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# Proper orientation of cacti ${ }^{\star}$ 

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#### Abstract

An orientation of a graph $G$ is proper if two adjacent vertices have different in-degrees. The proper-orientation number $\vec{\chi}(G)$ of a graph $G$ is the minimum maximum in-degree of a proper orientation of $G$.

In [1], the authors ask whether the proper orientation number of a planar graph is bounded.

We prove that every cactus admits a proper orientation with maximum indegree at most 7 . We also prove that the bound 7 is tight by showing a cactus having no proper orientation with maximum in-degree less than 7 . We also prove that any planar claw-free graph has a proper orientation with maximum in-degree at most 6 and that this bound can also be attained.


Keywords: proper orientation, graph coloring, cactus graph, claw-free graph, planar graph, block graph.

## 1. Introduction

For basic notions and notations on Graph Theory and Computational Complexity, the reader is referred to [2, 3]. All graphs in this work are considered to be simple.

An orientation $D$ of a graph $G=(V, E)$ is a digraph obtained from $G$ by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the in-degree of $v$ in $D$, denoted by $d_{D}^{-}(v)$, is the number of arcs with root $v$ in $D$. We use the notation $d^{-}(v)$ when the orientation $D$ is clear from the context. An orientation $D$ of $G$ is proper if $d^{-}(u) \neq d^{-}(v)$, for all $u v \in E(G)$. An orientation with maximum in-degree at most $k$ is called a $k$-orientation. The proper-orientation number of a graph

[^0]$G$, denoted by $\vec{\chi}(G)$, is the minimum integer $k$ such that $G$ admits a proper $k$-orientation.

This graph parameter was introduced by Ahadi and Dehghan [4]. They observed that this parameter is well-defined for any graph $G$ since one can always obtain a proper $\Delta(G)$-orientation. Note that every proper orientation of a graph $G$ induces a proper vertex coloring of $G$. Hence, we have the following sequence of inequalities:

$$
\omega(G)-1 \leq \chi(G)-1 \leq \vec{\chi}(G) \leq \Delta(G) .
$$

These inequalities are best possible since, for a complete graph $K_{n}$ :

$$
\omega\left(K_{n}\right)-1=\chi\left(K_{n}\right)-1=\vec{\chi}\left(K_{n}\right)=\Delta\left(K_{n}\right)=n-1 .
$$

In [4], the authors characterize the proper-orientation number of regular bipartite graphs, study other particular subclasses of regular graphs and prove the NP-hardness of the problem even when restricted to planar graphs.

Recently, it has been shown that the problem remains NP-hard for subclasses of planar graphs that are also bipartite and of bounded degree [1]. In the same paper, it is proved that the proper-orientation number of a tree is at most 4.

Theorem 1 ([1]). Every tree has proper-orientation number at most 4.
A natural question is to ask how this theorem can be generalized.
Problem 2. Which graph classes containing the trees have bounded properorientation number?

In [1], several generalizations are suggested: on the one hand, the authors ask whether the proper-orientation number of planar graphs is bounded; on the other hand, they asked whether the proper-orientation number can be bounded by a function of the treewidth. We pose a similar, but simpler, question.

Problem 3. Is there a constant $c$ such that $\vec{\chi}(G) \leq c$, for every outerplanar graph $G$ ?

Already this question seems highly non-trivial. One of the reasons is that, contrary to many other parameters like the chromatic number, the properorientation number is not monotonic. Recall that a graph parameter $\gamma$ is monotonic if $\gamma(H) \leq \gamma(G)$ for every (induced) subgraph $H$ of $G$. For example, the tree $T^{*}$, depicted in Figure 1, satisfies $\vec{\chi}\left(T^{*}\right)=2$, while $\vec{\chi}\left(T^{*} \backslash\{x\}\right)=3$ as $T^{*} \backslash\{x\}$ is exactly the tree $T_{3}$ mentioned in [1]. Its non-monotonicity makes it difficult to handle the proper-orientation number.

In this paper, we consider a standard graph class containing the trees, namely the cacti. A graph $G$ is a cactus if every 2 -connected component of $G$ is either an edge or a cycle. Clearly, every cactus is an outerplanar graph. We prove that the proper orientation of such graphs is bounded by 7 .

Theorem 4. If $G$ is a cactus, then $\vec{\chi}(G) \leq 7$.


Figure 1: Tree $T^{*}$ and a proper 2-orientation of it.

Furthermore, we show in Corollary 20 that this upper bound 7 is attained.
We conclude this section by introducing some definitions and previous results that we need in different sections of this work.

Let $S \subseteq V(G)$ be a subset of vertices of $G$ and $F \subseteq E(G)$ be a subset of its edges. We denote by $G[S]$ the subgraph of $G$ induced by $S$, by $G \backslash F$ the graph obtained from $G$ by removing the edges in $F$ from its edge set $E(G)$, and by $G-S$ the graph $G[V(G) \backslash S]$.

For any two adjacent vertices $u$ and $v$ of $G$, the edge $(u, v)$ is denoted by $u v$. Given an orientation $D$ of $G$, we denote the orientation of $u v$ towards $v$ by $(u, v)$.

Let $T$ be a tree. A leaf of $T$ is a vertex with degree 1. A $t w i g$ of $T$ is a vertex which is not a leaf and whose neighbors are all leaves except possibly one. A bough of $T$ is a vertex which is neither a leaf nor a twig and whose neighbors are all leaves or twigs except possibly one. A branch of $T$ is a vertex which is neither a leaf nor a twig nor a bough and whose neighbors are all leaves or twigs or boughs except possibly one.

The definitions above are the same as the ones used in [1] and we borrow from them. Let $G$ be a graph. The block tree associated to $G$ is the tree $T(G)$ with vertex set the set of blocks of $G$ such that two vertices are adjacent in $T(G)$ if and only if the blocks intersect. A block of order $i$ is said to be an $i$-block. A leaf block (resp. twig block, bough block, branch block) is a block which is a leaf (resp. twig, bough, branch) in $T(G)$. By the definitions in the previous paragraph, observe that if $B$ is of one of these types of blocks, then $B$ may have a neighbor in $T(G)$ that is an exception in its neighborhood. If such a neighbor $B^{\prime}$ exists and $u \in B$ separates $B$ from $B^{\prime}$, then we call $u$ the root of $B$. Otherwise, we pick any vertex of $B$ to be the root of $B$. If $B$ is a twig block with root $r$, then the twig subgraph of $G$ with root $r$ is the union of $B$ and all leaf blocks with root in $V(B) \backslash\{r\}$. If $B$ is a bough block with root $r$, then the bough subgraph of $G$ with root $r$ is the union of $B$ and all twig subgraphs with root in $V(B) \backslash\{r\}$. Observe that twig and bough subgraphs are connected.

Let $B$ be a block in $G$. For any vertex $v \in B$ we denote by $G_{B}\langle v\rangle$ the connected component of $G \backslash E(B)$ containing $v$. If the block $B$ is clear from the context, we often drop the subscript $B$.

## 2. Proper 7-orientation of cacti

In this section, we prove Theorem 4 by considering a minimum counterexample. Such a counter-example is a cactus $G$ that admits no proper 7orientation, and such that every cactus $H$ with fewer vertices than $G$ has a proper 7 -orientation. Observe that such a counter-example $G$ is clearly a connected graph.

The idea of the proof is to analyse the structure of the leaf, twig and bough subgraphs of $G$ and observe that there is always one such subgraph in $G$ with root $r$ such that any proper 7-orientation of $G\langle r\rangle$ (which exists by the minimality of $G$ ) can be extended in a proper 7 -orientation of $G$, which is a contradiction.

If $B$ is a block of $G$ with vertex set $\left\{v_{1}, \ldots, v_{p}\right\}$ appearing in this order on the cycle (or edge), then we write $B$ as $\left\langle v_{1}, \ldots, v_{p}\right\rangle$.

