14}$$

and

$$p_0^a(z) \in \left[\underline{p}_0^a, \bar{p}_0^a\right]. \tag{3.15}$$

In that case, if we define

$$S_b^u = \left[\underline{p}_0^a, \bar{p}_0^a\right] \times [\underline{\tau}_0, \bar{\tau}_0] \times \left[\underline{p}, \bar{p}\right] \times [\underline{\tau}, \bar{\tau}], \tag{3.16}$$

then for every $z \in A^u$, we have $(p_0(z), \tau_0(z), p(z), \tau(z)) \in S_b^u$, which is a compact and convex set.

For each allocation $z \in A^u$ we need to be able to identify the incomes of the consumers. Define an income map for consumer h as the map $r^{uh} : A^u \times \mathcal{S}_g \times [0,1] \to \mathbf{R}$, which for every $z \in A^u$, $\pi = (p_0^u, t_0, p, t) \in \mathcal{S}_b^u$, and $\theta_h \in [0,1]$ has image

$$r^{uh}(z, \pi, \theta_h) = \theta_h \left[p_0^u y_0(z) - C(y_0(z), p) + \Pi^c(p)\right] + \\ \frac{1}{H}\left[t_0 y_0(z) + t^T[y^c(z) - y^m(z) - y^g(z))] - p^T y^g(z)\right] + (p_0^u + t_0)\omega_0^h + (p^T + t^T)\omega^h \tag{3.17}$$

[17] If (3.7) is not satisfied then there is no ad valorem tax that that yields the same profit-maximizing output as the given unit tax $t_0(z)$, for as seen below, violation of (3.7) would imply that $p_0^a(z)$ is either less than zero or does not exist.

[18] Note normalization rules such as the unit hemisphere will ensure the restriction for $\nabla_{y_0} C(y_0(z), p(z))$, below, as such a normalization implies $\nabla_{y_0} C \geq 0 = \underline{p}_0^u$=0.

and so

$$\begin{aligned}\sum_h r^{uh}\left(z,\psi_p^u(z),\theta_h^u(z)\right) &= \Big[p_0^u(z)y_0(z) - p(z)^T y^m(z) + p^T(z)y^c(z)\Big]\sum_h \theta_h^u(z)\\ &+ \Big[t_0(z)y_0(z) + t^T(z)[y^c(z) - y^m(z) - y^g(z))] - p^T(z)y^g(z)\Big] + q_0(z)\omega_0 + q^T(z)\omega\\ &= q_0(z)y_0(z) - q(z)^T y^m(z) + q(z)^T y^c(z) - q^T(z)y^g(z) + q_0(z)\omega_0 + q^T(z)\omega\end{aligned} \tag{3.18}$$

The last equality in (3.18) is obtained by noting that $q_0(z) = p_0^u(z) + t_0(z)$ and $q(z) = p(z) + t(z)$.

Define the mapping $\psi_p^a(z) := (p_0^a(z), \tau_0(z), p(z), \tau(z))$. ψ_p^a identifies the ad valorem taxes and prices associated with an allocation $z \in A^u$ that results in the same output decisions as in the unit tax equilibrium. For all $h \in \mathcal{H}$ let $r^{ah} : A^u \times S_b^u \times [0,1] \to \mathbf{R}$ be defined so that

$$\begin{aligned}r^{ah}\left(z,\psi_p^a(z),\theta_h\right) &= \theta_h\Big[p_0^a(z)y_0(z) - p(z)^T y^m(z) + p^T(z)y^c(z)\Big]\\ &+ \frac{1}{H}\Big[\tau_0(z)p_0^a(z)y_0(z) + \tau^T(z)\mathbf{p}(z)\left[y^c(z) - y^m(z) - y^g(z))\right] - p^T(z)y^g(z)\Big].\end{aligned} \tag{3.19}$$

The maps $\langle r^{ah}\rangle$ generate the incomes of consumers at any allocation $z \in A^u$ using the equivalent ad valorem price-tax configuration and arbitrary ownership shares $\langle\theta_h\rangle$. Note that since $p_0^a(z)(1+\tau_0(z)) = p_0^u(z)+t_0(z) = q_0(z)$ and $p^a(z)(1+\tau(z)) = p(z)+t(z) = q(z)$, we have from (3.18) and (3.19)

$$\begin{aligned}\sum_h r^{ah}\left(z,\psi_p^a(z),\theta_h\right) &= \Big[p_0^a(z)y_0(z) - p(z)^T y^m(z) + p^T(z)y^c(z)\Big]\sum_h \theta_h\\ &+ \Big[\tau_0(z)p_0^a(z)y_0(z) + \tau^T(z)\mathbf{p}(z)\left[y^c(z) - y^m(z) - y^g(z))\right] - p^T(z)y^g(z)\Big]\\ &+ q_0(z)\omega_0 + q^T(z)\omega\\ = q_0(z)y_0(z) &- q^T(z)y^m(z) + q^T(z)y^c(z) - q^T(z)y^g(z) + q_0(z)\omega_0 + q^T(z)\omega\\ &= \sum_h r^{uh}\left(z,\psi_p^u(z),\theta_h\right) = \sum_h w^h(z).\end{aligned} \tag{3.20}$$

This demonstrates that the aggregate income at allocation z under unit-taxation and income rule $\langle\theta_h^u(z)\rangle$ is the same as the aggregate income at z with equivalent ad valorem taxes and any arbitrary income rule $\langle\theta_h\rangle$.

Next, for every $h \in \mathcal{H}$ define

$$\underline{r}^{hu} =: \min_{z\in A^u}\ r^{uh}\left(z,\psi_p^u(z),\theta_h^u(z)\right) \tag{3.21}$$

and let $\underline{z}_h^u \in A^u$ be the solution to (3.21). This is the allocation on A^u which yields the least income to h. The consumption bundle associated with it for consumer h is $(x_0^h(\underline{z}_h^u), x^h(\underline{z}_h^u)) =: (\underline{x}_0^h, \underline{x}^h)$.

Assumption 6: For $h \in \mathcal{H}$, $\underline{z}_h^u$ is unique, $u_h(x_0^h(\underline{z}_h^u), x^h(\underline{z}_h^u)) \leq u_h(x_0^h(z), x^h(z))$ for all $z \in A^u$, and $\underline{r}^{hu} \leq r^{ah}(z, \psi_p^u(z), \theta_h)$ for all $\theta_h \in [0,1]$.

This assumption implies, that, for every h, $\underline{z}_h^u$ is the allocation on A^u which is the worst unit-tax equilibrium for h.

The following theorem proves the existence of an ad valorem equilibrium of a given private ownership economy on the envelope of the Pareto frontiers of all private ownership unit economies.[19]

Theorem 1: *Let $E = (\langle (X^h, u^h)\rangle, Y^0, Y^c, Y^g, \langle(\omega_0^h, \omega^h)\rangle)$ be an economy. Fix the profit shares as $\langle\theta_h\rangle$, renormalize utility functions $\langle u_h\rangle$ such that $u_h(x_0^h(\underline{z}_h^u), x^h(\underline{z}_h^u)) = 0$ for all h, and suppose the following are true*[20]*:*

(i) assumptions 1 through 4 and 6 hold;

(ii) the mapping $\rho^u : A^u \to \rho^u(A^u)$ is bijective;

(iii) b is a normalization rule such that $\mathcal{S}_b^u$ is compact, and the mapping $\psi_p^a : A^u \to S_b^u$ is a continuous function;

(iv) A^u is compact and $\rho^u(A^u)$ is a $H-1$ dimensional manifold;

(v) for every $z \in A^u$

$$\frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))} > -1; \tag{3.22}$$

(vi) the mapping $\overset{*}{\theta}{}^u : \rho^u(A^u) \to \Delta_{H-1}$ is a function.

Then

(a) there exists a $\overset{*}{z} \in A^u$ such that $r^{uh}(\overset{*}{z}, \psi_p^u(\overset{*}{z}), \theta_h^u(\overset{*}{z})) = r^{ah}(\overset{*}{z}, \psi_p^a(\overset{*}{z}), \theta_h)$ for $h \in \mathcal{H}$;

(b) $\overset{*}{z}$ is also an allocation underlying an ad valorem tax equilibrium of the private ownership economy with shares $\langle\theta_h\rangle$;

(c) $\theta_h^u(\overset{*}{z}) = \theta_h$ for all $h \in \mathcal{H}$ if and only if $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$;

(d) $\rho^u(\overset{*}{z}) \in U^u(\langle\theta_h\rangle) := \{\langle u_h\rangle \in \mathbf{R}^H | u_1 \leq \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle)\}$ if and only if $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$.

Proof: See appendix A.

Conclusions (a) and (b) of the above theorem imply that given a ownership profile $\langle\theta_h\rangle$ there exists an allocation $\overset{*}{z}$ such that $\rho^u(\overset{*}{z})$ lies on the unit envelope and $\overset{*}{z}$ is also supported as an equilibrium of the ad valorem $\langle\theta_h\rangle$ economy. (c) says that unless $\theta_h = \frac{1}{H}$ for all h, the private ownership unit economy that is tangent to the unit envelope at $\overset{*}{u} = \rho^u(\overset{*}{z})$, is not the same as the $\langle\theta_h\rangle$ unit economy. (d) says that unless $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$, the utility imputation $\overset{*}{u}$ will never belong to the utility possibility frontier corresponding to the $\langle\theta_h\rangle$ unit economy. All these conclusions imply that (unless $\theta_h = \frac{1}{H}$

[19] The proof is motivated by the works of Guesnerie [1980] and Quinzii [1992] on non-convex economies. The current strategy is similar to proving the existence of an efficient marginal cost pricing equilibrium in a non-convex economy with a given income-distribution map.

[20] The importance of Assumption 6 and Assumptions (ii) and (iv) for proving Theorem 1 is discussed at the end of its proof.

for all $h \in \mathcal{H}$) though $\overset{*}{u}$ belongs to the utility possibility set corresponding to the $\langle\theta_h\rangle$ *ad valorem* economy, it will not belong to the utility possibility set corresponding to the $\langle\theta_h\rangle$ unit economy.

Note that Assumption 6 is sufficient to ensure against the situation where, given a $\theta_h \in [0,1]$, we have $\underline{r}^{hu} > r^{ah}(z, \psi_p^u(z), \theta_h)$ for all $z \in A^u$. In such a case, no unit-tax equilibrium in A^u can be expressed as an ad valorem-tax equilibrium of a private ownership economy where the share of h is θ_h. Lemmas B5 to B8 in the appendix show that assumption (ii) and latter part of assumption (iv) will hold if the solution mappings to the problems (3.2) and (3.1) are functions. The absence of these assumptions may create discontinuities in the mapping ρ^u, which creates problem for applying the Kakutani's fixed point theorem.[21] In addition, it may imply that, at the fixed point, say $\overset{*}{u}$, corresponding to $\langle\theta_h\rangle$ share profile, the unit envelope is tangent to Pareto frontiers of two or more private ownership economies, one of which, say $\langle\psi_h\rangle$ will result in $\overset{*}{u}$ being attainable in the $\langle\theta_h\rangle$ ownership ad valorem economy. However, there may also be a tangency between the envelope and the Pareto frontier of a $\langle\theta_h\rangle$ ownership unit-economy at that point, in which case we cannot conclusively prove that the utility possibility set of the ad valorem economy, $U^a(\langle\theta_h\rangle)$, is not a subset of $U^u(\langle\theta_h\rangle)$.

3.2. A Unit-Tax Private-Ownership Equilibrium on the Envelope of Ad Valorem-Tax Utility Possibility Frontiers.

Arguments for proving that, for any private ownership economy, the ad valorem utility possibility set is not a subset of the unit utility possibility set, are similar to the ones in the previous section. The Pareto manifold for a private ownership economy with ad valorem taxes can be derived in a manner similar to (3.1). We will denoted its image by $u_1 = \mathcal{U}^a(u_2, \ldots, u_H, \langle\theta_h\rangle)$ for shares $\langle\theta_h\rangle \in \Delta_{H-1}$. An envelope for all Pareto manifolds of private ownership economies with ad valorem taxes is obtained by solving the following problem, where we choose the shares $\langle\theta_h\rangle \in \Delta_{H-1}$ to solve

$$\begin{gathered} \hat{\mathcal{U}}^a(u_2, \ldots, u_H) := \max_{\langle\theta_h\rangle} \mathcal{U}^a(u_2, \ldots, u_H, \langle\theta_h\rangle) \\ \text{subject to} \\ \sum_h \theta_h = 1 \\ \theta_h \in [0,1], \forall h. \end{gathered} \tag{3.23}$$

We denote the solution to this problem by

$$\langle\overset{*}{\theta}{}^a_h\rangle = \langle\overset{*}{\theta}{}^a_h(u_2, ..., u_h)\rangle. \tag{3.24}$$

Under assumptions analogous to the ones in the previous subsection, a theorem analogous to Theorem 1 can be proved to show that, for every allocation of shares $\langle\theta_h\rangle \in \Delta_{H-1}$, there exists a unit-tax equilibrium of a $\langle\theta_h\rangle$ ownership economy on the ad valorem-tax envelope.

