Minimum-Density Identifying Codes in Square Grids

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Abstract. An identifying code in a graph G is a subset of vertices with the property that for each vertex $v \in V(G)$, the collection of elements of C at distance at most 1 from v is non-empty and distinct from the collection of any other vertex. We consider the minimum density $d^*(\mathcal{S}_k)$ of an identifying code in the square grid $\mathcal{S}_k$ of height k (i.e. with vertex set $\mathbb{Z} \times \{1, \dots, k\}$). Using the Discharging Method, we prove $\frac{7}{20} + \frac{1}{20k} \leq d^*(\mathcal{S}_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}$, and $d^*(\mathcal{S}_3) = \frac{7}{18}$.

Keywords: identifying code, square grid, discharging method

1 Introduction

The *two-way infinite path*, denoted $P_{\mathbb{Z}}$, is the graph with vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} \ : \ i \in \mathbb{Z}\}$. For every positive integer k, the *finite path of length* $k-1$, denoted P_k, is the subgraph of $P_{\mathbb{Z}}$ induced by $\{1, 2, \dots, k\}$.

The *cartesian product* of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y) \mid$ either $(a = b$ and $xy \in E(H))$ or $(ab \in E(G)$ and $x = y)$ $\}$. A *square grid* is the cartesian product of two paths, which can be finite or infinite. The *square lattice* is the cartesian product $P_{\mathbb{Z}} \Box P_{\mathbb{Z}}$ of two two-way infinite paths and is denoted by $\mathcal{G}$. For every positive integer k, we denote by $\mathcal{S}_k$ the square grid $P_{\mathbb{Z}} \Box P_k$

Let G be a graph. The *closed neighbourhood* of v, denoted $N[v]$, is the set of vertices that are either v or adjacent to v in G. A set $C \subseteq V(G)$ is an *identifying code* in G if for every vertex $v \in V(G)$, $N[v] \cap C \neq \emptyset$, and for any two distinct vertices $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$.

Let G be a (finite or infinite) graph. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G, that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C, G)$, is defined by

$$d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} \ ,$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code in G.

The problem of finding identifying codes of small density was introduced in [10] in relation to fault diagnosis in arrays of processors. Identifying codes are also used in [11] to model a location detection problem with sensor networks. Identifying codes of the grids have been studied [2], [5], [9], [10] as well as variations where instead of considering the closed neighbourhood to identify a vertex, the the ball of radius r (for some fixed r) is considered [2], [8]. The closely related problem of finding a locating-dominating set with minimum density has also been studied [12].

In this paper, we are interested in identifying codes of square grids, and more specifically the $\mathcal{S}_k$. The tile depicted in Figure 1 was given in [3]. It generates a

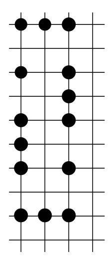

Fig. 1. Tile generating an optimal identifying code of the grid.

periodic tiling of the plane with periods $(0, 10)$ and $(4, 1)$, yielding an identifying code $C^*_\mathcal{G}$ of the square lattice with density $\frac{7}{20}$. Ben-Haim and Litsyn [1] proved that this density is optimal, that is $d^*(\mathcal{G}) = \frac{7}{20}$.

Daniel, Gravier, and Moncel [6] showed that $d^*(\mathcal{S}_1) = \frac{1}{2}$ and $d^*(\mathcal{S}_2) = \frac{3}{7}$. For larger value of k, they proved the following lower and upper bound on $d^*(\mathcal{S}_k)$: $\frac{7}{20} - \frac{1}{2k} \leq d^*(\mathcal{S}_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{2}{k}\right\}$. In this paper, we improve on both the lower and upper bounds of $d^*(\mathcal{S}_k)$. We prove

$$\frac{7}{20} + \frac{1}{20k} \leq d^*(\mathcal{S}_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

The upper bound is obtain by deriving an identifying code of $\mathcal{S}_k$ with density $\frac{7}{20} + \frac{3}{10k}$ from the optimal identifying code $C^*_\mathcal{G}$ of the square lattice.

The lower bound is obtained using the Discharging Method and proceeds in two phases. The first one is a rewriting of the proof of Ben-Haim and Litsyn [1] as a Discharging Method proof. Doing so, it becomes clear that it extends to any square grid, and so that $d^*(G) \geq \frac{7}{20}$ for any square grid. It makes it also possible to improve on this bound when $G = \mathcal{S}_k$ with $k \geq 3$ in a second phase.

We strongly believe that both our upper and lower bounds may be improved using the same general techniques. In fact, to obtain the upper bound, we only alter the code $C^*_{\mathcal{G}}$ on the top two rows and the bottom two rows of $\mathcal{S}_k$. Looking for alterations on more rows, possibly with the help of a computer, will certainly yield codes with smaller density. We made no attempt to optimize the second phase in the lower bound proof. Doing more complicated discharging rules, based on more complicated properties of identifying codes will surely give better bounds. However, we do not see any way to make the two bounds meet for all k. Nevertheless, we are able to do it for $k = 3$: we show that $d^*(\mathcal{S}_3) = \frac{7}{18}$.

2 General upper bounds

Theorem 1. *For all $k \geq 7$, we have* $d^*(\mathcal{S}_k) \leq \dfrac{7}{20} + \dfrac{3}{10k}$.

