On Witten Laplacians and Brascamp–Lieb's Inequality on Manifolds with Boundary

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On Witten Laplacians and Brascamp-Lieb's inequality on manifolds with boundary

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Abstract

In this paper, we derive from the supersymmetry of the Witten Laplacian Brascamp-Lieb's type inequalities for general differential forms on compact Riemannian manifolds with boundary. In addition to the supersymmetry, our results essentially follow from suitable decompositions of the quadratic forms associated with the Neumann and Dirichlet self-adjoint realizations of the Witten Laplacian. They moreover imply the usual Brascamp-Lieb's inequality and its generalization to compact Riemannian manifolds without boundary.

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Brascamp-Lieb's inequality, Witten Laplacian, Riemannian manifolds with boundary, Supersymmetry, Bakry-Émery tensor.

1 Introduction

1.1 Context and aim of the paper

Let $V \in \mathcal{C}^2(\mathbf{R}^n, \mathbf{R})$ be a strictly convex function such that $e^{-V} \in L^1(\mathbf{R}^n)$ and let ν be the probability measure defined by $d\nu := \frac{e^{-V}}{\int_{\mathbf{R}^n} e^{-V} dx}\, dx$. The classical Brascamp-Lieb's inequality proven in [7] states that every smooth compactly supported function ω satisfies the estimate

$$\int_{\mathbf{R}^n} \Big|\omega - \Big(\int_{\mathbf{R}^n} \omega \, d\nu\Big)\Big|^2 d\nu \;\leq\; \int_{\mathbf{R}^n} \big(\operatorname{Hess} V\big)^{-1}(\nabla\omega, \nabla\omega)\, d\nu\,. \tag{1.1}$$

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This inequality and suitable variants have since been e.g. used in works such as [1–3,10,12,21,22] studying correlation asymptotics in statistical mechanics. The latter works exploit in particular crucially some relations of the following type and which at least go back to the work of Helffer and Sjöstrand [12]:

$$\Big\| \eta - \langle \eta, \frac{e^{-\frac{V}{2}}}{\|e^{-\frac{V}{2}}\|}\rangle \frac{e^{-\frac{V}{2}}}{\|e^{-\frac{V}{2}}\|} \Big\|^2 \;=\; \langle\, (\Delta^{(1)}_{\frac{V}{2}})^{-1} \big(d_{\frac{V}{2}}\, \eta \big)\,,\, d_{\frac{V}{2}}\, \eta \,\rangle, \qquad (1.2)$$

where $\eta \in \mathcal{C}^\infty_c(\mathbf{R}^n)$, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ stand for the usual $L^2(dx)$ inner product and norm, $d_{\frac{V}{2}} := d + d\frac{V}{2}$ and $\Delta^{(1)}_{\frac{V}{2}}$ is the Witten Laplacian acting on 1-forms (or equivalently on vector fields) which is given by

$$\Delta^{(1)}_{\frac{V}{2}} \;:=\; \Delta^{(0)}_{\frac{V}{2}} \otimes \mathrm{Id} + \mathrm{Hess}\, V \;=\; \big(-\Delta + |\nabla \frac{V}{2}|^2 - \Delta \frac{V}{2} \big) \otimes \mathrm{Id} + \mathrm{Hess}\, V\,. \quad (1.3)$$

In the last relation,

$$\Delta^{(0)}_{\frac{V}{2}} \;:=\; -\Delta + |\nabla \frac{V}{2}|^2 - \Delta \frac{V}{2} \;=\; \big(-\mathrm{div} + \nabla \frac{V}{2} \big)\big(\nabla + \nabla \frac{V}{2} \big) \;=\; d^*_{\frac{V}{2}}\, d_{\frac{V}{2}} \quad (1.4)$$

denotes the Witten Laplacian acting on functions (or equivalently on 0-forms). The Witten Laplacian, initially introduced in [24], is more generally defined on the full algebra of differential forms and is nonnegative and essentially self-adjoint (when acting on smooth compactly supported forms) on the space of $L^2(dx)$ differential forms. It is moreover supersymmetric, which essentially amounts, when restricting our attention to the interplay between $\Delta^{(0)}_{\frac{V}{2}}$ and $\Delta^{(1)}_{\frac{V}{2}}$, to the intertwining relation

$$\forall\, \eta \,\in\, \mathcal{C}^\infty_c(\mathbf{R}^n)\,, \quad d_{\frac{V}{2}}\, \Delta^{(0)}_{\frac{V}{2}}\, \eta \;=\; \Delta^{(1)}_{\frac{V}{2}}\, d_{\frac{V}{2}}\, \eta\,,$$

which enables to prove relations of the type (1.2) (when $\Delta^{(1)}_{\frac{V}{2}}$ is invertible). The nonnegativity of $\Delta^{(0)}_{\frac{V}{2}}$ together with the relations (1.2) and (1.3) then easily leads to (1.1) when V is strictly convex (at least formally) taking finally $\omega := e^{\frac{V}{2}}\eta$. To connect to some spectral properties of $\Delta^{(0)}_{\frac{V}{2}}$, the relation (1.2) together with the lower bound $\Delta^{(1)}_{\frac{V}{2}} \geq c$ for some $c > 0$ – which is in particular satisfied if $\mathrm{Hess}\, V \geq c$ – implies, according to formula (1.4), a spectral gap greater or equal to c for $\Delta^{(0)}_{\frac{V}{2}}$ (its kernel being $\mathrm{Span}\{e^{-\frac{V}{2}}\}$ as it can be seen from (1.4)). In addition to the already mentioned [10,22] making extra assumptions on V, we refer especially to the very complete [14] for precise statements and proofs in relation with the above discussion.