[21] See Quinzii, p. 50.

4. Unit-Tax Versus Ad Valorem-Tax Envelopes.

In the previous sections, we used the unit and the ad valorem envelopes to compare the utility possibility sets of an arbitrarily given private ownership economy with unit and ad valorem taxes. We established conditions under which neither Pareto dominates the other. In this section, we compare the unit and ad valorem envelopes themselves. A precurser to this study is an understanding of the relation between these envelopes and the first-best Pareto frontier.

GL established that any first-best allocation can be decentralized as an equilibrium of an economy with a monopoly, unit taxation, and *personalized* lump-sum transfers. The optimum tax on the monopoly will be a subsidy, the tax on the competitive goods will be zero (under suitable price normalization), and the personalized lump-sum transfer to any consumer is the value of his consumption bundle at the existing shadow prices. Note, that it is possible to find a profile of profit shares such that the personalized income to each consumer at the given allocation can be expressed as a sum of his profit and endowment incomes and a demogrant. This means that it is possible to decentralize a first-best as a unit-tax equilibrium of some private ownership economy. Such profit share profiles will vary from allocation to allocation on the first best frontier, and in general, there may exist share profiles where some of the shares are negative.[22] Intuitively, the first-best is an envelope of Pareto frontiers of unit-tax economies corresponding to all possible private ownership economies (including economies where some profit shares could well be negative).[23] This implies that if a first-best allocation can be decentralized as a unit-tax equilibrium of a private ownership economy with a non-negative share profile, then it must also lie on the unit envelope. In other words, points on the unit envelope where the inequality constraints in problem (3.2) are non-binding are first-best. Theorem 2 below formalizes this intuition. An exactly similar argument holds for ad valorem taxation and the relation between the ad valorem envelope and the first-best Pareto frontier.

An understanding of the relation of the first-best frontier to the unit and ad valorem envelopes is helpful for understanding where on the unit (ad valorem) envelope, does the fixed point in Theorem 1 (or its analogue using the ad valorem envelope) occur for each share profile $\langle\theta_h\rangle \in \Delta_{H-1}$. Theorems 3 and 4 address this issue. Together, Theorems 2 to 4 allow us to make some conjectures on the position of the ad valorem envelope viz-a-viz the unit envelope. In general, we find that there are regions of tangency between the two (these occur at some first-best points or at points where the monopoly tax is zero), but neither is a subset of the other. This establishes the fact that even in the bigger set of

[22] This may be true, for example, when at some first-best allocation, the value of the consumption bundle of some consumer at the existing shadow prices is smaller than the sum of the demogrant and his endowment income.

[23] It can be shown that the first-best Pareto frontier is a solution to the following problem:

$$\max_{\langle\theta_h\rangle} \mathcal{U}^u(u_2,\ldots,u_H,\langle\theta_h\rangle) \quad \text{subject to} \quad \sum_h \theta_h = 1. \tag{4.1}$$

tax equilibria of all possible private ownership economies with non-negative profit shares, neither unit taxation nor ad valorem taxation dominates the other.

4.1. Relation Between the First-Best Frontier and the Unit-Tax Envelope.

Using (3.1), the programme (3.2) can be rewritten as

$$\begin{aligned}
\hat{\mathcal{U}}^u(u_2,\ldots,u_H) := \max_{p_0^u,p,t_0,t,\langle w_h\rangle,\langle\theta_h\rangle} & V^1(p_0^u+t_0,p+t,w_1) \\
& \text{subject to} \\
& V^h(p_0^u+t_0,p+t,w_h)\ge u_h,\ \forall h=2,\ldots,H, \\
& -x(p_0^u+t_0,p+t,\langle w_h\rangle)+y^c(p)-y^m(p,y_0^u)-y^g\ge 0 \\
& -x_0(p_0^u+t_0,p+t,\langle w_h\rangle)+y_0^u\ge 0 \\
& p_0^u-P_0^u(p,t_0,t,\langle w_h\rangle)=0, \\
w_h=\theta_h[p_0^uy_0^u-C(y_0^u,p)+\Pi^c(p)]+\frac{1}{H}\ & [t^T(y^c-y^m-y^g)+t_0y_0-py^g\ ],\ \forall h\in\mathcal{H}, \\
& F(y^g)-g\ge 0, \\
& \sum_h\theta_h=1,\quad\text{and} \\
& \theta_h\in[0,1],\Leftrightarrow\theta_h\ge 0 \text{ and } \theta_h\le 1,\ \forall h\in\mathcal{H}.
\end{aligned} \tag{4.2}$$

Theorem 2: *Given Assumptions 1–4, if, for a utility profile on the unit-tax envelope, $\langle u_h\rangle$, $\langle\overset{*}{\theta}{}_h^u(u_2,\ldots,u_H)\rangle$ is such that $\overset{*}{\theta}{}_h^u(u_2,\ldots,u_H)>0$ for all $h\in\mathcal{H}$, and the $(N+1)\times(N+1)$-dimensional Slutsky matrix of aggregate consumer demands is of rank N, then, $\langle u_h\rangle$ lies on the first-best frontier.*[24]

Proof: See Appendix A.

A similar theorem can be proved for the ad valorem envelope.

4.2. The Difference in Unit and Ad Valorem Demogrants is the Difference in the Unit and Ad Valorem Tax or Subsidy on the Monopolist.

Consider an allocation z that has both a unit equilibrium and an ad valorem equilibrium representation. Suppose the shares that make it a unit equilibrium are $\langle\theta_h^u\rangle\in\Delta_{H-1}$, and the shares that make it an ad valorem equilibrium are $\langle\theta_h^a\rangle\in\Delta_{H-1}$. Letting $M_G^u(z)$ and $M_G^a(z)$ be total government revenue at $z\in A^u$ under the unit and ad valorem representations of z respectively, $\frac{1}{H}M_G^u(z)$ and $\frac{1}{H}M_G^a(z)$ are the demogrants under the two

[24] Recall that solution for the optimal shares for programme (4.2) is denoted by $\langle\theta_h=\overset{*}{\theta}{}_h^u(u_2,\ldots,u_H)\rangle$.

regimes. We calculate the following difference, recalling the Suits and Musgrave relation $\tau_0(z) = \frac{t_0(z)}{\nabla_{y_0} C(z)}$ and (3.7)–(3.10).

$$\begin{aligned} M_G^u(z) - M_G^a(z) =& [t_0(z) - \tau_0(z) p_0^a(z)] y_0(z) \\ =& [t_0(z) - \frac{t_0(z) p_0^a(z)}{\nabla_{y_0} C(z)}] y_0(z) \\ =& [\frac{\nabla_{y_0} C(z) - p_0^a(z)}{\nabla_{y_0} C(z)}] t_0(z) y_0(z). \end{aligned} \tag{4.3}$$

Some remarks follow:
Remark 1: Since, under monopoly, $\nabla_{y_0} C(z) - p_0^a(z) < 0$, from (4.3) it follows that the unit demogrant is bigger than (smaller than, equal to) the ad valorem demogrant iff $t_0(z) < 0$ ($t_0(z) > 0$, $t_0(z) = 0$).
Remark 2 (From GL): If z is a first-best allocation and has both a unit and ad valorem tax representation, then $t_0(z) < 0$, and hence $\tau_0(z) = \frac{t_0(z)}{\nabla_{y_0} C(z)} < 0$.[25]

4.3. The Nature of the Mapping of Ad Valorem Equilibria Onto the Unit Envelope.

Suppose assumptions of Theorem 1 hold for every $\langle \theta_h \rangle \in \Delta_{H-1}$. Let us see where on the unit envelope do the ad valorem equilibria corresponding to each such share profile map into.

Theorem 3: *Suppose $\langle \theta_h \rangle \in \Delta_{H-1}$ and assumptions of Theorem 1 hold. Suppose z is the allocation on the unit envelope, which is supported as the ad valorem tax equilibrium of the $\langle \theta_h \rangle$ ownership economy (that is z is the fixed point in Theorem 1). Then the following are true.*
(i) if $\theta_h > 0$ for all $h \in \mathcal{H}$, then the utility profile corresponding to z, $\rho^u(z)$, either lies on first-best frontier, or it lies below the ad valorem envelope.
(ii) if there exists h' such that $\theta_{h'} = 0$ then $t_0(z) = 0$ and $\theta_h^u(z) = \theta_h$ for all $h \in \mathcal{H}$.

Proof: See Appendix A.

4.4. The Nature of the Mapping of Unit Equilibria Onto the Ad Valorem Envelope.

As stated in the introduction and the previous section, a result analogous to Theorem 1 can be proved to demonstrate that ad valorem taxation does not dominate unit taxation for any given private ownership economy. To prove such a result would require making regularity assumptions analogous to the ones made in Theorem 1 about the envelope of the Pareto frontiers of all possible private-ownership economies with ad valorem taxes. Suppose such assumptions hold. We now investigate the location of the unit-tax equilibria on the ad valorem-tax envelope for a fixed share profile.

[25] Note that every first-best allocation on the unit-envelope lies also on the ad valorem envelope, but the reverse may not be true. Because of the subsidy to the monopolist and Remark 1 above, decentralizing a first-best allocation on the ad valorem envelope as an equivalent unit-tax equilibrium of some private ownership economy may involve negative shares.

Theorem 4: *Suppose* $\langle\theta_h\rangle \in \Delta_{H-1}$ *and assumptions about the ad valorem envelope that are analogous to those made in Theorem 1 for the unit envelope hold. Suppose* z *is the allocation on the ad valorem envelope, which is supported as the unit tax equilibrium of the* $\langle\theta_h\rangle$ *ownership economy. Then the following are true.*

(*i*) *if* $\theta_h > 0$ *for all* $h \in \mathcal{H}$*, then the utility profile corresponding to* z*,* $\rho^a(z)$*, either lies on first-best frontier, or it lies below the unit envelope.*

(*ii*) *if there exists* h' *such that* $\theta_{h'} = 0$ *then either* $t_0(z) = 0$ *and* $\theta^a_h(z) = \theta_h$ *for all* $h \in \mathcal{H}$ *or the utility profile* $\rho^a(z)$ *lies on the first-best with* $\theta^a_h(z) > 0$ *for all* $h \in \mathcal{H}$*.*

Proof: See Appendix A.

These theorems help us make some conjectures about the relative positions of the unit and ad valorem envelopes. One such conjecture is shown in Figure 1. The points where the two envelopes are tangent correspond either to some first-best situations (and hence with negative monopoly taxes) or to second-best situations where the monopoly tax is zero. There are situations where the unit envelope lies above the ad valorem envelope. These are associated with positive monopoly taxes. Then there are also situations where the reverse is true, and these are associated with negative monopoly taxes.

5. Concluding Remarks

In a general equilibrium model with a monopoly sector we have shown that the set of Pareto optima in a unit-tax economy neither dominates the set of ad valorem-tax Pareto optima nor is it dominated by it. If the shares in the private sector profits are equal for all consumers (which is equivalent to one hundred per cent profit taxation) the two sets of Pareto optima coincide. This conclusion is at odds with most of the existing literature relating unit taxation to ad valorem taxation.

Earlier claims that equilibria in unit-tax economies are dominated by equilibria in ad valorem-tax economies did not deal with the fact that the monopoly profits must be redistributed to consumers either via government taxation and a uniform lump-sum transfer or via the private ownership of firms. Nevertheless, the move from a unit-tax equilibrium to an ad valorem one is not simply an accounting identity as it is in a competitive economy.

The technical problems encountered arise because it is not possible to make a direct comparison of the unit-tax and ad valorem-tax equilibria for a given profile of profit shares because the utilities of the consumers are different in the two regimes (unless the shares are equal). To circumvent this problems we have resorted to an indirect and somewhat novel procedure which draws heavily on earlier work by Guesnerie [1980] and Quinzi [1992] in a somewhat different context of economies with increasing returns .

6. Appendix A

This appendix contain the proofs of all of the theorems in the paper. The following appendix rationalizes some of the assumptions in Theorems 1 and 2 in terms of the underlying primitives of the problem in so far as possible.