Proof. Let K_k be the code of $\mathcal{S}_k$ obtained from $C^*_{\mathcal{G}}$ on $\mathcal{S}_k$ by replacing the rows $\mathbb{Z} \times \{1\}$ and $\mathbb{Z} \times \{2\}$ by the rows depicted in Figure 2 and the rows $\mathbb{Z} \times \{k-1\}$ and $\mathbb{Z} \times \{k\}$ by the ones obtained symmetrically.

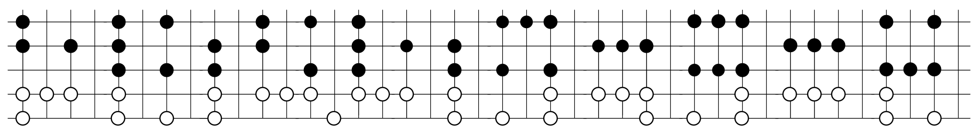

Fig. 2. The bottom rows (white disks) of K_k.

We claim the code K_k is identifying. Indeed since $C^*_{\mathcal{G}}$ is an identifying code of the square lattice, it suffices to check that for every vertex $v \in \mathbb{Z} \times \{1, 2, 3, k-2, k-1, k\}$, there is no vertex w such that $N[v] \cap K_k = N[w] \cap K_k$. This can be easily done.

The density of K_k on $\mathbb{Z} \times \{1, 2, k, k-1\}$ is $\frac{34}{80}$. So, the density of K_k is $\frac{7}{20}(1 - \frac{4}{k}) + \frac{34}{80} \times \frac{4}{k} = \frac{7}{20} + \frac{3}{10k}$.

3 Lower bounds on $d^*(S_k)$

The aim of this section is to show that $d^*(\mathcal{S}_k) \geq \frac{7}{20} + \frac{1}{20k}$.

The general idea is to consider an identifying code C in a square grid G. We assign an initial weight $w(v)$ to each vertex where $w(v) = 1$ if $v \in C$ and $w(v) = 0$ otherwise. We then apply some local discharging rules. In such rules, some vertices send part of their weight to some other vertices at distance at most s, for some fixed integer s. We then prove that the final weight $w^*(v)$ of every vertex v is at least d^*. We claim that it implies $d(C, G) \geq d^*$. This is trivial if G is bounded. Suppose now that $G = \mathcal{S}_k$. Since a vertex sends at most 1 to vertices at distance at most s, a charge of at most $|B_{r+s}(v_0) \setminus B_r(v_0)| \leq 2sk$ enters $B_r(v_0)$ during the discharging phase. Thus

$$|C \cap B_r(v_0)| = \sum_{v \in B_r(v_0)} w(v) \geq \sum_{v \in B_r(v_0)} w^*(v) - |B_{r+s}(v_0) \setminus B_r(v_0)| \geq d^* \cdot |B_r(v_0)| - 2sk .$$

But $|B_r(v_0)| \geq (2r+1)k - k^2$, thus $d(C, \mathcal{S}_k) \geq \limsup_{r \to +\infty} \left(d^* - \frac{2sk}{(2r+1)k - k^2} \right) = d^*$. This proves our claim. We then deduce $d^*(S_k) \geq d^*$.

Let C be an identifying code in a square grid G. We denote by U the set of vertices not in C. For $1 \leq i \leq 5$, we define $L_i = \{v \in G \mid |N[v] \cap C| = i\}$, and we set $C_i = L_i \cap C$ and $U_i = L_i \cap U$. Observe that U_5 is empty. For $X \in \{C, L, U\}$ we set $X_{\geq i} = \bigcup_{j=i}^{5} X_j$ and $X_{\leq i} = \bigcup_{j=1}^{i} X_j$.

For every set $S \subseteq V(G)$, a vertex in S is called an *S-vertex*.

The following proposition is a direct consequence of the definition of identifying code.

Proposition 1. *Let C be an identifying code in a square grid G.*

(i) Every vertex in C has at most one neighbour in U_1.
(ii) Every vertex in C_1 has no neighbour in U_1.
(iii) Two vertices in C_2 are not adjacent.

Let C' be the set of vertices in C_1 that have four neighbours in G that belong all to $U_{\leq 2}$. Let $\tilde{L}_3$ be the set of vertices in C_3 having at least one neighbour in $C_{\geq 3}$. Set $\overline{L}_3 = L_3 \setminus \tilde{L}_3$ and $\overline{C}_3 = C_3 \setminus \tilde{L}_3$

Proposition 2 (Ben-Haim and Litsyn [1]). *Let C be an identifying code in a square grid G. There is a bipartite graph H with bipartition $(C', L_{\geq 3})$ such that*

(i) the degree of every element of C' is at least 4,
(ii) the degree of every element of $\overline{L}_3$ is at most 2,
(iii) the degree of every element of $\tilde{L}_3$ is at most 6, and
(iii) the degree of every element of $L_{\geq 4}$ is at most 4.

