Infinitary proof theory : the multiplicative additive case

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Abstract

Infinitary and regular proofs are commonly used in fixed point logics. Being natural intermediate devices between semantics and traditional finitary proof systems, they are commonly found in completeness arguments, automated deduction, verification, etc. However, their proof theory is surprisingly underdeveloped. In particular, very little is known about the computational behavior of such proofs through cut elimination. Taking such aspects into account has unlocked rich developments at the intersection of proof theory and programming language theory. One would hope that extending this to infinitary calculi would lead, *e.g.*, to a better understanding of recursion and corecursion in programming languages. Structural proof theory is notably based on two fundamental properties of a proof system: cut elimination and focalization. The first one is only known to hold for restricted (purely additive) infinitary calculi, thanks to the work of Santocanale and Fortier; the second one has never been studied in infinitary systems. In this paper, we consider the infinitary proof system $\mu\mathsf{MALL}^\infty$ for multiplicative and additive linear logic extended with least and greatest fixed points, and prove these two key results. We thus establish $\mu\mathsf{MALL}^\infty$ as a satisfying computational proof system in itself, rather than just an intermediate device in the study of finitary proof systems.

1 Introduction

Proof systems based on non-well-founded derivation trees arise naturally in logic, even more so in logics featuring fixed points. A prominent example is the long line of work on tableaux systems for modal μ-calculi, *e.g.*, [16, 24, 14, 11], which have served as the basis for analysing the complexity of the satisfiability problem, as well as devising practical algorithms for solving it. One key observation in such a setting, and many others, is that one needs not consider arbitrary infinite derivations but can restrict to *regular* derivation trees (also known as *circular* proofs) which are finitely representable and amenable to algorithmic manipulation. Because infinitary systems are easier to work with than the finitary proof systems (or axiomatizations) based on Kozen-Park (co)induction schemes, they are often found in completeness arguments for such finitary systems [16, 27, 28, 29, 15, 12]. We should note, however, that those arguments are far from being limited to translations from (regular) infinitary to finitary proofs, since such translations are very complex and only known to work in limited cases. There are many other uses of infinite (or regular) derivations, *e.g.*, to study the relationship between induction and infinite descent in first-order arithmetic [8], to generate invariants for program verification in separation logic [7], or as an intermediate between ludics' designs and proofs in linear logic with fixed points [5]. Last but not least, Santocanale introduced circular proofs [22] as a system for representing morphisms in μ-bicomplete categories [21, 23], corresponding to simple computations on (co)inductive data.

Surprisingly, despite the elegance and usefulness of infinitary proof systems, few proof theoretical studies are directly targetting these objects. More precisely, we are concerned with an analysis of proofs that takes into account their computational behaviour in terms of cut elimination. In other words, we would hope that the Curry-Howard correspondence extends nicely to infinitary proofs. In this line of proof-theoretical study, two main properties stand out: cut elimination and focalization; we shall see that they have been barely addressed in infinitary proof systems. The idea of cut elimination is as old as sequent calculus, and at the heart of the proof-as-program viewpoint, where the process of eliminating cuts (indirect reasoning) in proofs is seen as computation. Considering logics with least and greatest fixed points, the computational behavior of induction and coinduction is recursion and corecursion, two important and complex programming principles that would deserve a logical understanding. Note that the many completeness results for infinitary proof systems (*e.g.*, for modal μ-calculi) only imply cut admissibility, but say nothing about the computational process of cut elimination. To our knowledge, leaving aside an early and very restrictive result of Santocanale [22], cut elimination has only been studied by Fortier and Santocanale [13] who considered an infinitary sequent calculus for lattice logic (purely additive linear logic with least and greatest fixed points) and showed that certain cut reductions converge to a limit cut-free derivation. Their proof involves a mix of combinatorial and topological arguments. So far, it has resisted attempts to extend it beyond the purely additive case. The second key property, much more recently identified than cut elimination, is focalization. It has appeared in the work of [3] on proof search and logic programming in linear logic, and is now recognized as one of the deep outcomes of linear logic, putting to the foreground the role of *polarity* in logic. In a way, focalization generalizes the invertibility results that are notably behind most deductive systems for classical μ-calculi, by bringing some key observations about non-invertible connectives. Besides its deep impact on proof search and logical frameworks, focalization resulted in important advances in all aspects of computational proof theory: in the game-semantical analysis of logic [17, 19], the understanding of evaluation order of programming languages, CPS translations, or semantics of pattern matching [10, 30], the space compression in computational complexity [26, 6], etc. Briefly, one can say that while proof nets have led to a better understanding of phenomena related to parallelism with proof-theoretical methods, polarities and focalization have led to a fine-grained understanding of sequentiality in proofs and programs. To the best of our knowledge, while reversibility has since long been a key-ingredient in completeness arguments based on infinitary proof systems, focalization has simply never been studied in such settings.

Organization and contributions of the paper. In this paper, we consider the logic $\mu\mathsf{MALL}$, that is multiplicative additive linear logic extended with least and greatest fixed point operators. It has been studied in finitary sequent calculus [4]: it notably enjoys cut elimination, and focalization has been shown to extend nicely (though not obviously) to it. We give in Section 2 a natural infinitary proof system for $\mu\mathsf{MALL}$, called $\mu\mathsf{MALL}^\infty$, which notably extends that of Santocanale and Fortier [13]. The system $\mu\mathsf{MALL}^\infty$ is also related to $\mu\mathsf{MALL}$ in the sense that any $\mu\mathsf{MALL}$ derivation can be turned into a $\mu\mathsf{MALL}^\infty$ proof, with cuts. We study the focalization of $\mu\mathsf{MALL}^\infty$ in Section 3. We find out that, even though fixed point polarities are not forced in the finitary sequent calculus for $\mu\mathsf{MALL}$, they are uniquely determined in $\mu\mathsf{MALL}^\infty$. Despite some novel aspects due to the infinitary nature of our calculus, we are able to re-use the generic *focalization graph* argument [20] to prove that focalized proofs are complete. We then turn to cut elimination in Section 4 and show that (fair) cut reductions converge to an infinitary cut free derivation. We could not apply any standard cut elimination technique (*e.g.*, induction on formulas and proofs, reducibility arguments, topological arguments as in [13]) and propose instead an unusual argument in which a coarse truth semantics is used to show that the cut elimination process cannot go wrong. We also note here that, even for the regular fragment of $\mu\mathsf{MALL}^\infty$, it would be highly non-trivial to obtain cut elimination from the result for μMALL, since it is not known whether regular $\mu\mathsf{MALL}^\infty$ derivations can be translated to μMALL derivations (even without requiring that this translation preserves the computational behaviour of proofs). We conclude in Section 5 with directions for future work. Appendices provide technical details, proofs, and additional background material.

2 μMALL and its infinitary proof system $\mu\mathsf{MALL}^\infty$

In this section we introduce multiplicative additive linear logic extended with least and greatest fixed point operators, and an infinitary proof system for it.

▶ **Definition 1.** Given an infinite set of propositional variables $\mathcal{V} = \{X, Y, \dots\}$, $\mu\mathsf{MALL}^\infty$ ***pre-formulas*** are built over the following syntax:

$$\varphi, \psi ::= \mathbf{0} \mid \top \mid \varphi \oplus \psi \mid \varphi \& \psi \mid \bot \mid \mathbf{1} \mid \varphi ⅋ \psi \mid \varphi \otimes \psi \mid \mu X.\varphi \mid \nu X.\varphi \mid X \quad \text{with} \quad X \in \mathcal{V}.$$

The connectives μ and ν bind the variable X in φ. From there, bound variables, free variables and capture-avoiding substitution are defined in a standard way. The subformula ordering is denoted $\leq$ and $\mathrm{fv}(\bullet)$ denotes free variables. Closed pre-formulas are simply called ***formulas***. Note that negation is not part of the syntax, so that we do not need any positivity condition on fixed point expressions.

▶ **Definition 2.** ***Negation*** is the involution on pre-formulas written $\varphi^\perp$ and satisfying $(\varphi ⅋ \psi)^\perp = \psi^\perp \otimes \varphi^\perp$, $(\varphi \oplus \psi)^\perp = \psi^\perp \& \varphi^\perp$, $\bot^\perp = \mathbf{1}$, $\mathbf{0}^\perp = \top$, $(\nu X.\varphi)^\perp = \mu X.\varphi^\perp$, $X^\perp = X$.

Having $X^\perp = X$ might be surprising, but it is harmless since our proof system will only deal with closed pre-formulas. Our definition yields, *e.g.*, $(\mu X.X)^\perp = (\nu X.X)$ and $(\mu X.\mathbf{1} \oplus X)^\perp = (\nu X.X \& \bot)$, as expected [4]. Note that we also have $(\varphi[\psi/X])^\perp = \varphi^\perp[\psi^\perp/X]$.

Sequent calculi are sometimes presented with sequents as sets or multisets of formulas, but most proof theoretical observations actually hold in a stronger setting where one distinguishes between several *occurrences* of a formula in a sequent, which gives the ability to precisely *trace* the provenance of each occurrence. This more precise viewpoint is necessary, in particular, when one views proofs as programs. In this work, due to the nature of our proof system and because of the operations that we perform on proofs and formulas, it is also crucial to work with occurrences. There are several ways to formally treat occurrences; for the sake of clarity, we provide below a concrete presentation of that notion which is well suited for our needs.

▶ **Definition 3.** An ***address*** is a word over $\Sigma = \{l, r, i\}$, which stands for left, right and inside. We define a ***duality*** over Σ^* as the morphism satisfying $l^\perp = r$, $r^\perp = l$ and $i^\perp = i$. We say that α' is a ***sub-address*** of α when α is a prefix of α', written $\alpha \sqsubseteq \alpha'$. We say that α and β are ***disjoint*** when α and β have no upper bound wrt. $\sqsubseteq$.

▶ **Definition 4.** A ***(pre)formula occurrence*** (denoted by F, G, H) is given by a (pre)formula φ and an address α, and written φ_α. We say that occurrences are ***disjoint*** when their addresses are. The occurrences φ_α and ψ_β are ***structurally equivalent***, written $\varphi_\alpha \equiv \psi_\beta$, if $\varphi = \psi$. Operations on formulas are extended to occurrences as follows: $(\varphi_\alpha)^\perp = (\varphi^\perp)_{\alpha^\perp}$; for any $\star \in \{⅋, \otimes, \oplus, \&\}$, $F \star G = (\varphi \star \psi)_\alpha$ if $F = \varphi_{\alpha l}$ and $G = \psi_{\alpha r}$; for any $\sigma \in \{\mu, \nu\}$, $\sigma X.F = (\sigma X.\varphi)_\alpha$ if $F = \varphi_{\alpha i}$; we also allow ourselves to write units as formula occurrences without specifying their address, which can be chosen arbitrarily. Finally, ***substitution of occurrences*** forgets addresses: $(\varphi_\alpha)[\psi_\beta/X] = (\varphi[\psi/X])_\alpha$.

▶ **Example.** Let $F = \varphi_{\alpha l}$ and $G = \psi_{\alpha r}$. We have, on the one hand, $(F \otimes G)^\perp = ((\varphi \otimes \psi)_\alpha)^\perp = (\psi^\perp ⅋ \varphi^\perp)_{\alpha^\perp}$ and, on the other hand, $G^\perp ⅋ F^\perp = (\psi^\perp)_{\alpha^\perp l} ⅋ (\varphi^\perp)_{\alpha^\perp r} = (\psi^\perp ⅋ \varphi^\perp)_{\alpha^\perp}$. Thus,

$$\frac{\vdash F,\Gamma \quad \vdash G,\Gamma}{\vdash F \& G,\Gamma}\ (\&) \qquad \frac{\vdash F,G,\Gamma}{\vdash F ⅋ G,\Gamma}\ (⅋) \qquad \frac{\vdash F_i,\Gamma}{\vdash F_1 \oplus F_2,\Gamma}\ (\oplus_i) \qquad \frac{\vdash F,\Gamma \quad \vdash G,\Delta}{\vdash F \otimes G,\Gamma,\Delta}\ (\otimes)$$

$$\frac{}{\vdash \top,\Gamma}\ (\top) \qquad \frac{\vdash \Gamma}{\vdash \bot,\Gamma}\ (\bot) \qquad \text{(no rule for } \mathbf{0}) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{\vdash F[\mu X.F/X],\Gamma}{\vdash \mu X.F,\Gamma}\ (\mu) \qquad \frac{\vdash G[\nu X.G/X],\Gamma}{\vdash \nu X.G,\Gamma}\ (\nu) \qquad \frac{F \equiv G}{\vdash F,G^\perp}\ (\mathsf{Ax}) \qquad \frac{\vdash \Gamma,F \quad \vdash F^\perp,\Delta}{\vdash \Gamma,\Delta}\ (\mathsf{Cut})$$

Figure 1 Rules of the proof system $\mu\mathsf{MALL}^\infty$.

$(F \otimes G)^\perp = G^\perp ⅋ F^\perp$. We could have designed our system to obtain $(F \otimes G)^\perp = F^\perp ⅋ G^\perp$ instead; this choice is inessential for the present work but makes our definitions suitable, in principle, for a treatment of non-commutative logic.

▶ **Definition 5.** The ***Fischer-Ladner closure*** of a formula occurrence F, denoted by $\mathrm{FL}(F)$, is the least set of formula occurrences such that $F \in \mathrm{FL}(F)$ and, whenever $G \in \mathrm{FL}(F)$,

- $G_1, G_2 \in \mathrm{FL}(F)$ if $G = G_1 \star G_2$ for any $\star \in \{\oplus, \&, ⅋, \otimes\}$;
- $B[G/X] \in \mathrm{FL}(F)$ if $G = \sigma X.B$ for $\sigma \in \{\nu, \mu\}$.

We say that G is a ***sub-occurrence*** of F if $G \in \mathrm{FL}(F)$. Note that, for any F and α, there is at most one φ such that φ_α is a sub-occurrence of F.

We are now ready to introduce our infinitary sequent calculus. Details regarding formula occurrences can be ignored at first read, and will only make full sense when one starts permuting inferences and eliminating cuts.

▶ **Definition 6.** A ***sequent***, written $\vdash \Gamma$, is a finite set of pairwise disjoint, closed formula occurrences. A ***pre-proof*** of $\mu\mathsf{MALL}^\infty$ is a possibly infinite tree, coinductively generated by the rules of Figure 1, subject to the following conditions: any two formulas occurrences appearing in different branches must be disjoint except if the branches first differ right after a $(\&)$ inference; if φ_α and $\psi_{\alpha^\perp}$ occur in a pre-proof, they must be the respective sub-occurrences of the formula occurrences F and $F^\perp$ introduced by a (Cut) rule.