Lemma 5. Let $P=\left(v_{1}, \ldots, v_{n}\right)$ be a path on $n$ vertices, $n \neq 2$. Then, there exists a proper 2 -orientation of $P$ such that $v_{1}$ and $v_{n}$ have in-degree 0 .

Proof. If $n$ is odd, it suffices to orient the arcs of $P$ from vertices with odd indices towards vertices with even indices. This yields an alternating in-degree sequence of 0 's and 2's that starts and ends with 0 . If $n$ is even, orient $\left(v_{1}, \ldots, v_{n-1}\right)$ as above and $v_{n-1} v_{n}$ towards $v_{n-1}$ in order to obtain the desired orientation.

Now we show that, in $G$, every vertex of small degree has a neighbor of higher degree.

Proposition 6. Let $u$ be a vertex of $G$. If $d(u) \leq 7$, then there exists $v \in N(u)$ such that $d(v)>d(u)$.

Proof. Suppose for a contradiction that $d(u) \leq 7$ and all vertices in $N(u)$ have degree at most $d(u)$. Let $D$ be a proper 7 -orientation of $G-u$. For each $v \in N_{G}(u)$, since $d_{G-u}(v)=d_{G}(v)-1 \leq d_{G}(u)-1$, we know that $d_{D}^{-}(v)<d_{G}(u)$. Therefore, because $d_{G}(u) \leq 7$, one can extend $D$ by orienting every edge incident to $u$ in $G$ towards $u$ to obtain a proper 7 -orientation of $G$, a contradiction.

Proposition 7. Every leaf block of $G$ is either a 2-block or a 3-block.
Proof. Observe that, for any leaf block with at least four vertices, there must be at least one vertex of degree 2 whose neighbors also have degree 2 , contradicting Proposition 6.

Proposition 7 implies that a leaf block is either a 1-path (i.e. a path of length 1) or a triangle (i.e. a cycle of length 3). In Figure 2, we present every possible proper orientation of a leaf block.

Proposition 8. Every vertex of $G$ is contained in at most one leaf 2-block.
Proof. By contradiction, suppose that it is not the case and let $\langle u, v\rangle,\langle u, w\rangle$ be two leaf 2 -blocks containing $u$. Let $D$ be a proper 7 -orientation of $G-w$. If $d_{D}^{-}(u) \neq 1$, orienting $u w$ towards $w$ extends $D$ into a proper 7-orientation of $G$, a contradiction. Hence $d_{D}^{-}(u)=1$. Since $D$ is proper, the edge $u v \in E(G)$ must


Figure 2: Leaf blocks and their possible proper orientations.
be the only one oriented towards $u$ in $D$. Therefore all neighbors of $u$ distinct from $v$ and $w$ have in-degree greater than 1 in $D$. Reverting the orientation of $u v$ in $D$ and orienting $u w$ towards $w$, we obtain a 7 -orientation of $G$, which is proper because the in-degree of $u$ is 0 , hence different from the in-degree of all of its neighbors. This is a contradiction.

Proposition 9. Every twig block is a 2-block or a 3-block.
Proof. Let $B$ be a twig block of order $q$ at least 4 , say $B=\left\langle u_{1}, \ldots, u_{q}\right\rangle$ with $u_{1}$ the root of $B$.

Claim 9.1. $d\left(u_{i}\right) \neq 3$, for every $i \in\{2, \ldots, q\}$.
Subproof. By contradiction, suppose that there exists a vertex $u_{i} \in\left\{u_{2}, \ldots, u_{q}\right\}$ of degree 3 in $G$. Note that $u_{i}$ is contained in the block $B$ and in a leaf 2-block, say $\left\langle u_{i}, v\right\rangle$.

First suppose that $i \notin\{2, q\}$ and let $G^{\prime}=G-\left\{u_{i}, v\right\}$. By the minimality of $G$, there exists a proper 7 -orientation $D$ of $G^{\prime}$. If $\left\{d_{D}^{-}\left(u_{i-1}\right), d_{D}^{-}\left(u_{i+1}\right)\right\} \neq\{2,3\}$, then one could extend $D$ to a proper 7 -orientation of $G$ by orienting $u_{i} u_{i-1}$ and $u_{i} u_{i+1}$ towards $u_{i}$ and choosing the orientation of $u_{i} v$ according to the indegrees of $u_{i-1}$ and $u_{i+1}$ in $D$, a contradiction. Hence, without loss of generality, consider that $d_{D}^{-}\left(u_{i-1}\right)=2$ and $d_{D}^{-}\left(u_{i+1}\right)=3$.

Let us extend $D$ by orienting all the arcs incident to $u_{i}$ away from this vertex. The resulting orientation $D^{\prime}$ is not yet proper but we shall prove how to change it into a proper 7 -orientation of $G$. Problems could only appear in edges incident to $u_{i-1}$ or $u_{i+1}$ which had in-degree 3 and 4 respectively in $D$. Observe that these two vertices have degree more than 2 and thus belong to some other blocks which must be leaf blocks since $u_{1}$ is the root of $B$. One can reorient the leaf blocks containing $u_{i+1}$ using the orientations of Figure 2 so that the in-degree of $u_{i+1}$ becomes 3 again. Similarly, if $d\left(u_{i-1}\right)=4$, one can reorient the leaf blocks containing $u_{i-1}$ so that the in-degree of $u_{i-1}$ is in $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{i-2}\right)\right\}$, and if $d\left(u_{i-1}\right)=3$ (that is $u_{i-1}$ is in a unique leaf 2-block), one can reorient the leaf block containing $u_{i-1}$ so that the in-degree of $u_{i-1}$ becomes 2 again. The resulting orientation is then a proper 7 -orientation of $G$, a contradiction.

Suppose now that $i \in\{2, q\}$. Without loss of generality, we may assume that $i=2$. Let $G^{\prime}$ be the connected component of $G-u_{3}$ containing $u_{2}$. Let $D^{\prime}$ be a proper 7 -orientation of $G^{\prime}$. Clearly, $d_{D^{\prime}}^{-}\left(u_{2}\right) \leq 2$. By the previous paragraphs and because $q \geq 4$, we know that $d_{G}\left(u_{3}\right) \neq 3$. If $d_{G}\left(u_{3}\right)>3$, we
can obtain a proper orientation of $G$ by orienting edges $u_{2} u_{3}$ and $u_{3} u_{4}$ towards $u_{3}$ and orienting the leaf blocks containing $u_{3}$ in such a way that $d^{-}\left(u_{3}\right) \in$ $\{3,4\} \backslash d_{D}^{-}\left(u_{4}\right)$; this is a contradiction. Consequently, $d\left(u_{3}\right)=2$, and we can suppose that $2 \in\left\{d_{D}^{-}\left(u_{2}\right), d_{D}^{-}\left(u_{4}\right)\right\}$, as otherwise we get a contradiction by adding $\left(u_{2}, u_{3}\right)$ and ( $u_{4}, u_{3}$ ).

First, suppose that $d_{D}^{-}\left(u_{2}\right) \neq 2$, in which case one can verify that we can suppose that $d_{D}^{-}\left(u_{2}\right)=0$. If $d\left(u_{4}\right)>3$, we reorient the leaf blocks and $u_{3} u_{4}$ so that $u_{4}$ has in-degree in $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{5}\right)\right\}$, then we let $u_{3}$ have in-degree 1 or 2 , depending on the orientation of $u_{3} u_{4}$. This gives us a contradiction, and, because $d_{D}^{-}\left(u_{4}\right)=2$, we get that $d\left(u_{4}\right)=3$ and, by the previous paragraphs, that $q=4$. Let $v^{\prime} \in N\left(u_{4}\right) \backslash B$. We get a contradiction by reversing $\left(v^{\prime}, u_{4}\right)$ and adding $\left(u_{2}, u_{3}\right)$ and $\left(u_{3}, u_{4}\right)$.