More generally, in the case of a Riemannian manifold without boundary Ω, it is also well known that an inequality of the type (1.1) holds if one replaces $\operatorname{Hess} V$ (and the condition $\operatorname{Hess} V > 0$ everywhere) by the following quadratic form, sometimes called the Bakry-Émery (-Ricci) tensor,

$$\mathrm{Ric} + \operatorname{Hess} V \quad \text{(and if we assume its strict positivity everywhere)}\,,$$

Ric denoting the Ricci tensor. We refer for example to [5, Theorem 4.9.3] for a precise statement whose proof relies on the supersymmetry of the counterpart of the Witten Laplacian in the weighted space $L^2(\Omega, e^{-V} d\,\mathrm{Vol}_\Omega)$, sometimes called the weighted Laplacian and more precisely defined when acting on functions by

$$L_V^{(0)} \;:=\; e^{\frac{V}{2}}\big(-\Delta + |\nabla \frac{V}{2}|^2 - \Delta \frac{V}{2}\big)e^{-\frac{V}{2}} \;=\; -\Delta + \nabla V \cdot \nabla\,.$$

This operator, unitarily equivalent to $\Delta^{(0)}_{\frac{V}{2}}$, is an important model of the Bakry-Émery theory of diffusion processes and we refer especially in this direction to the pioneering work of Bakry and Émery [4] or to the book [5] for an overview of the concerned literature. On its side, the Bakry-Émery tensor $\mathrm{Ric} + \operatorname{Hess} V$ – named after [4] but first introduced by Lichnerowicz in [18] – is the natural counterpart of the Ricci tensor Ric in the weighted Riemannian manifold $(\Omega, e^{-V} d\,\mathrm{Vol}_\Omega)$ and we refer for example to [18, 19] for some of its geometric properties. Let us also mention e.g. [20] extending this notion to metric measure spaces.

In this paper, we derive from the supersymmetry of the Witten Laplacian Brascamp-Lieb's type inequalities for general differential forms on a Riemannian manifold with a boundary. In addition to the supersymmetry, our results essentially follow from suitable decompositions of the quadratic forms associated with the self-adjoint Neumann and Dirichlet realizations of the Witten Laplacian stated in Theorem 1.1. When restricting to the interplay between 0- and 1-forms, they imply in particular the already mentioned results in the case of $\mathbf{R}^n$ or of a compact manifold with empty boundary as well as some results recently obtained by Kolesnikov and Milman in [15] in the case of a compact manifold with a boundary (see indeed Corollaries 1.3 and 1.4 and the corresponding remarks).

1.2 Decomposition formulas

Let $(\Omega, g = \langle \cdot, \cdot \rangle)$ be a smooth n-dimensional oriented connected and compact Riemannian manifold with boundary $\partial\Omega$. The cotangent (resp. tangent) bundle of Ω is denoted by $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle by $\Lambda T^*\Omega = \oplus_{p=0}^{n}\Lambda^p T^*\Omega$. The fiber bundles $T^*\partial\Omega$, $T\partial\Omega$, and $\Lambda T^*\partial\Omega = \oplus_{p=0}^{n-1}\Lambda^p T^*\partial\Omega$ are defined similarly. The (bundle) scalar product on $\Lambda^p T^*\Omega$ inherited from g is denoted by $\langle\cdot,\cdot\rangle_{\Lambda^p}$. The space of $\mathcal{C}^\infty$, L^2, etc. sections of any of the above fiber bundles E, over $O = \Omega$ or $O = \partial\Omega$, are respectively denoted by $\mathcal{C}^\infty(O,E)$, $L^2(O,E)$, etc.. The more compact notation $\Lambda^p\mathcal{C}^\infty$, $\Lambda^p L^2$, etc. will also be used instead of $\mathcal{C}^\infty(\Omega,\Lambda^p T^*\Omega)$, $L^2(\Omega,\Lambda^p T^*\Omega)$, etc. and we will denote by $\mathcal{L}(\Lambda^p T^*\Omega)$ the space of smooth bundle endomorphisms of $\Lambda^p T^*\Omega$. The L^2 spaces are those associated with the respective unit volume forms μ and $\mu_{\partial\Omega}$ for the Riemannian structures on Ω and on $\partial\Omega$. The $\Lambda^p L^2$ scalar product and norm corresponding to μ will be denoted by $\langle\cdot,\cdot\rangle_{\Lambda^p L^2}$ and $\|\cdot\|_{\Lambda^p L^2}$ or more simply by $\langle\cdot,\cdot\rangle_{L^2}$ and $\|\cdot\|_{L^2}$ when no confusion is possible.

We denote by d the exterior differential on $\mathcal{C}^\infty(\Omega,\Lambda T^*\Omega)$ and by d^* its formal adjoint with respect to the L^2 scalar product. The Hodge Laplacian is then defined on $\mathcal{C}^\infty(\Omega,\Lambda T^*\Omega)$ by

$$\Delta \;:=\; \Delta_H \;:=\; d^*d + dd^* \;=\; (d+d^*)^2\,. \tag{1.5}$$

For a (real) smooth function f, the distorted differential operators d_f and d_f^* are defined on $\mathcal{C}^\infty(\Omega,\Lambda T^*\Omega)$ by

$$d_f \;:=\; e^{-f}\,d\,e^{f} \quad\text{and}\quad d_f^* \;:=\; e^{f}\,d^*\,e^{-f}\,, \tag{1.6}$$

and the Witten Laplacian Δ_f is defined on $\mathcal{C}^\infty(\Omega,\Lambda T^*\Omega)$ similarly as the Hodge Laplacian by

$$\Delta_f \;:=\; d_f^*d_f + d_f d_f^* \;=\; (d_f + d_f^*)^2\,. \tag{1.7}$$

Note the supersymmetry structure of the Witten Laplacian acting on the complex of differential forms: for every u in $\mathcal{C}^\infty(\Omega,\Lambda^p T^*\Omega)$, it holds

$$\Delta_f^{(p+1)} d_f^{(p)} u \;=\; d_f^{(p)}\Delta_f^{(p)} u \quad\text{and}\quad \Delta_f^{(p-1)} d_f^{(p-1),*} u \;=\; d_f^{(p-1),*}\Delta_f^{(p)} u\,. \tag{1.8}$$