The disjointness condition on sequents ensures that two formula occurrences from the same sequent will never engender a common sub-occurrence, *i.e.*, we can define traces uniquely. The disjointness condition on pre-proofs is there to ensure that the proof transformations used in focusing and cut elimination preserve the disjointness condition on sequents. Note that these conditions are not restrictive. Clearly, the condition on sequents never prevents the (backwards) application of a propositional rule. Moreover, there is an infinite supply of disjoint addresses, *e.g.*, $\{\ r^n l\ :\ n > 0\ \}$. One may thus pick addresses from that supply for the conclusion sequent of the derivation, and then carry the remaining supply along proof branches, splitting it on branching rules, and consuming a new address for cut rules.

Pre-proofs are obviously unsound: the pre-proof schema shown on the right allows to derive any formula. In order to obtain proper proofs from pre-proofs, we will add a validity condition. This condition will reflect the nature of our two fixed point connectives.

$$\frac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\vdots}{\vdash \nu X.X, F}\ (\nu)}{\vdash F}\ (\mathsf{Cut})$$

▶ **Definition 7.** Let $\gamma = (s_i)_{i \in \omega}$ be an infinite branch in a pre-proof of $\mu\mathsf{MALL}^\infty$. A ***thread*** t in γ is a sequence of formula occurrences $(F_i)_{i \in \omega}$ with $F_i \in s_i$ and $F_i \sqsubseteq F_{i+1}$. The set of formulas that occur infinitely often in $(F_i)_{i \in \omega}$ (when forgetting addresses) admits a minimum wrt. the subformula ordering, denoted by $\mathsf{min}(t)$. A thread t is ***valid*** if $\mathsf{min}(t)$ is a ν formula and the thread is not eventually constant, *i.e.*, the formulas F_i are always eventually principal.

▶ **Definition 8.** The ***proofs*** of $\mu\mathsf{MALL}^\infty$ are those pre-proofs in which every infinite branch contains a valid thread.

This validity condition has its roots in parity games and is very natural for infinitary proof systems with fixed points. It is somehow independent of the ambiant logic, and only deals with fixed points. It is commonly found in deductive systems for modal μ-calculi: see [11] for a closely related presentation, which yields a sound and complete sequent calculus for linear time μ-calculus. The validity conditions of Santocanale's circular proofs [22, 13], with and without cut, are also instances of the above notion.

In the rest of the paper, we work mostly with formula occurrences and will often simply call them formulas when it is not ambiguous. As usual in sequent calculus, (Ax) on a formula F can be expanded into axioms on its immediate subformulas. Repeating this process, one obtains an axiom-free and valid proof of the original sequent. In fact, this construction yields a *regular* derivation tree, the simplest kind of finitely representable infinite derivation.

▶ **Proposition 9.** *Rule* (Ax) *is admissible in* $\mu\mathsf{MALL}^\infty$*.*

This basic observation, proved in appendix A, justifies that the (Ax) rule will be ignored in the rest of the paper. In particular, we consider that axioms are expanded away before dealing with cut elimination. Our system $\mu\mathsf{MALL}^\infty$ is naturally equipped with the cut elimination rules of MALL, extended with the obvious principal and auxiliary rules for fixed point connectives (we do not show symmetric cases):

$$\dfrac{\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\,(\mu) \qquad \dfrac{\vdash F^\perp[\nu X.F^\perp/X],\Delta}{\vdash \nu X.F^\perp,\Delta}\,(\nu)}{\vdash\Gamma,\Delta}\,(\mathsf{Cut}) \quad\longrightarrow\quad \dfrac{\vdash \Gamma, F[\mu X.F/X] \qquad \vdash F^\perp[\nu X.F^\perp/X],\Delta}{\vdash\Gamma,\Delta}\,(\mathsf{Cut})$$

$$\dfrac{\dfrac{\vdash\Gamma,F[\mu X.F/X],G}{\vdash\Gamma,\mu X.F,G}\,(\mu)\qquad \vdash G^\perp,\Delta}{\vdash\Gamma,\mu X.F,\Delta}\,(\mathsf{Cut}) \quad\longrightarrow\quad \dfrac{\dfrac{\vdash\Gamma,F[\mu X.F/X],G \qquad \vdash G^\perp,\Delta}{\vdash\Gamma,F[\mu X.F/X],\Delta}\,(\mathsf{Cut})}{\vdash\Gamma,\mu X.F,\Delta}\,(\mu)$$

Natural numbers may be expressed as $\varphi_{\mathsf{nat}} := \mu X.\mathbf{1}\oplus X$. Occurrences of that formula will be denoted N, N', etc. We give below a few examples of proofs/computations on natural numbers, shown using two sided sequents for clarity: $F_1,\dots,F_n \vdash \Gamma$ should be read as $\vdash \Gamma, F_1^\perp,\dots,F_n^\perp$ as usual. The proof π_{succ}, shown on the right, computes the successor on natural numbers: if we cut it against a (necessarily finite) cut-free proof of N we obtain after a finite number of cut elimination steps a proof of N' which is the right injection (rule (μ) followed by $(\oplus_2)$, which represents the successor) of the original proof of N, relocated at the address of N''. Consider now the following pre-proof, called π_{dup}:

$$\dfrac{\dfrac{\dfrac{}{N\vdash N''}\,(\mathsf{Ax})}{N\vdash \mathbf{1}\oplus N''}\,(\oplus_2)}{N\vdash N'}\,(\mu)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash N_1}\,(\mu),(\oplus_1),(\mathbf{1}) \qquad \dfrac{}{\vdash N_2}\,(\mu),(\oplus_1),(\mathbf{1})}{\mathbf{1}\vdash N_1\otimes N_2}\,(\bot),(\otimes) \qquad \dfrac{\dfrac{(\star)}{N'\vdash N_1'\otimes N_2'} \qquad \dfrac{\pi_{\mathsf{succ}} \qquad \pi_{\mathsf{succ}}}{N_1'\otimes N_2'\vdash N_1\otimes N_2}\,(\parr),(\otimes)}{N'\vdash N_1\otimes N_2}\,(\mathsf{Cut})}{(\star)\qquad N\vdash N_1\otimes N_2}\,(\nu),(\&)$$

Here, $(\star)$ represents the cyclic repetition of the same proof, on a structurally equivalent sequent (same formulas, new adresses). The resulting pre-proof has exactly one infinite branch, validated by the thread starting with N. If we cut that proof against an arbitrary cut-free proof of N, and perform cut elimination steps, we obtain in finite time a cut-free proof of $N_1 \otimes N_2$ which consists of two copies (up-to adresses) of the original proof of N.

Now let $\varphi_{\mathsf{stream}} = \nu X.\varphi_{\mathsf{nat}} \otimes X$ be the formula representing infinite streams of natural numbers, whose occurrences will be denoted by S, S', etc. Let us consider the derivation shown on the right, where F is an arbitrary, useless formula occurrence for illustrative purposes.

$$\dfrac{\dfrac{\dfrac{\pi_{\mathsf{dup}}}{N \vdash N_1 \otimes N_2} \qquad \dfrac{\dfrac{\dfrac{}{N_1 \vdash N'}\,(\mathsf{Ax}) \qquad \dfrac{\dfrac{\pi_{\mathsf{succ}}}{N_2 \vdash N''} \qquad \dfrac{(\star)}{N'', F \vdash S'}}{N_2, F \vdash S'}\,(\mathsf{Cut})}{N_1, N_2, F \vdash N' \otimes S'}}{N_1 \otimes N_2, F \vdash N' \otimes S'}}{N, F \vdash N' \otimes S'}\,(\mathsf{Cut})}{(\star) \quad N, F \vdash S}$$

It is a valid proof thanks to the thread on S. By cut elimination, the computational behaviour of that proof is to take a natural number n, and some irrelevant f, and compute the stream $n :: (n+1) :: (n+2) :: \ldots$. However, unlike in the two previous examples, the result of the computation is not obtained in finite time; instead, we are faced with a productive process which will produce any finite prefix of the stream when given enough time. The presence of the useless formula F illustrates here that weakening may be admissible in $\mu\mathsf{MALL}^\infty$ under some circumstances, and that cutting against some formulas (F in this case) will form a redex that will be delayed forever. These subtleties will show up in the next two sections, devoted to showing our two main results.

3 Focalization

Focalization in linear logic. MALL connectives can be split in two classes: *positive* $(\otimes, \oplus, \mathbf{0}, \mathbf{1})$ and *negative* $(⅋, \&, \top, \bot)$ connectives. The distinction can be easily understood in terms of proof search: negative inferences $(⅋), (\&), (\top)$ and $(\bot)$ are *reversible* (meaning that provability of the conclusion transfers to the premisses) while positive inferences require choices (splitting the context in $(\otimes)$ or choosing between $(\oplus_1)$ and $(\oplus_2)$ rules) resulting in a possible of loss of provability. Still, positive inferences satisfy the ***focalization*** property [3]: in any provable sequent containing no negative formula, some formula can be chosen as a ***focus***, hereditarily selecting its positive subformulas as principal formulas until a negative subformula is reached. It induces the following complete proof search strategy:

Sequent Γ ***contains a negative*** formula	Sequent Γ ***contains no negative*** formula
Choose any negative formula (e.g. the leftmost one) and decompose it using the only possible negative rule.	Choose some positive formula and decompose it (and its subformulas) hereditarily until we get to atoms or negative subformulas.

Focalization graphs. Focused proofs are complete for proofs, not only provability: any linear proof is equivalent to a focused proof, up to cut-elimination. Indeed, focalization can be proved by means of proof transformations [18, 20, 6] preserving the denotation of the proof. A flexible, modular method for proving focalization that we shall apply in the next sections has been introduced by Miller and the third author [20] and relies on ***focalization graphs***. The heart of the focalization graph proof technique relies on the fact the positive inference, while not reversible, all permute with each other. As a consequence, if the positive layer of some positive formula is completely decomposed within the lowest part of the proof, below any negative inference, then it can be taken as a focus. Focalization graphs ensure that it is always possible: their acyclicity provides a source which can be taken as a focus.

Focusing infinitary proofs. The infinitary nature of our proofs interferes with focalization in several ways. First, while in μMALL μ and ν can be set to have an arbitrary polarity, we will see that in $\mu\mathsf{MALL}^\infty$, ν must be negative. Second, permutation properties of the negative inferences, which can be treated locally in μMALL, now require a global treatment due to infinite branches. Last, focalization graphs strongly rely on the finiteness of maximal positive subtrees of a proof: this invariant must be preserved in $\mu\mathsf{MALL}^\infty$.

For simplicity reasons, we restrict our attention to cut-free proofs in the rest of this section. The result holds for proofs with cuts thanks to the usual trick of viewing cuts as $\otimes$.

3.1 Polarity of connectives

Let us first consider the question of polarizing $\mu\mathsf{MALL}^\infty$ connectives. Unlike in μMALL, we are not free to set the polarity of fixed points formulas: consider the proof π of sequent $\vdash \mu X.X, \nu Y.Y$ which alternates inferences (ν) and (μ). Assigning opposite polarities to dual formulas (an invariant necessary to define properly cut-elimination in focused proof systems), this sequent contains a negative formula; each polarization of fixed points induces one focused pre-proof, either π_μ which always unrolls μ or π_ν which repeatedly unrolls ν. Only π_ν happens to be valid, leaving but one possible choice, $\nu X.F$ negative and $\mu X.F$ positive, resulting in the following polarization:

▶ **Definition 10.** ***Negative formulas*** are formulas of the form $\nu X.F$, $F \parr G$, $F \& G$, $\bot$ and $\top$, ***positive formulas*** are formulas of the form $\mu X.F$, $F \otimes G$, $F \oplus G$, $\mathbf{1}$ and $\mathbf{0}$. A $\mu\mathsf{MALL}^\infty$ sequent containing only positive formulas is said to be ***positive***. Otherwise, it is ***negative***.

The following proposition will be useful in the following:

▶ **Proposition 11.** *An infinite branch of a pre-proof containing only negative (resp. positive) rules is always valid (resp. invalid).*

3.2 Reversibility of negative inferences

$$\dfrac{\dfrac{(\star)}{\vdash F, P \parr Q} \qquad \dfrac{\dfrac{\pi}{\vdash F, P, Q}}{\vdash F, P \parr Q}\,(\parr)}{\dfrac{\vdash F \& F, P \parr Q}{\dfrac{\vdash (F \& F) \oplus \mathbf{0}, P \parr Q}{(\star)\ \vdash F, P \parr Q}\,(\nu)}\,(\oplus_1)}\,(\&)$$

The following example with $F = \nu X.(X \& X) \oplus \mathbf{0}$ shows that, unlike in (MA)LL, negative inferences cannot be permuted down locally: no occurrence of a negative inference $(\parr)$ on $P \parr Q$ can be permuted below a $(\&)$ since it is never available in the left premise. We thus introduce a global proof transformation (which could be realized by means of cut, as is usual).

Negative rules have a uniform structure: $\dfrac{(\vdash \Gamma, \mathcal{N}_i^N)_{1 \le i \le n}}{\vdash \Gamma, N}\,(\mathsf{r}_\mathsf{N})$. ***Sub-occurrence families*** of N are thus defined as $\mathcal{N}(N) = (\mathcal{N}_i^N)_{1 \le i \le n}$, its ***slicing index*** being $\mathsf{sl}(N) = \#\mathcal{N}(N)$.

N	$F_1 \parr F_2$	$\bot$	$F_1 \& F_2$	$\top$	$\nu X.F$
$\mathcal{N}(N)$	$\{1 \mapsto \{F_1, F_2\}\}$	$\{1 \mapsto \emptyset\}$	$\{1 \mapsto \{F_1\}, 2 \mapsto \{F_2\}\}$	$\emptyset$	$\{1 \mapsto \{F[\nu X.F/X]\}\}$

The following two definitions define what the reversibility of a proof π, $\mathsf{rev}(\pi)$, is:

▶ **Definition 12** ($\pi(i, N)$)**.** Let π be a proof of $\vdash \Gamma$ of last rule (r) and premises $\pi_1, \ldots, \pi_n$. If $1 \le i \le \mathsf{sl}(N)$, we define $\pi(i, N)$ coinductively:

- if N does not occur in $\vdash \Gamma$, $\pi(i, N) = \pi$;
- if r is the inference on N, then $\pi(i, N) = \pi_i$; (which is legal since in this case $n = \mathsf{sl}(N)$);
- if r is not the inference on N, then $\pi(i, N) = \dfrac{\pi_1(i, N) \quad \ldots \quad \pi_n(i, N)}{\vdash \Gamma, \mathcal{N}_i^N}\,(\mathsf{r})$.