Proof. Ben-Haim and Litsyn [1] proved Proposition 2 for another definition of the set $\tilde{L}_3$: their set is larger than ours. However it happens that using word for word the same rules as Ben-Haim and Litsyn (Steps 1 to 10 in [1]) for building the bipartite graph from an identifying code of the square lattice, regardless to the fact that $\tilde{L}_3$ is not the same set, we get a bipartite graph that may be different but has exactly the same degree properties ; the proof of this fact is exactly the same as the one of [1].

Furthermore, in the construction of Ben-Haim and Litsyn, the neighbours in H of an element c' of C' are always in the rectangle with corners c' and another vertex $c \in C$. Therefore it is in any square grid containing those two vertices, and so their proof works for any square grid. However, it might be possible that this vertex is not in G if this graph is not a square grid and the proof of Proposition 2 does not work for any induced subgraph G of $\mathcal{G}$.

Remark 1. The graph H in Proposition 2 may have some double edges.

Theorem 2. *Let G be a square grid. Then $d^*(G) \geq \frac{7}{20}$.*

Proof. Let C be an indentifying code in G and H be a bipartite graph associated to C as described in Proposition 2. We give an initial weight 1 to the vertices of C and 0 to the vertices in U. We then apply the following discharging rules, one after another. So if several rules must be applied to a same vertex, then it will send charge several times.

(R1) Every vertex of C sends $\frac{7}{20}$ to each neighbour in U_1 and $\frac{7}{40}$ to each neighbour in $U_{\geq 2}$.
(R2) Every vertex of $L_{\geq 3}$ sends $\frac{1}{20}$ to its neighbours in $C_{\leq 2}$.
(R3) Every vertex of $L_{\geq 3}$ sends $\frac{1}{80}$ to each C'-vertex to which it is adjacent in H by one edge and $\frac{2}{80}$ to each C'-vertex to which it is adjacent in H by two edges.

Let us prove that the final weight $w'(v)$ of each vertex v is at least 7/20.

If $v \in C'$, then its original weight is 1. By Proposition 1-(ii), it has no U_1 neighbour. Hence it sends $\frac{7}{40}$ to each of its four neighbours in U by ($R1$), and receives $\frac{1}{80}$ from each of its at least four edges in H by (R3). Hence $w'(v) \geq 1 - 4 \cdot \frac{7}{40} + 4 \cdot \frac{1}{80} = \frac{7}{20}$.

If $v \in C_1 \setminus C'$, then its original weight is 1. By Proposition 1-(ii), it has no U_1 neighbour. By definition of C', it has a neighbour in $L_{\geq 3}$ from which it receives $\frac{1}{20}$ by ($R2$). Hence $w'(v) \geq 1 - 4 \cdot \frac{7}{40} + \frac{1}{20} = \frac{7}{20}$.

If $v \in U_1 \cup U_2$, then its original weight is 0. It receives $\frac{7}{20}$ by (R1), and it does not send anything. Hence $w'(v) = \frac{7}{20}$.

If $v \in C_2$, then its original weight is 1. By Proposition 1-(i), it has at most one U_1 neighbour, so its sends at most $\frac{7}{20} + 2 \cdot \frac{7}{40}$ by (R1). Moreover, by Proposition 1-(iii), it has a neighbour in $C_{\geq 3}$ from which it receives $\frac{1}{20}$ by (R2). Hence $w'(v) \geq 1 - \frac{7}{20} - 2 \cdot \frac{7}{40} + \frac{1}{20} = \frac{7}{20}$.

If $v \in \tilde{L}_3$, then it is in C_3, so its original weight is 1. By Proposition 1-(i), it has at most one U_1 neighbour, so it sends at most $\frac{7}{20} + \frac{7}{40}$ by (R1). By definition of $\tilde{L}_3$, v has at most one neighbour in $C_{\leq 2}$, so its sends at most $\frac{1}{20}$ by (R2). Finally, it has degree at most 6 in H, so it sends at most $6 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 1 - \frac{7}{20} - \frac{7}{40} - \frac{1}{20} - 6 \cdot \frac{1}{80} = \frac{7}{20}$.

If $v \in \overline{C}_3$, then it is in C_3, so its original weight is 1. By Proposition 1-(i), it has at most one U_1 neighbour, so it sends at most $\frac{7}{20} + \frac{7}{40}$ by (R1). It has at most two neighbours in $C_{\leq 2}$, so its sends at most $2 \cdot \frac{1}{20}$ by (R2). And it has degree at most 2 in H, so it sends at most $2 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 1 - \frac{7}{20} - \frac{7}{40} - 2.\frac{1}{20} - 2 \cdot \frac{1}{80} = \frac{7}{20}$.

If $v \in U_3$, then its original weight is 0. It receives $3 \cdot \frac{7}{40}$ by (R1). It has at most three neighbours in $C_{\leq 2}$, so its sends at most $3 \cdot \frac{1}{20}$ by (R2). And it has degree at most 2 in H, so it sends at most $2 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 3 \cdot \frac{7}{40} - 3.\frac{1}{20} - 2 \cdot \frac{1}{80} = \frac{7}{20}$.

If $v \in C_4$, then its original weight is 1. It send at most $\frac{7}{20}$ to its unique U-neighbour by (R1). It has at most three neighbours in $C_{\leq 2}$, so its sends at most $3 \cdot \frac{1}{20}$ by (R2). And it has degree at most 4 in H, so it sends at most $4 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 1 - \frac{7}{20} - 3.\frac{1}{20} - 4 \cdot \frac{1}{80} = \frac{9}{20}$.