The Witten Laplacian $\Delta_f^{(p)}$ (the superscript (p) means that we are considering its action on differential p-forms) extends in the distributional sense into an operator acting on the Sobolev space $\Lambda^p H^2$ and is nonnegative and self-adjoint on the flat space $\Lambda^p L^2 = \Lambda^p L^2(d\mu)$ once endowed with appropriate Dirichlet or Neumann type boundary conditions (see indeed [11, 17] and Section 3). These self-adjoint extensions are respectively denoted by $\Delta_f^{\mathbf{t},(p)}$ and $\Delta_f^{\mathbf{n},(p)}$, their respective domains being given by

$$D(\Delta_f^{\mathbf{t},(p)}) \;=\; \big\{\omega\in\Lambda^p H^2\,,\ \mathbf{t}\omega = 0 \quad\text{and}\quad \mathbf{t}d_f^*\omega = 0 \quad\text{on}\quad \partial\Omega\big\} \tag{1.9}$$

and

$$D(\Delta_f^{\mathbf{n},(p)}) \;=\; \left\{\omega \in \Lambda^p H^2\,,\;\mathbf{n}\omega = 0 \quad \text{and} \quad \mathbf{n} d_f \omega = 0 \quad \text{on} \quad \partial\Omega\right\}\,. \qquad (1.10)$$

In the above two formulas, $\mathbf{n}\eta$ and $\mathbf{t}\eta$ stand respectively for the normal and tangential components of the form η, see (2.1) and (2.2) in the following section for a precise definition. For $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, the quadratic form associated with $\Delta_f^{\mathbf{b},(p)}$ is denoted by $\mathcal{D}_f^{\mathbf{b},(p)}$. Its domain is given by

$$\Lambda^p H^1_{\mathbf{b}} \;:=\; \left\{\omega \in \Lambda^p H^1\,,\;\mathbf{b}\omega = 0 \quad \text{on} \quad \partial\Omega\right\}\,, \qquad (1.11)$$

and we have, for every $\omega \in \Lambda^p H^1_{\mathbf{b}}$,

$$\mathcal{D}_f^{\mathbf{b},(p)}(\omega) \;:=\; \mathcal{D}_f^{\mathbf{b},(p)}(\omega,\omega) \;=\; \langle d_f\omega, d_f\omega\rangle_{L^2} + \langle d_f^*\omega, d_f^*\omega\rangle_{L^2}. \qquad (1.12)$$

More details about these self-adjoint realizations are given in Section 3.

The different Brascamp-Lieb's type inequalities stated in this work arise from the following integration by parts formulas relating the quadratic forms $\mathcal{D}_f^{\mathbf{t},(p)}$ and $\mathcal{D}_f^{\mathbf{n},(p)}$ with the geometry of Ω. In order to lighten this presentation, some notations involved in these formulas will only be precisely defined in the next section: $\partial_n f$ denotes the normal derivative of f along the boundary (see (2.3)), $\mathrm{Ric}^{(p)}$ and $\mathrm{Hess}^{(p)} f$ respectively denote the smooth bundle symmetric endormorphism of $\Lambda^p T^*\Omega$ defined from the Weitzenböck formula in (2.9) and the one canonically associated with $\mathrm{Hess}\, f$ (see (2.24)), and, for $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$, $\mathcal{K}_{\mathbf{b}}^{(p)} \in \mathcal{L}(\Lambda^p T^*\Omega\big|_{\partial\Omega})$ is defined by means of the second fundamental form of $\partial\Omega$ in (2.13)–(2.16).

Theorem 1.1. *Let $\omega \in \Lambda^p H^1_{\mathbf{b}}$ with $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$. It holds*

$$\begin{aligned}\mathcal{D}_f^{\mathbf{b},(p)}(\omega) \;=\; & \|e^f\omega\|^2_{\dot{H}^1(e^{-2f}d\mu)} + \langle\big(\mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)} f\big)\,\omega,\omega\rangle_{L^2} \\ & + \int_{\partial\Omega} \langle \mathcal{K}_{\mathbf{b}}^{(p)}\omega,\omega\rangle_{\Lambda^p}\; d\mu_{\partial\Omega} - 2\,\mathbf{1}_{\mathbf{t}}(\mathbf{b}) \int_{\partial\Omega} \langle\omega,\omega\rangle_{\Lambda^p}\,\partial_n f\; d\mu_{\partial\Omega}\,, \end{aligned} \qquad (1.13)$$

where $\mathbf{1}_{\mathbf{t}}(\mathbf{b}) = 1$ if $\mathbf{b} = \mathbf{t}$ and 0 if $\mathbf{b} = \mathbf{n}$, and

$$\|\cdot\|^2_{\dot{H}^1(e^{-2f}d\mu)} \;:=\; \|\cdot\|^2_{H^1(e^{-2f}d\mu)} - \|\cdot\|^2_{L^2(e^{-2f}d\mu)}\,.$$