Proof of Theorem 1:

Our utility normalization and Assumptions (i), (ii), and (iv) of the theorem imply that $\rho^u(A^u)$ is homeomorphic to Δ_{H-1}, where the homeomorphism is $\kappa : \rho^u(A^u) \to \Delta_{H-1}$ with image

$$\kappa(u) = \frac{u}{||u||}, \tag{6.1}$$

for every $u = (u_1, \ldots, u_H) \in \rho^u(A^u)$.[26] Note if $u \in \rho^u(A^u)$, then by our normalization $u \geq 0_H$.[27]

Define the inverse of κ as $\mathcal{K} : \Delta_{H-1} \to \rho^u(A^u)$.[28]

Define the correspondence $T : A^u \times S_b^u \to \Delta_{H-1}$ as

$$\begin{aligned} T(z, \pi) = \{\beta \in \Delta_{H-1} | \beta_h = 0 \text{ if there exists } h \text{ such that} \\ q_0 x_0^h(z) + q^T x^h(z) + q_0\omega_0^h + q^T\omega^h > r^{ah}(z, \pi, \theta_h)\}. \end{aligned} \tag{6.3}$$

We claim that T is non-empty, compact, convex valued, and upper-hemi continuous. It is trivial to prove that T is nonempty and convex valued. We now show that it is upper-hemi continuous, which will imply that it is compact valued, given Assumptions (iii) and (iv). Suppose $(z^v, \pi^v\rangle \to (z, \pi\rangle \in A^u \times S_b^u$ and $\beta^v \to \beta$ such that $\beta^v \in T(z^v, \pi^v)$ for all v. We need to show that $\beta \in T(z, \pi)$. If there exists h such that $q_0 x_0^h(z) + q^T x^h(z) + q_0\omega_0^h + q^T\omega^h - r^{uh}(z, \pi, \theta_h) > 0$ then, by the definition of the mapping T, we have $\beta^h = 0$. Since the functions r^{ah} are continuous for all h in z and π, there exists v' such that for all $v \geq v'$, we have $q_0^v x_0^h(z^v) + q^{vT} x^h(z^v) + q_0^v\omega_0^h + q^{vT}\omega^h - r^{ah}(z^v, \pi^v, \theta^h) > 0$. Hence $\beta^{hv} = 0$ for all $v \geq v'$. Therefore $\beta^v \to \beta$ implies that $\beta^h = 0$.

The idea of correspondence T is to penalize (reduce utility of consumers) whenever the allocation and producer prices and tax combination $(z, \pi\rangle$ is such that the imputation of consumption bundles at consumer prices exceeds the income made available through profit shares and demogrant, evaluated at producer prices and taxes corresponding to π.

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[26] See also Quinzii, p. 51.

[27] For $H > 1$, we can show that $u \neq 0_H$ if $u \in \rho^u(A^u)$. For, suppose $u = 0_H$. Then for any other $u' \in \rho^u(A^u)$ (such a u' exists otherwise $\rho^u(A^u)$ would be a singleton and hence zero dimensional manifold, contradicting assumption (iv)), there exists h such that $u'_h < u_h$ (by definition of Pareto optimality), and hence $u'_h < 0$, which is a contradiction to our normalization of the utility functions (for under that normalization, $u \geq 0$ for all $u \in \rho^u(A^u)$). Note also that, if $\kappa(\rho^u(\underline{z}_h^u)) = \alpha$, then $\alpha_h = 0$ and $\sum_{h' \neq h} \alpha_{h'} = 1$.

[28] Its image is

$$\mathcal{K}(\alpha) = \lambda(\alpha)\alpha \tag{6.2}$$

where $\lambda(\alpha) = \max\{\lambda \geq 0 | \lambda\alpha \in \rho^u(A^u)\}$.

Define the correspondence $K : \Delta_{H-1} \times S_b^u \to \Delta_{H-1} \times S_b^u$, as

$$K(\alpha, \pi) = (T(\rho^{u-1}(\mathcal{K}(\alpha)), \pi),\ \Psi_p^a(\rho^{u-1}(\mathcal{K}(\alpha)))). \tag{6.4}$$

Under the maintained assumptions of this theorem, this correspondence is convex valued and upper-hemi continuous. The Kakutani's fixed point theorem implies that there is a fixed point $(\overset{*}{\alpha}, \overset{*}{\pi}\rangle$ such that $\overset{*}{\alpha} \in T(\rho^{u-1}(\mathcal{K}(\overset{*}{\alpha})), \overset{*}{\pi})$ and $\overset{*}{\pi} \in \Psi_p^a(\rho^{u-1}(\mathcal{K}(\overset{*}{\alpha})))$.
Let $\overset{*}{z} := \rho^{u-1}(\mathcal{K}(\overset{*}{\alpha}))$. Hence, $\overset{*}{z} \in A$ and is unique (as ρ^u and $\mathcal{K}$ are bijective). From Assumption 6, we have

$$\underline{r}^{hu} \leq r^{ah}(z, \psi_p^u(z), \theta_h), \forall h \in \mathcal{H}, \text{and } \forall z \in A^u. \tag{6.5}$$

We now prove that

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) = q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})\omega_0^h + q^T(\overset{*}{z})\omega^h,\ \forall h \in \mathcal{H}. \tag{6.6}$$

If there exists h such that

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) < q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})\omega_0^h + q^T(\overset{*}{z})\omega^h, \tag{6.7}$$

then by the definition of the correspondence T, we have $\overset{*}{\alpha}_h = 0$. By the definition of the homeomorphism $\mathcal{K}$, this would imply $\overset{*}{u}^h = 0$, and by our utility normalization, we will have $(x_0^h(\overset{*}{z}), x^h(\overset{*}{z})) = (x_0^h(\underline{z}_h^u), x^h(\underline{z}_h^u))$, so that the right-hand side of (6.7) is $\underline{r}^{hu}$. This means that (6.7) contradicts (6.5). Hence, we have

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) \geq q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})\omega_0^h + q^T(\overset{*}{z})\omega^h,\ \forall h \in \mathcal{H}, \tag{6.8}$$

which implies

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) - \left[q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})\omega_0^h + q^T(\overset{*}{z})\omega^h\right] \geq 0,\ \forall h \in \mathcal{H}. \tag{6.9}$$

Now monotonicity of preferences (in Assumption (i)) implies that at the Pareto optimal allocation $\overset{*}{z}$, we will have

$$\begin{aligned} &\sum_h x^h(\overset{*}{z}) + \omega = y^c(\overset{*}{z}) - y^m(\overset{*}{z}) - y^g(\overset{*}{z}) + \omega \text{ and} \\ &\sum_h x_0^h(\overset{*}{z}) + \omega_0 = y_0(\overset{*}{z}) + \omega_0. \end{aligned} \tag{6.10}$$

From second last equality in (3.20) and by multiplying the system in (6.10) by $q(\overset{*}{z})$ and $q_0(\overset{*}{z})$ and adding, we have

$$\begin{aligned} \sum_h r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) =& q^T(\overset{*}{z})[y^c(\overset{*}{z}) - y^m(\overset{*}{z}) - y^g(\overset{*}{z})] + q_0(\overset{*}{z})y_0(\overset{*}{z}) + q^T(\overset{*}{z})\omega + q_0(\overset{*}{z})\omega_0 \\ =& q_0(\overset{*}{z}) \sum_h x_0^h(\overset{*}{z}^h) + q^T(\overset{*}{z}) \sum_h x^h(\overset{*}{z}) + q^T(\overset{*}{z})\omega + q_0(\overset{*}{z})\omega_0. \end{aligned} \tag{6.11}$$

This implies that

$$\sum_h \left[\, r^{ah}(\overset{*}{z},\overset{*}{\pi},\theta_h) - \big(q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q^T(\overset{*}{z})\omega^h + q_0(\overset{*}{z})\omega_0^h\big)\right] = 0 \tag{6.12}$$

Since (6.9) holds, (6.12) is true iff (6.9) holds as an equality, that is,

$$r^{ah}(\overset{*}{z},\overset{*}{\pi},\theta_h) = q^T(\overset{*}{z})x^h(\overset{*}{z}) + q_0(\overset{*}{z})x_0^h(\overset{*}{z}) + q_0(\overset{*}{z})\omega_0^h + q^T(\overset{*}{z})\omega^h = w^h(\overset{*}{z}),\ \forall h \in \mathcal{H}. \tag{6.13}$$

From (6.13) we have for all $h \in \mathcal{H}$

$$r^{uh}(\overset{*}{z},\psi_p^u(\overset{*}{z}),\theta_h^u(\overset{*}{z})) = w^h(\overset{*}{z}) = r^{ah}(\overset{*}{z},\psi_p^a(\overset{*}{z}),\theta_h) \tag{6.14}$$

This proves (a).

The price and the ad valorem tax configuration $\psi_p^a(\overset{*}{z}) = (p_0^a(\overset{*}{z}),\tau_0(\overset{*}{z}),p(\overset{*}{z}),\tau(\overset{*}{z}))$ and the income configuration $\langle r^{ah}(\overset{*}{z},\psi_p^a(\overset{*}{z}),\theta_h)\rangle$ define an ad valorem tax equilibrium of the private ownership economy $\langle\theta_h\rangle$, the underlying equilibrium allocation is $\overset{*}{z}$ and the consumer prices are $(q_0(\overset{*}{z}),q^T(\overset{*}{z})) = (p_0^a(\overset{*}{z})[1+\tau_0(\overset{*}{z})],p^T(\overset{*}{z})[I_N+\boldsymbol{\tau}(\overset{*}{z})]) = (p_0^u(\overset{*}{z})+t_0^u(\overset{*}{z}),p(\overset{*}{z})+t(\overset{*}{z}))$. This proves (b).
Denote $M_\Pi^u(\overset{*}{z}) = \Pi^{mu}(\overset{*}{z}) + \Pi^c(\overset{*}{z})$, $\quad M_\Pi^a(\overset{*}{z}) = \Pi^{ma}(\overset{*}{z}) + \Pi^c(\overset{*}{z})$, $\quad M_G^u(\overset{*}{z}) = t_0(\overset{*}{z})y_0(\overset{*}{z}) + t^T(\overset{*}{z})[y^c(\overset{*}{z})-y^m(\overset{*}{z})-y^g(\overset{*}{z})]-p^T(\overset{*}{z})y^g(\overset{*}{z})$, and $\quad M_G^a(\overset{*}{z}) = p_0^a(\overset{*}{z})\tau_0(\overset{*}{z})y_0(\overset{*}{z})+p^T(\overset{*}{z})\boldsymbol{\tau}(\overset{*}{z})[y^c(\overset{*}{z})-y^m(\overset{*}{z})-y^g(\overset{*}{z})]-p^T(\overset{*}{z})y^g(\overset{*}{z})$. At $\overset{*}{z}$ we know, from (3.20), that the sums of profits and government revenue are the same under the unit and ad valorem systems, that is

$$\begin{aligned} M_\Pi^u(\overset{*}{z}) + M_G^u(\overset{*}{z}) &= M_\Pi^a(\overset{*}{z}) + M_G^a(\overset{*}{z}) \\ \Leftrightarrow -\,[M_\Pi^a(\overset{*}{z}) - M_\Pi^u(\overset{*}{z})] &= M_G^a(\overset{*}{z}) - M_G^u(\overset{*}{z}). \end{aligned} \tag{6.15}$$

From conclusion (a) we have for all $h \in \mathcal{H}$

$$\begin{aligned} w^h(\overset{*}{z}) = \theta_h^u(\overset{*}{z})M_\Pi^u(\overset{*}{z}) + \frac{1}{H}M_G^u(\overset{*}{z}) &= \theta_h M_\Pi^a(\overset{*}{z}) + \frac{1}{H}M_G^a(\overset{*}{z}) \\ \Rightarrow \theta_h^u(\overset{*}{z})M_\Pi^u(\overset{*}{z}) - \theta_h M_\Pi^a(\overset{*}{z}) &= \frac{1}{H}[M_G^a(\overset{*}{z}) - M_G^u(\overset{*}{z})] \\ \Rightarrow \theta_h^u(\overset{*}{z})M_\Pi^u(\overset{*}{z}) - \theta_h M_\Pi^a(\overset{*}{z}) &= \frac{1}{H}[M_\Pi^u(\overset{*}{z}) - M_\Pi^a(\overset{*}{z})]. \end{aligned} \tag{6.16}$$