▶ **Definition 13** (rev(π))**.** Let π be a $\mu\mathsf{MALL}^\infty$ proof of $\vdash \Gamma$. $\mathsf{rev}(\pi)$ is a pre-proof non-deterministically defined as π if $\vdash \Gamma$ is positive and, otherwise, when $N \in \Gamma$ and $n = \mathsf{sl}(N)$, as $\mathsf{rev}(\pi) = \dfrac{\mathsf{rev}(\pi(1,N)) \quad \dots \quad \mathsf{rev}(\pi(n,N))}{\vdash \Gamma}\,(\mathsf{r_N})$.

Reversed proofs formalize the requirement for the whole negative layer to be reversed:

▶ **Definition 14.** ***Reversed pre-proofs*** are defined to be the largest set of pre-proofs such that: *(i)* every pre-proof of a positive sequent is reversed; *(ii)* a pre-proof of a negative sequent is reversed if it ends with a negative inference and if each of its premises is reversed.

▶ **Example 15.** rev is illustrated on the proof starting this subsection ($N = P \parr Q$, $\mathsf{sl}(N) = 1$) in Figure 2

▶ **Theorem 16.** *Let π be a $\mu\mathsf{MALL}^\infty$ proof. $\mathsf{rev}(\pi)$ is a reversed proof of the same sequent.*

$$\mathsf{rev}(\pi) = \dfrac{\pi(1,N)}{\vdash F, P \parr Q}\,(\parr)$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\star)}{\vdash F, P, Q} \quad \dfrac{\pi}{\vdash F, P, Q}}{\vdash F \& F, P, Q}\,(\&)}{\vdash (F \& F) \oplus \mathbf{0}, P, Q}\,(\oplus_1)}{(\star) \ \vdash F, P, Q}\,(\nu)}{\vdash F, P \parr Q}\,(\parr)$$

Figure 2 rev(π)

3.3 Focalization Graph

In this section, we adapt the focalization graphs introduced in [20] to our setting. Considering the permutability properties of positive inferences in $\mu\mathsf{MALL}^\infty$, finiteness of positive trunks and acyclicity of focalization graphs will be sufficient to make the proof technique of [20] applicable. In order to illustrate this subsection, an example is fully explained in appendix B.5

▶ **Definition 17** (Positive trunk, positive border, active formulas)**.** Let π be a $\mu\mathsf{MALL}^\infty$ proof of $\mathcal{S}$. The ***positive trunk*** π^+ of π is the tree obtained by cutting (finite or infinite) branches of π at the first occurrence of a negative rule. The ***positive border*** of π is the collection of lowest sequents in π which are conclusions of negative rules. ***P-active*** formulas of π are those formulas of $\mathcal{S}$ which are principal formulas of an inference in π^+.

▶ **Proposition 18.** *The positive trunk of a $\mu\mathsf{MALL}^\infty$ proof is always finite.*

▶ **Definition 19** (Focalization graph)**.** Given a $\mu\mathsf{MALL}^\infty$ proof π, we define its ***focalization graph*** $\mathcal{G}(\pi)$ to be the graph whose vertices are the P-active formulas of π and such that there is an edge from F to G iff there is a sequent $\mathcal{S}'$ in the positive border containing a negative sub-occurrence F' of F and a positive sub-occurrence G' of G.

$\mu\mathsf{MALL}^\infty$ positive inferences are those of MALL extended with (μ) which is not branching: this ensures both that any two positive inferences permute and that the proof of acyclicity of MALL focalization graphs can easily be adapted, from which we conclude that:

▶ **Proposition 20.** *Focalization graphs are acyclic.*

Acyclicity of the focalization graph implies in particular that it has a source, that is a formula P of the conclusion sequent such that whenever one of its subformulas F appears in a border sequent, F is negative. This remark, together with the fact that the trunk is finite ensures that the positive layer of P is completely decomposed in the positive trunk.

▶ **Definition 21** (foc(π, P))**.** Let π be a $\mu\mathsf{MALL}^\infty$ proof of $\vdash \Gamma, P$ with P a source of π's focalization graph. One defines $\mathsf{foc}(\pi, P)$ as the $\mu\mathsf{MALL}^\infty$ proof obtained by permuting down all the positive inferences on P and its positive subformulas (all occurring in π^+).

▶ **Proposition 22.** *Let $\mathcal{S}$ be a lowest sequent of* $\mathsf{foc}(\pi, P)$ *which is not conclusion of a rule on a positive subformula of P. Then $\mathcal{S}$ contains exactly one subformula of P, which is negative.*

3.4 Productivity and validity of the focalization process

Reversibility of the negative inferences and focalization of the positive inferences allow to consider the following (non-deterministic) proof transformation process:

Focalization Process: Let π be a $\mu\mathsf{MALL}^\infty$ proof of $\mathcal{S}$. Define $\mathsf{Foc}(\pi)$ as follows:

- **Asynchronous phase:** If $\mathcal{S}$ is negative, transform π into $\mathsf{rev}(\pi)$ which is reversed. At least one negative inference has been brought to the root of the proof. Apply (corecursively) the synchronous phase to the proofs rooted in the lowest positive sequents of $\mathsf{rev}(\pi)$.
- **Synchronous phase:** If $\mathcal{S}$ is positive, let $P \in \mathcal{S}$ be a source of the associated focalization graph. Transform π into a proof $\mathsf{foc}(\pi, P)$. At least one positive inference on P has been brought to the root of the proof. Apply (corecursively) the asynchronous phase to the proofs rooted in the lowest negative sequents of $\mathsf{foc}(\pi, P)$.

Each of the above phases produces one non-empty phase, the above process is thus productive. It is actually a pre-proof thanks to theorem 16 and by definition of $\mathsf{foc}(\pi, P)$. It remains to show that the resulting pre-proof is actually a proof. The following property is easily seen to be preserved by both transformations foc and rev and thus holds for $\mathsf{Foc}(\pi)$:

▶ **Proposition 23.** *Let π be a $\mu\mathsf{MALL}^\infty$ proof, r a positive rule occurring in π and r' be a negative rule occurring below r in π. If r occurs in $\mathsf{Foc}(\pi)$, then r' occurs in $\mathsf{Foc}(\pi)$, below r.*

▶ **Lemma 24.** *For any infinite branch γ of $\mathsf{Foc}(\pi)$ containing an infinite number of positive rules, there exists an infinite branch in π containing infinitely many positive rules of γ.*

▶ **Theorem 25.** *If π is a $\mu\mathsf{MALL}^\infty$ proof then $\mathsf{Foc}(\pi)$ is also a $\mu\mathsf{MALL}^\infty$ proof.*

Proof sketch, see appendix. An infinite branch γ of $\mathsf{Foc}(\pi)$ may either be obtained by reversibility only after a certain point, or by alternating infinitely often synchronous and asynchronous phases. In the first case it is valid by proposition 11 while in the latter case, lemma 24 ensures the existence of a branch δ of π containing infinitely many positive rules of γ, with a valid thread t of minimal formula F_m: every rule r of δ in which F_m is principal is below a positive rule occurring in γ. Thus r occurs in γ, which is therefore valid. ◀

4 Cut elimination

In this section, we show that any $\mu\mathsf{MALL}^\infty$ proof can be transformed into an equivalent cut-free derivation. This is done by applying the cut reduction rules described in Section 2, possibly in infinite reductions converging to cut-free proofs. As usual with infinitary reductions it is not the case that any reduction sequence converges: for instance, one could reduce only deep cuts in a proof, leaving a cut untouched at the root. We avoid this problem by considering a form of head reduction where we only reduce cuts at the root.

Cut reduction rules are of two kinds, *principal* reductions and *auxiliary* ones. In the infinitary setting, principal cut reductions do not immediately contribute to producing a cut-free pre-proof. On the contrary, auxiliary cut reductions are productive in that sense. In other words, principal rules are seen as internal computations of the cut elimination process, while auxiliary rules are seen as a partial output of that process. Accordingly, the former will be called *internal rules* and the latter *external rules*.

$$\cfrac{\cfrac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\,(\mathsf{Cut}) \quad \dots}{\vdash \Sigma}\,(\mathsf{mcut}) \longrightarrow \cfrac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta \quad \dots}{\vdash \Sigma}\,(\mathsf{mcut})$$

$$\cfrac{\cfrac{\vdash \Gamma, F}{\vdash \Gamma, F \oplus G} \quad \cfrac{\vdash G^\perp, \Delta \quad \vdash F^\perp, \Delta}{\vdash G^\perp \& F^\perp, \Delta} \quad \dots}{\vdash \Sigma}\,(\mathsf{mcut}) \longrightarrow \cfrac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta \quad \dots}{\vdash \Sigma}\,(\mathsf{mcut})$$

$$\cfrac{s_1 \dots s_n \quad \cfrac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \& G}\,(\&)}{\vdash \Sigma, F \& G}\,(\mathsf{mcut}) \longrightarrow \cfrac{\cfrac{s_1 \dots s_n \quad \vdash \Gamma, F}{\vdash \Sigma, F}\,(\mathsf{mcut}) \quad \cfrac{s_1 \dots s_n \quad \vdash \Gamma, G}{\vdash \Sigma, G}\,(\mathsf{mcut})}{\vdash \Sigma, F \& G}\,(\&)$$

Figure 3 (Cut)/(mcut) and $(\oplus_1)/(\&)$ internal reductions and $(\&)/$(mcut) external reduction.

When analyzing cut reductions, cut commutations can be troublesome. A common way to avoid this technicality [13], which we shall follow, is to introduce a ***multicut*** rule which merges multiple cuts, avoiding cut commutations.

$$\frac{s_1 \dots s_n}{s}\,(\mathsf{mcut})$$

▶ **Definition 26.** Given two sequents s and s', we say that they are cut-connected on a formula occurrence F when $F \in s$ and $F^\perp \in s'$. We say that they are cut-connected when they are connected for some F. We define the ***multicut*** rule as shown above with conclusion s and premisses $\{s_i\}_i$, where the set $\{s_i\}_i$ is connected and acyclic with respect to the cut-connection relation, and s is the set of all formula occurrences F that appear in some s_i but such that no s_j is cut-connected to s_j on F.

From now on we shall work with $\mu\mathsf{MALL}^\infty_\mathsf{m}$ derivations, which are $\mu\mathsf{MALL}^\infty$ derivations in which the multicut rule may occur, though only at most once per branch. The notions of thread and validity are unchanged. In $\mu\mathsf{MALL}^\infty_\mathsf{m}$ we only reduce multicuts, in a way that is naturally obtained from the cut reductions of $\mu\mathsf{MALL}^\infty$. A complete description of the rules is given in Definition 49, appendix C.1; only the (Cut)/(mcut) and $(\oplus_1)/(\&)$ internal reduction cases and the $(\&)/$(mcut) external reduction case are shown in figure 3. As is visible in the last reduction, applying an external rule on a multicut may yield multiple multicuts, though always on disjoint subtrees.

We will be interested in a particular kind of multicut reduction sequences, the ***fair*** ones, which are such that any redex which is available at some point of the sequence will eventually have disappeared from the sequence (being reduced or erased), details are provided in appendix C.1. We will establish that these reductions eliminate multicuts:

▶ **Theorem 27.** *Fair multicut reductions on $\mu\mathsf{MALL}^\infty_\mathsf{m}$ proofs produce $\mu\mathsf{MALL}^\infty$ proofs.*

Additionnally, if all cuts in the initial derivation are above multicuts, the resulting $\mu\mathsf{MALL}^\infty$ derivation must actually be cut-free: indeed, multicut reductions never produce a cut. Thus Theorem 27 gives a way to eliminate cuts from any $\mu\mathsf{MALL}^\infty$ proof π of $\vdash \Gamma$ by forming a multicut with conclusion $\vdash \Gamma$ and π as unique subderivation, and eliminating multicuts (and cuts) from that $\mu\mathsf{MALL}^\infty_\mathsf{m}$ proof. The proof of Theorem 27 is in two parts. We first prove that fair internal multicut reductions cannot diverge (Proposition 37), hence fair multicut reductions are productive, *i.e.*, reductions of $\mu\mathsf{MALL}^\infty_\mathsf{m}$ proofs converge to $\mu\mathsf{MALL}^\infty$ pre-proofs. We then establish that the obtained pre-proof is a valid proof (Proposition 38).

Regarding productivity, assuming that there exists an infinite sequence σ of internal cut-reductions from a given proof π of Γ, we obtain a contradiction by extracting from π a proof of the empty sequent in a suitably defined proof-system. More specifically, we observe that no formula of Γ is principal in the subtree π_σ of π visited by σ. Hence, by erasing every formula of Γ from π_σ, local correctness of the proof is preserved, resulting in a tree deriving the empty sequent. This tree can be viewed as a proof in a new proof-system $\mu\mathsf{MALL}^\infty_\tau$ which is shown to be sound (Proposition 34) with respect to the traditional boolean semantics of the μ-calculus, thus the contradiction. The proof of validity of the produced pre-proof is similar: instead of extracting a proof of the empty sequent from π we will extract, for each invalid branch of π, a $\mu\mathsf{MALL}^\infty_\tau$ proof of a formula containing neither $\mathbf{1}$, $\top$, nor ν formulas, contradicting soundness again.

4.1 Extracting proofs from reduction paths

We define now a key notion to analyze the behaviour of multicut-elimination: given a multicut reduction starting from π, we extract a (slightly modified) subderivation of π which corresponds to the part of the derivation that has been explored by the reduction. More precisely, we are interested in *reduction paths* which are sequences of proofs that end with a multicut rule, obtained by tracing one multicut through its evolution, selecting only one sibling in the case of $(\&)$ and $(\otimes)$ external reductions. Given such a reduction path starting with π, we consider the subtree of π whose sequents occur in the reduction path as premises of some multicut. This subtree is obviously not always a $\mu\mathsf{MALL}^\infty$ derivation since some of its nodes may have missing premises. We will provide an extension of $\mu\mathsf{MALL}^\infty$ where these trees can be viewed as proper derivations by first characterizing when this situation arises.