When $f = 0$, we recover Theorems 2.1.5 and 2.1.7 of [23] which were generalizing results in the boundaryless case due to Bochner for $p = 1$ and to Gallot and Meyer for general p's (see [6,9]). These results allow in particular to draw topological conclusions on the cohomology of Ω from its geometry. When the boundary $\partial\Omega$ is not empty, the relative and absolute cohomologies of Ω (corresponding respectively to the Dirichlet and Neumann boundary conditions) have to be considered (see [23, Section 2.6]). To be more precise, note from Theorem 1.1 that for any $p \in \{0, \dots, n\}$, the (everywhere) positivity of the quadratic form $\mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)} f$ together with the nonnegativity of $\mathcal{K}_{\mathbf{n}}^{(p)}$ (resp. of $\mathcal{K}_{\mathbf{t}}^{(p)} - 2\,\partial_n f$) implies the lower bounds (in the sense of quadratic forms)

$$\Delta_f^{\mathbf{b},(p)} \;\geq\; \mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)} f \;>\; 0 \quad (\,\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}\,)$$

for the Witten Laplacian and hence the triviality of its kernel which is isomorphic to the p-th absolute (resp. relative) cohomology group of Ω when $f = 0$.

1.3 Consequences: Brascamp-Lieb's type inequalities

We now define $V := 2f$, the probability measure ν associated with V by

$$d\nu \;:=\; \frac{e^{-V}}{\int_\Omega e^{-V} d\mu} d\mu \;=\; \frac{e^{-2f}}{\|e^{-f}\|_{L^2}^2} d\mu\,,$$

and the weighted Laplacian acting on p-forms $L_V^{(p)}$ by

$$L_V^{(p)} \;:=\; e^f\, \Delta_f^{(p)}\, e^{-f}\,. \tag{1.14}$$

The latter operator acting on the weighted space $\Lambda^p L^2(e^{-V} d\mu)$ is then unitarily equivalent to $\Delta_f^{(p)}$ (acting on the flat space) and we denote by $L_V^{\mathbf{t},(p)}$ and $L_V^{\mathbf{n},(p)}$ the nonnegative self-adjoint unbounded operators on $\Lambda^p L^2(e^{-V} d\mu)$ associated with $\Delta_f^{\mathbf{t},(p)}$ and $\Delta_f^{\mathbf{n},(p)}$ via (1.14). Their respective domains are easily deduced from (1.9), (1.10), and (1.14).

We denote moreover, for $p \in \{0, \dots, n\}$, by $\Lambda^p L^2(d\nu)$, $\Lambda^p H^1(d\nu)$, $\langle \cdot, \cdot \rangle_{L^2(d\nu)}$ and $\|\cdot\|_{L^2(d\nu)}$ the weighted Lebesgue and Sobolev spaces, L^2 scalar product and L^2 norm. We also denote by $\Lambda^p H^1_{\mathbf{b}}(d\nu)$ the set of the $\omega \in \Lambda^p H^1(d\nu)$ such that $\mathbf{b}\omega = 0$ on $\partial\Omega$, which is the domain of the quadratic form associated with $L_V^{\mathbf{b},(p)}$ according to (1.11) and (1.14). Since we are working on a compact manifold, note that $\Lambda^p H^1_{\mathbf{b}}(d\nu)$ is nothing but $\Lambda^p H^1_{\mathbf{b}}$ (algebraically and topologically).

Playing with the supersymmetry, we easily get from Theorem 1.1 the following Brascamp-Lieb's type inequalities for differential forms, where for any

$\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, $\pi_{\mathbf{b}} = \pi_{\mathbf{b}}^{(p)}$ denotes the orthogonal projection on $\operatorname{Ker}(L_V^{\mathbf{b},(p)})$.

Theorem 1.2 (Brascamp-Lieb's inequalities for differential forms)**.**

1. *Let $p \in \{0, \dots, n\}$ and let us assume that $\mathcal{K}_{\boldsymbol{n}}^{(p)} \geq 0$ everywhere on $\partial\Omega$ and that $\mathrm{Ric}_V^{(p)} := \mathrm{Ric}^{(p)} + \mathrm{Hess}^{(p)} V > 0$ everywhere on Ω (in the sense of quadratic forms). It then holds:*

   i) if $p > 0$, we have for every $\omega \in \Lambda^{p-1} H^1_{\mathbf{n}}(d\nu)$ such that $d_V^* \omega = 0$:

$$\|\omega - \pi_{\mathbf{n}}\omega\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \left\langle \left(\mathrm{Ric}_V^{(p)}\right)^{-1} d\omega \,,\, d\omega \right\rangle_{\Lambda^p} d\nu \,,$$

   ii) if $p < n$, we have for every $\omega \in \Lambda^{p+1} H^1_{\mathbf{n}}(d\nu)$ such that $d\omega = 0$:

$$\|\omega - \pi_{\mathbf{n}}\omega\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \left\langle \left(\mathrm{Ric}_V^{(p)}\right)^{-1} d_V^*\omega \,,\, d_V^*\omega \right\rangle_{\Lambda^p} d\nu \,.$$

2. *Assume similarly that $\mathcal{K}_{\boldsymbol{t}}^{(p)} - \partial_n V \geq 0$ everywhere on $\partial\Omega$ and that $\mathrm{Ric}_V^{(p)} > 0$ everywhere on Ω. It then holds:*

   i) if $p > 0$, we have for every $\omega \in \Lambda^{p-1} H^1_{\mathbf{t}}(d\nu)$ such that $d_V^* \omega = 0$:

$$\|\omega - \pi_{\mathbf{t}}\omega\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \left\langle \left(\mathrm{Ric}_V^{(p)}\right)^{-1} d\omega \,,\, d\omega \right\rangle_{\Lambda^p} d\nu \,,$$

   ii) if $p < n$, we have for every $\omega \in \Lambda^{p+1} H^1_{\mathbf{t}}(d\nu)$ such that $d\omega = 0$:

$$\|\omega - \pi_{\mathbf{t}}\omega\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \left\langle \left(\mathrm{Ric}_V^{(p)}\right)^{-1} d_V^*\omega \,,\, d_V^*\omega \right\rangle_{\Lambda^p} d\nu \,.$$