Finally, suppose that $d_{D}^{-}\left(u_{2}\right)=2$. Then we can also suppose that $d_{D}^{-}\left(u_{4}\right)=1$ as otherwise we can reverse $\left(v, u_{2}\right)$, orient $u_{2} u_{3}$ towards $u_{2}$, and orient $u_{3} u_{4}$ towards $u_{3}$ to obtain a proper 7 -orientation of $G$. By similar arguments, if $d\left(u_{4}\right)>3$, then we can change its in-degree to some $c \in\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{5}\right)\right\}$; hence $d\left(u_{4}\right) \in\{2,3\}$ and we analyse the cases below:

- $d\left(u_{4}\right)=3$ : let $v^{\prime} \in N\left(u_{4}\right) \backslash B$. Because $d^{-}\left(u_{4}\right)=1$, we know that $\left(v^{\prime}, u_{4}\right),\left(u_{4}, u_{1}\right) \in D$. Reverse $\left(v, u_{2}\right)$ and $\left(v^{\prime}, u_{4}\right)$, and add $\left(u_{3}, u_{2}\right)$ and $\left(u_{4}, u_{3}\right)$ to obtain a contradiction;
- $d\left(u_{4}\right)=2$ : if $q=4$, because $d_{D}^{-}\left(u_{2}\right)=2$ we know that $d_{D}^{-}\left(u_{1}\right) \neq 2$. Reverse $\left(v, u_{2}\right)$ and add $\left(u_{3}, u_{2}\right)$ and $\left(u_{3}, u_{4}\right)$ to obtain a contradiction. Otherwise, by similar arguments we can suppose that $d_{D}^{-}\left(u_{5}\right)=2$. Suppose that $d\left(u_{5}\right)>3$ and reorient the leaf blocks containing $u_{5}$ and $u_{4} u_{5}$ so that $u_{5}$ has in-degree in $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{6}\right)\right\}$. After this, $u_{4}$ has in-degree either 0 or 1 , in which case we reverse $\left(v, u_{2}\right)$, add $\left(u_{3}, u_{2}\right)$ and either $\left(u_{4}, u_{3}\right)$ or $\left(u_{3}, u_{4}\right)$, depending on $u_{4}$. Finally, we can suppose that $d\left(u_{5}\right)=3$ and $q=5$. Let $v^{\prime} \in N\left(u_{5}\right) \backslash B$. Reverse $\left(v, u_{2}\right)$ and $\left(v^{\prime}, u_{5}\right)$, and add $\left(u_{3}, u_{2}\right)$, $\left(u_{4}, u_{5}\right)$ and $\left(u_{3}, u_{4}\right)$ to get a contradiction.

Now we return to the proof of the proposition. By the minimality of $G$, there is a proper 7 -orientation $D$ of $G\left\langle u_{1}\right\rangle$.

We shall extend $D$ into a proper 7 -orientation of $G$, which gives us the desired contradiction. We first add $\left(u_{1}, u_{2}\right),\left(u_{1}, u_{q}\right)$. We then distinguish some cases according to $d_{D}^{-}\left(u_{1}\right)$.

Assume first $d_{D}^{-}\left(u_{1}\right) \notin\{2,4\}$. Add $\left(u_{3}, u_{2}\right),\left(u_{q-1}, u_{q}\right)$ and orient the path $\left(u_{3}, \ldots, u_{q-1}\right)$ according to Lemma 5. So far the vertices $u_{2}, \ldots, u_{q}$ have in-degree 0,1 , or 2 in $B$. For each $i \in\{2, \ldots, q\}$, if $u_{i}$ is contained in some leaf block, then by Claim $9.1 d\left(u_{i}\right) \geq 4$. Thus, by Proposition $8, u_{i}$ is in at least one leaf 3 -block. If $u_{i}$ has in-degree 0 in $B$, then we orient all the leaf blocks containing $u_{i}$ with $A_{1}$ or $T_{1}$, so that $u_{i}$ still has in-degree 0 . If $u_{i}$ has in-degree 1 (resp. 2) in $B$, we orient one leaf 3 -block according to
$T_{3}$ and all other blocks according to $A_{1}$ and $T_{1}$, so that its in-degree is 3 (resp. 4). It is now a simple matter to check that the obtained orientation is a proper 7 -orientation of $G$.

Assume now $d_{D}^{-}\left(u_{1}\right) \in\{2,4\}$. If $q=4$, add $\left(u_{2}, u_{3}\right)$ and $\left(u_{4}, u_{3}\right)$, and one can verify that we can get a contradiction again by orienting the leaf blocks containing vertices in $B$ in the same way as above. So, suppose that $q \geq 6$. Add $\left(u_{2}, u_{3}\right),\left(u_{4}, u_{3}\right),\left(u_{q}, u_{q-1}\right)$, and $\left(u_{q-2}, u_{q-1}\right)$. Furthermore, if $q=7$ then add $\left(u_{4}, u_{5}\right)$, and if $q>7$ apply Lemma 5 to orient the path $\left(u_{4}, \ldots, u_{q-2}\right)$. We then orient the leaf blocks containing vertices in $B$ in the same way as above to get a contradiction.
Therefore, we can consider $q=5$. Add the $\operatorname{arcs}\left(u_{1}, u_{2}\right),\left(u_{1}, u_{5}\right),\left(u_{3}, u_{4}\right)$, and $\left(u_{5}, u_{4}\right)$ to $D$.

If $d\left(u_{2}\right)>2$, then $u_{2}$ is in a leaf 3-block. Add $\left(u_{3}, u_{2}\right)$, and orient one leaf 3 -block containing $u_{2}$ with $T_{2}$ and the other leaf blocks with $A_{1}$ or $T_{1}$ so that $u_{2}$ has in-degree 3 . For $j \in\{3,4,5\}$, if $u_{j}$ is contained in some leaf block, orient its leaf blocks so that the in-degree of $u_{j}$ increases by 2 (using one $T_{3}$ and possibly some $A_{1}$ and $T_{1}$ ). It is simple matter to check that it gives a proper 7 -orientation of $G$. By symmetry, we get the result if $d\left(u_{5}\right)=2$.

Finally, consider $d\left(u_{2}\right)=d\left(u_{5}\right)=2$, and since $B$ is not a leaf block, we can suppose, without loss of generality, that $d\left(u_{3}\right)>2$. In this case, add $\left(u_{2}, u_{3}\right),\left(u_{3}, u_{4}\right)$ and $\left(u_{5}, u_{4}\right)$, orient the leaf block(s) containing $u_{3}$ so that its in-degree is 3 and, if $d\left(u_{4}\right)>2$, orient the leaf block(s) containing $u_{4}$ so that its in-degree is 4 .

Proposition 10. Let $B$ be a twig block with root $u_{1}$.
(a) If $B=\left\langle u_{1}, u_{2}\right\rangle$, then either $d\left(u_{2}\right)=2$ or $u_{2}$ belongs exactly to $B$ and to a leaf 3-block.
(b) If $B=\left\langle u_{1}, u_{2}, u_{3}\right\rangle$, then, for each $j \in\{2,3\}$, $u_{j}$ belongs exactly to $B$ and either a leaf 2-block or a leaf 3-block.

Proof. (a) Assume that $d\left(u_{2}\right)>2$. Let $D$ be a proper 7 -orientation of $G\left\langle u_{1}\right\rangle$. We can suppose that $d\left(u_{2}\right)=3$, as otherwise we extend $D$ to a proper 7 -orientation of $G$ by orienting $u_{1} u_{2}$ towards $u_{2}$ and orienting the leaf blocks with root $u_{2}$ in such a way that its in-degree belongs to $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{1}\right)\right\}$. Consequently, by Proposition 7 and Proposition 8, we obtain that $u_{2}$ is contained exactly in $B$ and in a leaf 3-block.
(b) Suppose first that one vertex of $\left\{u_{2}, u_{3}\right\}$, say $u_{3}$, is in no leaf block, since $B$ is a twig block, so $d\left(u_{2}\right) \geq 3$.