▶ **Definition 28** (Useless sequents, distinguished formula)**.** Let $\mathcal{R}$ be a reduction path starting with π. A sequent $s = (\vdash \Gamma, F)$ of π is said to be ***useless*** with ***distinguished formula*** F when in one of the following cases:

1. The sequent eventually occurs as a premise of all multicuts of $\mathcal{R}$ and F is the principal formula of s in π. (Note that the distinguished formula F of a useless sequent s of sort (1) must be a sub-occurrence of a cut formula in π. Otherwise, the fair reduction path $\mathcal{R}$ would eventually have applied an external rule on s. Moreover, $F^\perp$ never becomes principal in the reduction path, otherwise by fairness the internal rule reducing F and $F^\perp$ would have been applied.)
2. At some point in the reduction, the sequent is a premise of $(\&)$ on $F \& F'$ or $F' \& F$ which is erased in an internal $(\&)/(\oplus)$ multicut reduction. (In the $(\oplus_1)/(\&)$ internal reduction of figure 3, the sequent $\vdash G^\perp, \Delta$ is useless of sort (2).)
3. The sequent is ignored at some point in the reduction path because it is not present in the selected multicut after a branching external reduction on $F \star F'$ or $F' \star F$, for $\star \in \{\otimes, \&\}$. (In the $(\&)/(\mathsf{mcut})$ external reduction of figure 3, if one is considering a reduction path that follows the multicut having $\vdash \Gamma, F$ as a premise, then the sequent $\vdash \Gamma, G$ is useless of sort (3), and vice versa.)
4. The sequent is ignored at some point in the reduction path because a $(\otimes)/(\mathsf{mcut})$ external reduction distributes s to the multicut that is not selected in the path. This case will be illustrated next, and is described in full details in appendix C.1.

Note that, although the external reduction for $\top$ erases sequents, we do not need to consider such sequents as useless: indeed, we will only need to work with useless sequents in infinite reduction paths, and the external reduction associated to $\top$ terminates a path.

▶ **Example.** Consider a multicut composed of the last example of Section 2 and an arbitrary proof of $\vdash F, \Delta$ where F is principal. In the reduction paths which always select the right premise of an external $(\otimes)/(\mathsf{mcut})$ corresponding to the $N'\otimes S'$ formulas, the sequent $\vdash F, \Delta$ will always be present and thus useless by case (1). In the reduction paths which eventually select a left premise, the sequent $N_2, F \vdash S'$ is useless of sort (3) with S' distinguished, and $\vdash F, \Delta$ is useless of sort (4) with F distinguished.

In order to obtain a proper pre-proof from the sequents occurring in a reduction path, we need to close the derivation on useless sequents. This is done by replacing distinguished formulas by $\top$ formulas. However, a usual substitution is not appropriate here as we are really replacing formula occurrence, which may be distributed in arbitrarily complex ways among sub-occurrences.

▶ **Definition 29.** A ***truncation*** τ is a partial function from Σ^* to $\{\top, \mathbf{0}\}$ such that:

- For any $\alpha \in \Sigma^*$, if $\alpha \in \mathtt{Dom}(\tau)$, then $\alpha^\perp \in \mathtt{Dom}(\tau)$ and $\tau(\alpha) = \tau(\alpha^\perp)^\perp$.
- If $\alpha \in \mathtt{Dom}(\tau)$ then for any $\beta \in \Sigma^+$, $\alpha.\beta \notin \mathtt{Dom}(\tau)$.

▶ **Definition 30** (Truncation of a reduction path)**.** Let $\mathcal{R}$ be a reduction path. The truncation τ associated to $\mathcal{R}$ is defined by setting $\tau(\alpha) = \top$ and $\tau(\alpha^\perp) = \mathbf{0}$ for every formula occurrence φ_α that is distinguished in some useless sequent of $\mathcal{R}$.

The above definition is justified because F and $F^\perp$ cannot both be distinguished, by fairness of $\mathcal{R}$. We can finally obtain the pre-proof associated to a reduction path, in a proof system slightly modified to take truncations into account.

▶ **Definition 31** (Truncated proof system)**.** Given a truncation τ, the infinitary proof system $\mu\mathsf{MALL}^\infty_\tau$ is obtained by taking all the rules of $\mu\mathsf{MALL}^\infty$, with the proviso that they only apply when the address of their principal formula is not in the domain of τ, with the following extra rule:

$$\dfrac{\vdash \tau(\alpha)_{\alpha i}, \Delta}{\vdash F, \Delta}\ (\tau) \quad \text{if } \alpha \in \mathtt{Dom}(\tau)$$

The adress $\alpha.i$ associated with $\tau(\alpha)$ in the rule (τ) forbids loops on a (τ) rule. Indeed if $\alpha \in \mathtt{Dom}(\tau)$ then $\alpha.i \notin \mathtt{Dom}(\tau)$.

▶ **Definition 32** (Truncated proof associated to a reduction path)**.** Let $\mathcal{R}$ be a fair infinite reduction path starting with π and τ be the truncation associated to it. We define $TR(\mathcal{R})$ to be the $\mu\mathsf{MALL}^\infty_\tau$ proof obtained from π by keeping only sequents that occur as premise of some multicut in $\mathcal{R}$, using the same rules as in π whenever possible, and deriving useless sequents by rules (τ) and $(\top)$.

This definition is justified by definition of τ and because only useless sequents may be selected without their premises (in π) being also selected. Notice that the dual $F^\perp$ of a distinguished formula F may only occur in $\mathcal{R}$ for distinguished formulas of type (1) and (4); in these cases $F^\perp$ is never principal in $\mathcal{R}$ by fairness. Thus, there is no difficulty in constructing $TR(\mathcal{R})$ with a truncature defined on the address of $F^\perp$. Finally, note that $TR(\mathcal{R})$ is indeed a valid $\mu\mathsf{MALL}^\infty_\tau$ pre-proof, because its infinite branches are infinite branches of π.

▶ **Example.** Continuing the previous example, we consider the path where the left premise of the tensor is selected immediately. The associated truncation is such that $\tau(S') = \top$ and $\tau(F) = \top$ by (3) and (4) respectively. The derivation $TR(\mathcal{R})$ is shown below, where Π_{ax} denotes the expansion of the axiom given by Prop 9.

$$\dfrac{\dfrac{}{\vdash F, \Delta}\,(\tau),(\top) \qquad \dfrac{\dfrac{\dfrac{\Pi_{\mathsf{dup}}}{N \vdash N_1 \otimes N_2} \qquad \dfrac{\dfrac{\dfrac{\Pi_{\mathsf{ax}}}{N_1 \vdash N'} \qquad \dfrac{}{N_2, F \vdash S'}\,(\tau),(\top)}{N_1, N_2, F \vdash N'\otimes S'}}{N_1 \otimes N_2, F \vdash N' \otimes S'}}{N, F \vdash N' \otimes S'}\,(\mathsf{Cut})}{N, F \vdash S}}{N \vdash S, \Delta}\,(\mathsf{mcut})$$

4.2 Truncated truth semantics

We fix a truncation τ and define a truth semantics with respect to which $\mu\mathsf{MALL}^{\infty}_{\tau}$ will be sound. The semantics is classical, assigning a boolean value to formula occurrences. For convenience, we take $\mathcal{B} = \{\mathbf{0}, \top\}$ as our boolean lattice, with $\wedge$ and $\vee$ being the usual meet and join operations on it. The following definition provides an interpretation of $\mu\mathsf{MALL}$ formulas which consists in the composition of the standard interpretation of μ-calculus formulas with the obvious linearity-forgetting translation from $\mu\mathsf{MALL}$ to classical μ-calculus.

▶ **Definition 33.** Let φ_α be a pre-formula occurrence. We call ***environment*** any function $\mathcal{E}$ mapping free variables of φ to (total) functions of $E := \Sigma^* \to \mathcal{B}$. We define $[\varphi_\alpha]^{\mathcal{E}} \in \mathcal{B}$, the ***interpretation*** of φ_α in the environment $\mathcal{E}$, by $[\varphi_\alpha]^{\mathcal{E}} = \tau(\alpha)$ if $\alpha \in \mathtt{Dom}(\tau)$, and otherwise:

- $[X_\alpha]^{\mathcal{E}} = \mathcal{E}(X)(\alpha)$, $[\top_\alpha]^{\mathcal{E}} = [\mathbf{1}_\alpha]^{\mathcal{E}} = \top$ and $[\mathbf{0}_\alpha]^{\mathcal{E}} = [\bot_\alpha]^{\mathcal{E}} = \mathbf{0}$.
- $[(\varphi \owedge \psi)_\alpha]^{\mathcal{E}} = [\varphi_{\alpha.l}]^{\mathcal{E}} \wedge [\psi_{\alpha.r}]^{\mathcal{E}}$, for $\owedge \in \{\&, \otimes\}$.
- $[(\varphi \ovee \psi)_\alpha]^{\mathcal{E}} = [\varphi_{\alpha.l}]^{\mathcal{E}} \vee [\psi_{\alpha.r}]^{\mathcal{E}}$, for $\ovee \in \{\oplus, \parr\}$.
- $[(\mu X.\varphi)_\alpha]^{\mathcal{E}} = \mathrm{lfp}(f)(\alpha)$ and $[(\nu X.\varphi)_\alpha]^{\mathcal{E}} = \mathrm{gfp}(f)(\alpha)$ where $f : E \to E$ is given by $f : h \mapsto \beta \mapsto \big(\tau(\beta)$ if $\beta \in \mathtt{Dom}(\tau)$ and $[\varphi_{\beta.i}]^{\mathcal{E}::X\mapsto h}$ otherwise$\big)$.

When F is closed, we simply write $[F]$ for $[F]^{\emptyset}$.

We refer the reader to the appendix for details on the construction of the interpretation. We simply state here the main result about it.

▶ **Proposition 34.** *If $\vdash \Gamma$ is provable in $\mu\mathsf{MALL}^{\infty}_{\tau}$, then $[F] = \top$ for some $F \in \Gamma$.*

We only sketch the soundness proof (see appendix C for details) which proceeds by contradiction. Assuming we are given a proof π of a formula F such that $[F] = \mathbf{0}$, we exhibit a branch β of π containing only formulas interpreted by $\mathbf{0}$. A validating thread of β unfolds infinitely often some formula $\nu X.\varphi$. Since the interpretation of $\nu X.\varphi$ is defined as the gfp of a monotonic operator f we have, for each occurrence $(\nu X.\varphi)_\alpha$ in β, an ordinal λ such that $[(\nu X.\varphi)_\alpha] = f^{\lambda}(\bigvee E)(\alpha)$, where $\bigvee E$ is the supremum of the complete lattice E. We show that this ordinal can be forced to decrease along β at each fixed point unfolding, contradicting the well-foundedness of the class of ordinals.

▶ **Definition 35.** A truncation τ is ***compatible*** with a formula φ_α if $\alpha \notin dom(\tau)$ and, for any $\alpha \sqsubseteq \beta.d \in \mathtt{Dom}(\tau)$ where $d \in \{l, r, i\}$, we have that φ_α admits a sub-occurrence ψ_β with $\otimes$ or $\&$ as the toplevel connective of ψ, $d \in \{l, r\}$, and $\alpha.d' \notin \mathtt{Dom}(\tau)$ for any $d \neq d'$.

In other words, a truncation τ is compatible with a formula F if it truncates only sons of $\otimes$ or $\&$ nodes in the tree of the formula F and at most one son of each such node.

▶ **Proposition 36.** *If F is a formula compatible with τ and containing no ν binders, no $\top$ and no $\mathbf{1}$, then $[F] = \mathbf{0}$.*

4.3 Proof of cut elimination

Multicut reduction is shown productive and then to result in a valid cut-free proof.

▶ **Proposition 37.** *Any fair reduction sequence produces a $\mu\mathsf{MALL}^{\infty}$ pre-proof.*

Proof. By contradiction, consider a fair infinite sequence of internal multicut reductions. This sequence is a fair reduction path $\mathcal{R}$. Let τ and $TR(\mathcal{R})$ be the associated truncations and truncated proof. Since no external reduction occurs, it means that conclusion formulas of $TR(\mathcal{R})$ are never principal in the proof, thus we can transform it into a proof of the empty sequent, which contradicts soundness of $\mu\mathsf{MALL}^{\infty}_{\tau}$. ◀

▶ **Proposition 38.** *Any fair mcut-reduction produces a* $\mu\mathsf{MALL}^\infty$ *proof.*

Proof. Let π be a $\mu\mathsf{MALL}^\infty_\mathsf{m}$ proof of conclusion $\vdash \Gamma$, and π' the cut-free pre-proof obtained by Prop. 37, *i.e.*, the limit of the multicut reduction process. Any branch of π' corresponds to a multicut reduction path. For the sake of contradiction, assume that π' is invalid. It must thus have an invalid infinite branch, corresponding to an infinite reduction path $\mathcal{R}$. Let τ and $\theta := TR(\mathcal{R})$ be the associated truncation and truncated proof in $\mu\mathsf{MALL}^\infty_\tau$.

We first observe that formulas of Γ cannot have suboccurrences of the form $\mathbf{1}_\alpha$ or $\top_\alpha$ that are principal in π'. Indeed, this could only be produced by an external rule $(\top)/(\mathsf{mcut})$ in the reduction path $\mathcal{R}$, but that would terminate the path, contradicting its infiniteness.

Next, we claim that all threads starting from formulas in Γ are invalid. Indeed, all rules applied to those formulas are transferred (by means of external rules) to the branch produced by the reduction path. The existence of a valid thread starting from the conclusion sequent in θ would thus imply the existence of a valid thread in our branch of π'.

By the first observation, we can replace all $\mathbf{1}$ and $\top$ subformulas of Γ by $\mathbf{0}$ without changing the derivation, and obviously without breaking its validity. By the second observation, we can further modify Γ by changing all ν combinators into μ combinators. The derivation is easily adapted (using rule (μ) instead of (ν)) and it remains valid, since the validity of θ could not have been caused by a valid thread starting from the root. We thus obtain a valid pre-proof θ' of $\vdash \Gamma'$ in $\mu\mathsf{MALL}^\infty_\tau$, where Γ' contains no ν, $\mathbf{1}$ and $\top$.

We finally show that τ is compatible with any formula occurrence from Γ. Indeed, if $\tau(\beta)$ is defined for some suboccurrence ψ_β of a formula $\varphi_\alpha \in \Gamma$, then it can only be because of a useless sequent of sort (3), *i.e.*, a truncation due to the fact that the reduction path has selected only one sibling after a branching external rule. We thus conclude, by Proposition 36, that all formulas of Γ are interpreted as $\mathbf{0}$ in the truncated semantics associated to τ, which contradicts the validity of θ' and Proposition 34. ◀

5 Conclusion

We have established focalization and cut elimination for $\mu\mathsf{MALL}^\infty$, the infinitary sequent calculus for $\mu\mathsf{MALL}$. Our cut elimination result extends that of Santocanale and Fortier [13], but this extension has required the elaboration of a radically different proof technique.