The last equality follows from (6.15). Hence, (6.16) implies that $\theta_h^u(\overset{*}{z}) = \theta_h$ for all $h \in \mathcal{H}$ iff $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $M_\Pi^u(\overset{*}{z}) - M_\Pi^a(\overset{*}{z}) = 0$. The latter is true when $t_0(\overset{*}{z}) = 0$. Thus, (c) is true.
We now prove (d). Let $\overset{*}{u} := \rho^u(\overset{*}{z})$.
If $\overset{*}{u} \in U^u(\langle\theta_h\rangle) := \{\langle u_h\rangle \in \mathbf{R}^H | u_1 \le \mathcal{U}^u(u_2,\ldots,u_H,\langle\theta_h\rangle)\}$, then since $\overset{*}{u} \in \rho^u(A)$, we have, because of Assumption (vi), the unique solution to (3.2) as

$$\overset{*}{\theta}{}^u(\overset{*}{u}) = \langle\theta_h\rangle. \tag{6.17}$$

From (c) this is true iff $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$.
If $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$, then again (6.17) follows from (c), and we have $\overset{*}{u} \in U^u(\langle\theta_h\rangle) := \{\langle u_h\rangle \in \mathcal{H} | u_1 \leq \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle)\}$. ■

Proof of Theorem 2: We write the Lagrangian of problem (4.2) as

$$\begin{aligned} L = &- \sum_h \bar{s}_h[u_h - V^h()] - \bar{v}^T[x() - y^c() + y^m() + y^g] - \bar{v}_0[x_0() - y_0^u] - \bar{\beta}[p_0^u - P_0^u()] \\ &- \sum_h \bar{\alpha}_h\big[w_h - \theta_h[p_0^u y_0^u - C() + \Pi^c()] - \frac{1}{H}\ [t^T(y^c - y^m - y^g) + t_0 y_0 - p y^g\ ]\ \big] \\ &- \bar{r}[g - F(y^g)] - \bar{\gamma}[\sum_h \theta_h - 1] - \sum_h \bar{\phi}_h[\theta_h - 1], \end{aligned} \tag{6.18}$$

where $\bar{s}_1 = 1$. Assuming interior solutions for variables p_0^u, p, t_0, t, and $\langle w_h\rangle$, the first-order conditions include

$$- \sum_h \bar{s}_h x_0^h - \bar{v}^T \nabla_{q_0} x - \bar{v}_0 \nabla_{q_0} x_0 + \sum_h \bar{\alpha}_h y_0^u \theta_h - \bar{\beta} = 0, \tag{6.19}$$

$$- \sum_h \bar{s}_h x_0^h - \bar{v}^T \nabla_{q_0} x - \bar{v}_0 \nabla_{q_0} x_0 + \sum_h \bar{\alpha}_h y_0^u \frac{1}{H} + \bar{\beta} \nabla_{t_0} P_0^u = 0, \tag{6.20}$$

$$\begin{aligned} &- \sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x + \bar{v}^T [\nabla_p y^c - \nabla_p y^m] - \bar{v}_0 \nabla_q^T x_0 \\ &\quad + \sum_h \bar{\alpha}_h \big[\theta_h[-\nabla_p^T C + \nabla_p^T \Pi^c] + \frac{1}{H}[t^T(\nabla_p y^c - \nabla_p y^m) - y^{gT}]\ \big] + \bar{\beta} \nabla_p^T P_0^u = 0, \end{aligned} \tag{6.21}$$

$$- \sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x - \bar{v}_0 \nabla_q^T x_0 + \sum_h \bar{\alpha}_h \frac{1}{H}[y^{cT} - y^{mT} - y^{gT}] + \bar{\beta} \nabla_t^T P_0^u = 0, \tag{6.22}$$

$$\bar{s}_h - \bar{v}^T \nabla_{w_h} x^h - \bar{v}_0 \nabla_{w_h} x_0^h - \bar{\alpha}_h + \bar{\beta} \nabla_{w_h} P_0^u = 0, \text{ for } h \in \mathcal{H}, \tag{6.23}$$

$$\bar{v}^T = \bar{r} \nabla_{y^g}^T F - \sum_h \bar{\alpha}_h \frac{1}{H}[t^T + p^T], \tag{6.24}$$

and

$$\bar{v}^T \nabla_{y_0^u} y^m = \bar{v}_0 + \sum_h \bar{\alpha}_h \left[\theta_h[p_0^u - \nabla_{y_0^u} C] + \frac{1}{H}[-t\ ^T \nabla_{y_0^u} y^m + t_0\ ]\right]. \tag{6.25}$$

We also have the following Kuhn-Tucker conditions for $\langle\theta_h\rangle$ and the Lagrange multipliers $\langle\bar{\phi}_h\rangle$

$$\begin{aligned} &\bar{\alpha}_h[p_0^u y_0^u - C() + \Pi^c()] - \bar{\gamma} + \bar{\phi}_h \leq 0,\ \theta_h \geq 0 \text{ and} \\ &\theta_h[\bar{\alpha}_h[p_0^u y_0^u - C() + \Pi^c()] - \bar{\gamma} + \bar{\phi}_h] = 0,\ \forall h \in \mathcal{H}, \end{aligned} \tag{6.26}$$

and

$$\theta_h - 1 \leq 0, \;\; \bar{\phi}_h \geq 0, \;\text{and}\; \bar{\phi}_h[\theta_h - 1] = 0, \;\; \forall h \in \mathcal{H}. \tag{6.27}$$

The system can be simplified further. Subtract (6.20) from (6.19) to get

$$\bar{\beta}[1 + \nabla_{t_0} P_0^u] = y_0^u \sum_h \bar{\alpha}_h[\theta_h - \frac{1}{H}]. \tag{6.28}$$

Subtract (6.22) from (6.21) to get

$$[\bar{v}^T + \sum_h \frac{\bar{\alpha}_h}{H} t^T][\nabla_p y^c - \nabla_p y^m] + [y^{cT} - y^{mT}] \sum_h \bar{\alpha}_h[\theta_h - \frac{1}{H}] + \bar{\beta}[\nabla_p^T P_0^u - \nabla_t^T P_0^u] = 0. \tag{6.29}$$

Let $\langle \theta_h \rangle = \langle \overset{*}{\theta}{}_h^u(u_2, \ldots, u_H) \rangle$. Then $\theta_h > 0$ for all $h \in \mathcal{H}$. Hence, from (6.27), we have $\phi_h = 0$ for all h, and (6.26) implies that

$$\bar{\alpha}_h[p_0^u y_0^u - C() + \Pi^c()] = \bar{\gamma}, \;\; \forall h \in \mathcal{H}. \tag{6.30}$$

which correspond to variables $(\theta_h)_h$. Now, (6.30) implies

$$\bar{\alpha}_h = \frac{\bar{\gamma}}{p_0^u y_0^u - C() + \Pi^c()} =: K, \;\; \forall h \tag{6.31}$$

Given that $\sum_h \theta_h = 1$, (6.31) implies that at any optimum,

$$\sum_h \bar{\alpha}_h[\theta_h - \frac{1}{H}] = K \sum_h [\theta_h - \frac{1}{H}] = 0. \tag{6.32}$$

Thus, (6.32) and (6.28), and the assumption that $\nabla_{t_0} P_0^u \neq -1$ imply that

$$\bar{\beta} = 0, \tag{6.33}$$

i.e., the monopoly constraint is non-binding at the solution to problem (4.2). From (6.29), (6.32), and (6.33), we have

$$[\bar{v}^T + \sum_h \frac{\bar{\alpha}_h}{H} t^T][\nabla_p y^c - \nabla_p y^m] = 0. \tag{6.34}$$

Homogeneity of degree zero in p of the competitive supplies and the monopolist's cost minimizing input demands implies

$$[\bar{v}^T + \sum_h \frac{\bar{\alpha}_h}{H} t^T] = \mu p^T, \tag{6.35}$$

which from (6.24), implies

$$\bar{r} \nabla_{y^g}^T F = (\mu + K) p^T. \tag{6.36}$$

Now, (i) using (6.33) and (6.31), (ii) post multiplying (6.22) by x_0^h, summing up over all h, and subtracting from (6.20), (iii) post multiplying (6.22) by x^{hT}, summing up over all

h, and subtracting from (6.22), and (iv) recalling that at a tight equilibrium, we have $\sum_h x^h = y^c - y^m - y^g$ and $y_0^u = \sum_h x_0^h$, we obtain

$$\begin{bmatrix} \bar{v}^T & \bar{v}_0 \end{bmatrix} \sum_h \begin{bmatrix} \nabla_{q_0} x^h + \nabla_{w_h} x^h x_0^h & \nabla_q x^h + \nabla_{w_h} x^h x^{hT} \\ \nabla_{q_0} x_0^h + \nabla_{w_h} x_0^h x_0^h & \nabla_q x_0^h + \nabla_{w_h} x_0^h x^{hT} \end{bmatrix} = \begin{bmatrix} 0^T & 0 \end{bmatrix}. \tag{6.37}$$

But the second matrix on the left-handside of (6.37) is the sum over all $h \in \mathcal{H}$ of Slutsky matrices of price derivatives of compensated demands of the consumers. Since, by assumption, each of these matrices has rank N, we have

$$\begin{bmatrix} \bar{v}^T & \bar{v}_0 \end{bmatrix} = \kappa \begin{bmatrix} q^T & q_0 \end{bmatrix} = \kappa \begin{bmatrix} p^T + t^T & p_0^u + t_0 \end{bmatrix}. \tag{6.38}$$

Employing (6.31), (6.35), and (6.38), we have

$$\begin{aligned} \kappa[p^T + t^T] + Kt^T &= \mu p^T \\ \Rightarrow \kappa[p^T + t^T] + K[t^T + p^T] &= [\mu + K] p^T \\ \Rightarrow \frac{\kappa + K}{\mu + K} q^T &= p^T \end{aligned} \tag{6.39}$$

From (6.31), (6.38), and (6.25), and exploiting the homogeneity properties of the cost function, we have

$$\begin{aligned} \mu \nabla_{y_0^u} C &= \kappa q_0 + K[q_0 - \nabla_{y_0^u} C] \\ \Rightarrow \frac{\kappa + K}{\mu + K} q_0 &= \nabla_{y_0^u} C \end{aligned} \tag{6.40}$$

By choosing $\bar{r}, \kappa, \mu$, and K such that $\frac{\kappa+K}{\mu+K} = 1$ and $\bar{r} = \kappa + K$, we obtain from (6.36), (6.39), and (6.40)

$$\begin{aligned} &\nabla_{y^g}^T F = q^T = p^T \text{ and} \\ &\nabla_{y_0^u} C = q_0. \end{aligned} \tag{6.41}$$

Thus (6.41) is reflective of joint consumption and production efficiency at the solution of program (4.2). Hence, the allocation corresponding to a solution of program (4.2) is a first best Pareto optimal allocation.■

Proof of Theorem 3:

(ii) Suppose $\exists h'$ such that $\theta_{h'} = 0$. The unit shares that make z a unit tax equilibrium (lying on the unit envelope) are given by

$$\theta_h^u(z) = \frac{\theta_h M_\Pi^a(z) + \frac{1}{H}[M_G^a(z) - M_G^u(z)]}{M_\Pi^u(z)} \geq 0, \ \forall h \in \mathcal{H}. \tag{6.42}$$

So for h', we have

$$\theta_{h'}^u(z) = \frac{\frac{1}{H}[M_G^a(z) - M_G^u(z)]}{M_\Pi^u(z)} \geq 0. \tag{6.43}$$

From (4.3) this implies that $t_0(z) \geq 0$. We prove that $t_0(z) = 0$. Suppose not. Then $t_0(z) > 0$. This means, from (4.3), that

$$\theta_h^u(z) = \frac{\theta_h M_\Pi^a(z) + \frac{1}{H}[M_G^a(z) - M_G^u(z)]}{M_\Pi^u(z)} > 0, \forall h \in \mathcal{H}. \tag{6.44}$$

Thus z is on the unit envelope with $\theta_h^u(z) > 0$ for all $h \in \mathcal{H}$. This implies from Theorem 2 that z is first-best. But this contradicts Remark 2 based on GL, which says $t_0(z) < 0$ for a first-best allocation with a unit-tax representation. Hence $t_0(z) = 0$. This means $M_\Pi^a(z) = M_\Pi^u(z)$. Combined with (4.3) and (6.42) we get $\theta_h = \theta_h^u(z)$ for all $h \in \mathcal{H}$.

(i) Suppose $\theta_h > 0$ for all $h \in \mathcal{H}$. Two case are possible from viewing (6.42).

(a) $\theta_h^u(z) > 0$ for all $h \in \mathcal{H}$. Thus z is on the unit envelope with $\theta_h^u(z) > 0$ for all $h \in \mathcal{H}$. This implies from Theorem 2 that z is first-best. Remark 2 based on GL, implies $t_0(z) < 0$.