Suppose $d\left(u_{2}\right)>3$ and let $D$ be a proper 7 -orientation of $G\left\langle u_{1}\right\rangle$. One can orient the edges $u_{1} u_{2}$ and $u_{1} u_{3}$ from $u_{1}$ to its neighbors and then orient the
leaf block(s) containing $u_{2}$ and the edge $u_{2} u_{3}$ in such a way that the in-degree of the pair $\left(u_{2}, u_{3}\right)$ is $(3,2)$, in case $d_{D}^{-}\left(u_{1}\right) \notin\{2,3\}$, or $(4,1)$, otherwise. This results in a proper 7 -orientation of $G$, a contradiction.

If $d\left(u_{2}\right)=3$, then let $D$ be a proper 7-orientation of $G-v$, where $v$ is the neighbor of $u_{2}$ not in $B$. Since $u_{2}$ and $u_{3}$ are symmetric in $G-v$, we can suppose that $d_{D}^{-}\left(u_{2}\right) \neq 1$, in which case we can extend $D$ into a proper 7 -orientation of $G$ by orienting $u_{2} v$ towards $v$. This is a contradiction.

Suppose now that $d\left(u_{2}\right)>2$, and $d\left(u_{3}\right)>2$. If $d\left(u_{2}\right) \geq 5$, let $G^{\prime}$ be the component of $G-u_{2}$ containing $u_{1}$. Let $D$ be a proper 7 -orientation of $G^{\prime}$. One could then extend $D$ to a proper 7 -orientation of $G$ by orienting the edges $u_{1} u_{2}$ and $u_{2} u_{3}$ towards $u_{2}$ and orienting the leaf blocks containing $u_{2}$ in such a way that its in-degree belongs to $\{3,4,5\} \backslash\left\{d_{D}^{-}\left(u_{1}\right), d_{D}^{-}\left(u_{3}\right)\right\}$. By symmetry, we get a contradiction in the same way if $d\left(u_{3}\right) \geq 5$. Therefore $d\left(u_{2}\right) \leq 4$ and $d\left(u_{3}\right) \leq 4$. Then, the proposition follows by Proposition 7 and by Proposition 8.

The 2-path, the kite, the bull, the elk, and the moose are the rooted graphs depicted in Figure 3 where the root is the white vertex.


Figure 3: The five possible twig subgraphs.
Propositions 8,9 , and 10 imply directly the following.
Corollary 11. Every twig subgraph in $G$ is either a 2-path, or a kite, or a bull, or an elk, or a moose.

In the following we will very often use this corollary without referring explicitly to it.

All the possible (partial) proper orientations of the twig subgraphs are depicted in Figures 4 to 8. In these figures, the notation $i-j$ means that the corresponding vertex can have any in-degree in this range, depending on the orientation given to the non-oriented edges.

Proposition 12. Let $B$ be a bough block with root u. Every vertex $v$ in $V(B) \backslash$ $\{u\}$ with degree at least 3 is the root of a twig subgraph or a leaf block that is neither a kite nor a moose.

Proof. Let $v$ be a vertex in $V(B) \backslash\{u\}$ with degree at least 3 . It must be the root of at least one twig subgraph or leaf block. Suppose the contrary that $v$ is only root of kites and moose. Let $S$ be the set of vertices that belong to such


Figure 4: Proper orientations of the 2-path.


Figure 5: Proper orientations of the kite.


Figure 6: Proper orientations of the bull.


Figure 7: Proper orientations of the elk.


Figure 8: Proper orientations of the moose.
kites and moose rooted at $v$. Let $D$ be a proper 7 -orientation of the subgraph of $G$ induced by $(V(G) \backslash S) \cup\{v\}$. Observe that $d_{D}^{-}(v) \leq 2$. Thus, one could extend $D$ to $G$ by orienting the kites and moose rooted at $v$ according to $K_{2}$ or $M_{3}$.

Proposition 13. Let $B=\left\langle u_{1}, \ldots, u_{q}\right\rangle$ be a bough block with root $u_{1}$. For all $i \in\{2, \ldots, q\}, d\left(u_{i}\right) \leq 4$.

Proof. Let $i \in\{2, \ldots, q\}$. Let $G^{\prime}$ be the connected component containing $u_{1}$ in $G-u_{i}$. By the minimality of $G, G^{\prime}$ admits a proper 7 -orientation $D$. Set $F=\left\{d_{D}^{-}\left(u_{i-1}\right), d_{D}^{-}\left(u_{i+1}\right)\right\}$. Add the $\operatorname{arcs}\left(u_{i-1}, u_{i}\right)$ and $\left(u_{i+1}, u_{i}\right)$.

If $d\left(u_{i}\right) \geq 7$, then one can properly orient the twig subgraphs and leaf blocks with root $u_{i}$ in such a way that $u_{i}$ has in-degree in $\{5,6,7\} \backslash F$. Observe that all other vertices of those graphs have in-degree at most 4 , so we obtain a proper 7 -orientation of $G$, a contradiction.

If $d\left(u_{i}\right)=6$, by Proposition 12, it is contained in at most one moose. Therefore, one can orient the twig and leaf subgraphs containing $u_{i}$ so that $u_{i}$ has in-degree in $\{4,5,6\} \backslash F$, taking care to use $M_{1}$ for the possible moose. Observe that every other possible twig can avoid a 4 from appearing in $N\left(u_{i}\right)$. Hence, we have a proper 7 -orientation of $G$, a contradiction.

Thus, we can suppose that $d\left(u_{i}\right) \leq 5$, for all $i \in\{2, \ldots, q\}$.
Assume now for a contradiction that there is some $i \in\{2, \ldots, q\}$ such that $d\left(u_{i}\right)=5$.

If $q=2$, then $|F|=1$ and one can extend $D$ to a proper 7-orientation of $G$ by orienting the twig and leaf blocks containing $u_{i}$ so that the in-degree of $u_{i}$ belongs to $\{4,5\} \backslash F$. This is a contradiction so $q \geq 3$.

Observe that if $\{4,5\} \neq F$, then one can extend $D$ to $G$ by orienting the twig and leaf blocks containing $u_{i}$ in such a way that its in-degree belong to $\{4,5\} \backslash F$. Consequently, we can assume that $F=\{4,5\}$. But $d\left(u_{j}\right) \geq d_{D}^{-}\left(u_{j}\right)+1$. So one vertex in $\left\{u_{i-1}, u_{i+1}\right\}$ is $u_{1}$. Free to relabel the vertices in the other sense around $B$, we may assume that $i=2$. Hence $d_{D}^{-}\left(u_{1}\right)=5$ and $d_{D}^{-}\left(u_{3}\right)=4$. So $d\left(u_{3}\right)=5$. Applying the same reasoning to $u_{3}$, we obtain that $q=3$.

Claim 13.1. There is a proper 7 -orientation $D^{\prime}$ of $G^{\prime}$ such that $d_{D^{\prime}}^{-}\left(u_{3}\right) \in$ $\{2,3\}$.

Subproof. The idea is to start form $D$ and to reorient the edges of the leaf blocks and twig subgraphs with root $u_{3}$. Observe that in $D$ all the edges incident to $u_{3}$ are directed towards $u_{3}$. In particular $\left(u_{1}, u_{3}\right)$ is an arc of $D$.

By Propositions 7, 9 and 10, $u_{3}$ is the root of:

1. two subgraphs, $H_{1}$ and $H_{2}$, with $H_{1}$ being a triangle, a bull, an elk or a moose, and $\mathrm{H}_{2}$ being a 1-path, a 2-path, or a kite; or
2. three subgraphs, $H_{1}, H_{2}$ and $H_{3}$, each of them being a 1-path, a 2-path, or a kite.