An obvious direction for future work is now to go beyond linear logic, and notably handle structural rules in infinitary cut elimination. But many interesting questions are also left in the linear case. First, it will be natural to relax the hypothesis on fairness in the cut-elimination result. Other than cut elimination, the other long standing problem regarding $\mu\mathsf{MALL}^\infty$ and similar proof systems is whether regular proofs can be translated, in general, to finitary proofs. Further, one can ask the same question, requiring in addition that the computational content of proofs is preserved in the translation. It may well be that regular $\mu\mathsf{MALL}^\infty$ contains more computations than $\mu\mathsf{MALL}$; even more so if one considers other classes of finitely representable infinitary proofs. It would be interesting to study how this could impact the study of programming languages for (co)recursion, and understanding links with other approaches to this question [1, 2]. In this direction, we will be interested in studying the computational interpretation of focused cut-elimination, providing a logical basis for inductive and coinductive matching in regular and infinitary proof systems.

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Contents

A Appendix relative to Section 2

In this appendix we provide a proof of Proposition 9, but also supplementary material that may be useful to better understand μMALL$^\infty$, its validity condition and its relationship to μMALL. Most of this material is adapted directly from classical observations about μ-calculi, with the exception of the translation from μMALL to μMALL$^\infty$: it is unpublished, but we view it more as folklore than as a contribution of this paper.

A.1 Details on the validity condition

We first provide more details and intuitions about the notion of valid thread. If a thread $(F_i)_{i\in\omega}$ is eventually constant in terms of formula occurrences, it simply means that it traces a formula that is never principal in the branch: this formula plays no role in the proof, and there is no reason to declare the thread valid. Otherwise, addresses keep growing along the thread: at any point in the thread there is a later point where the address increases. Forgetting addresses and considering the set S of formulas that appear infinitely often in the thread, we immediately see that any two formulas $\varphi, \psi \in S$ are *co-accessible*, *i.e.*, $\psi \in \mathrm{FL}(\varphi)$. Indeed, if $F_i = \varphi_\alpha$, there must be some $j > i$ such that $F_j = \psi_\beta$. In that case, the thread is valid iff the minimum of S wrt. the subformula ordering is a ν-formula. As we shall see, this definition makes sense because that minimum is always defined. Moreover, it is always a fixed point formula, so what the definition really says is that this minimum fixed point must be a greatest fixed point for the thread to be valid. All this is justified by the following classical observation about μ-calculi, which we restate next in our setting.

▶ **Proposition 39.** *Let $t = (F_i)_{i\in\omega}$ be a thread that is not eventually constant. The set S of formulas that occur infinitely often in t admits a minimum with respect to the subformula ordering, and that minimum is a fixed point formula.*

Proof. We assume that all formulas of t occur infinitely often in t, and that $F_i = \psi_\alpha$ implies $F_{i+1} = \psi'_{\alpha a}$ for some $a \in \Sigma$, *i.e.*, F_{i+1} is an immediate descendant of F_i. This is without loss of generality, by extracting from t the infinite sub-thread of occurrences F_i whose formulas are in S and which are principal, *i.e.*, for which $F_{i+1} \not\equiv F_i$.

Let $|\varphi|$ be the size of a formula, *i.e.*, the number of connectives used to construct the formula. Take any $\varphi \in S$ that has minimum size, *i.e.*, $|\varphi| \leq |\psi|$ for all $\psi \in S$. We shall establish that φ must in fact be a minimum for the subformula ordering, *i.e.*, $\varphi \leq \psi$ for all $\psi \in S$. It suffices to prove that if $F_i = \psi_\alpha$ and $F_j = \varphi_{\alpha\beta}$, then $\varphi \leq \psi$. We proceed by induction on β. The result is obvious if β is empty, since one then has $\varphi = \psi$. Otherwise, we distinguish two cases:

- If $\psi = \psi^l \star \psi^r$ and $F_{i+1} = (\psi^a)_{\alpha a}$ for some $a \in \{l, r\}$, we have $\beta = a\beta'$. By induction hypothesis (with $\alpha := \alpha a$ and $\beta := \beta'$) we obtain that $\varphi \leq \psi^a$, and thus $\varphi \leq \psi$.
- Otherwise, $\psi = \sigma X.\psi'$, $F_{i+1} = (\psi'[\psi/X])_{\alpha i}$ and $\beta = i\beta'$. By induction hypothesis, $\varphi \leq \psi'[\psi/X]$. Since $|\varphi| \leq |\psi|$, φ is a subformula of $\psi'[\psi/X]$ which cannot strictly contain ψ. Thus we either have $\varphi = \psi$ or $\varphi \leq \psi'$. In both cases, we conclude immediately.

We finally show that φ must be a fixed point formula. Take any i such that $F_i = \varphi_\alpha$. We have $F_{i+1} = \psi_{\alpha a}$. Assuming that φ is not a fixed point expression, it would be of the form $\varphi_1 \star \varphi_2$ with $\psi = \varphi_i$ for some $1 \leq i \leq 2$, contradicting $|\varphi| \leq |\psi|$. ◀

A.2 Admissibility of the axiom

We now prove the admissibility of (Ax), by showing that infinite η-expansions are valid.

▶ **Proposition** (9). *Rule* (Ax) *is admissible in* μMALL$^\infty$.

Proof. As is standard, any instance of (Ax) can be expanded by introducing two dual connectives and concluding by (Ax) on the sub-occurrences. For instance, (Ax) on $\vdash (\varphi \otimes \psi)_\alpha, (\psi^\perp \parr \varphi^\perp)_\beta$ is expanded by using rules ($\parr$), ($\otimes$), and then axioms on $\vdash \varphi_{\alpha l}, \varphi^\perp_{\beta r}$ and $\vdash \psi_{\alpha r}, \psi^\perp_{\beta l}$. In μMALL$^\infty$ we can co-iterate this expansion to obtain an axiom-free pre-proof from any instance of (Ax) on $\vdash F, G^\perp$. On any infinite branch of that pre-proof, there are exactly two threads and they are not eventually constant. Let $t = (F_i)_{i\in\omega}$ and $t' = (G_i)_{i\in\omega}$ be the corresponding sequences of distinct sub-occurrences, *i.e.*, keeping an occurrence only when it is principal. We actually have that, for all i, $F_i \equiv G_i^\perp$. The minimum of a thread that is not eventually constant is necessarily a fixed point formula, thus $\mathsf{min}(t)$ is a ν formula iff $\mathsf{min}(t')$ is a μ, and one of the two threads validates the branch. ◀

A.3 Translating from μMALL to μMALL$^\infty$

Generalizing the previous construction, we now introduce the functoriality construction, which shall be useful to present the translation from the finitary sequent calculus μMALL to its infinitary counterpart μMALL$^\infty$.

▶ **Definition 40.** Let F be a pre-formula such that $\mathrm{fv}(F) \subseteq \{X_i\}_{1\le i\le n}$, and let $\vec{\Pi} = (\Pi_i)_{1\le i\le n}$ be a collection of pre-proofs of respective conclusions $\vdash P_i, Q_i$. We define coinductively the pre-proof $F(\vec{\Pi})$ of conclusion $\vdash F^\perp[P_i/X_i]_{1\le i\le n}, F[Q_i/X_i]_{1\le i\le n}$ as follows:

- If $F = X_i$ then $F(\vec{\Pi}) = \Pi_i$ up to relocalization, *i.e.*, changing the addresses of occurrences in Π_i to match the required ones.
- If $F = F_1 \otimes F_2$, then $F(\vec{\Pi})$ is:

$$\dfrac{\dfrac{\dfrac{F_1(\vec{\Pi})}{\vdash F_1^\perp[P_i/X_i]_i, F_1[Q_i/X_i]_i} \qquad \dfrac{F_2(\vec{\Pi})}{\vdash F_2^\perp[P_i/X_i]_i, F_2[Q_i/X_i]_i}}{\vdash F_2^\perp[P_i/X_i]_i, F_1^\perp[P_i/X_i]_i, (F_1 \otimes F_2)[Q_i/X_i]_i}\,(\otimes)}{\vdash (F_2^\perp \parr F_1^\perp)[P_i/X_i]_i, (F_1 \otimes F_2)[Q_i/X_i]_i}\,(\parr)$$

- If $F = F_1 \oplus F_2$, then $F(\vec{\Pi})$ is:

$$\dfrac{\dfrac{\dfrac{F_1(\vec{\Pi})}{\vdash F_1^\perp[P_i/X_i]_i, F_1[Q_i/X_i]_i}}{\vdash F_1^\perp[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i}\,(\oplus_1) \qquad \dfrac{\dfrac{F_2(\vec{\Pi})}{\vdash F_2^\perp[P_i/X_i]_i, F_2[Q_i/X_i]_i}}{\vdash F_2^\perp[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i}\,(\oplus_2)}{\vdash (F_2^\perp \& F_1^\perp)[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i}\,(\&)$$

- If $F = \mu X.G$ then $F(\vec{\Pi})$ is obtained from applying functoriality on G with $F(\vec{\Pi})$ as the derivation for the new free variable $X_{n+1} := X$:

$$\dfrac{\dfrac{\dfrac{G(\vec{\Pi}, F(\vec{\Pi}))}{\vdash G^\perp[(\nu X.G^\perp)/X][P_i/X_i]_i, G[(\mu X.G)/X][Q_i/X_i]_i}}{\vdash G^\perp[(\nu X.G^\perp)/X][P_i/X_i]_i, (\mu X.G)[Q_i/X_i]_i}\,(\mu)}{\vdash (\nu X.G^\perp)[P_i/X_i]_i, (\mu X.G)[Q_i/X_i]_i}\,(\nu)$$

- If $F = \mathbf{0}$ then $F(\vec{\Pi})$ is directly obtained by applying ($\top$) on $F^\perp[P_i/X_i]_i$.
- If $F = \mathbf{1}$ then $F(\vec{\Pi})$ is obtained by applying rule ($\perp$) followed by ($\mathbf{1}$).

- Other cases are treated symmetrically.

As said above, the construction $F(\vec{\Pi})$ is a generalization of the infinitary η-expansion, where the derivations Π_i are plugged where free variables are encountered. In fact, if F is a closed pre-formula, then $F()$ is the derivation constructed in the proof of Proposition 9.

Also note that, since only finitely many sequents may arise in the process of constructing $F(\vec{\Pi})$, and since the construction is entirely guided by its end sequent, the derivation $F(\vec{\Pi})$ is actually regular as long as the derivations Π_i are regular as well.

An infinite branch of $F(\vec{\Pi})$ either has an infinite branch of some Π_i as a suffix, or is only visiting sequents of $F(\vec{\Pi})$ that are not sequents of the input derivations $\vec{\Pi}$. In the former case, the branch is valid provided that the input derivations are valid. In the latter case, the branch contains exactly two dual threads (as in the proof of Proposition 9), one of which must be valid. Thus, $F(\vec{\Pi})$ is a proof provided that the input derivations are proofs. This result is however not usable directly to prove the validity of a pre-proof in which we make repeated use of functoriality, *i.e.*, one where branches may go through infinitely many successive uses of functoriality.

We now make use of functoriality to translate finitary μMALL proofs (corresponding to the propositional fragment of [4]) to infinitary derivations.

▶ **Definition 41** (μMALL sequent calculus)**.** The sequent calculus for the propositional fragment of μMALL is a finitary sequent calculus whose rules are the same as those of μMALL$^\infty$, except that the ν rule is as follows:

$$\frac{\vdash S^\perp, F[S/X]}{\vdash S^\perp, \nu X.F}$$

The ν rule corresponds to reasoning by coinduction. In [4] it is found in a slightly different form, which can be obtained from the above version by means of cut:

$$\frac{\vdash \Gamma, S \qquad \vdash S^\perp, F[S/X]}{\vdash \Gamma, \nu X.F}$$

▶ **Definition 42** (Translation from μMALL to μMALL$^\infty$)**.** Given a μMALL proof Π of $\vdash \Gamma$, we define coinductively the μMALL$^\infty$ pre-proof Π^i of $\vdash \Gamma$, as follows:

- If Π starts with an inference that is present in μMALL$^\infty$, we use the same inference and proceed co-recursively. For instance,

$$\Pi \quad = \quad \dfrac{\dfrac{\Pi_1}{\vdash \Gamma', F} \qquad \dfrac{\Pi_2}{\vdash G, \Gamma''}}{\vdash \Gamma', F \otimes G, \Gamma''} \qquad \text{yields} \qquad \Pi^i \quad = \quad \dfrac{\dfrac{\Pi_1^i}{\vdash \Gamma', F} \qquad \dfrac{\Pi_2^i}{\vdash G, \Gamma''}}{\vdash \Gamma', F \otimes G, \Gamma''} \ .$$

- Otherwise, Π starts with an instance of the ν rule of μMALL:

$$\Pi \quad = \quad \dfrac{\dfrac{\Pi_1}{\vdash S^\perp, F[S/X]}}{\vdash S^\perp, \nu X.F}$$

We transform it as follows, where (F) denotes a use of the functoriality construction:

$$\Pi^i \quad = \quad \dfrac{\dfrac{\dfrac{\Pi_1^i}{\vdash S^\perp, F[S/X]} \qquad \dfrac{\dfrac{\Pi^i}{\vdash S^\perp, \nu X.F}}{\vdash F^\perp[S^\perp/X], F[(\nu X.F)/X]}\,(\mathrm{F})}{\vdash S^\perp, F[(\nu X.F)/X]}\,(\mathsf{Cut})}{\vdash S^\perp, \nu X.F}$$

This construction induces infinite branches, some of which being contained in the functoriality construct, and some of which that encounter infinitely often the sequent $\vdash S^\perp, \nu X.F$ (up-to structural equivalence). Note that a branch that eventually goes to the left of the above (Cut) cannot cycle back to $\vdash S^\perp, \nu X.F$ anymore. It may still be infinite, going through other cycles obtained from the translation of other coinduction rules in Π_1.

As a side remark, note that if Π is cut-free, then so is Π^i. Of course, if Π is cut-free but uses the version of the ν rule that embeds a cut, this is not true anymore.

▶ **Proposition 43.** *For any μMALL derivation Π, its translation Π^i is a $\mu\mathsf{MALL}^\infty$ proof.*

Proof sketch. We have to check that all infinite branches of Π^i are valid. Consider one such infinite branch. After a finite prefix, the branch must be contained in the pre-proof obtained from the translation of a coinduction rule (second case in the above definition). If the branch is eventually contained in a functoriality construct, then it contains two dual threads, and is thus valid. Otherwise, the branch visits infinitely often (up-to structural equivalence) the sequent $\vdash S^\perp, \nu X.F$ corresponding the our translated coinduction rule. The branch in Π^i contains a thread that contains the successive sub-occurrences of $\nu X.F$ in those sequents. More specifically, that formula is principal infinitely often in the thread. It only remains to show that it is minimal among formulas that appear infinitely often: this simply follows from the fact that formulas encountered along the thread inside the functoriality construct (F) all contain $\nu X.F$ as a subformula. ◀

B Appendix relative to Section 3

In this appendix, we first prove results corresponding to Section 3 and then develop a complete example of focusing process, in order to examplify the different concepts and objects defined in Section 3:

- reversibility of negative inference;
- focalization graph;
- focusing on positive inference;
- stepwise construction, by alternation of the two above – asynchronous and synchronous – phases, of a focusing proof from any given proof.