If $v \in U_4$, then its original weight is 0. It receives $4 \cdot \frac{7}{40}$ by (R1). It has at most four neighbours in $C_{\leq 2}$, so its sends at most $4 \cdot \frac{1}{20}$ by (R2). And it has degree at most 4 in H, so it sends at most $4 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 4 \cdot \frac{7}{40} - 4.\frac{1}{20} - 4 \cdot \frac{1}{80} = \frac{9}{20}$.

If $v \in C_5$, then its original weight is 1. It has no U-neighbour. It has at most four neighbours in $C_{\leq 2}$, so its sends at most $4 \cdot \frac{1}{20}$ by (R2). And it has degree at most 4 in H, so it sends at most $4 \cdot \frac{1}{80}$ by (R3). Hence $w'(v) \geq 1 - 4.\frac{1}{20} - 4 \cdot \frac{1}{80} = \frac{15}{20}$.

Thus at the end, $w'(v) \geq \frac{7}{20}$ for all vertex v, so $d(C, G) \geq \frac{7}{20}$.

Theorem 2 is tight because $d^*(\mathcal{G}) = \frac{7}{20}$. However, for $\mathcal{S}_k$, we can improve on 7/20.

Theorem 3. *For any $k \geq 3$, $d^*(\mathcal{S}_k) = \frac{7}{20} + \frac{1}{20k}$.*

Proof. Let us first give some definition. In $\mathcal{S}_k$, the *row* of index i, denoted R_i, is the set of vertices $\mathbb{Z} \times \{i\}$, the *column* of index j, denoted Q_j, is the set of vertices $\{j\} \times \{1, \dots, k\}$. The *border vertices* are those of $R_1 \cup R_k$.

Let C be an identifying code in $\mathcal{S}_k$.

We first apply the discharging phase as in the proof of Theorem 2. At the end of this phase every vertex has weight at least 7/20. But some of them may have a larger weight.

It is for example the case of C_4-vertices which have weight at least 9/20. Let D_3 be the set of vertices of C_3 having no neighbour in $C_{\leq 2}$. Observe that $D_3 \subseteq \tilde{L}_3$. A vertex of D_3 do not send anything by (R2), hence its weight is at least $\frac{8}{20}$. Set $D = D_3 \cup C_4$.

Consider also border C-vertices. Such vertices are missing one neighbour, so for any $1 \leq i \leq 4$, border C_i-vertices gives to one U-neighbour less than non-border C_i-vertices by (R_1). It follows that if v is a border C-vertex, then $w(v) \geq \frac{7}{20} + \frac{7}{40}$.

The following claim shows that there are many vertices in $R_1 \cup R_2$ with a weight larger than 7/20.

Claim. Let C be a code of $\mathcal{S}_k$. If $\{(a-3,1), (a-2,1), (a-1,1), (a,1), (a+1,1), (a+2,1), (a+3,1)\} \cap C = \emptyset$, then $(a,2)$ is in D.

Proof. If $\{(a-3,1), (a-2,1), (a-1,1), (a,1), (a+1,1), (a+2,1), (a+3,1)\} \cap C = \emptyset$, then necessarily $(a-2,2)$, $(a-1,2)$, $(a,2)$, $(a+1,2)$, and $(a+2,2)$ are in C, because each vertex has a neighbour in C. Therefore $(a-1,2)$, $(a,2)$, and $(a+1,2)$ are in $C_{\geq 3}$ and so $(a,2)$ is in D.

We then proceed to a second discharging phase. Set $S_j = Q_{j-3} \cup Q_{j-2} \cup Q_{j-1} \cup Q_j \cup Q_{j+1} \cup Q_{j+2} \cup Q_{j+3}$.

(R4) Every vertex in D gives $\frac{1}{20k}$ to every vertex in its column.
(R5) Every border C-vertex in column Q_j gives $\frac{1}{40k}$ to every vertex in S_j.

Let us examine the weight $w^*(v)$ of a vertex v after this phase.

Observe first that every vertex receives at least $\frac{1}{20k}$ during this second phase. Indeed, if $v = (a, b)$ has a D-vertex in its column, then it receives $\frac{1}{20k}$ from it by (R4). If it has no D-vertex in its column, then by Claim 3, a vertex in $\{(a-3,1),(a-2,1),(a-1,1),(a,1),(a+1,1),(a+2,1),(a+3,1)\}$ is a border C-vertex, and symmetrically, a vertex in $\{(a_3,k),(a-2,k),(a-1,k),(a,k),(a+1,k),(a+2,k),(a+3,k)\}$ is a border C-vertex. And these two vertices send $\frac{1}{40k}$ each to v by (R5), so v receives at least $\frac{1}{20k}$ in total.

If $v \in D$, then $w'(v) \geq \frac{8}{20}$. By (R4), it sends $\frac{1}{20k}$ to the k vertices of its column. Hence it sends $\frac{1}{20}$. Since it received at least $\frac{1}{20k}$, $w^*(v) \geq \frac{7}{20} + \frac{1}{20k}$.