In the case $p = 1$, the points 1.i) and 2.i) of Theorem 1.2 take a simpler form. Every $\omega \in \Lambda^0 H^1(d\nu)$ satisfies indeed $d_V^*\omega = 0$. Moreover, we have simply

$$\Lambda^0 H^1_{\mathbf{n}}(d\nu) \;=\; H^1(d\nu) \quad \text{and} \quad \operatorname{Ker}(L_V^{\mathbf{n},(p)}) \;=\; \mathrm{Span}\{1\}$$

as well as

$$\Lambda^0 H^1_{\mathbf{t}}(d\nu) = H^1_0(d\nu) \quad \text{and} \quad \operatorname{Ker}(L_V^{\mathbf{t},(p)}) \;=\; \{0\} \,.$$

Defining the mean of $u \in L^2(d\nu)$ by $\langle u \rangle_\nu := \langle u, 1 \rangle_{L^2(d\nu)}$, we then immediately get from Theorem 1.2 (together with (2.13) and (2.15)) the following (where $\mathcal{K}_1$ denotes the shape operator defined in the next section, in (2.12))

Corollary 1.3. *i) Assume that the shape operator $\mathcal{K}_1$ is nonpositive everywhere on $\partial\Omega$ and that* $\mathrm{Ric} + \mathrm{Hess}\, V > 0$ *everywhere on Ω. It then holds: for every $\omega \in H^1(d\nu)$,*

$$\|\omega - \langle\omega\rangle_\nu\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \big(\mathrm{Ric} + \mathrm{Hess}\, V\big)^{-1}(\nabla\omega, \nabla\omega)\; d\nu\,. \tag{1.15}$$

ii) Assume similarly that $-\,\mathrm{Tr}\,(\mathcal{K}_1) - \partial_n V \geq 0$ everywhere on $\partial\Omega$ and that $\mathrm{Ric}+\mathrm{Hess}\, V > 0$ *everywhere on Ω. It then holds: for every $\omega \in H^1_0(d\nu)$,*

$$\|\omega\|^2_{L^2(d\nu)} \;\leq\; \int_\Omega \big(\mathrm{Ric} + \mathrm{Hess}\, V\big)^{-1}(\nabla\omega, \nabla\omega)\; d\nu\,. \tag{1.16}$$

When $\Omega \setminus \partial\Omega$ appears to be a smooth open subset of $\mathbf{R}^n$, Ric and $\mathrm{Ric}^{(p)}$ vanish and the latter corollary as well as Theorem 1.2 then write in a simpler way just relying on a control from below of $\mathrm{Hess}\, V$ or $\mathrm{Hess}^{(p)}V$ instead of $\mathrm{Ric}^{(p)}_V = \mathrm{Ric}^{(p)} + \mathrm{Hess}^{(p)}V$. One recovers in particular the usual Brascamp-Lieb's inequality when $\Omega = \mathbf{R}^n$: even if Ω has been assumed compact here, we recover the estimate (1.1) for a probability measure $d\nu$ on $\mathbf{R}^n$ using the first point of Corollary 1.3 for the family of measures $\Big(\frac{1}{\nu(B(0,N))} d\nu\big|_{B(0,N)}\Big)_{N\in\mathbf{N}}$ and letting $N \to +\infty$ since $B(0,N)$ is convex; see also [14].

The above results can be useful for semiclassical problems involving the low spectrum of semiclassical Witten Laplacians (or equivalently of semiclassical weighted Laplacians) in large dimension, such as problems dealing with correlation asymptotics, under some suitable (and uniform in the dimension) estimates on the eigenvalues of $\mathrm{Hess}\, V$ (and then of $\mathrm{Hess}^{(p)}V$) on some parts of Ω. We refer for example to [1–3, 12] or to the more recent [8] for some works exploiting this kind of estimates. Let us recall that we consider in this setting, for a small parameter $h > 0$, $\frac{f}{h}$ and $\frac{V}{h}$ instead of f and V, and $h^2\Delta^{(p)}_{\frac{f}{h}}$ instead of $\Delta^{(p)}_f$ for the usual semiclassical Schrödinger operator form. Note then from $\mathrm{Ric}^{(p)}_{\frac{V}{h}} = h^{-1}(h\,\mathrm{Ric}^{(p)} + \mathrm{Hess}^{(p)}V)$ that the curvature effects due to $\mathrm{Ric}^{(p)}$ become negligible at the semiclassical limit $h \to 0^+$ under the condition $\mathrm{Hess}^{(p)}V > 0$ everywhere on Ω. To apply Theorem 1.2 for any small $h > 0$ in the Neumann case under this condition then only requires the additional h-independent condition $\mathcal{K}^{(p)}_{\mathbf{n}} \geq 0$ everywhere on $\partial\Omega$. In the Dirichlet case, the required additional condition becomes $h\,\mathcal{K}^{(p)}_{\mathbf{t}} - \partial_n V \geq 0$, which requires in particular $\partial_n V \leq 0$ everywhere on $\partial\Omega$. The point ii) of Corollary 1.3 is thus irrelevant in this case.