B.1 Polarity of connectives

▶ **Proposition** (11). *An infinite branch of a pre-proof containing only negative (resp. positive) rules is always valid (resp. invalid).*

Proof. An infinite negative branch contains only greatest fixed points. Among the threads, some are not eventually constant and their minimal formulas are ν-formulas: they are valid threads.

An infinite positive branch cannot be valid since for any non-constant thread t, $\mathsf{min}(t)$, its minimal formula, is a μ-formula. ◀

B.2 Reversibility

Before proving that rev actually builds a reversed proof, we first consider a simplified proof transformation for a proof π of a sequent $\vdash \Gamma, N$, $\mathsf{rev}_0(\pi, N)$, the effect of which being to reverse only the topmost connective of N. It is defined similarly to rev except that the procedure is not called on the subproofs contrarily to definition 13.

▶ **Definition 44** ($\mathsf{rev}_0(\pi, N)$)**.** We define $\mathsf{rev}_0(\pi, N)$ to be the pre-proof

$$\dfrac{\pi(1, N) \qquad \ldots \qquad \pi(\mathsf{sl}(N), N)}{\vdash \Gamma, N} \ (\mathsf{r_N}).$$

▶ **Proposition 45.** *Let π be a μMALL$^\infty$ proof of $\vdash \Gamma, N$. $\mathsf{rev}_0(\pi, N)$ is a μMALL$^\infty$ proof.*

Proof. The reader will easily check that any infinite branch β of $\mathsf{rev}_0(\pi, N)$ is obtained from a branch α of π, either of the form $(\mathsf{r_N}) \cdot \alpha$ when α does not contain an inference on N or $(\mathsf{r_N}) \cdot \alpha_1 \ldots \alpha_{n-1} \cdot \alpha_{n+1} \ldots$ where α_n has N a principal formula (occurrence). Validating threads are therefore preserved. ◀

We can now consider the general case of rev:

▶ **Theorem** (16)**.** *Let π be a μMALL$^\infty$ proof. $\mathsf{rev}(\pi)$ is a reversed proof of the same sequent.*

Proof. rev is obviously productive: each recursive call is guarded. Inferences of $\mathsf{rev}(\pi)$ are locally valid: if π is a preproof, so is $\mathsf{rev}(\pi)$.

If moreover π is a proof, infinite branches of $\mathsf{rev}(\pi)$ are valid: indeed, infinite branches of $\mathsf{rev}(\pi)$ are either fully negative (and therefore valid) or after a certain point they coincide with inferences of an infinite branch of π and their validity follows that of π.

The resulting proof is obviously shown to be reversed: we do not find any positive inference on any branch of $\mathsf{rev}(\pi)$, until the first positive sequent is reached. ◀

B.3 Focalization graphs

▶ **Proposition** (18)**.** *The positive trunk of a μMALL$^\infty$ proof is always finite.*

Proof. The positive trunk of a proof cannot have infinite branches, because they would be infinite positive branches of the original proof, thus necessarily invalid by proposition 11. ◀

▶ **Proposition** (20)**.** *Focalization graphs are acyclic.*

Even though the proof directly adapts the argument from [20], we provide it for completeness:

Proof. We prove the result by *reductio ad absurdum*. Let $\mathcal{S}$ be a positive sequent with a proof π. Let π^+ be the corresponding positive trunk and $\mathcal{G}$ the associated Focalization Graph. Suppose that $\mathcal{G}$ has a cycle and consider such a cycle of minimal length $(F_1 \to F_2 \to \cdots \to F_n \to F_1)$ in $\mathcal{G}$ and let us consider $\mathcal{S}_1, \ldots, \mathcal{S}_n$ sequents of the border justifying the arrows of the cycle.

These sequents are actually uniquely defined or the exact same reason as in MALL [20]. With the same idea we can immediately notice that the cycle is necessarily of length $n \geq 2$ since two $\prec$-subformulas of the same formula can never be in the same sequent in the border of the positive trunk.

Let $\mathcal{S}_0$ be $\bigwedge_{i=1}^{n} \mathcal{S}_i$ be the highest sequent in π such that all the $\mathcal{S}_i$ are leaves of the tree rooted in $\mathcal{S}_0$. We will obtain the contradiction by studying $\mathcal{S}_0$ and we will reason by case on the rule applied to this sequent $\mathcal{S}_0$:

- the rule cannot be $(\mathbf{1})$ rule since this rule produces no premiss and thus we would have an empty cycle which is non-sens. Any rule with no premiss would lead to the same contradiction.

- If the rule is one of ($\oplus_i$) or (μ), then the premiss $\mathcal{S}'_0$ of the rule would also satisfy the condition required for $\mathcal{S}_0$ (all the $\mathcal{S}_i$ would be part of the proof tree rooted in $\mathcal{S}'_0$) contradicting the maximality of $\mathcal{S}_0$. If the rule is any other non-branching rule, maximality of $\mathcal{S}_0$ would also be contradicted.
- Thus the rule shall be branching: it shall be a ($\otimes$). Write $\mathcal{S}_L$ and $\mathcal{S}_R$ for the left and right premisses of $\mathcal{S}_0$. Let $G = G_L \otimes G_R$ be the principal formula in $\mathcal{S}_0$ and let F be the active formula of the Trunk such that $F \prec G$.
  There are two possibilities:

  (i) either $F \in \{F_1, \dots, F_n\}$ and F is the only formula of the cycle having at the same time $\prec$-subformulas in the left premiss and in the right premiss,

  (ii) or $F \notin \{F_1, \dots, F_n\}$ and no formula of the cycle has $\prec$-subformulas in both premisses.

  Let thus I_L (resp. I_R) be the sets of indices of the active formulas of the root $\mathcal{S}$ having ($\prec$-related) subfomulas only in the left (resp. right) premiss. Clearly neither I_L nor I_R is empty since it would contradict the maximality of $\mathcal{S}_0$. Indeed if $I_L = \emptyset$, then $\mathcal{S}_R$ satisfies the condition of being dominated by all the $\mathcal{S}_i, 1 \leq i \leq n$ and $\mathcal{S}_0$ is not maximal anymore. By definition of the two sets of indices we have of course $I_L \cap I_R = \emptyset$ and the only formula of the cycle possibly not in $I_L \cup I_R$ is F if we are in the case (i): all other formulas in the cycle have their index either in I_L or in I_R.
  As a consequence there must be an arrow in the cycle (and thus in the graph) from a formula in I_L to a formula in I_R (or the opposite). Let $i \in I_L$ and $j \in I_R$ be such indexes (say for instance $F_i \to F_j$ in $\mathcal{G}$) and let $\mathcal{S}'$ be the sequent of the border responsible for this edge. $\mathcal{S}'$ contains F'_i and F'_j and by definition of the sets I_L and I_R, $\mathcal{S}'$ cannot be in the tree rooted in $\mathcal{S}_0$ which is in contradiction with the way we constructed $\mathcal{S}_0$.

Then there cannot be any cycle in the focalization graph. ◀

▶ **Proposition** (22). *Let $\mathcal{S}$ be a lowest sequent of* $\mathsf{foc}(\pi, P)$ *which is not conclusion of a rule on a positive subformula of P. Then $\mathcal{S}$ contains exactly one subformula of P, which is negative.*

Proof. $\mathsf{foc}(\pi, P)$ is such that the maximal prefix containing only rules applied to P and its positive subformulas decomposes P up to its negative subformulas. Uniqueness of the subformula in the case of MALL, treated in [20], can be directly adapted here. ◀

B.4 Productivity and validity of the focalization process

▶ **Proposition** (23). *Let π be a μMALL$^\infty$ proof,* r *a positive rule occurring in π and* r′ *be a negative rule occurring below* r *in π. If* r *occurs in* $\mathsf{Foc}(\pi)$*, then* r′ *occurs in* $\mathsf{Foc}(\pi)$*, below* r*.*

Proof. The proposition amounts to the simple remark that none of the transformation we do, for foc and rev, will ever permute a positive ***below*** a negative.

The proposition is thus satisfied by both transformations foc and rev and thus holds for $\mathsf{Foc}(\pi)$ which results from the iteration of the reversibility and focalization processes. ◀

▶ **Lemma** (24). *For any infinite branch γ of* $\mathsf{Foc}(\pi)$ *containing an infinite number of positive rules, there exists an infinite branch in π containing infinitely many positive rules of γ.*

Proof. The lemma results from a simple application of Koenig's lemma. ◀

▶ **Theorem** (25). *If π is a μMALL$^\infty$ proof then* $\mathsf{Foc}(\pi)$ *is also a μMALL$^\infty$ proof.*

Proof. Let γ be an infinite branch of $\mathsf{Foc}(\pi)$. If, at a certain point, γ is obtained by reversibility only, then it contains only negative rules and is therefore valid.

Otherwise, γ has been obtained by alternating infinitely often focalization phases foc and reversibility phases rev as described above. It therefore contains infinitely many positive inferences. By Lemma 24, there exists an infinite branch δ of π containing an infinite number of positive rules of γ. Since δ is valid, it contains a valid thread t.

Let F_m be the minimal formula of thread t, a ν-formula, and $(\mathsf{r_i})_{i\in\omega}$ the rules of δ in which F_m is the principal formula.

For any i, there exists a positive rule $\mathsf{r_i'}$ occurring in γ which is above $\mathsf{r_i}$ and $\mathsf{r_i}$ therefore also appears in γ by Proposition 23, which is therefore valid. ◀

B.5 An Example of Focalization

To conclude this section of the appendices, we present a detailed example of a focalization process in order to illustrate the material developped in the section of the paper devoted to focalization.

Let us consider the following proof of sequent

$$\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), (\mu X.X) \otimes \mathbf{1}.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\,(\nu) \quad \dfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\,(\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}}\,(\otimes)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} ⅋ \mathbf{0}}\,(⅋) \quad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mu X.X}\,(\nu),(\mu)}{\vdash \nu X.X, \mu X.X}\,(\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\mu) \quad \dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (r)), (s) \otimes (\mathbf{1} ⅋ \mathbf{0}), (\mu X.X) \otimes \mathbf{1}}\,(\otimes)$$

The Positive Trunk corresponding to this proof is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} ⅋ \mathbf{0} \quad \vdash \nu X.X, \mu X.X}{\vdash (\nu X.X) \otimes (\nu X.X), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), \mu X.X}\,(\mu) \quad \dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0}), (\mu X.X) \otimes \mathbf{1}}\,(\otimes)$$

and the Border is made of only two sequents:

$$\{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} ⅋ \mathbf{0} \quad ; \quad \vdash \nu X.X, \mu X.X\}$$

the Active Formulas of the positive trunk are thus:

- $\mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X))$
- $(\nu X.X) \otimes (\mathbf{1} ⅋ \mathbf{0})$
- $(\mu X.X) \otimes \mathbf{1}$

the Focalization Graph, which has thus those three formulas as vertices, is the following:

$$(\mu X.X)\otimes\mathbf{1} \longleftarrow (\nu X.X)\otimes(\mathbf{1}\parr\mathbf{0}) \longrightarrow \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X))$$

which is indeed acyclic and has a single source, $(\nu X.X)\otimes(\mathbf{1}\parr\mathbf{0})$, which we pick as focus. By rewriting the Prositive Trunk we arrive at

$$\dfrac{\dfrac{\pi_1}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}\parr\mathbf{0}} \qquad \dfrac{\pi_2}{\vdash \nu X.X, (\mu X.X)\otimes\mathbf{1}}}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), (\nu X.X)\otimes(\mathbf{1}\parr\mathbf{0}), (\mu X.X)\otimes\mathbf{1}}\ (\otimes)$$

with

$$\pi_1 = \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\ (\nu) \qquad \dfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\ (\nu)}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}, \mathbf{0}}\ (\otimes)}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}\parr\mathbf{0}}\ (\parr)}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}\parr\mathbf{0}}\ (\oplus_2) \quad \text{and} \quad \pi_2 = \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mu X.X}\ (\nu),(\mu)}{\vdash \nu X.X, \mu X.X}\ (\nu)}{\vdash \nu X.X, \mu X.X}\ (\mu) \qquad \dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1})}{\vdash \nu X.X, (\mu X.X)\otimes\mathbf{1}}\ (\otimes)$$

and we continue by focalizing π_1 and π_2.
As for π_1, its conclusion is a negative sequent, so that one first considers $\mathsf{rev}(\pi_1)$:

$$\mathsf{rev}(\pi_1) = \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\ (\nu) \qquad \dfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\ (\nu)}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}, \mathbf{0}}\ (\otimes)}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}, \mathbf{0}}\ (\oplus_2)}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}\parr\mathbf{0}}\ (\parr)$$

$\mathsf{rev}(\pi_1)$ is actually already focused: the conclusion of

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\ (\nu) \qquad \dfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\ (\nu)}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}, \mathbf{0}}\ (\otimes)}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}, \mathbf{0}}\ (\oplus_2)$$

is a positive sequent and its positive trunk is:

$$\dfrac{\dfrac{\vdash \nu X.X, \mathbf{1} \qquad \vdash \nu X.X, \mathbf{0}}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}, \mathbf{0}}\ (\otimes)}{\vdash \mathbf{0}\oplus((\nu X.X)\otimes(\nu X.X)), \mathbf{1}, \mathbf{0}}\ (\oplus_2)$$

This positive trunk contains only one active formula which therefore is automatically chosen as a focus (and the positive trunk actually already focused on it).
Subproofs

$$\dfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\ (\nu) \qquad \dfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\ (\nu)$$

are infinite negative branches and therefore reversed, focused proofs.