If v is a border C-vertex, then $w'(v) \geq \frac{7}{20} + \frac{7}{40}$. By (R4), it sends $\frac{1}{40k}$ to the $7k$ vertices of S_j. Hence it sends $\frac{7}{40}$. It also receives at least $\frac{1}{20k}$. So $w^*(v) \geq \frac{7}{20} + \frac{1}{20k}$.

If v is neither a border C-vertex, nor a D-vertex, then it does not send anything. So $w^*(v) \geq w'(v) + \frac{1}{20k} \geq \frac{7}{20} + \frac{1}{20k}$.

To conclude, after the second phase, each vertex has weight at least $\frac{7}{20} + \frac{1}{20k}$. Thus $d(C, \mathcal{S}_k) \geq \frac{7}{20} + \frac{1}{20k}$.

4 Optimal identifying code in $\mathcal{S}_3$.

Theorem 4. $d^*(\mathcal{S}_3) = \frac{7}{18}$.

It is straightforward to check that repeating the tile of Figure 3 with period $(12, 0)$, we obtain an identifying code of density $\frac{7}{18}$.

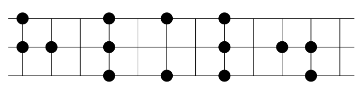

Fig. 3. Tile of a minimum-density identifying code in $\mathcal{S}_3$.

It remains to show that every identifying code in $\mathcal{S}_3$ has density at least $\frac{7}{18}$. We again use the Discharging Method: the technical details are more complicated than in the previous section, but the general framework is the same.

4.1 Properties of codes in $\mathcal{S}_3$

The *lower row*, (resp. *central row*, *upper row*) of $\mathcal{S}_3$, is the set of vertices in $\mathbb{Z} \times \{1\}$, (resp. $\mathbb{Z} \times \{2\}$, $\mathbb{Z} \times \{3\}$). A *border vertex* is a vertex on the upper or lower row. A *central vertex* is a vertex on the central row. The *column* of index a is the set $\{(a,1),(a,2),(a,3)\}$.

For convenience, instead of using the set C_i we use the set B_i, which is defined as follows. A vertex is in B_i if it is in C and adjacent to i vertices in U. Hence, a border vertex in B_i is in C_{4-i} and a central vertex in B_i is in C_{5-i}.

Similarly to Proposition 1, we get the following proposition.

Proposition 3. *Let C be an identifying code in $\mathcal{S}_3$. Every border B_3-vertex has no neighbour in U_1.*

Proposition 4. *Let C be an identifying code in $\mathcal{S}_3$. Every vertex in B_4 has a neighbour in $U_{\geq 3}$.*

Proof. Let x be a vertex in B_4. Necessarily x must be a central vertex, that is, $x = (a, 2)$ for some a. Assume for a contradiction that x has no neighbour in $U_{\geq 3}$. Then by Proposition 1, its four neighbours are in U_2. Consider $u = (a, 3)$: one of its neighbours y is in C. By symmetry, we may assume $y = (a-1, 3)$. Now the two vertices u and $v = (a-1, 2)$ are both adjacent to x and y. Hence, since u and v are in U_2, we obtain $N[u] \cap C = \{x, y\} = N[v] \cap C$, a contradiction.

Proposition 5. *Let C be an identifying code in $\mathcal{S}_3$. Every border C-vertex adjacent to a central B_3 is in $B_0 \cup B_1$.*

Proof. Assume for a contradiction that a border B_2-vertex y is adjacent to a central B_3-vertex x. Then $N[x] \cap C = N[y] \cap C = \{x, y\}$, a contradiction.

4.2 Establishing the lower bound

We use the Discharging Method. Let C be an indentifying code in $\mathcal{S}_3$. We give an initial weight 1 to the vertices of C and 0 to the vertices in U. We will then apply some discharging rules. Our aim is to prove that at the end the final weight of each vertex will be at least $\frac{7}{18}$.

For sake of clarity and to simplify the proof, we will perform these discharging rules in two stages.

A *generous* vertex is either a B_0-vertex or a border vertex in B_1 having its central neighbour in C. We first apply the following rules.

(R0) For $1 \leq i \leq 4$, every vertex of C gives $\frac{7}{18 \times i}$ to each of its neighbours in U_i.
(R1) Every generous vertex gives $\frac{3}{18}$ to its central neighbour(s).

Let us denote by $w_1(v)$ the weight of the vertex v after applying (R0–R1).

Observe that after (R0–R1) all the vertices of U have weight exactly $\frac{7}{18}$. Indeed for all $1 \leq i \leq 4$, every vertex u in U_i receives $\frac{7}{18 \times i}$ from each of its i neighbours. Hence in total it receives $\frac{7}{18}$ and so $w_1(u) = \frac{7}{18}$.

The weight of the vertices of U will not change anymore and the charge will now only move from C-vertices to other C-vertices.

We define the *excess* of a vertex v of C as $\epsilon(v) = w_1(v) - \frac{7}{18}$. Informally, if it is positive, the excess of v measures how much weight v has above $\frac{7}{18}$ and thus can give to other vertices. If it is negative, the excess of v measures the quantity of weight v must receives from others to get weight $\frac{7}{18}$.