Let us lastly underline that to prove Theorem 1.2 (and then Corollary 1.3), we only use the supersymmetry structure and the relation

$$\Delta_f^{\mathbf{b},(p)} \;\geq\; \mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)} f > 0$$

implied by Theorem 1.1 together with the hypotheses of Theorem 1.2. However, a control from below of the restriction $\Delta_f^{\mathbf{b},(p)}\big|_{\mathrm{Ran}\, d_f}$ for the points 1.i) and 2.i) (resp. of $\Delta_f^{\mathbf{b},(p)}\big|_{\mathrm{Ran}\, d_f^*}$ for the points 1.ii) and 2.ii)) would actually be sufficient as it can be seen by looking for example at the further relation (4.8) generalizing (1.2) (see also Proposition 3.3 for more details about the latter restrictions). The specific form of the nonnegative first term in the r.h.s. of the integration by parts formula (1.13) stated in Theorem 1.1 is moreover not used, i.e. only its nonnegativity comes into play. When $p = 1$, we can easily slightly improve Corollary 1.3 taking advantage of this nonnegative term which allows to compare $\Delta_f^{\mathbf{b},(1)}\big|_{\mathrm{Ran}\, d_f}$ (or equivalently $L_{V,h}^{\mathbf{b},(1)}\big|_{\mathrm{Ran}\, d}$) with the so-called N-dimensional Bakry-Émery tensor

$$\mathrm{Ric}_{V,N} \;:=\; \mathrm{Ric} \,+\, \mathrm{Hess}\, V \,-\, \frac{1}{N-n}\, dV \otimes dV\,, \tag{1.17}$$

where $N \in (-\infty, +\infty]$ and, when $N = n$, $\mathrm{Ric}_{V,n}$ is defined iff V is constant. The hypotheses of Corollary 1.3 require in particular the (everywhere) positivity of $\mathrm{Ric}_{V,+\infty}$ and we have more generally the

Corollary 1.4. *In the following, we assume that* $N \in (-\infty, 0] \cup [n, +\infty]$.

i) Assume that $\mathcal{K}_1 \leq 0$ *everywhere on* $\partial\Omega$ *and that* $\mathrm{Ric}_{V,N} > 0$ *everywhere on* Ω. *It then holds: for every* $\omega \in H^1(d\nu)$,

$$\|\omega - \langle\omega\rangle_\nu\|^2_{L^2(d\nu)} \;\leq\; \frac{N-1}{N} \int_\Omega \big(\mathrm{Ric}_{V,N}\big)^{-1}(\nabla\omega, \nabla\omega)\; d\nu\,.$$

ii) Assume similarly that $-\,\mathrm{Tr}\,(\mathcal{K}_1) - \partial_n V \geq 0$ *everywhere on* $\partial\Omega$ *and that* $\mathrm{Ric}_{V,N} > 0$ *on* Ω. *It then holds: for every* $\omega \in H^1_0(d\nu)$,

$$\|\omega\|^2_{L^2(d\nu)} \;\leq\; \frac{N-1}{N} \int_\Omega \big(\mathrm{Ric}_{V,N}\big)^{-1}(\nabla\omega, \nabla\omega)\; d\nu\,.$$

Note that $\frac{1}{N}$ appears here as a natural parameter and that $N \in (-\infty, 0] \cup [n, +\infty]$ is equivalent to $\frac{1}{N} \in [-\infty, \frac{1}{n}]$ with the convention $\frac{1}{0} = -\infty$.

This result corresponds to the cases (1) and (2) of Theorem 1.2 in the recent article [15] to which we also refer for more details and references concerning the N-dimensional Bakry-Émery tensor and its connections with the Bakry-Émery operators Γ and Γ_2 (see (2.21) and (2.22) in the following section, and also [5]). The authors derive these formulas from the so-called generalized Reilly formula stated in Theorem 1.1 there, which somehow generalizes, in the weighted space setting, the statement given by Theorem 1.1 when $p = 1$ and ω has the form $d_f\eta$, to arbitrary $\omega = d_f\eta$ which are not assumed tangential nor normal. We also mention the related work [16] of the same authors.

Note lastly that for $N > n$, Corollary 1.4 does not provide any improvement in comparison with Corollary 1.3 in the semiclassical setting, that is when V is replaced by $\frac{V}{h}$ where $h \to 0^+$, because of the term $-\frac{1}{(N-n)\,h^2}\, dV \otimes dV$ involved in $\mathrm{Ric}_{\frac{V}{h},N}$ (see indeed (1.17)).

1.4 Plan of the paper

In the following section, we recall general definitions and properties related to the Riemannian structure and to the Witten and weighted Laplacians. We then give the basic properties of the self-adjoint realizations $\Delta_f^{\mathbf{t},(p)}$ and $\Delta_f^{\mathbf{n},(p)}$ in Section 3. Lastly, in Section 4, we prove Theorem 1.1, Theorem 1.2, and Corollary 1.4.

2 Geometric setting

2.1 General definitions and properties

Let us begin with the notion of local orthonormal frame that will be frequently used in the sequel. A local orthonormal frame on some open set $U \subset \Omega$ is a family $(E_1, \dots, E_n)$ of smooth sections of $T\Omega$ defined on U such that

$$\forall\, i, j \in \{1, \dots, n\}\,,\ \forall x \in U\ ,\quad \langle\, E_i, E_j\,\rangle_x \;=\; \delta_{i,j}\,.$$

According for example to [23, Definition 1.1.6] and to the related remarks, it is always possible to cover Ω with a finite family (since Ω is compact) of opens sets U's such that there exists a local orthonormal frame $(E_1, \dots, E_n)$ on each U. Such a covering is called a nice cover of Ω.