If Case 1 occurs, then we are in one of the following subcases.
1.1. $H_{1}$ is a moose. Orient it using $M_{3}$ and $H_{2}$ using $A_{2}, P_{2}$ or $K_{1}$ (with the in degree of its neighbor 0 ). This yields the desired proper orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=2$.
1.2. $H_{1}$ is an elk or a bull. If $H_{2}$ is a 1-path or 2-path, then orient $H_{1}$ with $E_{7}$ or $B_{3}$ and $H_{2}$ with $A_{1}$ or $P_{1}$ to obtain the desired orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=3$. If not, then $H_{2}$ is a kite. Orient $H_{1}$ with $E_{3}$ or $B_{2}$ (with the neighbor of $u_{3}$ having in degree different from 2) and $H_{2}$ with $K_{2}$ to obtain the desired orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=2$.
$1.3 H_{1}$ is a triangle. Orient $H_{1}$ with $T_{2}$ and $H_{2}$ with $A_{2}, P_{2}$ or $K_{1}$ to obtain the desired orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=3$.

If Case 2 occurs, without loss of generality, we are in one of the following subcases.
2.1 $H_{1}$ and $H_{2}$ are kites. Orient $H_{1}$ and $H_{2}$ using $K_{2}$ and $H_{3}$ using $A_{2}, P_{2}$ or $K_{1}$, to obtain the desired orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=2$.
$2.2 H_{1}$ is a kite or a 1-path or a 2-path, and $H_{2}$ and $H_{3}$ are 1-path or a 2-path. Orient $H_{1}$ using $K_{1}$ or $A_{2}$ or $P_{2}, H_{2}$ using $A_{2}$ or $P_{2}$, and $H_{3}$ using $A_{1}$ or $P_{1}$, to obtain the desired orientation $D^{\prime}$ with $d_{D^{\prime}}^{-}\left(u_{3}\right)=3$.

Now apply the above reasoning with the orientation $D^{\prime}$ given by Claim 13.1: we have $F \neq\{4,5\}$ because $d_{D^{\prime}}^{-}\left(u_{3}\right) \in\{2,3\}$. Therefore, we obtain a proper 7 -orientation of $G$, a contradiction.

Proposition 6 implies the following.
Proposition 14. Let $u$ be a vertex in $G$.
(a) if $u$ is the root of a kite or a bull, then $d(u) \geq 4$;
(b) if $u$ is the root of an elk or a moose, then $d(u) \geq 5$.

Proposition 15. Every bough block is a 3-block.

Proof. Let $B=\left\langle u_{1}, \ldots, u_{q}\right\rangle$ be a block with root $u_{1}$.
Assume first that $q=2$. Let $D$ be a proper 7 -orientation of $G\left\langle u_{1}\right\rangle$. By Proposition 13, we know that $d\left(u_{2}\right) \leq 4$. If $d\left(u_{2}\right)=4$, we can orient the remaining edges in such a way that $u_{2}$ has in-degree in $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{1}\right)\right\}$ taking care that all kites are oriented using $K_{1}$. This is possible because $u_{2}$ is the root of at most two kites thanks to Proposition 12. This yields a proper 7-orientation of $G$, a contradiction.

Henceforth, since $B$ is a bough block, $u_{2}$ is the root of a twig subgraph $H_{1}$. In particular, $d\left(u_{2}\right)=3$, and by Proposition $14, H_{1}$ is a 2-path, say $\left(u_{2}, x, x^{\prime}\right)$. Vertex $u_{2}$ must also be the root of another subgraph $H_{2}$ that is either a 2-path $\left(u_{2}, y, y^{\prime}\right)$ or a 1-path $(u, y)$. Add the $\operatorname{arc}\left(u_{1}, u_{2}\right)$. If $d_{D}^{-}\left(u_{1}\right) \neq 3$, one can orient $H_{1}$ and $H_{2}$ using $P_{2}$ and $A_{2}$ so that $u_{2}$ get in-degree 3 . This yields a proper 7 -orientation of $G$, a contradiction. Assume $d_{D}^{-}\left(u_{1}\right)=3$. If $H_{2}$ is a 2-path, then orient $H_{1}$ and $H_{2}$ using $P_{1}$ so that $u_{2}$ get in-degree 1. If $H_{2}$ is a 1-path, then orient $H_{1}$ using $P_{2}$ and $H_{2}$ using $A_{1}$ so that $u_{2}$ get in-degree 2. In both cases, it results in a proper 7 -orientation of $G$, a contradiction.

Now, suppose that $q \geq 4$. Note that Propositions 6 and 13 imply $d\left(u_{3}\right) \leq 3$, and that Proposition 14 implies that $u_{3}$ is not root of a kite. So, either $d\left(u_{3}\right)=2$ or $u_{3}$ is the root of a 1 -path or a 2 -path.

Suppose first that $d\left(u_{2}\right)=4$. Let $D$ be a proper orientation of $G\left\langle u_{2}\right\rangle-u_{2}$. Because $d\left(u_{3}\right) \leq 3$, we get that $d_{D}^{-}\left(u_{3}\right) \leq 2$. Add the $\operatorname{arcs}\left(u_{1}, u_{2}\right)$ and $\left(u_{3}, u_{2}\right)$. By Proposition 12, $u_{2}$ is neither the root of a moose nor of two kites. Therefore, one can orient the twig subgraphs and leaf blocks with root $u_{2}$ so that its indegree belongs to $\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{1}\right)\right\}$. This results in a proper 7-orientation of $G$, a contradiction.

Similarly, we get a contradiction if $d\left(u_{q-1}\right)=4$, so we can assume that: $(\star)$ $d\left(u_{i}\right) \leq 3$, for all $i \in\{2, \ldots, q\}$.

Now by Proposition 14-(a), if $d\left(u_{i}\right)=3$ for some $i \in\{2, \ldots, q\}$, it is the root of a 1-path or a 2-path. Consequently, by ( $\star$ ), for all $i \in\{3, \ldots, q-1\}$, it has no neighbor of degree more than 3. Thus, by Proposition 6, we get $d\left(u_{i}\right)=2$, for every $i \in\{3, \ldots, q-1\}$, and $q \leq 5$, for otherwise $u_{4}$ has degree 2 and no neighbor of degree more than 2.

Since $B$ is a bough block and not a twig block, one of its vertices distinct from the root $u_{1}$ must be the root of a twig subgraph. Necessarily, it must be $u_{2}$ or $u_{q}$ as all other vertices have degree 2 . By symmetry, we may assume that it is $u_{2}$. Furthermore, since $d\left(u_{2}\right)=3$, by Proposition 14-(a), $u_{2}$ is necessarily the root of a 2-path, say $\left(u_{2}, x, x^{\prime}\right)$.

Let $D$ be a proper 7 -orientation of $G\left\langle u_{1}\right\rangle$. Orient the edges $u_{1} u_{2}, u_{1} u_{q}$ and $u_{2} u_{3}$ towards $u_{2}, u_{q}$ and $u_{3}$, respectively. We now describe how to extend this orientation in a proper 7 -orientation of $G$, yielding the contradiction. We distinguish two cases depending on whether $q=4$ or $q=5$.

- $q=4$. Assume first $d\left(u_{4}\right)=2$. If $d_{D}^{-}\left(u_{1}\right) \neq 2$, add $\left(u_{3}, u_{4}\right),\left(x, u_{2}\right)$ and $\left(x, x^{\prime}\right)$; otherwise, add their reverses. So suppose that $d\left(u_{4}\right)=3$. Then $u_{4}$ is the root of either a 1-path $\left(u_{4}, y\right)$ or a 2 -path $\left(u_{4}, y, y^{\prime}\right)$ by Proposition 14. If $d_{D}^{-}\left(u_{1}\right) \neq 3$, then $D$ can be extended to $G$ by reversing
$u_{2} u_{3}$ and orienting the remaining edges so that the in-degrees of $u_{2}$ and $u_{4}$ will be 3 . If $d_{D}^{-}\left(u_{1}\right)=3$. Add $\left(u_{4}, y\right)$. If $u_{4}$ is the root of a 1-path, add $\left(u_{3}, u_{4}\right),\left(x, u_{2}\right)$ and $\left(x, x^{\prime}\right)$. Otherwise, $u_{4}$ is the root of a 2-path : add $\left(u_{4}, u_{3}\right),\left(u_{2}, x\right),\left(x^{\prime}, x\right)$, and $\left(y^{\prime}, y\right)$.
- $q=5$. By Proposition 6, we have $d\left(u_{5}\right)=3$. So $u_{5}$ is the root of either a 1-path $\left(u_{5}, y\right)$ or a 2 -path $\left(u_{5}, y, y^{\prime}\right)$ by Proposition 14 . If $d_{D}^{-}\left(u_{1}\right) \neq 3$, reverse $u_{2} u_{3}$ and orient properly the remaining edges in a way that the in-degrees of $u_{2}$ and $u_{5}$ is 3 . If $d_{D}^{-}\left(u_{1}\right)=3$, first add $\left(u_{2}, x\right),\left(x^{\prime}, x\right)$ and $\left(u_{4}, u_{3}\right)$ to $D$. If $u_{5}$ is the root of a 1-path, then add $\left(u_{5}, y\right)$ and $\left(u_{4}, u_{5}\right)$; otherwise, $u_{5}$ is the root of a 2-path : add $\left(y, u_{5}\right),\left(u_{5}, u_{4}\right)$ and $\left(y, y^{\prime}\right)$.