Observe that B_0-, B_1-, B_2- and border B_3-vertices have positive excess:

– If $v \in B_0$, then it gives nothing to U-vertices, and it gives $\frac{3}{18}$ to each (at most two) central neighbours. Thus $\epsilon(v) \geq 1 - \frac{7}{18} - \frac{6}{18} = \frac{5}{18}$.

– If $v \in B_1$, then it gives at most $\frac{7}{18}$ to its U-neighbour. So if it is not generous, $\epsilon(v) \geq 1 - \frac{7}{18} - \frac{7}{18} = \frac{2}{9}$ and if v is generous, $\epsilon(v) \geq 1 - \frac{7}{18} - \frac{7}{18} - \frac{3}{18} = \frac{1}{18}$.
– If $v \in B_2$, then it is adjacent to at most one U_1-vertex by Proposition 1. Hence $\epsilon(v) \geq 1 - \frac{7}{18} - \frac{7}{18} - \frac{7}{36} = \frac{1}{36}$.
– If v is a border B_3-vertex, then by Proposition 3, it is adajcent to no U_1-vertex. Hence it gives at most $\frac{7}{36}$ to each of its U-neighbours. So $\epsilon(v) \geq 1 - \frac{7}{18} - 3 \times \frac{7}{36} = \frac{1}{36}$.

On the opposite, some vertices of $B_3 \cup B_4$ may have negative excess. Such vertices of $B_3 \cup B_4$ will be called *defective*. Observe that defective vertices are on the central line. Moreover it is easy to check that a defective vertex has no generous neighbour. Indeed if a defective vertex x has a generous neighbour y, then it is in B_3. Since x has at most one U_1-neighbour, it sends at most $\frac{7}{18} + 2 \times \frac{7}{36} = \frac{14}{18}$ to its U-neighbours. But it also receives $\frac{3}{18}$ from its generous neighbour. Hence $\epsilon(x) \geq 1 - \frac{7}{18} - \frac{14}{18} + \frac{3}{18} = 0$.

Simple calculations and Propositions 1, 3 and 4 show that a defective vertex is of one of the following kinds:

– a B_4-vertex with at least two U_2-neighbours;
– a central B_3-vertex with one U_1-neighbour and no generous neighbour.

We will now apply some new discharging rules in order to give charge to the defective vertices so that the final excess $\epsilon^*(v)$ of every vertex v is non-negative. The rules are applied one after another, so if several rules must be applied to a same vertex then it will send charge several times.

For $S \in \{C, U\}$, an S*-column* is a column all vertices of which are in S. A *right barrier* (resp. *left barrier*) is a C-column such that the right (resp. left) neighbours of its two border vertices are in U_1. A *lonely barrier* is a barrier such that the columns to its right and its left are U-columns. Let x be a C-vertex. Its *right pal* (resp. *left pal*) is the closest central C-vertex to its right (resp. left). A pal is *good* if it is defective or in a lonely barrier.

(R2) Every border C-vertex x whose central neighbour is not in C sends $\epsilon(x)/2$ to each of its good pals, if it has two of them and $\epsilon(x)$ to its good pal, if it has exactly one.
(R3) Every border vertex x in a right (resp. left) barrier sends $\epsilon(x)$ to its right (resp. left) pal. Every central vertex x of a right (resp. left) barrier sends to its right (resp. left) pal $\epsilon(x)$ if it is in B_2 and $\frac{1}{18}$ if it is in B_1.
(R4) Every generous B_1-vertex not in a barrier sends $\frac{1}{36}$ to each of its pals.
(R5) Every central B_1-vertex whose left (resp. right) neighbour is not in C sends $\frac{3}{18}$ to its right (resp. left) neighbour.
(R6) Every border B_2-vertex whose central neighbour is in B_2 and adjacent to a central B_3 sends $\frac{1}{36}$ to this later vertex.
(R7) Every central B_2-vertex with a border C-neighbour and a central C-neighbour sends $\frac{1}{36}$ to its central C-neighbour.
(R8) Every central B_0-vertex or central B_1-vertex with its two central neighbours in C sends $\frac{1}{18}$ to each of its central neighbours.

(R9) Every central vertex in a right (resp. left) barrier resend to its right (resp. left) pal all the charge its receives from border vertices to its left (rep. right) by (R2).

(R10) A central B_2-vertex with a B_3-neighbour to its right (resp. left) and a B_2-neighbour to its left (resp. right) sends $\frac{1}{36}$ and everything it gets from the left (resp. right) to its right (resp. left) neighbour.

It is routine to check that every non-defective vertex sends at most its excess and that its final excess is non-negative. We now consider defective vertices. Let $v = (a, 2)$ be a defective vertex. Let us show that its final excess $\epsilon^*(v)$ is non-negative.

We first consider the case when v is in B_4.