(b) There exists h' such that $\theta_{h'}^u(z) = 0$. (6.42) implies that

$$0 = \theta_{h'} M_\Pi^a(z) + \frac{1}{H}[M_G^a(z) - M_G^u(z)]. \tag{6.45}$$

Since $\theta_h > 0$ for all $h \in \mathcal{H}$, including h' by assumption and profits are not zero, this implies

$$\theta_{h'} M_\Pi^a(z) = \frac{1}{H}[M_G^u(z) - M_G^a(z)] > 0. \tag{6.46}$$

From (4.3), this means $t_0(z) < 0$. So either z is a first-best (with constraints in Theorem 2 just binding) or is an ad valorem equilibrium with positive shares on the unit envelope. The analogue of Theorem 1 for the ad valorem-tax envelope implies that z does not lie on the ad valorem envelope (as any ad valorem equilibrium on the ad valorem envelope with positive shares is a first-best by the analogue of Theorem 2, which gives the relation between the first-best frontier and the ad valorem-tax envelope). Hence the ad valorem envelope lies above the unit envelope for the utility profile $\rho^u(z)$.■

Proof of Theorem 4:

(i) Suppose $\theta_h > 0$ for all $h \in \mathcal{H}$. The ad valorem shares that make z an ad valorem tax equilibrium (lying on the ad valorem envelope) are given by

$$\theta_h^a(z) = \frac{\theta_h M_\Pi^u(z) + \frac{1}{H}[M_G^u(z) - M_G^a(z)]}{M_\Pi^a(z)} \geq 0, \ \forall h \in \mathcal{H}. \tag{6.47}$$

Two cases are possible from (6.47):

(a) $\theta_h^a(z) > 0$ for all $h \in \mathcal{H}$. Since we are on an ad valorem envelope, the analogue of Theorem 2 for ad valorem-tax envelope implies that z is a first-best.

(b) There exists h' such that $\theta_{h'}^a(z) = 0$. (6.47) implies that

$$\theta_{h'}^a(z) = 0 = \theta_{h'} M_\Pi^u(z) + \frac{1}{H}[M_G^u(z) - M_G^a(z)]. \tag{6.48}$$

Which implies, because $\theta_h > 0$ for all $h \in \mathcal{H}$ in case (i) of this theorem, that

$$\theta_{h'} M^u_{\Pi}(z) = \frac{1}{H}[M^a_G(z) - M^u_G(z)] > 0. \tag{6.49}$$

From (4.3), this means that $\tau_0(z) > 0$ or $t_0(z) > 0$. So from Remark 2, we cannot be on a first-best, at this point on the ad valorem envelope. Since this is a unit-tax equilibrium with positive shares, which is not on the first-best, from Theorem 2, we cannot be on the unit envelope. Hence the unit envelope lies above the ad valorem envelope at this utility profile $\rho^a(z)$.

(ii) Suppose $\exists h'$ such that $\theta_{h'} = 0$. Then from (6.47)

$$\theta^a_{h'}(z) = \frac{\frac{1}{H}[M^u_G(z) - M^a_G(z)]}{M^a_{\Pi}(z)} \geq 0, \ \forall h \in \mathcal{H}. \tag{6.50}$$

From (4.3), this means that $\tau_0(z) \leq 0$ or $t_0(z) \leq 0$. Two case are possible:

(a) $\tau_0(z) < 0$. From (4.3), this would mean $M^u_G(z) - M^a_G(z) > 0$, and hence from (6.50), this would mean $\theta^a_h(z) > 0$ for all $h \in \mathcal{H}$. Since we are on the ad valorem envelope, from the analogue of Theorem 2 for the ad valorem envelope, this would mean that z is first-best.

(b) $\tau_0(z) = 0$. This means $M^a_{\Pi}(z) = M^u_{\Pi}(z)$. Combined with (4.3) and (6.50) we get $\theta_h = \theta^a_h(z)$ for all $h \in \mathcal{H}$.■

7. Appendix B

The discussion and proofs in this appendix are for economies with unit taxes. Similar discussions and results can be obtained for the ad valorem tax case.

A tight unit-tax equilibrium is obtained by replacing the inequalities in (2.6) to (2.10) by equalities. We focus only on tight unit-tax equilibria. The domain of the vector of variables $(p^u_0, p, t_0, t, \langle w_h \rangle, y_0, y^g)$ is taken to be $\mathbf{R}^{N+1}_{++} \times \mathbf{R}^{N+1} \times \mathbf{R}^{H+N+1}_{+}$, which we denote by Ω_E.

Note that the equilibrium system (2.6) to (2.10) is homogeneous of degree zero in the variables p^u_0, p, t_0, t, and $\langle w_h \rangle$.[29] So we can adopt a normalization rule to uniquely determine prices, taxes, and incomes corresponding to equilibrium allocations.

A function $b : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^N_+ \times \mathbf{R}^N \times \mathbf{R}^H_+ \to \mathbf{R}$ defines a price-normalization rule

$$b(p_0, t_0, p, t, \langle w_h \rangle) = 0 \tag{7.1}$$

if it is continuous and increasing and there exists a function $B_b : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^N_+ \times \mathbf{R}^N \to \mathbf{R}_{++}$, with image $B_b(\pi_{p_0}, \pi_{t_0}, \pi_p, \pi_t)$, such that for every $(\pi_{p_0}, \pi_{t_0}, \pi_p, \pi_t, \langle \pi_{w_h} \rangle) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^N_+ \times \mathbf{R}^N \times \mathbf{R}^H_+$,

$$b(\frac{\pi_{p_0}}{B_b()}, \frac{\pi_{t_0}}{B_b()}, \frac{\pi_p}{B_b()}, \frac{\pi_t}{B_b()}, \langle \frac{\pi_{w_h}}{B_b()} \rangle) = 0 \tag{7.2}$$

[29] Recall, that the function $P^u_0(p, t_0, t, \langle w_h \rangle)$ is homogeneous of degree one in p, t_0, t, and $\langle w_h \rangle$.

Monotonicity of the function b implies that the function B_b is unique for a given function b.[30] Let us choose an increasing and differentiable function $b : \Omega_N \to \mathbf{R}$, which defines a valid price normalization rule in the sense of that defined in the earlier section[31]

$$b(p_0, t_0, p, t, \langle w_h \rangle) = 0. \tag{7.3}$$

Under such a normalization rule and some regularity assumptions, the set of all (tight) unit tax equilibria can be shown to be generically a $2N-1$ dimensional manifold. Lemma A1 below demonstrates this. Define the function: $\mathcal{F} : \Omega_E \to \mathbf{R}^{N+H+4}$ with image $\mathcal{F}(p_0, p, t, t_0, \langle w_h \rangle, y_0, y^g)$ as

$$\begin{aligned}
&-x(p_0+t_0, p+t, \langle w_h \rangle) + y^c(p) - y^m(y_0^u, p) - y^g \\
&-x_0(p_0+t_0, p+t, \langle w_h \rangle) + y_0^u \\
&p_0^u - P_0^u(p, t_0, t, \langle w_h \rangle) \\
&w_h - \big[\theta_h[p_0^u y_0^u - C(y_0^u, p) + \Pi^c(p)] + \frac{1}{H}[t^T(y^c(p) - y^m(y_0^u, p) - y^g) + t_0 y_0^u - p y^g]\big], \ \forall h \\
&F(y^g) - g \\
&b(p_0, t_0, p, t, \langle w_h \rangle).
\end{aligned} \tag{7.4}$$

Lemma A1: *Suppose $\mathcal{F}$ is a differentiable mapping and there exists a neighborhood $\mathcal{E}$ around 0 in $\mathbf{R}^{N+H+4}$ such that for all $\nu \in \mathcal{E}$, $\mathcal{F}^{-1}(\nu) \neq \emptyset$. Then for almost all $\nu \in \mathcal{E}$ (that is, except for a set of measure zero in $\mathcal{E}$), $\mathcal{F}^{-1}(\nu)$ is a manifold of dimension $2N-1 = 3(N+1) + H - [N+H+4]$.*

The proof follows from Sard's theorem.[32] This lemma implies that the set of regular economies which differ from the original one only in terms of endowments is very large (this set is dense).

Suppose $\mathcal{F}$ is differentiable. Denote the derivative of $\mathcal{F}$, evaluated at $v \in \Omega_E$ as the linear mapping $\partial\mathcal{F}_v : \Omega_E \to \mathbf{R}^{N+H+4}$ where $\partial\mathcal{F}_v$ is the Jacobian matrix, evaluated at v

$$\partial\mathcal{F}_v = \begin{bmatrix} J_{E1} \\ J_{E2} \\ J_{E3} \\ J_{E4} \\ J_{E5} \end{bmatrix}, \tag{7.5}$$

[30] Some examples:
(i) $b(p_0, t_0, p, t, \langle w_h \rangle) = \| \langle p_0, t_0, p, t \rangle \| - 1$ and $B_b(p_0, t_0, p, t) = \| \langle p_0, t_0, p, t \rangle \|$
(ii) $b(p_0, t_0, p, t, \langle w_h \rangle) = p_1 - 1$ and $B_b(p_0, t_0, p, t) = p_1$
(iii) $b(p_0, t_0, p, t, \langle w_h \rangle) = p_0 + \sum_{k=1}^N p_k - 1$ and $B_b(p_0, t_0, p, t) = p_0 + \sum_{k=1}^N p_k$.

[31] $\Omega_N := \mathbf{R}_{++}^{N+1} \times \mathbf{R}^{N+1} \times \mathbf{R}_+^H$.

[32] For a discussion of the method of proof, see Guesnerie [1995; pp. 106-107].

with

$$J_{E1} = [J^1_{E1}\ J^2_{E1}]$$
$$J^1_{E1} = \begin{bmatrix} -\nabla_{q_0}x & -\nabla_{q_0}x & -\nabla_q x & -\nabla_q x - \nabla^T_p y^m + \nabla_p y^c & -\nabla_{w_1}x^1 & \dots & -\nabla_{w_H}x^H \\ -\nabla_{q_0}x_0 & -\nabla_{q_0}x_0 & -\nabla_q x_0 & -\nabla_q x & -\nabla_{w_1}x^1_0 & \dots & -\nabla_{w_H}x^H_0 \end{bmatrix}$$
$$J^2_{E1} = \begin{bmatrix} -\nabla_{y_0}y^m & -I \\ 1 & 0^T_N \end{bmatrix}. \tag{7.6}$$

$$J_{E2} = [J^1_{E2}\ J^2_{E2}]$$
$$J^1_{E2} =$$
$$\begin{bmatrix} -\frac{1}{H}y_0 & -\theta_1 y^u_0 & -\frac{1}{H}[y^{cT} - y^{mT} - y^g] & -\theta_1[y^{cT} - y^{mT}] + \frac{1}{H}y^{gT} & 1 & 0 & \dots & 0 \\ \vdots \\ -\frac{1}{H}y_0 & -\theta_H y^u_0 & -\frac{1}{H}[y^{cT} - y^{mT} - y^g] & -\theta_H[y^{cT} - y^{mT}] + \frac{1}{H}y^{gT} & 0 & 0 & \dots & 1 \end{bmatrix}$$
$$J^2_{E2} = \begin{bmatrix} \theta_1\nabla_{y_0}C - \frac{1}{H}t_0 & \frac{1}{H}P^T \\ \vdots \\ \theta_H\nabla_{y_0}C - \frac{1}{H}t_0 & \frac{1}{H}P^T \end{bmatrix}. \tag{7.7}$$

$$J_{E3} = \begin{bmatrix} -\nabla_{t_0}P^u_0 & 1 & -\nabla^T_t P^u_0 & -\nabla^T_p P^u_0 & -\nabla_{h_1}P^u_0 & \dots & -\nabla_{w_H}P^u_0 & 0 & 0^T_N \end{bmatrix}, \tag{7.8}$$

$$J_{E4} = \begin{bmatrix} 0 & 0 & 0^T_N & 0^T_N & 0 & \dots & 0 & 0 & \nabla^T_{y^g}F \end{bmatrix}, \tag{7.9}$$

and

$$J_{E5} = \begin{bmatrix} \nabla_{t_0}b & \nabla_{p_0}b & \nabla^T_t b & \nabla^T_p b & \nabla_{w_1}b & \dots & \nabla_{w_H}b & 0 & 0^T_N \end{bmatrix}. \tag{7.10}$$

By stacking the matrices above and looking at the structure of $\partial\mathcal{F}_v$, it can be seen that $\partial\mathcal{F}_v$, which is of dimension $(N+H+4)\times 3(N+1)+H$, has at least rank $N+H+2$ for all $v \in \Omega_E$. There are at least $N+H+2$ columns in $\partial\mathcal{F}_v$ which are linearly independent for all $v \in \Omega_E$. These are columns that correspond to the variables $y_g,\ y_0,\ \langle w_h\rangle$, and p_0.