A reindeer is the graph depicted in Figure 9, where the root is the white vertex. It also depicts all possible orientations of the reindeer.


Figure 9: The reindeer and its possible orientations. The dashed edge may or may not exist.

Proposition 16. Every bough subgraph is a reindeer.
Proof. Let $H$ be a bough subgraph rooted at $u_{1}$. It contains a bough block $B$. By Proposition $15, B$ is a 3 -block, say $B=\left\langle u_{1}, u_{2}, u_{3}\right\rangle$. By Proposition 13 , $d\left(u_{2}\right) \leq 4$ and $d\left(u_{3}\right) \leq 4$.

Let $G^{\prime}$ be the connected component of $G-u_{2}$ containing $u_{1}$. Let $D$ be a proper 7 -orientation of $G^{\prime}$.

Assume $d\left(u_{2}\right)=4$. By Proposition 14, $u$ is the root of no moose nor elk, and by Proposition 12 , it is the root of at most one kite. If $\left\{d_{D}^{-}\left(u_{1}\right), d_{D}^{-}\left(u_{3}\right)\right\} \neq\{3,4\}$, then adding $\left(u_{1}, u_{2}\right)$ and $\left(u_{3}, u_{2}\right)$ and using appropriate orientations of the twig subgraphs and leaf blocks with root $u_{2}$, one can get an orientation of $D$ such that $d^{-}\left(u_{2}\right) \in\{3,4\} \backslash\left\{d_{D}^{-}\left(u_{1}\right), d_{D}^{-}\left(u_{3}\right)\right\}$. This is a proper 7-orientation of $D$, a contradiction. Consequently, $\left\{d_{D}^{-}\left(u_{1}\right), d_{D}^{-}\left(u_{3}\right)\right\}=\{3,4\}$, and so $d_{D}^{-}\left(u_{3}\right)=$ $d_{G}\left(u_{3}\right)-1=3$. Let $x$ be a neighbor of $u_{3}$ not in $B$ and let $H$ be the twig subgraph or leaf block with root $u_{3}$ containing $x$. By Proposition 12, one can choose $x$ so that $H$ is not in a kite. Add $\left(u_{2}, u_{3}\right)$ and use $A_{1}, T_{2}, P_{1}$, or $B_{2}$ to reverse $\left(x, u_{3}\right)$. If $u_{2}$ is not the root of two 2 -paths, we can orient the twig subgraphs and leaf blocks with root $u_{2}$ so that its in-degree becomes 2 by using orientations $A, T_{2}, P_{2}, K$ or $B_{2}$. If $u_{2}$ is the root of two 2-paths, we can orient
these 2-paths using $P_{2}$ so that $u_{2}$ gets in-degree 1 . In both cases, we obtain a proper 7 -orientation of $D$, a contradiction.

Similarly, we get a contradiction if $d\left(u_{3}\right)=4$. Therefore $d\left(u_{2}\right) \leq 3$ and $d\left(u_{3}\right) \leq 3$. Since $B$ is a bough block, $u_{2}$ or $u_{3}$ must be the root of a twig subgraph. Without loss of generality, we may assume that $u_{2}$ is. By Proposition 14, $u_{2}$ must be the root of a 2-path, say $\left(u_{2}, x, x^{\prime}\right)$.

Assume $d\left(u_{3}\right)=2$. If $d^{-}\left(u_{1}\right) \notin\{1,2\}$, add $\left(u_{2}, x\right),\left(u_{2}, u_{3}\right),\left(x^{\prime}, x\right)$, and if $\left(u_{3}, u_{1}\right) \in D$, reverse it and add $\left(u_{2}, u_{1}\right)$; otherwise, add $\left(u_{1}, u_{2}\right)$. And if $d^{-}\left(u_{1}\right) \in\{1,2\}$, add $\left(u_{1}, u_{2}\right),\left(u_{3}, u_{2}\right),\left(x, u_{2}\right)$ and $\left(x, x^{\prime}\right)$. In both cases, it results in a proper 7 -orientation of $D$, a contradiction.

Hence $d\left(u_{3}\right)=3$, which by Proposition 12 implies that $u_{3}$ is the root of either a 2-path or a 1-path. Therefore $H$ is a reindeer.

We can finally prove the main result of this paper.
Proof of Theorem 4. If $G$ has no branch blocks, then there exists a vertex $u$ such that $G$ is the union of bough subgraphs, twig subgraphs and leaf blocks with root $u$. In this case, one may obtain a proper 4-orientation of $G$ by orienting all bough subgraphs, twig subgraphs and leaf blocks so that the in-degree of $u$ is 0 .

Thus, $G$ contains a branch block $B$. It must contain a vertex $u$ which is the root of a bough subgraph $R$. By Proposition $16, R$ is a reindeer, and by Proposition 6 , we have $d(u) \geq 4$. Denote by $Q$ the subgraph rooted at $u$ containing exactly all the bough, twig and leaf blocks rooted at $u$.

Let $H$ be the component of $G-u$ that contains $B-u$; then $u$ has at most 2 neighbors in $H$. By minimality of $G, H$ has a proper 7 -orientation $D$. Let $F$ be the set of in-degrees of neighbors of $u$ in $H$. Orient the edges of $H$ incident to $u$ towards $u$.

If $d(u) \geq 7$, we can orient $G\langle u\rangle$ in such a way that $u$ has in-degree in $\{5,6,7\} \backslash F$ and no vertex in $Q$ has in-degree more than 4 . This gives a proper 7 -orientation of $G$, a contradiction.

Assume $d(u)=6$. Let $\alpha$ be an integer in $\{4,5,6\} \backslash F$. We can orient $Q$ in such a way that $u$ has in-degree $\alpha$ and no vertex of $Q-u$ has in-degree $\alpha$. This is possible because no vertex has in-degree 5 in the orientations depicted in Figures 2, 4-8 and 9 and $u$ is in at most two moose, so if $\alpha=4$, we can orient the moose first using $M_{1}$ or $M_{2}$. This gives a proper 7 -orientation of $G$, a contradiction.

Assume $d(u)=4$. If $u$ has two neighbors in $H$, then $Q=R$. Let $\alpha$ be an integer in $\{2,3,4\} \backslash F$. If $\alpha=2$, then orient $R$ with $R_{1}$; if $\alpha=3$, then orient $R$ with $R_{2}$; if $\alpha=4$, then orient $R$ with $R_{3}$. In each case, this yields a proper 7 -orientation of $G$, a contradiction. If $u$ has a unique neighbor in $H$, then $Q$ is the union of $R$ and either a 1-path, or a 2-path, or a kite. Orient that subgraph using $A_{2}, P_{2}$ or $K_{1}$. Now, since $|F|=1$, we can orient $R$ using $R_{2}$ or $R_{3}$ so that the in-degree of $u$ in $\{3,4\} \backslash F$. This yields a proper 7-orientation of $G$, a contradiction.