As for π_2, its conclusion is also a negative sequent so that we build $\mathsf{rev}(\pi_2)$ which turns out to be focused as it is reduced to an infinite negative branch of (ν) rules:

$$\mathsf{rev}(\pi_2) = \cfrac{\vdots}{\vdash \nu X.X, (\mu X.X)\otimes\mathbf{1}}\ (\nu)$$

To sum up, the focused proof associated with our starting proof object is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\vdash \nu X.X, \mathbf{1}}\ (\nu) \quad \cfrac{\vdots}{\vdash \nu X.X, \mathbf{0}}\ (\nu)}{\vdash (\nu X.X)\otimes(\nu X.X), \mathbf{1}, \mathbf{0}}\ (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X)\otimes(\nu X.X)), \mathbf{1}, \mathbf{0}}\ (\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X)\otimes(\nu X.X)), \mathbf{1}\parr\mathbf{0}}\ (\parr) \quad \cfrac{\vdots}{\vdash \nu X.X, (\mu X.X)\otimes\mathbf{1}}\ (\nu)}{\vdash \mathbf{0} \oplus ((\nu X.X)\otimes(\nu X.X)), (\nu X.X)\otimes(\mathbf{1}\parr\mathbf{0}), (\mu X.X)\otimes\mathbf{1}}\ (\otimes)$$

C Appendix relative to Section 4

C.1 Detailed definitions

We first give a detailed description of the multicut reduction rules. In order to treat the external reduction for the tensor, we first need to introduce a few preliminary definitions. Given a sequent $\vdash \Gamma, \Delta, F\otimes G$ that is a premise of a multicut, we need to define which part of the multicut is connected to Γ and which part is connected to Δ. These two sub-nets, respectively called $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Delta$, will be split apart in the external tensor reduction.

▶ **Definition 46.** We call ***cut net*** any set of sequents $\{s_i\}_i$ that forms a valid set of premises for the multicut rule, *i.e.*, a connected acyclic graph for the cut-connection relation. The conclusion of a cut net is the conclusion that the multicut rule would have with the cut net as premise, *i.e.*, the set of formula occurrences that appear in the net but not as cut formulas.

▶ **Definition 47.** Let $\mathcal{M}$ be a cut net, and F be a formula occurrence appearing in some $s \in \mathcal{M}$. We define $\mathcal{C}_F \subseteq \mathcal{M}$ as follows. If $F^\perp \in s'$ for some $s' \in \mathcal{M}$, then $\mathcal{C}_F$ is the connected component of $\mathcal{M} \setminus \{s\}$ containing s'. Otherwise, $\mathcal{C}_F = \emptyset$. If Δ is a set of formula occurrences, we define $\mathcal{C}_\Delta := \bigcup_{F\in\Delta} \mathcal{C}_F$.

▶ **Proposition 48.** *Let $s = \vdash F, \Delta, \Gamma$ be a sequent, and $\mathcal{M} = \{s\} \cup \mathcal{C}$ be a cut net of conclusion $\vdash F, \Sigma$. One has $\mathcal{C} = \mathcal{C}_\Delta \uplus \mathcal{C}_\Gamma$. Moreover, $\{\vdash \Gamma\} \cup \mathcal{C}_\Gamma$ and $\{\vdash \Delta\} \cup \mathcal{C}_\Delta$ are cut nets and, if Σ_Γ and Σ_Δ are their respective conclusions, we have $\Sigma = \Sigma_\Delta \uplus \Sigma_\Gamma$.*

▶ **Definition 49** (Multicut reduction rules)**.** ***Principal and external reductions*** are respectively defined in Figure 4 and 5. ***Internal reduction*** is the union of merge and principal reductions. ***Merge reduction*** is defined as follows, with $r = (\mathsf{merge}, \{F, F^\perp\})$:

$$\cfrac{\mathcal{C} \quad \cfrac{\vdash \Delta, F \quad \vdash \Gamma, F^\perp}{\vdash \Delta, \Gamma}\ (\mathsf{Cut})}{\vdash \Sigma}\ (\mathsf{mcut}) \quad \underset{r}{\longrightarrow} \quad \cfrac{\mathcal{C} \quad \vdash \Delta, F \quad \vdash \Gamma, F^\perp}{\vdash \Sigma}\ (\mathsf{mcut})$$

We can now provide more explicit notions of reduction sequences and fairness.

▶ **Definition 50.** A ***multicut reduction sequence*** is a finite or infinite sequence $\sigma = (\pi_i, r_i)_{i\in\lambda}$, with $\lambda \in \omega + 1$, where the π_i, r_i are pairs of $\mu\mathsf{MALL}^\infty_\mathsf{m}$ proofs and r_i is label identifying a multicut reduction rule and, whenever $i + 1 \in \lambda$, $\pi_i \underset{r_i}{\longrightarrow} \pi_{i+1}$.

The following definition of fair reduction is standard from rewriting theory (see for instance chapter 9 of [25], definition 4.9.10):

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F \quad \vdash \Gamma, G}{\vdash \Delta, \Gamma, F \otimes G}(\otimes) \quad \dfrac{\vdash \Theta, G^{\perp}, F^{\perp}}{\vdash \Theta, G^{\perp} \parr F^{\perp}}(\parr)}{\vdash \Sigma}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\mathcal{C} \quad \vdash \Delta, F \quad \vdash \Gamma, G \quad \vdash \Theta, G^{\perp}, F^{\perp}}{\vdash \Sigma}(\mathsf{mcut})$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F_2 \quad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1}(\&) \quad \dfrac{\vdash \Gamma, F_i^{\perp}}{\vdash \Gamma, F_1^{\perp} \oplus F_2^{\perp}}(\oplus_i)}{\vdash \Sigma}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\mathcal{C} \quad \vdash \Delta, F_i \quad \vdash \Gamma, F_i^{\perp}}{\vdash \Sigma}(\mathsf{mcut})$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}(\mu) \quad \dfrac{\vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Gamma, \nu X.F^{\perp}}(\nu)}{\vdash \Sigma}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\mathcal{C} \quad \vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Sigma}(\mathsf{mcut})$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}(\bot) \quad \dfrac{}{\vdash \mathbf{1}}(\mathbf{1})}{\Sigma}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\mathcal{C} \quad \vdash \Gamma}{\Sigma}(\mathsf{mcut})$$

Figure 4 Principal reductions, where $r = (\mathsf{principal}, \{F, F^{\perp}\})$ with $\{F, F^{\perp}\}$ the principal formulas that have been reduced.

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F \quad \vdash \Gamma, G}{\vdash \Delta, \Gamma, F \otimes G}(\otimes)}{\vdash \Sigma_{\Delta}, \Sigma_{\Gamma}, F \otimes G}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\dfrac{\mathcal{C}_{\Delta} \quad \vdash \Delta, F}{\vdash \Sigma_{\Delta}, F}(\mathsf{mcut}) \quad \dfrac{\mathcal{C}_{\Gamma} \quad \vdash \Gamma, G}{\vdash \Sigma_{\Gamma}, G}(\mathsf{mcut})}{\vdash \Sigma_{\Delta}, \Sigma_{\Gamma}, F \otimes G}(\otimes)$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}(\parr)}{\vdash \Sigma, F \parr G}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\dfrac{\mathcal{C} \quad \vdash \Delta, F, G}{\vdash \Sigma, F, G}(\mathsf{mcut})}{\vdash \Sigma, F \parr G}(\parr)$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F \quad \vdash \Delta, G}{\vdash \Delta, F \& G}(\&)}{\vdash \Sigma, F \& G}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\dfrac{\mathcal{C} \quad \vdash \Delta, F}{\vdash \Sigma, F}(\mathsf{mcut}) \quad \dfrac{\mathcal{C} \quad \vdash \Delta, G}{\vdash \Sigma, G}(\mathsf{mcut})}{\vdash \Sigma, F \& G}(\&)$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta, F_i}{\vdash \Delta, F_1 \oplus F_2}(\oplus_i)}{\vdash \Sigma, F_1 \oplus F_2}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\dfrac{\mathcal{C} \quad \vdash \Delta, F_i}{\vdash \Sigma, F_i}(\mathsf{mcut})}{\vdash \Sigma, F_1 \oplus F_2}(\oplus_i)$$

$$\dfrac{\mathcal{C} \quad \dfrac{}{\vdash \Delta, \top_{\alpha}}(\top)}{\vdash \Sigma, \top_{\alpha}}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{}{\vdash \Sigma, \top_{\alpha}}(\top)$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Delta}{\vdash \Delta, \bot_{\alpha}}(\bot)}{\vdash \Sigma, \bot_{\alpha}}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{\dfrac{\mathcal{C} \quad \vdash \Delta}{\vdash \Sigma}(\mathsf{mcut})}{\vdash \Sigma, \bot_{\alpha}}(\bot) \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}(\mathbf{1})}{\vdash \mathbf{1}}(\mathsf{mcut}) \underset{r}{\longrightarrow} \dfrac{}{\vdash \mathbf{1}}(\mathbf{1})$$

Figure 5 External reductions rules, where $r = (\mathsf{ext}, F)$ and F is the formula occurrence that is principal after the rule application.

▶ **Definition 51** (Fair reduction sequences). A ***multicut reduction sequence*** $(\pi_i, r_i)_{i\in\lambda}$ is ***fair*** if for every $i \in \lambda$ and r such that $\pi_i \underset{r}{\longrightarrow} \pi'$, there is some $j \geq i$, $j \in \lambda$, such that π_j contains no residual of r.

Fairness is defined in the same way for a reduction path rather than a reduction sequence. In that case, fairness can be rephrased in a simpler way: A ***multicut reduction path*** $(\pi_i, r_i)_{i\in\lambda}$ is ***fair*** if for every $i \in \lambda$ and r such that $\pi_i \underset{r}{\longrightarrow} \pi'$, there is some $j \geq i$, $j \in \lambda$, such that r has disappeared from π_{j+1} (or: r_j is r or r_j erases r).

Note that reduction paths issued from a fair reduction sequence are always fair.

We end this section with more details on definition 28, which defines useless sequents. Useless sequents of sort (3) and (4) are useless only because we are considering a reduction path and not a reduction sequence. Writing $\Rightarrow$ for the reduction steps associated to reduction paths, we can more explicitly say that the sequent $\vdash \Gamma, F_i$ is useless of sort (3) with distinguished formula F_i if, at some point in the reduction path, one of the following reductions is performed (with $\{i,j\} = \{1,2\}$):

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Gamma, F_1 \quad \vdash \Gamma, F_2}{\vdash \Gamma, F_1 \& F_2}\,(\&)}{\vdash \Sigma, F_1 \& F_2}\,(\mathsf{mcut}) \quad \underset{r_i}{\Longrightarrow} \quad \dfrac{\mathcal{C} \quad \vdash \Gamma, F_j}{\vdash \Sigma, F_j}\,(\mathsf{mcut})$$

$$\dfrac{\mathcal{C} \quad \dfrac{\vdash \Gamma, F_i \quad \vdash \Delta, F_j}{\vdash \Delta, \Gamma, F_1 \otimes F_2}\,(\otimes)}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F_1 \otimes F_2}\,(\mathsf{mcut}) \quad \underset{r_i}{\Longrightarrow} \quad \dfrac{\mathcal{C}_\Delta \quad \vdash \Delta, F_j}{\vdash \Sigma_\Delta, F_j}\,(\mathsf{mcut})$$

Moreover, the second reduction renders all sequents of $\mathcal{C}_\Gamma$ useless of sort (4). Their distinguished formulas are cut formulas, chosen based on a traversal of the acyclic graph $\mathcal{C}_\Gamma$, in a way which ensures that G and $G^\perp$ are never both distinguished. In particular, for each $s' \in \mathcal{C}_\Gamma$ that is cut-connected to $\vdash \Gamma, F_i$ on G, we choose $G^\perp$ as the distinguished formula of s'. More precisely, we define the distigiushed formulas of $\mathcal{C}_\Gamma$ inductively as follows:

- The distinguished formula of Γ, F_i is F_i.
- If the distinguished formula of a sequent s has been defined, and if s' cut-connected to s on $G \in s'$, we choose G as the distinguished formula of s'.

Notice that two dual cut formulas G and $G^\perp$ can never both be distinguished.

C.2 Truncated truth semantics

In order to develop the soundness argument for the interpretation of truncated formula occurrences, we need to work with a slightly enriched notion of formula. We thus introduce below a generalization of formulas and of the interpretation of Definition 33.

▶ **Definition 52.** ***Marked pre-formulas*** are built over the following syntax, where θ is an ordinal:

$$\varphi, \psi ::= \mathbf{0} \mid \top \mid \varphi \oplus \psi \mid \varphi \& \psi \mid \bot \mid \mathbf{1} \mid \varphi \parr \psi \mid \varphi \otimes \psi \mid \mu X.\varphi \mid \nu^\theta X.\varphi \mid X \text{with } X \in \mathcal{V}.$$

A marked formula is a marked pre-formula with no free variables. A marked formula occurrence is given by a marked formula φ and an address α and is written φ_α.

▶ **Definition 53.** Let $\bigvee E$ be the truncation $\alpha \mapsto \top$. Let f be an operator over E. We define the iterations of f starting from $\bigvee E$ by:

- $f^0(\bigvee E) = \bigvee E$;
- $f^\delta(\bigvee E) = f(f^\lambda(\bigvee E))$ for every successor ordinal $\delta = \lambda + 1$;
- $f^\delta(\bigvee E) = \bigcap_{\lambda<\delta} f^\lambda(\bigvee E)$ for every limit ordinal δ.

We define the interpretation of a marked formula occurrence as follows, generalizing Definition 33:

▶ **Definition 54.** Let φ_α be a marked formula occurrence and $\mathcal{E}$ be an environment, *i.e.*, a function mapping every free variable of φ to an element of E. We define $[\varphi_\alpha]^\mathcal{E} \in \mathcal{B}$, the interpretation of φ_α in the environment $\mathcal{E}$ as follows: if $\alpha \in \mathtt{Dom}(\tau)$ then $[\varphi_\alpha]^\mathcal{E} = \tau(\alpha)$; otherwise:

- $[X_\alpha]^\mathcal{E} = \mathcal{E}(X)(\alpha)$, $[\top_\alpha]^\mathcal{E} = \top$, $[\mathbf{0}_\alpha]^\mathcal{E} = \mathbf{0}$, $[\mathbf{1}_\alpha]^\mathcal{E} = \top$ and $[\bot_\alpha]^\mathcal{E} = \mathbf{0}$.
- $[(\varphi \owedge \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \wedge [\psi_{\alpha.r}]^\mathcal{E}$, for $\owedge \in \{\&, \otimes\}$.
- $[(\varphi \ovee \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \vee [\psi_{\alpha.r}]^\mathcal{E}$, for $\ovee \in \{\oplus, \parr\}$.
- $[(\mu X.\varphi)_\alpha]^\mathcal{E} = \mathrm{lfp}(f)(\alpha)$ and $[(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} = f^\theta(\bigvee E)(\alpha)$ where $f : E \to E$ is defined by:

$$f : h \mapsto \beta \mapsto \begin{cases} \tau(\beta) & \text{if } \beta \in \mathtt{Dom}(\tau) \\ [\varphi_{\beta.i}]^{\mathcal{E}, X \mapsto h} & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{O}(\varphi, X, \mathcal{E})$ the operator f and we set $[\varphi]^\mathcal{E} := (\alpha \mapsto [\varphi_\alpha]^\mathcal{E})$.