Lemma A2: *Suppose $\mathcal{F}$ is a differentiable mapping, $\mathcal{F}^{-1}(0) \neq \emptyset$, and 0 is a regular value of $\mathcal{F}$ (that is, the rank of $\partial\mathcal{F}_v$ is $N+H+4$ for all $v \in \mathcal{F}^{-1}(0)$). Then $\mathcal{F}^{-1}(0)$ (the set of all tight tax equilibria) is a manifold of dimension $2N-1$.*

The proof follows from the pre-image theorem.
Assumption A1: The set of tight tax equilibria $\mathcal{F}^{-1}(0)$ is a subset of the interior of Ω_E and is a manifold of dimension $2N-1$.

Consider problem that identifies the Pareto manifold for a $\langle\theta_h\rangle \in \Delta_{H-1}$ economy. Using (3.1) and our normalization rule, the programme can be rewritten as

$$\begin{aligned}
\mathcal{U}^u(u_2,\ldots,u_H,\langle\theta_h\rangle) := &\max_{p_0^u,p,t_0,t,\langle w_h\rangle,y_0,y^g} V^1(p_0^u+t_0,p+t,w_1)\\
&\text{subject to}\\
&V^h(p_0^u+t_0,p+t,w_h)\ge u_h,\ \forall h=2,\ldots,H,\\
&-x(p_0^u+t_0,p+t,\langle w_h\rangle)+y^c(p)-y^m(p,y_0^u)-y^g\ge 0\\
&-x_0(p_0^u+t_0,p+t,\langle w_h\rangle)+y_0^u\ge 0\\
&p_0^u-P_0^u(p,t_0,t,\langle w_h\rangle)=0,\\
w_h=\theta_h[p_0^u y_0^u-C(y_0^u,p)+\Pi^c(p)]+&\frac{1}{H}\ [t^T(y^c-y^m-y^g)+t_0y_0-py^g\ ],\ \forall h,\\
&F(y^g)-g\ge 0,\\
&b(p_0^u,t_0,p,t,\langle w_h\rangle)=0.
\end{aligned} \tag{7.11}$$

We write the Lagrangian as

$$\begin{aligned}
L=&-\sum_h \bar{s}_h[u_h-V^h()]-\bar{v}^T[x()-y^c()+y^m()+y^g]-\bar{v}_0[x_0()-y_0^u]-\bar{\beta}[p_0^u-P_0^u()]\\
&-\sum_h\bar{\alpha}_h\big[w_h-\theta_h[p_0^uy_0^u-C()+\Pi^c()]-\frac{1}{H}\ [t^T(y^c-y^m-y^g)+t_0y_0-py^g\ ]\ \big]\\
&-\bar{r}[g-F(y^g)]-\bar{\delta}\ b(p_0,t_0,p,t,\langle w_h\rangle),
\end{aligned} \tag{7.12}$$

with $\bar{s}_1=1$, $\bar{s}_h\ge 0$ for all $h=2,\ldots,H$, $\bar{v}\ge 0_N$, $\bar{v}_0\ge 0$, $\bar{r}\ge 0$, and the signs of the other Lagrange multipliers (those corresponding to equality constraints) being unrestricted.[33]

Suppose Assumption A1 holds and all solutions to (7.11) involve *tight* tax equilibria. The first order necessary conditions of this problem, for any utility profile $(u_2,\ldots,u_H)$ for which solution exists, include
(a) those ($3(N+1)+H$ of them) obtained by taking the derivatives of the Lagrangian with respect to the choice variables,

$$-\sum_h \bar{s}_hx_0^h-\bar{v}^T\nabla_{q_0}x-\bar{v}_0\nabla_{q_0}x_0+\sum_h\bar{\alpha}_hy_0^u\theta_h-\bar{\beta}-\bar{\delta}\nabla_{p_0}b=0, \tag{7.13}$$

$$-\sum_h \bar{s}_hx_0^h-\bar{v}^T\nabla_{q_0}x-\bar{v}_0\nabla_{q_0}x_0+\sum_h\bar{\alpha}_hy_0^u\frac{1}{H}+\bar{\beta}\nabla_{t_0}P_0^u-\bar{\delta}\nabla_{t_0}b=0, \tag{7.14}$$

[33] Note, in general, the signs of the Lagrange multipliers are specific to the way in which one sets up the optimization. If we were optimizing consumer h's utility keeping utility of consumers $1,\ldots,h-1,h+1,\ldots,H$ fixed then the sign restrictions on the vector $\bar{s}$ would be different ($\bar{s}_h=1$ and $\bar{s}_{h'}\ge 0$, $\forall h'\ne h$. As an example, in the context of tax reforms as in Guesnerie [1998], the sign restrictions we need to impose when the optimum corresponds to one where there exist no directions of change that improve welfare of each consumer are $0_H\ne\bar{s}\ge 0_H$.

$$
-\sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x + \bar{v}^T [\nabla_p y^c - \nabla_p y^m] - \bar{v}_0 \nabla_q^T x_0 + \sum_h \bar{\alpha}_h \big[\theta_h [-\nabla_p^T C + \nabla_p^T \Pi^c]
$$

$$
+ \frac{1}{H}[t^T(\nabla_p y^c - \nabla_p y^m) - y^{gT}] \,\big] + \bar{\beta} \nabla_p^T P_0^u - \bar{\delta} \nabla_p^T b = 0, \tag{7.15}
$$

$$
-\sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x - \bar{v}_0 \nabla_q^T x_0 + \sum_h \bar{\alpha}_h \frac{1}{H}[y^{cT} - y^{mT} - y^{gT}] + \bar{\beta} \nabla_t^T P_0^u - \bar{\delta} \nabla_t^T b = 0, \tag{7.16}
$$

$$
\bar{s}_h - \bar{v}^T \nabla_{w_h} x^h - \bar{v}_0 \nabla_{w_h} x_0^h - \bar{\alpha}_h + \bar{\beta} \nabla_{w_h} P_0^u - \bar{\delta} \nabla_{w_h} b = 0, \text{ for } h = 1, \dots, H, \tag{7.17}
$$

$$
\bar{v}^T = \bar{r} \nabla_{y^g}^T F - \sum_h \bar{\alpha}_h \frac{1}{H}[t^T + p^T], \tag{7.18}
$$

and

$$
\bar{v}^T \nabla_{y_0^u} y^m = \bar{v}_0 + \sum_h \bar{\alpha}_h \left[\theta_h [p_0^u - \nabla_{y_0^u} C] + \frac{1}{H}[-t^{\ T} \nabla_{y_0^u} y^m + t_0 \ ] \right]. \tag{7.19}
$$

(b) the equilibrium conditions (2.6) to (2.10) ($N + H + 3$ of them) written as equalities,
(c)

$$
V^h(p_0^u + t_0, p + t, w_h) = u_h, \ \forall h = 2, \dots, H, \tag{7.20}
$$

and
(d) the normalization rule

$$
b(p_0, t_0, p, t, \langle w_h \rangle) = 0. \tag{7.21}
$$

We can prove that the normalization constraint is non-binding at the optimum, that is, $\bar{\delta} = 0$.

Lemma A3: *At any solution of problem (7.11), $\bar{\delta} = 0$.*

Proof: Multiplying (7.13) to (7.17) by $p_0^u, \ t_0, \ p, \ t$, and $\langle w_h \rangle$, respectively, adding, and making use of (i) the homogeneity properties of consumer demands, the competitive output supplies, and the input demands of the monopolist, (ii) the fact that the consumers budget constraints hold at the optimum, (iii) the tight equilibrium conditions $x = y^c - y^m - y^g$ and $x_0 = y_0$, and (iv) the Shephard's and the Hotelling's lemma, we obtain

$$
\bar{\delta}[\nabla_{p_0} b \ p_0^u + \nabla_{t_0} b \ t_0 + \nabla_p^T b \ p + \nabla_t^T b \ t + \sum_h \nabla_{w_h} b \ w_h] = 0. \tag{7.22}
$$

Clearly b being an increasing function (as required when it is a valid normalization rule) cannot be homogeneous of degree zero. This means (7.22) implies that $\bar{\delta} = 0$.■

This demonstrates the invariance of the Pareto optimal allocations to the normalization rule that is adopted.

Further, at the optimum, the Lagrange multipliers on the inequality constraints– $\langle \bar{s}_h \rangle$, $(\bar{v}_0, \bar{v})$, and $\bar{r}$– are all non-negative with $\bar{s}_1 = 1$, the multipliers on the equality constraints– $\bar{\alpha}$, $\bar{\delta}$, and $\bar{\beta}$– are unrestricted in sign ($\bar{\delta}$ of course takes a zero value at the optimum).

Note that there may exist parameter values $(u_2, \ldots, u_H)$ for which the constraint set of problem (3.1) is empty, and hence no solution exists to this problem. We would like to know the set of parameter values for which the solution exists and, once we know this, we would like to know whether the Pareto frontier (the utility possibility frontier) for this private ownership economy is a manifold and, if so, of what dimension. Can the dimension of this manifold be $H-1$?

Denote $\{1\} \times \mathbf{R}_+^{H-1} \times \mathbf{R}_+^{N+1} \times \mathbf{R}^{H+3} =: \Omega_L$. This is the space in which the Lagrange multipliers $s_1 = 1, s_2, \ldots, s_H, v_0, v, \langle \alpha_h \rangle, r, \beta$, and δ lie. Thus, Ω_L is a $N+2H+3$-dimensional manifold in a $N+2H+4$-dimensional Euclidean space. Suppose $c := (p_0, p, t_0, t, \langle w_h \rangle, y_0, y^g, \langle s_h \rangle, v_0, v, \langle \alpha_h \rangle, \beta, r, \delta) \in \Omega_E \times \Omega_L$ are a configuration of choice variables and the Lagrange multipliers that solve equations (7.13) to (7.19), (2.6) to (2.10) and (7.21). (Note, we are excluding here equations (7.20)). Then it would mean that c offer a solution to problem (3.1) for parameter values $u_h = V^h(p_0^u + t_0, p+t, w_h)$ for all $h = 2, \ldots, H$. Hence problem (3.1) is well defined for parameter values $(u_2, \ldots, u_H)$. We use this method to try to find the set of all parameter values for which the solution to (7.6) exists. Thus, we first find the set of all configurations $c \in \Omega_E \times \Omega_L$ that solve equations (7.13) to (7.19), (2.6) to (2.10), and (7.21). Recall, Ω_E lies in a $3(N+1)+H$-dimensional space, while Ω_L is a $N+2H+3$-dimensional manifold in a $N+2H+4$-dimensional Euclidean space. Thus, $\Omega_E \times \Omega_L$ is a $4N+3H+6$-dimensional manifold in a $4N+3H+7$-dimensional Euclidean space.

Define the mapping: $\mathcal{P}^u : \Omega_E \times \Omega_L \to \mathbf{R}^{4N+2H+7}$ as one with image

$$\mathcal{P}^u(p_0, t_0, p, t, \langle w_h \rangle, y_0, y^g, \langle s_h \rangle, \langle \alpha_h \rangle, v, v_0, \beta, \delta) \tag{7.23}$$

given by the vector of the following $4N+2H+7$ functions,

$$
\begin{aligned}
&-\sum_h \bar{s}_h x_0^h - \bar{v}^T \nabla_{q_0} x - \bar{v}_0 \nabla_{q_0} x_0 + \sum_h \bar{\alpha}_h y_0^u \theta_h - \bar{\beta} - \bar{\delta} \nabla_{p_0} b,\\
&-\sum_h \bar{s}_h x_0^h - \bar{v}^T \nabla_{q_0} x - \bar{v}_0 \nabla_{q_0} x_0 + \sum_h \bar{\alpha}_h y_0^u \frac{1}{H} + \bar{\beta} \nabla_{t_0} P_0^u - \bar{\delta} \nabla_{t_0} b,\\
&-\sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x + \bar{v}^T [\nabla_p y^c - \nabla_p y^m] - \bar{v}_0 \nabla_q^T x_0\\
&\qquad + \sum_h \bar{\alpha}_h \big[\theta_h [-\nabla_p^T C + \nabla_p^T \Pi^c] + \frac{1}{H} [t^T (\nabla_p y^c - \nabla_p y^m) - y^{gT}]\ \big] + \bar{\beta} \nabla_p^T P_0^u - \bar{\delta} \nabla_p^T b,\\
&-\sum_h \bar{s}_h x^{hT} - \bar{v}^T \nabla_q x - \bar{v}_0 \nabla_q^T x_0 + \sum_h \bar{\alpha}_h \frac{1}{H} [y^{cT} - y^{mT} - y^{gT}] + \bar{\beta} \nabla_t^T P_0^u - \bar{\delta} \nabla_t^T b,\\
&\bar{s}_h - \bar{v}^T \nabla_{w_h} x^h - \bar{v}_0 \nabla_{w_h} x_0^h - \bar{\alpha}_h + \bar{\beta} \nabla_{w_h} P_0^u - \bar{\delta} \nabla_{w_h} b \text{ for } h = 1, \ldots, H,\\
&\bar{v}^T - [\bar{r} \nabla_{y^g}^T F - \sum_h \bar{\alpha}_h \frac{1}{H} [t^T + p^T]],\\
&\bar{v}^T \nabla_{y_0^u} y^m - [\bar{v}_0 + \sum_h \bar{\alpha}_h \left[\theta_h [p_0^u - \nabla_{y_0^u} C] + \frac{1}{H} [-t\ ^T \nabla_{y_0^u} y^m + t_0\ ]\right]],\\
&-x(p_0 + t_0, p + t, \langle w_h \rangle) + y^c(p) - y^m(y_0^u, p) - y^g,\\
&-x_0(p_0 + t_0, p + t, \langle w_h \rangle) + y_0^u,\\
&-p_0^u + P_0^u(p, t_0, t, \langle w_h \rangle),\\
&w_h - [\theta_h [p_0^u y_0^u - C(y_0^u, p) + \Pi^c(p)] + \frac{1}{H} [t^T (y^c(p) - y^m(y_0^u, p) - y^g) + t_0 y_0^u - p y^g]],\ \forall h,\\
&F(y^g) - g,\ \text{and}\\
&b(p_0, t_0, p, t, \langle w_h \rangle).
\end{aligned}
\tag{7.24}
$$