Finally assume $d(u)=5$. If $F \neq\{4,5\}$, we can orient the edges of $Q$ so that the in-degree of $u$ is some $\alpha \in\{4,5\} \backslash F$, and no vertex of $Q-u$ has in-degree
$\alpha$. If $\alpha=4$, this is possible because $u$ is in at most one moose, and we can start orienting the moose with $M_{2}$. This yields a proper 7 -orientation of $G$, a contradiction. If $F=\{4,5\}$, then $Q$ is the union of $R$ and either a 1-path or a 2-path or a kite. In the first two cases, orient the 1-path or 2-path by using $A_{1}$ or $P_{1}$, and $R$ with $R_{2}$, so that vertex $u$ has in-degree 3 . In the latter case, orient the kite with $K_{2}$ and $R$ with $R_{1}$, so that vertex $u$ has in-degree 2 . In both cases, we obtain a proper 7 -orientation of $G$, a contradiction.

## 3. A tight example

Recall that a block graph is a graph such that each block is a clique. In the sequel, we find a tight example for Theorem 4. As a drawback, we obtain another tight example for Theorem 1 different to the one the authors in [1] propose and an example of a planar graph whose proper orientation must be at least 10 .

Theorem 17. Let $k$ be a positive integer. There exists a block graph $G(k)$ such that $\omega(G)=k$ and $\vec{\chi}(G) \geq 3 k-2$.

Let $G$ be a connected graph, and $K$ be a clique in $G$. We say that $K$ is a pending clique of $G$ if there exists $u \in K$ such that there are no edges between $K-u$ and $V(G)-u$. We say that $u$ is the root of $K$.

Lemma 18. Let $G$ be a connected graph, $K$ be a pending clique of $G$ with size $k$ and root $u$, and $D$ be a proper orientation of $G$. If $u$ has an in-neighbor in $V(G) \backslash K$, then $d_{D}^{-}(u) \geq k$.

Proof. By contradiction, suppose that $u$ has an in-neighbor in $V(G) \backslash K$ and that $d_{D}^{-}(u)=d \in\{1, \ldots, k-1\}$. Because $d \in\{1, \ldots, k-1\}$, and $d(v)=k-1$ for every $v \in K \backslash\{u\}$, we necessarily have that $\left\{d_{D}^{-}(v) \mid v \in K\right\}=\{0, \ldots, k-1\}$. Consequently, there exist $d$ vertices $k_{i_{0}}, \ldots, k_{i_{d-1}}$ in $K$ such that $d_{D}^{-}\left(k_{i_{j}}\right)=j$, for every $j \in\{0, \ldots, d-1\}$. Define $k_{i_{d}}=u$, similarly. Observe that all edges $k_{i_{j}} u$ must be oriented towards $u$, since $k_{i_{j}} k_{i_{\ell}}$ must be oriented towards $k_{i_{\ell}}$, whenever $0 \leq j<\ell \leq d$. This is a contradiction, because $u$ has another in-neighbor that does not belong to $K$ and thus $d_{D}^{-}(u) \geq d+1$.

A $k$-chandelier is the graph obtained from a $k$-clique $K=\left\{v_{0}, \ldots, v_{k-1}\right\}$ by adding $k-1$ pending $k$-cliques in the vertices $v_{1}, \ldots, v_{k-1}$. We say $v_{0}$ is the root of the $k$-chandelier and $K$ is its base.

Lemma 19. Let $G$ be a $k$-chandelier with root $v_{0}$ and base $K=\left\{v_{0}, \ldots, v_{k-1}\right\}$. If $D$ is a proper orientation of $G$ such that $\left(v_{0}, v_{i}\right) \in A(D)$ for every $i \in$ $\{1, \ldots, k-1\}$, then $d_{D}^{-}\left(v_{0}\right) \notin\{k, \ldots, 2 k-2\}$.

Proof. Consider any $i \in\{1, \ldots, k-1\}$. Since $\left(v_{0}, v_{i}\right) \in A(D)$, Lemma 18 yields $d_{D}^{-}\left(v_{i}\right) \geq k$. In addition $d_{D}^{-}\left(v_{i}\right) \leq 2 k-2$, because $d\left(v_{i}\right)=2 k-2$. Therefore, we must have $\left\{d_{D}^{-}\left(v_{i}\right) \mid v_{i} \in K-v_{0}\right\}=\{k, \ldots, 2 k-2\}$ and the lemma follows, because $D$ is a proper orientation.

Proof of Theorem 17. Let $G(k)$ be the graph obtained as follows: we start with a $k$-clique $K=\left\{v_{1}, \ldots, v_{k}\right\}$ and then we add $2 k-1$ pending $k$-cliques $C_{i, j}$ on each $v_{i}$, for every $i \in\{1, \ldots, k\}$ and $j \in\{1, \ldots, 2 k-1\}$. Define

$$
B=\bigcup_{i=1}^{k} \bigcup_{j=1}^{2 k-1} C_{i, j}
$$

Note that at this point $B$ contains all vertices we have added to $G(k)$ so far. Then, for each $u \in B$, we add $3 k-2$ copies of a $k$-chandelier and 2 pending $k$-cliques, all of them with root $u$. This finishes the construction of $G(k)$.

Suppose for a contradiction that there exists a proper orientation $D$ of $G(k)$ such that $\Delta^{-}(D) \leq 3 k-3$.

We claim that, for every $u \in B$, we have that $d_{D}^{-}(u) \notin\{1, \ldots, 2 k-2\}$. Indeed, suppose that $d_{D}^{-}(u) \neq 0$ and thus that $u$ has an in-neighbor $v$. One of the two $k$-cliques pending in $u$ does not contain $v$, so by Lemma $18, d_{D}^{-}(u) \geq k$. Now recall that $d_{D}^{-}(u) \leq \Delta^{-}(D) \leq 3 k-3$. Therefore, $u$ has no in-neighbors in at least one of the $3 k-2 k$-chandeliers with root $u$. Hence, by Lemma 19, $d_{D}^{-}(u) \notin\{k, \ldots, 2 k-2\}$. This proves our claim.

Therefore the in-degrees of the vertices of $B$ are in $\{0,2 k-1, \ldots, 3 k-3\}$. There are exactly $k$ values in this set, so each $k$-clique in $B$ must have exactly one vertex of each in-degree in this set. In particular, each of these cliques of $B$ must contain a vertex of in-degree 0 . Consider the vertex $v_{i} \in K$ such that $d_{D}^{-}\left(v_{i}\right)=2 k-1$. Let $u_{0} \in K$ be such that $d_{D}^{-}\left(u_{0}\right)=0$, and, for each $j \in\{1, \ldots, 2 k-1\}$, let $u_{j} \in C_{i, j}$ be such that $d_{D}^{-}\left(u_{j}\right)=0$. Since all edges $u_{j} v_{i}$ are oriented towards $v_{i}$, we have that $d_{D}^{-}\left(v_{i}\right) \geq 2 k$, a contradiction.

One may see that Theorem 17 provides a tight example for Theorem 1 when $k=2$ and a tight example for Theorem 4 for $k=3$.

Corollary 20. There exist cacti $G$ such that $\vec{\chi}(G) \geq 7$.
Since every block graph $G$ with $\omega(G)=4$ is planar, we also have the following corollary:

Corollary 21. There exist planar graphs $G$ such that $\vec{\chi}(G) \geq 10$.

## 4. Further Research

### 4.1. Proper-orientation number of planar graphs

We believe that Problem 3 must be answered in the affirmative: outerplanar graphs have proper-orientation number bounded by a constant $c$. If such a $c$ exists, then $c \geq 7$, since cacti (and in particular, the one described in Section 3) are outerplanar. A first step would be to established the result for 2-connected outerplanar graphs. We actually believe that in this case this constant should be smaller than 7 and that it should not be much greater than 3 . One can easily attain 3 as a lower bound using the following lemma.