- Assume first that v has three U_2-neighbours and one neighbour in $U_3 \cup U_4$. Then its original excess $\epsilon(v)$ is at least $1 - \frac{7}{18} - 3 \times \frac{7}{36} - \frac{7}{54} = -\frac{11}{108}$.
  By symmetry, we may assume that $(a, 3) \in U_2$, $(a-1, 3) \in C$ and $(a+1, 3) \in U$. Hence $(a-1, 2)$ is in $U_3 \cup U_4$ because $N[(a-1, 2)] \cap C \neq N[(a, 3] \cap C$. Thus $(a, 1)$ and $(a+1, 2)$ are in U_2. Since $N[(a, 1)] \cap C \neq N[(a+1, 2] \cap C$, $(a-1, 1)$ and $(a+2, 2)$ are in C and $(a+1, 1) \in U$. But $(a+1, 1)$ and $(a+1, 3)$ must have a neighbour in C, so $(a+2, 1)$ and $(a+2, 3)$ are in C. Hence the column of index $a+2$ is a left barrier. So v receives at least $3 \times \frac{1}{36}$ by (R3) from the vertices of this barrier and $\frac{1}{72}$ from each of $(a-1, 1)$ and $(a-1, 3)$ by (R2). Hence $\epsilon^*(v) \geq -\frac{11}{108} + 3 \times \frac{1}{36} + 2 \times \frac{1}{72} > 0$.
- Assume now that v has two U_2-neighbours and two neighbours in $U_3 \cup U_4$. Then its original excess $\epsilon(v)$ is at least $1 - \frac{7}{18} - 2 \times \frac{7}{36} - 2 \times \frac{7}{54} = -\frac{1}{27}$. Observe that $(a, 1)$ and $(a, 3)$ may not both be in U_3 for otherwise $(a-1, 2)$ and $(a+1, 2)$ would also be in U_3. Hence without loss of generality, we are in one of the following two subcases:
  - $\{(a-1, 1), (a-1, 3), (a+1, 3)\} \subseteq C$ and $(a+1, 1)$ is in U. Then $(a+2, 1)$ must be in C. Hence the three vertices $(a-1, 1)$, $(a-1, 3)$, $(a+1, 3)$ send each $\frac{1}{72}$ to v by (R2). Hence $\epsilon^*(v) \geq -\frac{1}{27} + 3 \times \frac{1}{72} > 0$.
  - $\{(a-2, 2), (a-1, 3), (a+1, 1), (a+2, 2)\} \subseteq C$ and $(a-1, 1)$ and $(a+1, 3)$ are in U. Then $(a-2, 1)$ and $(a+2, 3)$ must be in C. Observe that the columns of index $a-2$ and $a+2$ are not barriers since $(a-1, 1)$ and $(a+1, 3)$ are in U_1 and $(a-1, 3)$ and $(a+1, 1)$ are not in U_1. If $(a-2, 2)$ is not defective, then $(a-1, 3)$ sends at least $\frac{1}{36}$ to v by (R2) and $(a+1, 1)$ sends at least $\frac{1}{72}$ to v by (R2). Hence $\epsilon^*(v) \geq -\frac{1}{27} + \frac{1}{36} + \frac{1}{72} > 0$. If $(a-2, 2)$ is defective, then it is in B_3. Since the code is identifying $(a-3, 1)$ is in C, so $(a-2, 1)$ is a border B_1-vertex. So it sends $\frac{1}{36}$ to v by (R4). As v receives at least $\frac{1}{72}$ from each of $(a-1, 3)$ and $(a+1, 1)$, we have $\epsilon^*(v) \geq -\frac{1}{27} + \frac{1}{36} + 2 \times \frac{1}{72} > 0$.

We now consider the case when $v \in B_3$. Let w be its C-neighbour. w is a central vertex, for otherwise w would be a generous vertex by Proposition 5 and thus v would not be defective. By symmetry, we may assume that $w = (a-1, 2)$.

– Assume first that v has one U_1-neighbour and two U_2-neighbours. Up to symmetry, the U_1-neighbour z is either $(a+1,2)$ or $(a,3)$.

- If $z=(a+1,2)$, then $(a+1,1)$ and $(a+1,3)$ are in U and so $(a-1,1)$ and $(a-1,3)$ are in C. Hence w is in B_1 because a defective vertex has no generous neighbour. Thus w sends $\frac{3}{18}$ to v by (R5). So $\epsilon^*(v) \geq 0$.
- Assume that $z=(a,3)$. Then $(a-1,3)$ and $(a+1,3)$ are in U. Moreover $(a+1,2)$ and $(a,1)$ are in U_2 and so $(a+1,1)$ is not in C. It follows that $(a-1,1)$, $(a+2,2)$, $(a+2,1)$ and $(a+2,3)$ are in C. The column of index $a+2$ is a left barrier.
  We claim that v receives at least $\frac{2}{18}$ from its right and at least $\frac{1}{18}$ from its left. This yields $\epsilon^*(v) \geq 0$.
  Let us show that v receives $\frac{2}{18}$ from its right. If one vertex of the column of index $a+2$ is in B_1, the vertices of the barrier send at least $\frac{1}{18}+2\times\frac{1}{36}=\frac{2}{18}$ to v. Hence we may assume that $(a+3,1)$, $(a+3,2)$ and $(a+3,3)$ are in U. Furthermore by Proposition 1, $(a+3,1)$ and $(a+3,3)$ are in U_2 so $(a+4,1)$ and $(a+4,3)$ are in C. If $(a+4,2)$ is in C, then $(a+3,2) \in U_2$ and so $\epsilon((a+2,2))=\frac{2}{9}$. Hence v receives at least $\frac{2}{9}+2\times\frac{1}{36}>\frac{2}{18}$ from its right. If $(a+4,2)$ is not in C, then by (R2) $(a+4,1)$ and $(a+4,3)$ send in total $\frac{1}{36}$ to $(a+2,2)$ which redirect it to v by (R9). In addition, the barrier send at least $3\times\frac{1}{36}$ to v by (R3). Hence v receives at least $\frac{2}{18}$ from its right.
  Now, either $(a-1,2)$ is in B_1 in which case it sends $\frac{1}{18}$ to v by (R8), or $(a-1,2)$ is in B_2 and sends $\frac{1}{36}$ to v by (R7) and $(a-1,1)$ is in B_2 or B_1 and sends $\frac{1}{36}$ to v by (R6) or (R4). In both cases, v receives $\frac{1}{18}$ from its left.