Suppose $\mathcal{P}^{u-1}(0) \neq \emptyset$. This, would then be the set of all configurations $(t_0, p_0, t, p, \langle w_h \rangle, y_0, y^g)$ and $(\langle s_h \rangle, \langle \alpha_h \rangle, v, v_0, \beta, \delta)$ that solve equations (7.13) to (7.19) and (2.6) to (2.10), and (7.21). Hence if $v := (p_0, p, t_0, t, \langle w_h \rangle, y_0, y^g, \langle s_h \rangle, v_0, v, \langle \alpha_h \rangle, \beta, r, \delta) \in \mathcal{P}^{u-1}(0)$ then it offers a solution to the problem (3.1) for parameter values $u_h = V^h(p_0^u + t_0, p + t, w_h)$ for all $h = 2, \ldots, H$. Thus $V^1(p_0^u + t_0, p + t, w_1) = \mathcal{U}^u(u_2, \ldots, u_H, \langle \theta_h \rangle)$.

Lemma A4: *Suppose Assumptions 1 to 6 hold, and A1 hold and $\mathcal{P}^{u-1}(0) \neq \emptyset$. Define the following projection mapping*

$$
Proj : \mathcal{P}^{u-1}(0) \to \mathcal{F}^{-1}(0). \tag{7.25}
$$

If zero is a regular value of $\mathcal{P}^u$ then the dimension of the manifold $Proj(\mathcal{P}^{u-1}(0)) \leq min\{2N - 1, H - 1\}$.[34]

[34] As in Guesnerie [1995], this is a generic phenomenon.

Proof: Note that ${\mathcal{P}^u}^{-1}(0) \subset \mathcal{F}^{-1}(0) \times \Omega_L$ (as the system $\mathcal{P}^u = 0$ contains the equilibrium equations). Hence, dimension of $Proj({\mathcal{P}^u}^{-1}(0)) \leq$ dimension of $Proj(\mathcal{F}^{-1}(0) \times \Omega_L) = 2N-1$. If zero is regular value of the mapping $\mathcal{P}^u$ then, from the pre-image theorem, it follows that ${\mathcal{P}^u}^{-1}(0)$ is a $H-1$-dimensional manifold. Hence dimension of $Proj({\mathcal{P}^u}^{-1}(0)) \leq H-1$. Therefore, If zero is a regular value of $\mathcal{P}^u$ then the dimension of the manifold $Proj({\mathcal{P}^u}^{-1}(0)) \leq min\{2N-1, H-1\}$.■

Define the mapping $V : \hat{\Omega}_E \to \mathbf{R}^H$, where $\hat{\Omega}_E \subset \Omega_E$ is the set of all $(p_0^u, p, t_0, t, \langle w_h \rangle, y_0, y^g) \in \Omega_E$ such that $p_0 + t_0 > 0$ and $p + t \gg 0$ as

$$V := (V^1(p_0 + t_0, p + t, w_1), \ldots, V^H(p_0 + t_0, p + t, w_H)). \tag{7.26}$$

Applying the Roy's theorem, the Jacobian matrix ∇V is given by

$$\nabla V := \begin{bmatrix} -\lambda^1 x_0^1 & -\lambda^1 x^1 & -\lambda^1 x_0^1 & -\lambda^1 x^1 & \lambda^1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & & & & & & & \\ -\lambda^H x_0^H & -\lambda^H x^H & -\lambda^H x_0^H & -\lambda^H x^H & 0 & 0 & \ldots & \lambda^H & 0 & \ldots & 0 \end{bmatrix}. \tag{7.27}$$

It is clear that if preferences of all consumers were monotonic, then ∇V has rank H. Define the mapping $\mathcal{V}^u : \hat{\Omega}_E \times \Omega_L \to \mathbf{R}^H$ to be an obvious extension of the mapping V to the bigger space $\hat{\Omega}_E \times \Omega_L$.[35]

Lemma A5: *Suppose Assumptions 1 to 6 and A1 hold, zero is a regular value of $\mathcal{P}^u$, and ${\mathcal{P}^u}^{-1}(0) \neq \emptyset$. Then*

(a) the image of the restricted mapping $\mathcal{V}^u : {\mathcal{P}^u}^{-1}(0) \to \mathbf{R}^H$ (the utility possibility frontier of economy $\langle \theta_h \rangle$) is a $H-1$-dimensional manifold, and

(b) if ${\mathcal{P}^u}^{-1}(0)$ is compact, then for any $u \in \mathcal{V}^u({\mathcal{P}^u}^{-1}(0))$, the set $\{a \in {\mathcal{P}^u}^{-1}(0) \big| \mathcal{V}^u(a) = u\}$ is finite and its cardinality is locally constant as a function of u. If the cardinality of this set is one for all $u \in \mathcal{V}^u({\mathcal{P}^u}^{-1}(0))$, then $\mathcal{V}^u$ is a diffeomorphism between ${\mathcal{P}^u}^{-1}(0)$ and $\mathcal{V}^u({\mathcal{P}^u}^{-1}(0))$ and the solution mappings (optimal values of the choice variables) of problem (3.1) are smooth functions.[36]

Proof: Let $\mathcal{V}^u({\mathcal{P}^u}^{-1}(0)) =: V^u$. We need to show that V^u is a $H-1$-dimensional manifold. Pick $u \in V^u$. There exists $a \in {\mathcal{P}^u}^{-1}(0)$ such that $u = \mathcal{V}^u(a)$. The structure of the Jacobian ∇V implies that $\mathcal{V}^u : {\mathcal{P}^u}^{-1}(0) \to \mathbf{R}^H$ is an immersion at every point in its domain. Therefore, there exist (i) sets O and W open relative to ${\mathcal{P}^u}^{-1}(0)$ and $\mathbf{R}^H$, respectively, with $a \in O$ and $u \in W$ (ii) sets S_1 and V_1 open in $\mathbf{R}^{H-1}$ and $\mathbf{R}^H$, respectively, and (iii) diffeomorphisms $\phi_1 : O \to S_1$ and $\phi_2 : W \to V_1$ such that $\phi_2 \circ \mathcal{V}^u \circ {\phi_1}^{-1} : S_1 \to V_1$ is the standard immersion $i : S_1 \to V_1$. This implies that $i(S_1) \subset V_1$ is a $H-1$-dimensional

[35] If preferences of consumers are monotonic and $v \in {\mathcal{P}^u}^{-1}(0)$, then $v \in \hat{\Omega}_E$.

[36] Note, we can argue that the dimension of the utility possibility frontier being $H-1$ is a phenomenon that is generically true.

manifold and hence ${\phi^2}^{-1}(i(S_1)) = \mathcal{V}^u(O) \subset W$ is a $H-1$-dimensional manifold.[37] We show that $\mathcal{V}^u(O)$ is open relative to V^u. Suppose not. This means that there exists no set $\hat{W}$ open relative $\mathbf{R}^H$ such that $\hat{W} \cap V^u = \mathcal{V}^u(O)$. Pick a set $\bar{W}$ open relative to $\mathbf{R}^H$ such that the image of $Proj^1 : (\bar{W} \cap V^u) \rightarrow \mathbf{R}^{H-1}$ is the same as the image of $Proj^2 : \mathcal{V}^u(O) \rightarrow \mathbf{R}^{H-1}$, where $Proj^1$ and $Proj^2$ are the projection operators, where the projections are taken into the space formed by coordinates $u_2, \ldots, u_H$.[38] Since, $\mathcal{V}^u(O)$ is not open relative to V^u by our supposition, there exists a point $u = (u_1, \ldots, u_2) \in \bar{W} \cap V^u$ such that $u \notin \mathcal{V}^u(O)$. But $Proj^1(u) \in\ Proj^2(\mathcal{V}^u(O))$. This means there exists $\hat{u} \in \mathcal{V}^u(O)$ such that $\hat{u}_h = \bar{u}_h$ for all $h = 2, \ldots, H$ but $\hat{u}_1 \neq \bar{u}_1$. This means that the value function of the problem (3.1), $\mathcal{U}^u$ takes two different values for parameters $(u_2, \ldots, u_H)$. This is not possible. Hence we have a contradiction. Therefore $\mathcal{V}^u(O)$ is open relative to V^u. Thus, we found an open neighborhood around u in V^u that is diffeomorphic to an open set in $\mathbf{R}^{H-1}$. Since u was arbitrarily chosen, this is true for all $u \in V^u$. Hence, V^u is a $H-1$-dimensional manifold.

Since, $\mathcal{V}^u : {\mathcal{P}^u}^{-1}(0) \rightarrow \mathbf{R}^H$, is an immersion, for all $a \in {\mathcal{P}^u}^{-1}(0)$, we have $\nabla \mathcal{V}^u : T{\mathcal{P}^u}^{-1}(0)_a \rightarrow TV^u_u$ is an isomorphism, where $T{\mathcal{P}^u}^{-1}(0)_a$ and TV^u_u are the tangent spaces of ${\mathcal{P}^u}^{-1}(0)$ and V^u, respectively, at a and $\mathcal{V}^u(a) = u$, respectively. Hence, every u in V^u is a regular value. The cardinality conclusion in (b) about the set $\{a \in {\mathcal{P}^u}^{-1}(0) \big| \ \mathcal{V}^u(a) = u\}$ follows from arguments in Milnor [1931; pp. 8]. This, along with conclusion (a) of this lemma, implies the conclusion that $\mathcal{V}^u$ is a diffeomorphism between ${\mathcal{P}^u}^{-1}(0)$ and $\mathcal{V}^u({\mathcal{P}^u}^{-1}(0))$ if the cardinality of that set is one for all $u \in V^u$. ${\mathcal{P}^u}^{-1}(0)$ is the set of all possible solution vectors of problem (3.1) for all parameter values $(u_2, \ldots, u_H)$ for which solution exists. Suppose $\mathcal{V}^u$ is diffeomorphic and $v \neq v'$ are such that they both solve (3.1) for some $(u_2, \ldots u_H)$. Then $\mathcal{V}^u(v) = \mathcal{V}^u(v')$ contradicting the bijectiveness of $\mathcal{V}^u$. Hence, the solution mappings (optimal values of the choice variables) of problem (3.1) are functions. Since $\mathcal{V}^u : {\mathcal{P}^u}^{-1}(0) \rightarrow \mathcal{V}^u({\mathcal{P}^u}^{-1}(0))$, is a diffeomorphism, its inverse exists and is smooth. Hence, the solution mappings (optimal values of the choice variables) of problem (3.1) are smooth functions of the parameters of the problem (3.1). ■

The allocation corresponding to any $v \in {\mathcal{P}^u}^{-1}(0)$ is obtained by the mapping $\mathcal{A}(v)$ by using consumer demand and producer supply functions. Denote the image of this mapping by $A(\langle \theta_h \rangle) := \mathcal{A}({\mathcal{P}^u}^{-1}(0))$. $A(\langle \theta_h \rangle)$ is the set of allocations $(\langle x^h_0, x^h \rangle, y_0, y^m, y^c, y^g)$ underlying the solutions to (3.1).