As is standard, the least fixed point of f is guaranteed to exist in the above definition because $[\varphi]^\mathcal{E}$ is a monotonic operator in the complete lattice E, obtained by lifting the lattice $\mathcal{B}$ where $\mathbf{0} \leq \top$ with a pointwise ordering.

▶ **Proposition 55** (Cousot & Cousot)**.** *Let λ the least ordinal such that the class $\{\delta : \delta \in \lambda\}$ has a cardinality greater than the cardinality $Card(E)$. Let f be a monotonic operator over E. The sequence $(f^\delta(\bigvee E))_{\delta \in \lambda}$ is a stationary decreasing chain, its limit $f^\lambda(\bigvee E)$ is the greatest fixed point of f.*

Let $\overline{F}$ be the marked formula occurrence obtained from F by marking every ν binder by λ. As a consequence of Proposition 55, one has that $[F] = [\overline{F}]$.

▶ **Lemma 56.** *Let φ, ψ be marked pre-formulas such that $X \notin \mathrm{fv}(\psi)$. One has:*

$$[\varphi_\alpha]^{\mathcal{E}, X \mapsto [\psi]^\mathcal{E}} = [(\varphi[\psi/X])_\alpha]^\mathcal{E}.$$

Proof. The proof is by induction on φ. We treat only the cases where φ is a fixed point formula; the other cases are immediate.

Suppose that $\varphi = \nu Y^\theta.\xi$ and let $f = \mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^\mathcal{E})$ and $g = \mathcal{O}(\xi[\psi/X], Y, \mathcal{E})$. By induction hypothesis one has $f^\theta(\bigvee E) = g^\theta(\bigvee E)$, which concludes this case.

Suppose now that $\varphi = \mu Y.\xi$, then we have:

$$\begin{array}{rcl} [(\mu Y.\xi)_\alpha]^{\mathcal{E}, X \mapsto [\psi]^\mathcal{E}} & = & \mathrm{lfp}(\mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^\mathcal{E}))(\alpha) \\ & \overset{*}{=} & \mathrm{lfp}(\mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^{\mathcal{E}, Y \mapsto h}))(\alpha) \\ & \overset{IH}{=} & \mathrm{lfp}(\mathcal{O}(\xi[\psi/X], Y, \mathcal{E}))(\alpha) \\ & = & [(\mu Y.\xi[\psi/X])_\alpha]^\mathcal{E} \end{array}$$

(*) We are considering capture-free substitutions, hence $Y \notin fv(\psi)$ and $[\psi]^{\mathcal{E}, Y \mapsto f} = [\psi]^\mathcal{E}$. ◀

An immediate consequence of this proposition is that the interpretation of a least fixed point formula is equal to the interpretation of its unfolding:

▶ **Lemma 57.** *If $\alpha \notin \texttt{Dom}(\tau)$, $[(\mu X.\varphi)_\alpha]^{\mathcal{E}} = [(\varphi[\mu X.\varphi/X])_{\alpha.i}]^{\mathcal{E}}$*

Proof. We set $f = \mathcal{O}(\varphi, X, \mathcal{E})$. Let us notice first that for all $\alpha \in \Sigma^*$, one has $[(\mu X.\varphi)_\alpha]^{\mathcal{E}} = \mathrm{lfp}(f)(\alpha)$. Indeed, one has the equality by definition when $\alpha \notin \texttt{Dom}(\tau)$ and it is easy to prove it when $\alpha \in \texttt{Dom}(\tau)$ since both sides are equal to $\tau(\alpha)$.

$$\begin{array}{rcl} [(\mu X.\varphi)_\alpha]^{\mathcal{E}} & = & \mathrm{lfp}(f)(\alpha) \\ & = & [\varphi_{\alpha.i}]^{\mathcal{E}, X\mapsto \mathrm{lfp}(f)} \\ & = & [\varphi_{\alpha.i}]^{\mathcal{E}, X\mapsto [\mu X.\varphi]^{\mathcal{E}}} \\ & = & [(\varphi[\mu X.\varphi/X])_{\alpha.i}]^{\mathcal{E}} \end{array}$$

◀

▶ **Lemma 58.** *If $[(\nu X^\theta.\varphi)_\alpha]^{\mathcal{E}} = \mathbf{0}$ and $\alpha \notin \texttt{Dom}(\tau)$ then there is an ordinal $\gamma < \theta$ s.t. $[(\varphi[\nu X^\gamma.\varphi/X])_{\alpha.i}]^{\mathcal{E}} = \mathbf{0}$.*

Proof. We set $f = \mathcal{O}(\varphi, X, \mathcal{E})$. If θ is a successor ordinal $\delta + 1$, then:

$$\begin{array}{rcl} [(\nu X^\theta.\varphi)_\alpha]^{\mathcal{E}} & = & f^{\delta+1}(\bigvee E)(\alpha) \\ & = & [\varphi_{\alpha.i}]^{\mathcal{E}, X\mapsto f^\delta(\bigvee E)} \\ & = & [\varphi_{\alpha.i}]^{\mathcal{E}, X\mapsto [\nu X^\delta.\varphi]^{\mathcal{E}}} \\ & = & [(\varphi[\nu X^\delta.\varphi/X])_{\alpha.i}]^{\mathcal{E}} \end{array}$$

We take γ to be the ordinal δ and we have obviously that $[(\varphi[\nu X^\gamma.\varphi/X])_{\alpha.i}]^{\mathcal{E}} = \mathbf{0}$.

If θ is a limit ordinal, then:

$$\begin{array}{rcl} [(\nu X^\theta.\varphi)_\alpha]^{\mathcal{E}} & = & f^\theta(\bigvee E)(\alpha) \\ & = & \bigcap_{\beta<\theta} f^\beta(\bigvee E) \\ & = & \bigcap_{\delta+1<\theta} f^{\delta+1}(\bigvee E) \end{array}$$

Hence there is a successor ordinal $\delta + 1$ such that $[(\nu X^\theta.\varphi)_\alpha]^{\mathcal{E}} = f^{\delta+1}(\bigvee E)(\alpha)$ and we continue as before. ◀

We prove easily the following lemma by induction on F:

▶ **Lemma 59.** *Let F be an (unmarked) formula occurrence. One has $[F^\perp] = [F]^\perp$.*

We can finally establish our soundness result:

▶ **Proposition** (34). *If $\vdash \Gamma$ is provable in $\mu\mathsf{MALL}^\infty_\tau$, then $[F] = \top$ for some $F \in \Gamma$.*

Proof. If F is a marked formula occurrence, we denote by F^* the formula occurrence obtained by forgetting the marking information.

Suppose that $\vdash \Gamma$ has a $\mu\mathsf{MALL}^\infty_\tau$ proof π and that $[F] = \mathbf{0}$ for all $F \in \Gamma$. We will construct a branch $\gamma = s_0 s_1 \ldots$ of π and a sequence of functions $f_0, f_1, \ldots$ where f_i maps every formula occurrence G of s_i to a marked formula occurrence $f_i(G)$ such that $[f_i(G)] = \mathbf{0}$ and $f_i(G)^* = G$ unless $G = \varphi_{\alpha.i}$ with $\alpha \in \texttt{Dom}(\tau)$. We set $s_0 = \Gamma$ and $f_0(F) = \overline{F}$. One has $[\overline{F}] = [F] = \mathbf{0}$. Suppose that we have constructed s_i and f_i. We construct s_{i+1} depending on the rule applied to s_i:

- If the rule is a logical rule, G being principal in s_i, we set $G_m := f_i(G)$, we have the following cases:

- If $G = H \parr K$, then G_m is of the form $G_m = H_m \parr K_m$. We set s_{i+1} to be the unique premise of s_i, $f_{i+1}(H) = H_m$ and $f_{i+1}(K) = K_m$. Since $[G_m] = \mathbf{0}$ and $[G_m] = [H_m] \vee [K_m]$, one has $[G_m] = \mathbf{0}$ and $[K_m] = \mathbf{0}$. For every other formula occurrence L of s_{i+1} we set $f_{i+1}(L) = f_i(L)$.
- If $G = H \oplus K$, we proceed exactly in the same way as above.
- If $G = H \otimes K$, then G_m is of the form $G_m = H_m \otimes K_m$. Since $[G_m] = \mathbf{0}$ and $[G_m] = [H_m] \wedge [K_m]$, one has $[H_m] = \mathbf{0}$ or $[K_m] = \mathbf{0}$. Suppose wlog that $[H_m] = \mathbf{0}$. We set s_{i+1} to be the premise of s_i that contains H and $f_{i+1}(H) = H_m$. For every other formula occurrence L of s_{i+1} we set $f_{i+1}(L) = f_i(L)$.
- If $G = H \& K$, we proceed exactly in the same way as above.
- If $G = \mu X.K$, then G_m is of the form $G_m = \mu X.K_m$. We set s_{i+1} to be the unique premise of s_i and $f_{i+1}(K[G/X]) = K_m[G_m/X]$. By Corollary 57 and since $[G_m] = \mathbf{0}$, one has $[K_m[G_m/X]] = \mathbf{0}$. For every other formula occurrence L of s_{i+1}, we set $f_{i+1}(L) = f_i(L)$.
- If $G = \nu X.H$, then G_m is of the form $G_m = \nu X^\theta.K_m$. Let s_{i+1} be the unique premise of s_i. By corollary 58 and since $[G_m] = \mathbf{0}$, there is an ordinal $\delta < \theta$ such that $[K_m[\nu X^\delta.K_m/X]] = \mathbf{0}$. We set $f_{i+1}(H[G/X]) = K_m[\nu X^\delta.K_m/X]$ and for every other formula occurrence L of s_{i+1}, we set $f_{i+1}(L) = f_i(L)$.

- Suppose that the rule applied to s_i is a cut on the formula occurrence G. By Lemma 59, either $[G] = \mathbf{0}$ or $[G^\perp] = \mathbf{0}$, suppose wlog that $[G] = \mathbf{0}$. We set s_{i+1} to be the premise of s_i containing G, $f_{i+1}(G) \equiv \overline{G}$ and for every other formula occurrence L of s_{i+1}, we set $f_{i+1}(L) \equiv f_i(L)$.
- If the rule applied to s_i is the rule (τ) with a principal formula $G = \varphi_\alpha$, then $\alpha \in \mathtt{Dom}(\tau)$ and $f_i(G) = \psi_\alpha$ where $\psi^* = \varphi$. Hence $[f_i(G)] = \tau(\alpha)$. By construction $[f_i(G)] = \mathbf{0}$, hence $\tau(\alpha) = \mathbf{0}$ and $[\tau(\alpha)_{\alpha.i}] = \mathbf{0}$. We set s_{i+1} to be the unique premise of s_i.

Since π is a valid pre-proof, its branch γ must contain a valid thread $t = F_0F_1\ldots$. Let $\nu X.\varphi$ be the minimal formula of t and $i_0 i_1 \ldots$ be the sequence of indices where $\nu X.\varphi$ gets unfolded. By construction, for all $k > 0$ one has $f_{i_k}(F_{i_k}) = \nu X^{\theta_k}.G_k$ and the sequence of ordinals $(\theta_k)_k$ is strictly decreasing, which contradicts the well-foundedness of ordinals. ◀

We finally prove Proposition 36, generalized as follows:

▶ **Proposition 60.** *Let φ_α be a pre-formula occurrence compatible with τ and containing no ν binders, no $\top$ and no $\mathbf{1}$ subformulas. Let $\mathcal{E}$ be an environment such that for all $\beta \notin \mathtt{Dom}(\tau)$, $\mathcal{E}(X)(\beta) = \mathbf{0}$. We have $[\varphi_\alpha]^{\mathcal{E}} = \mathbf{0}$.*

Proof. The proof is by induction on φ.

- The cases when $\varphi = \mathbf{0}$ or $\perp$ are trivial.
- If $\varphi = X$, then $[X_\alpha]^{\mathcal{E}} = \mathcal{E}(X)(\alpha) = \mathbf{0}$ by hypothesis on $\mathcal{E}$ and since $\alpha \notin \mathtt{Dom}(\tau)$ by compatibility with τ.
- If $\varphi = \xi \ovee \psi$, where $\ovee \in \{\oplus, \parr\}$, then $[(\xi \ovee \psi)_\alpha]^{\mathcal{E}} = [\xi_{\alpha.l}]^{\mathcal{E}} \vee [\psi_{\alpha.r}]^{\mathcal{E}}$. Since $(\xi \ovee \psi)_\alpha$ is compatible with τ, one has $\alpha.l \notin \mathtt{Dom}(\tau)$ and $\alpha.r \notin \mathtt{Dom}(\tau)$. Indeed, if a formula is compatible with a truncation τ, then τ cannot truncate a son of $\oplus$ or a $\parr$ node. We can thus apply our induction hypothesis, obtaining $[\xi_{\alpha.l}]^{\mathcal{E}} = [\psi_{\alpha.r}]^{\mathcal{E}} = \mathbf{0}$, hence $[(\xi \ovee \psi)_\alpha]^{\mathcal{E}} = \mathbf{0}$.
- If $\varphi = \xi \owedge \psi$, where $\owedge \in \{\&, \otimes\}$, then $[(\xi \owedge \psi)_\alpha]^{\mathcal{E}} = [\xi_{\alpha.l}]^{\mathcal{E}} \wedge [\psi_{\alpha.r}]^{\mathcal{E}}$. Since $(\xi \owedge \psi)_\alpha$ is compatible with τ, one has $\alpha.l \notin \mathtt{Dom}(\tau)$ or $\alpha.r \notin \mathtt{Dom}(\tau)$. Indeed, if a formula is compatible with a truncation τ, then τ cannot truncate both sons of a $\&$ or a $\otimes$ node. We conclude by induction as before on the subformula that is not truncated, and which is thus still compatible with τ.

- If $\varphi = \mu X.\psi$, then $[\mu X.B]^{\mathcal{E}} = \mathrm{lfp}(f)(\tau)$ where f is as in the definition 33. By Cousot's theorem [9], $[(\mu X.B)_\alpha]^{\mathcal{E}} = \bigvee_{\delta<\lambda} \varphi^\delta(\bigwedge E)(\alpha)$. We show by an easy transfinite induction that for all $\delta < \lambda$ and $\beta \notin \texttt{Dom}(\tau)$, we have $\varphi^\delta(\bigwedge E)(\beta) = \mathbf{0}$. This concludes the proof. ◀