The outgoing normal vector field will be denoted by $\vec{n}$ and the orientation is chosen such that

$$\mu_{\partial\Omega} \;=\; \mathbf{i}_{\vec{n}}\, \mu\,,$$

where $\mathbf{i}$ denotes the interior product. Owing to the Collar Theorem stated in [23, Theorem 1.1.7], the vector field $\vec{n} \in \mathcal{C}^\infty(\partial\Omega, T\Omega\big|_{\partial\Omega})$ can be extended to a smooth vector field on a neighborhood of the boundary $\partial\Omega$. Moreover, taking maybe a finite refinement of a nice cover of Ω as defined previously, one can always assume that the local orthonormal frame $(E_1, \dots, E_n)$ corresponding to any of its elements U meeting $\partial\Omega$ is such that $E_n\big|_{\partial\Omega} = \vec{n}$.

For any $\omega \in \Lambda^p \mathcal{C}^\infty$, the tangential part of ω on $\partial\Omega$ is the form $\mathbf{t}\omega \in \mathcal{C}^\infty(\partial\Omega, \Lambda^p T^*\Omega\big|_{\partial\Omega})$ defined by:

$$\forall \sigma \in \partial\Omega\,, \quad (\mathbf{t}\omega)_\sigma(X_1, \dots, X_p) \;:=\; \omega_\sigma(X_1^T, \dots, X_p^T)\,, \tag{2.1}$$

with the decomposition $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$ into the tangential and normal components to $\partial\Omega$ at σ. More briefly, it holds $\mathbf{t}\omega = \mathbf{i}_{\vec{n}}(\vec{n}^\flat \wedge \omega)$. The normal part of ω on $\partial\Omega$ is then defined by:

$$\mathbf{n}\omega \;:=\; \omega|_{\partial\Omega} - \mathbf{t}\omega \;=\; \vec{n}^\flat \wedge (\mathbf{i}_{\vec{n}}\omega) \quad \in\; \mathcal{C}^\infty(\partial\Omega, \Lambda^p T^*\Omega\big|_{\partial\Omega}). \tag{2.2}$$

Here and in the sequel, the notation $\flat : X \mapsto X^\flat$ stands for the inverse isomorphism of the canonical isomorphism $\sharp : \xi \mapsto \xi^\sharp$ from $T^*\Omega$ onto $T\Omega$ (defined by the relation $\langle \xi^\sharp, X\rangle := \xi(X)$ for every $X \in T\Omega$).

For a (real) smooth function f and a smooth vector field X, we will use the notation

$$\nabla_X f \;:=\; X \cdot f \;=\; df(X)\,,$$

the normal derivative of f along the boundary being in particular defined by

$$\partial_n f \;:=\; \langle\, \nabla f, \vec{n}\,\rangle \;=\; \nabla_{\vec{n}} f\,. \tag{2.3}$$

We will also denote by $\nabla : \mathcal{C}^\infty(\Omega, T\Omega) \times \mathcal{C}^\infty(\Omega, T\Omega) \to \mathcal{C}^\infty(\Omega, T\Omega)$ the Levi-Civita connection on Ω and by $\nabla_X(\cdot)$ the covariant derivative (in the direction of X) of vector fields as well as the induced covariant derivative on $\Lambda^p T^*\Omega$. The second covariant derivative is then the bilinear mapping on $T\Omega$ defined, for $X, Y \in \mathcal{C}^\infty(\Omega, T\Omega)$ by

$$\nabla^2_{X,Y} \;:=\; \nabla_X \nabla_Y - \nabla_{\nabla_X Y}\,.$$

When f is a smooth function, $\nabla^2_{X,Y} f$ is simply the Hessian of f. It is in this case a symmetric bilinear form and has the simpler writing

$$\operatorname{Hess} f(X,Y) \;:=\; \nabla^2_{X,Y} f \;=\; (\nabla_X\, df)(Y) \;=\; \langle \nabla_X \nabla f, Y\rangle\,. \tag{2.4}$$

The Bochner Laplacian Δ_B on $\mathcal{C}^\infty(\Omega, \Lambda T^*\Omega)$ is defined as minus the trace of the bilinear mapping $(X,Y) \mapsto \nabla^2_{X,Y}$. For any local orthonormal frame $(E_1, \dots, E_n)$ on $U \subset \Omega$, Δ_B is in particular given on U by

$$\Delta_B \;=\; -\sum_{i=1}^n \left(\nabla_{E_i}\nabla_{E_i} - \nabla_{\nabla_{E_i} E_i}\right). \tag{2.5}$$

The Hodge and Bochner Laplacians $\Delta^{(p)}$ and $\Delta_B^{(p)}$ are related by the Weitzenböck formula: there exists a smooth bundle symmetric endormorphism $\mathrm{Ric}^{(p)}$ belonging to $\mathcal{L}(\Lambda^p T^*\Omega)$ such that (see [23, p. 26] where the opposite convention of sign is adopted)

$$\Delta_B^{(p)} \;=\; \Delta^{(p)} - \mathrm{Ric}^{(p)} \,. \tag{2.6}$$

This operator vanishes on 0-forms (i.e. on functions) and $\mathrm{Ric}^{(1)}$ is the element of $\mathcal{L}(\Lambda^1 T^*\Omega)$ canonically identified with the Ricci tensor Ric. We recall that Ric is the symmetric $(0,2)$-tensor defined, for $X, Y \in T\Omega$, by

$$\mathrm{Ric}(X,Y) \;:=\; \mathrm{Tr}\left(Z \longmapsto R(Z,X)Y \right), \tag{2.7}$$

where R denotes the Riemannian curvature tensor which is defined, for every $X, Y, Z \in T\Omega$, by

$$R(X,Y)Z \;:=\; \left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)Z \;=\; \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \,. \tag{2.8}$$