Lemma 22 (1]). Let $k$ be a positive integer, and let $G$ be a graph containing a clique $K$ of size $k+1$. In any proper $k$-orientation of $G$, all edges between $V(K)$ and $V(G) \backslash V(K)$ are oriented from $V(K)$ to $V(G) \backslash V(K)$.

Proposition 23. There exists a 2-connected outerplanar graph $G$ such that $\vec{\chi}(G)=3$.

Proof. Let $G$ be the graph on six vertices defined by $V(G)=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ and $E(G)=\left\{v_{1} v_{2}, v_{2} v_{3}, v_{1} v_{3}, v_{4} v_{5}, v_{5} v_{6}, v_{4} v_{6}, v_{1} v_{4}, v_{2} v_{5}\right\}$. Suppose by way of contradiction that $G$ has a proper 2-orientation $D$. Observe that the sets $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $\left\{v_{4}, v_{5}, v_{6}\right\}$ are cliques in $G$. Thus Lemma 22 implies that the edges $v_{1} v_{4}$ and $v_{2} v_{5}$ must be oriented in both ways, a contradiction.

To the more general case of planar graphs, similarly, it would be interesting to find a constant $c^{\prime}$, if it exists, satisfying $\vec{\chi}(G) \leq c^{\prime}$, for every planar graph $G$. We provided in Section 3 a planar graph whose proper orientation number is 10 and thus $c^{\prime} \geq 10$.

## 4.2. $\vec{\chi}$-bounded families of graphs

Gyárfás [5] introduced the concept of $\chi$-bounded graph classes. A class of graph $\mathcal{G}$ is said to be $\chi$-bounded if there is a function $f$ such that $\chi(G) \leq$ $f(\omega(G))$ for every $G \in \mathcal{G}$. Similarly, one can define $\vec{\chi}$-bounded graph classes. A class of graph $\mathcal{G}$ is said to be $\vec{\chi}$-bounded if there is a function $f$ such that $\vec{\chi}(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Because $\chi \leq \vec{\chi}$, a $\vec{\chi}$-bounded graph class is also $\chi$-bounded. Conversely, one might wonder which $\chi$-bounded graph classes are also $\vec{\chi}$-bounded.

The $\chi$-boundedness of graph classes defined by forbidden induced subgraphs have been particularly investigated. For a fixed graph $F$, let us denote by Forb $(F)$ the class of graphs that do not contain $F$ as an induced subgraph. Erdős [6] showed that there are graphs with arbitrarily high girth and chromatic number. This implies that if $F$ contains a cycle, then $\operatorname{Forb}(F)$ is not $\chi$-bounded. Conversely, Gyárfás [7] and Sumner [8] independently made the following beautiful and difficult conjecture

Conjecture 24 ([7] and [8]). For every tree T, the class $\operatorname{Forb}(T)$ is $\chi$-bounded.
It is natural to ask whether this conjecture generalizes to proper orientations.
Problem 25. Is the class $\operatorname{Forb}(T) \vec{\chi}$-bounded for all tree $T$ ?
Gyárfás [5] establishes Conjecture 24 for stars by showing that a graph in Forb $\left(K_{1, n}\right)$ has maximum degree $R(n, \omega(G))$, where $R(p, q)$ denotes the Ramsey number $(p, q)$. In particular, this shows that $\operatorname{Forb}\left(K_{1, n}\right)$ is also $\vec{\chi}$-bounded.

In particular, if $G$ is a planar claw-free graph (recall that the claw is the graph $K_{1,3}$ ), Gyárfás result gives us that $\vec{\chi}(G) \leq \Delta(G) \leq R(3,4)=9$. This is also a partial answer to whether planar graphs have bounded proper orientation number. However, this bound is not tight, as we show next. In [9], Plummer showed that any claw-free 3 -connected planar graph has maximum degree at most 6 . His result can be extended to any claw-free planar graph.

Theorem 26. If $G$ is a claw-free planar graph, then $\Delta(G) \leq 6$.
Proof. The proof is by induction on the number of vertices of $G$. If $G$ is disconnected, then, by the induction hypothesis, each connected component of $G$ has maximum degree at most 6 and so $\Delta(G) \leq 6$.

Assume that $G$ has a cut-vertex $u$. As $G$ is claw-free, $G-u$ has exactly two components $C_{i}, i=1,2$, and the neighborhood of $u$ in each $C_{i}$ is a clique $N_{i}$. Observe that $N_{i} \cup\{u\}$ is a clique, which has size at most 4 because $G$ is planar, so $\left|N_{i}\right| \leq 3$. Hence $d(u)=\left|N_{1}\right|+\left|N_{2}\right| \leq 6$. Now by the induction hypothesis applied to $G\left[V\left(C_{1}\right) \cup\{u\}\right]$ and $G\left[V\left(C_{2}\right) \cup\{u\}\right]$, we obtain that every vertex distinct from $u$ has degree at most 6 . Therefore $\Delta(G) \leq 6$. Henceforth we may assume that $G$ is 2 -connected.

Assume that $G$ has a 2 -cut $\{u, v\}$ (that is $G-\{u, v\}$ is disconnected). The graph $G^{\prime}=G-v$ is connected with cut-vertex $u$. As above, $G^{\prime}-u$ has exactly two components, $C_{1}$ and $C_{2}$, and $N_{i}=N(u) \cap C_{i}$ is a clique, for $i=1,2$ of size at most 3. We claim that $d(u) \leq 6$. If $u v \notin E(G)$, then $d(u)=\left|N_{1}\right|+\left|N_{2}\right|$, so $d(u) \leq 6$. If $u v \in E(G)$, then $d(u)=\left|N_{1}\right|+\left|N_{2}\right|+1$. But $\left|N_{1}\right|+\left|N_{2}\right| \leq 5$ for otherwise there exist $u_{1} \in N_{1}$ and $u_{2} \in N_{2}$ non-adjacent to $v$ (because $G$ has no clique of size 5$)$, so $G\left[\left\{u, v, u_{1}, u_{2}\right\}\right]$ is a claw, a contradiction. Therefore $d(u) \leq 6$. Similarly, one proves $d(v) \leq 6$. Now by the induction hypothesis applied to $G[V(C) \cup\{u, v\}]$ for each connected component of $G-\{u, v\}$, we obtain that every vertex distinct from $u$ and $v$ has degree at most 6 ; hence $\Delta(G) \leq 6$.

Henceforth, we may assume that $G$ is 3 -connected and the result follows by Plummer [9].


Figure 10: A planar claw-free graph $G^{*}$ with maximum degree 6 and proper orientation number 6 .

Theorem 26 is tight as shown by the graph $G$ depicted in Figure 10 which is claw-free, planar and has maximum degree 6 . Moreover, Theorem 26 implies that every planar claw-free graph has proper-orientation number at most 6 . This is tight as shown by the following proposition.

Proposition 27. The graph $G^{*}$, depicted in Figure 10, has proper orientation number equal to 6 .
Proof. The graph $G^{*}$ is made of 5 blocks isomorphic to $K_{4}$. One one them (in the center of the figure), denoted by $C$ intersects the four others. For every vertex
$v$ of $C$, let $B(v)$ be the block intersecting $C$ in $v$. Assume for a contradiction that $G$ has a proper 5 -orientation $D$. There are two vertices $v_{1}$ and $v_{2}$ in $C$, such that $d_{D}^{-}\left(v_{i}\right) \in\{0,1,2,3\}$. Now the set of in-degrees of the other vertices of $B\left(v_{i}\right)$ is exactly $\{0,1,2,3\} \backslash\left\{d_{D}^{-}\left(v_{i}\right)\right\}$. Thus inside $B(v)$ there are exactly $6-\left(0+1+2+3-d_{D}^{-}\left(v_{i}\right)\right)=d_{D}^{-}\left(v_{i}\right)$ arcs towards $v$. Hence all the edges such that $v_{i}$ is an endpoint are oriented from $v_{i}$ to its neighbors in $C$. This is a contradiction, because the edge $v_{1} v_{2}$ cannot be oriented both ways.

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