– Assume that v has one U_1-neighbour, one U_2-neighbour and one neighbour in $U_3 \cup U_4$. Then $\epsilon(v) \geq 1-\frac{7}{18}-\frac{7}{18}-\frac{7}{36}-\frac{7}{54}=-\frac{11}{108}$. Let t be the neighbour of v in $U_3 \cup U_4$. By symmetry, we may assume that $t=(a,3)$ or $t=(a+2,2)$.

- Assume that $t=(a,3)$. Then $(a-1,3)$, $(a+1,3)$ are in C and $(a-1,1)$, $(a+1,1)$ and $(a+2,2)$ are in U. Thus $(a+2,1)$ is in C.
  If $(a-1,2)$ is in B_1, then it sends $\frac{1}{18}$ to v by (R8). If not $(a-1,2)$ is in B_2 and thus sends $\frac{1}{36}$ to v by (R7). Moreover $(a-1,3)$ is in $B_1 \cup B_2$ and thus sends $\frac{1}{36}$ by (R4) or (R6). Hence v receives at least $\frac{1}{18}$ from its left.
  Let us now show that v receives at least $\frac{5}{54}$ from its right. Since $N[(a+1,1)] \cap C \neq N[(a+2,1)] \cap C$, we have $(a+3,1) \in C$. Since $N[(a+1,3)] \cap C \neq N[(a+2,3)] \cap C$, we have $(a+3,3) \in C$. If $(a+3,2)$ is in C, then this vertex is not good. So $(a+1,3)$ sends all its excess to v by (R2). This excess is at least $\frac{5}{54}$. If $(a+3,2)$ is not in C, then $(a+2,3)$ is in C, because $N[(a+2,2)] \cap C \neq N[(a+1,1)] \cap C$. Hence $(a+2,3)$ is in B_1 and it is not generous. So its excess is at least $\frac{2}{9}$ and by (R2), it sends at least $\frac{1}{9}$ to v.
  Hence v receives at least $\frac{1}{18}$ from its left and $\frac{5}{54}$ from its right. Thus $\epsilon^*(v) \geq -\frac{11}{108}+\frac{1}{18}+\frac{5}{54}>0$.
- Assume that $t=(a+1,2)$. By symmetry, we may assume that $(a,3) \in U_1$. Then $(a-1,3)$, $(a+1,3)$ and $(a-1,1)$ are in U and $(a+1,1)$, $(a+2,2)$ are in C.

Since $N[v] \cap C \neq N[w] \cap C$, necessarily $(a-2,2) \in C$. Since $(a-1,1)$ and $(a-1,3)$ have different closed neighbourhoods, then a vertex y in $\{(a-2,1),(a-2,3)\}$ is in C. If $(a-2,2)$ is in B_0, then it sends $\frac{1}{18}$ to w by (R8) ; if $(a-2,2)$ is in B_1, then it sends at least $\frac{1}{18}$ to w by (R5) or (R8); if $(a-2,2)$ is in B_2, then $(a-2,2)$ sends $\frac{1}{36}$ to w by (R7). In any case, w receives at least $\frac{1}{36}$ from $(a-2,2,)$ which it redirects to v with an additional $\frac{1}{36}$ by (R10). So w sends at least $\frac{1}{18}$ to v.
Now $(a+1,1)$ has excess at least $\frac{5}{54}$ since it is adjacent to no U_1 and at least one U_3. Hence it sends at least $\frac{5}{108}$ to v by (R2).
Thus $\epsilon^*(v) \geq -\frac{11}{108} + \frac{1}{18} + \frac{5}{108} = 0$.

– Assume that v has one U_1-neighbour and two neighbours in $U_3 \cup U_4$. Then $\epsilon(v) \geq 1 - \frac{7}{18} - \frac{7}{18} - 2 \times \frac{7}{54} = -\frac{1}{27}$.
Without loss of generality, $(a-1,3)$, $(a+1,3)$ and $(a+2,2)$ are in C, and $(a-1,1)$, $(a+1,1)$ are in U. Then the vertex $(a+1,3)$ had excess at least $\frac{17}{108}$ since it is adjacent to two U_3 and no U_1. Thus by (R2) it sends at least $\frac{17}{216}$ to v. So $\epsilon^*(v) \geq -\frac{1}{27} + \frac{17}{216} > 0$.

Hence at the end, all the C-vertices have non-negative final excess and final weight at least $\frac{7}{18}$. This finishes the proof of Theorem 4.

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