Define the mapping $\rho^u : A(\langle \theta_h \rangle) \rightarrow \rho^u(A(\langle \theta_h \rangle))$ with image

$$(u_1(x^1_0, x^1), u_2(x^2_0, x^2), \ldots, u_2(x^H_0, x^H)) \qquad (7.28)$$

[37] Note, locally, $\mathcal{V}^u$ is diffeomorphic, that is $\mathcal{V}^u : O \rightarrow \mathcal{V}^u(O)$ is a diffeomorphism. This is because, ${\mathcal{V}^u}^{-1} : \mathcal{V}^u(O) \rightarrow O$ is defined by $\phi_1^{-1} \circ i^{-1} \circ \phi_2$, which exists and is smooth.

[38] $\bar{W}$ exists. For every $a \in \mathcal{V}^u(O)$, there exists $\epsilon_a > 0$ such that $N_{\epsilon_a}(a) \cap \mathcal{V}^u(O)$ is diffeomorphic to some open set V_a of $\mathbf{R}^{H-1}$. Here $N_{\epsilon_a}(a)$ is the usual open ball around a in $\mathbf{R}^H$. Choose an open (relative to $\mathbf{R}^H$) rectangle $R_a \subset N_{\epsilon_a}(a)$ such that $a \in R_a$ and $Proj^1(R_a) = Proj^2(R_a \cap \mathcal{V}^u(O))$. Then $\bar{W} = \cup_{a \in \mathcal{V}^u(O)} R_a$.

for every $(\langle x_0^h, x^h\rangle, y_0, y^m, y^c, y^g) \in A(\langle\theta_h\rangle)$. ρ^u gives the utility imputations associated with allocations in $A(\langle\theta_h\rangle)$. Clearly, for every $a \in A(\langle\theta_h\rangle)$, there exists $v \in \mathcal{P}^{u-1}(0)$ such that $a = \mathcal{A}(v)$, so that $\rho^u(a) = \mathcal{V}^u(v)$.

Lemma A6: *Suppose Assumptions 1 to 6 hold and $\mathcal{V}^u : \mathcal{P}^{u-1}(0) \to \mathcal{V}^u(\mathcal{P}^{u-1}(0))$ is a diffeomorphism. Then the mapping*

$$\rho^u : A(\langle\theta_h\rangle) \to \rho^u(A(\langle\theta_h\rangle)) \tag{7.29}$$

is bijective.

Proof: If $\mathcal{V}^u$ is diffeomorphic then the allocation mapping $\mathcal{A}$ is one-to-one, under the maintained assumptions. So if $a, a' \in A(\langle\theta_h\rangle)$ such that $a \neq a'$, then $v = \mathcal{A}^{-1}(a) \neq \mathcal{A}^{-1}(a') = v'$ (note v and v' are unique under the maintained assumptions). This implies that $\rho(a) = \mathcal{V}^u(v) \neq \mathcal{V}^u(v') = \rho(a')$. ■

The unit envelope of all possible private ownership economies with unit taxes is obtained by programme (3.2) as the set

$$\hat{U}^u := \{\langle u_h\rangle \in \mathbf{R}^H \big| \ \exists(u_2 \dots, u_H) \in \mathbf{R}^{H-1} \text{ such that } u_1 = \hat{\mathcal{U}}^u(u_2, \dots, u_H)\}. \tag{7.30}$$

The solution of the problem (3.2) is given by (3.3).

Lemma A7: *Suppose assumptions of Lemma A5 hold. Then $\hat{U}^u$ is a manifold of dimension $H-1$.*

Proof: Let $\overset{*}{u} \in \hat{U}^u$. Then there exists $\langle\overset{*}{\theta}_h\rangle \in \Delta_{H-1}$ such that $\overset{*}{u}_1 = \mathcal{U}^u(\overset{*}{u}_2, \dots, \overset{*}{u}_H, \langle\overset{*}{\theta}_h\rangle)$. We need to find a neighborhood around $\overset{*}{u}$ open relative to $\hat{U}^u$ which is diffeomorphic to an open set in $\mathbf{R}^{H-1}$. Under our maintained assumptions, it follows from Lemma A5, that the set

$$U^u(\langle\overset{*}{\theta}_h\rangle) := \{\langle u_h\rangle \in \mathbf{R}^H \big| \ \exists(u_2 \dots, u_H) \in \mathbf{R}^{H-1} \text{ such that } u_1 = \mathcal{U}^u(u_2, \dots, u_H, \langle\overset{*}{\theta}_h\rangle)\} \tag{7.31}$$

is a manifold of dimension $H-1$. Hence there exists a set U open in $\mathbf{R}^H$ with $\overset{*}{u} \in U$ such that $W := U \cap U^u(\langle\overset{*}{\theta}_h\rangle)$ is diffeomorphic to some open set in $\mathbf{R}^{H-1}$. We prove that the mapping $Proj : W \to Proj(W)$ with image $Proj(u) = (u_2, \dots, u_H)$ for every $u \in W$ is such a diffeomorphism. The mapping $Proj : W \to \mathbf{R}^{H-1}$is a standard submersion and hence an open mapping (maps open subsets of W into open sets in $\mathbf{R}^{H-1}$). Further, this is an injective mapping. (For suppose not. Then, there exist u and $\bar{u}$ in W with $u \neq \bar{u}$ such that $Proj(u) = Proj(\bar{u}) =: (u_2, \dots, u_H)$, say. This contradicts the fact that $u_1 = \mathcal{U}^u(u_2, \dots, u_H, \langle\overset{*}{\theta}_h\rangle)$ is unique for parameter values $(u_2, \dots, u_H)$.) Hence, $Proj : W \to Proj(W)$ is a diffeomorphism and $Proj(W)$ is a $H-1$-dimensional manifold. Let the inverse mapping of $Proj : W \to Proj(W)$ be denoted by g. Then $g(\overset{*}{u}_2, \dots, \overset{*}{u}_H) = (\mathcal{U}^u(\overset{*}{u}_2, \dots, \overset{*}{u}_H, \langle\overset{*}{\theta}_h\rangle), \overset{*}{u}_2, \dots, \overset{*}{u}_H) = \overset{*}{u}$. Define a mapping $f : Proj(W) \to$

$\mathbf{R}^H$ by $f(u_2,\ldots,u_H) = (\hat{\mathcal{U}}^u(u_2,\ldots,u_H), u_2,\ldots,u_H)$. f is a well defined mapping as g is well defined in $Proj(W)$, so that for every $(u_2,\ldots,u_H) \in Proj(W)$, the constraint set of problem (3.2) is not empty. In particular, $f(\overset{*}{u}_2,\ldots,\overset{*}{u}_H) = g(\overset{*}{u}_2,\ldots,\overset{*}{u}_H) = \overset{*}{u}$. By an application of the envelope theorem $\nabla g_{\overset{*}{u}} = \nabla f_{\overset{*}{u}}$. By its definition, $\nabla g_{\overset{*}{u}} : \mathbf{R}^{H-1} \to \mathbf{R}^H$ is an immersion. Therefore $\nabla f_{\overset{*}{u}} : \mathbf{R}^{H-1} \to \mathbf{R}^H$ is also an immersion. Hence, following steps similar to the last part of proof of part (a) of Lemma A5, we can show (i) that there exists a neighborhood U around $(\overset{*}{u}_2,\ldots,\overset{*}{u}_H)$ in $Proj(W)$ such that $f(U)$ is a manifold of dimension $H-1$ and (ii) that $f(U)$ is open relative to both $f(Proj(W))$ and $\hat{U}^u$. Thus, we have found a neighborhood open relative to $\hat{U}^u$ around $\overset{*}{u}$ that is diffeomorphic to an open set in $\mathbf{R}^{H-1}$. This is true for any $\overset{*}{u} \in \hat{U}^u$. Hence, the conclusion of the theorem follows.■

Lemma A8: *Suppose assumptions of Lemma A5 hold and $\mathcal{V}^u : \mathcal{P}^{u-1}(0) \to \mathcal{V}^u(\mathcal{P}^{u-1}(0))$ is a diffeomorphism for all $\langle\theta_h\rangle \in \Delta_{H-1}$ economies and the solution mapping (3.3) of the problem (3.2) is a function.*[39] *Then the mapping*

$$\rho^u : A \to \rho^u(A) \tag{7.32}$$

is bijective.

Proof: Proof follows from the fact that, since the solution mapping of the problem (3.2) is a function, every $u \in \rho^u(A)$ corresponds to a tangency between the unit envelope and a Pareto frontier of a *unique* $\langle\theta_h\rangle$ economy. Under the maintained assumptions, Lemma A6 implies the uniqueness of the allocation underlying u.■

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[39] There is an abuse of notation here. The mapping $\mathcal{V}^u$ and the set $\mathcal{P}^{u-1}(0))$ are $\langle\theta_h\rangle$ specific.

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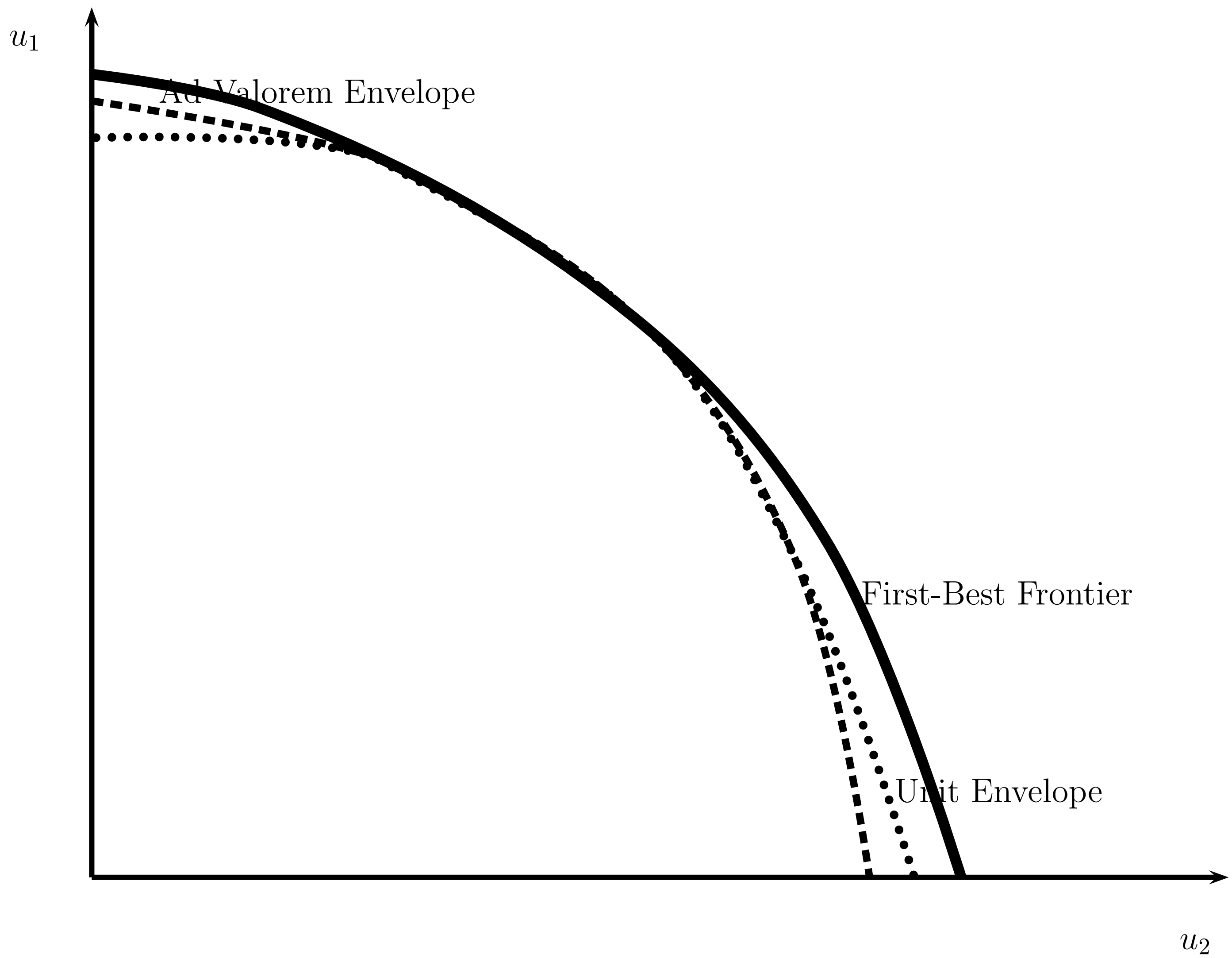
u_1
Ad Valorem Envelope
First-Best Frontier
Unit Envelope
u_2

FIGURE 1