More generally, for any local orthonormal frame $(E_1, \dots, E_n)$ on $U \subset \Omega$, $\mathrm{Ric}^{(p)}$ is defined on U for any $p \in \{1, \dots, n\}$ by

$$\begin{aligned} &\left(\mathrm{Ric}^{(p)}\omega\right)(X_1, \dots, X_p) \\ &:= -\sum_{i=1}^{n}\sum_{j=1}^{p} \left(\left(R(E_i,X_j)\right)^{(p)}\omega\right)(X_1, \dots, X_{j-1}, E_i, X_{j+1}, \dots, X_p)\,, \end{aligned} \tag{2.9}$$

where $\left(R(E_i,X_j)\right)^{(1)} \in \mathcal{L}(\Lambda^1 T^*\Omega)$ is canonically identified with $R(E_i,X_j)$ via $\left(\left(R(E_i,X_j)\right)^{(1)}\omega\right)(X) = \omega\left(\, R(E_i,X_j)X \,\right)$ and

$$\left(R(E_i,X_j)\right)^{(p)} \;=\; \left(\left(R(E_i,X_j)\right)^{(1)}\right)^{(p)},$$

where for any $A \in \mathcal{L}(\Lambda^1 T^*\Omega)$, $(A)^{(p)}$ is the element of $\mathcal{L}(\Lambda^p T^*\Omega)$ satisfying the following relation on decomposable p-forms:

$$(A)^{(p)}\left(\omega_1 \wedge \cdots \wedge \omega_p\right) \;=\; \sum_{i=1}^{p} \omega_1 \wedge \cdots \wedge A\omega_i \wedge \cdots \wedge \omega_p \,. \tag{2.10}$$

To end up this part, we recall the definition of the second fundamental form of $\partial\Omega$ before defining the operators $\mathcal{K}_{\mathbf{b}}^{(p)}$, $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, involved in Theorem 1.1 and in its corollaries. The second fundamental form $\mathcal{K}_2$ of $\partial\Omega$ is the symmetric bilinear mapping defined by

$$\mathcal{K}_2 \;:\; \begin{array}{ccc} T\partial\Omega \times T\partial\Omega & \longrightarrow & T\Omega\Big|_{\partial\Omega} \\ (U,V) & \longmapsto & (\nabla_U V)^{\perp} := \langle \nabla_U V, \vec{n}\rangle\, \vec{n} \end{array} \tag{2.11}$$

and it satisfies:

$$\forall\,(U,V)\,\in\,T\partial\Omega\times T\partial\Omega\;,\quad \langle\mathcal{K}_1(U),V\rangle\,\vec{n}\;=\;\mathcal{K}_2(U,V)\,,$$

where $\mathcal{K}_1\in\mathcal{L}(T\partial\Omega)$ is the shape operator of $\partial\Omega$ which is defined by:

$$\forall\,U\,\in\,T\partial\Omega\;,\quad \mathcal{K}_1(U)\;:=\;-\nabla_U\,\vec{n}\,. \tag{2.12}$$

The mean curvature of $\partial\Omega$ is defined as the trace of the bilinear mapping $(U,V)\mapsto\langle\mathcal{K}_2(U,V),\vec{n}\rangle$ or equivalently as the trace of the shape operator $\mathcal{K}_1$. Note also that with our choice of orientation for $\vec{n}$, Ω is locally convex iff $\langle\mathcal{K}_2(\cdot,\cdot),\vec{n}\rangle$ (or equivalently $\mathcal{K}_1$, in the sense of quadratic forms) is nonpositive.

Lastly, the smooth bundle endormophisms $\mathcal{K}^{(p)}_{\mathbf{b}}\in\mathcal{L}(\Lambda^pT^*\Omega\big|_{\partial\Omega})$, where $\mathbf{b}\in\{\mathbf{n},\mathbf{t}\}$ and $p\in\{0,\dots,n\}$, are defined by means of $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows:

1. For any $p\in\{0,\dots,n\}$, $\mathcal{K}^{(p)}_{\mathbf{n}}\in\mathcal{L}(\Lambda^pT^*\Omega\big|_{\partial\Omega})$ vanishes on 0-forms and:

   i) for any $\omega\in\Lambda^1T^*\Omega$, $\mathcal{K}^{(1)}_{\mathbf{n}}\omega$ is tangential and
   $$(\mathcal{K}^{(1)}_{\mathbf{n}}\omega)(X^T+x^\perp\vec{n})=-\omega\big(\,\mathcal{K}_1(X^T)\,\big)=\omega\big(\,\nabla_{X^T}\,\vec{n}\,\big)\,, \tag{2.13}$$
   where $\mathcal{K}_1$ is the shape operator defined in (2.12),

   ii) for any $p\in\{1,\dots,n\}$ and $\omega\in\Lambda^pT^*\Omega$, $\mathcal{K}^{(p)}_{\mathbf{n}}\omega$ is tangential and for any $X^T_1,\dots,X^T_p\in T\partial\Omega$,
   $$\big(\mathcal{K}^{(p)}_{\mathbf{n}}\omega\big)(X^T_1,\dots,X^T_p)\;=\;\big((\mathcal{K}^{(1)}_{\mathbf{n}})^{(p)}\omega\big)(X^T_1,\dots,X^T_p)\,, \tag{2.14}$$
   where the notation $(A)^{(p)}$ has been defined in (2.10).

2. For any $p\in\{0,\dots,n\}$, $\mathcal{K}^{(p)}_{\mathbf{t}}\in\mathcal{L}(\Lambda^pT^*\Omega\big|_{\partial\Omega})$ vanishes on 0-forms and:

   i) for any $\omega\in\Lambda^1T^*\Omega$, $\mathcal{K}^{(1)}_{\mathbf{t}}\omega$ is normal and
   $$(\mathcal{K}^{(1)}_{\mathbf{t}}\omega)(X^T+x^\perp\vec{n})=-x^\perp\;\mathrm{Tr}\,(\mathcal{K}_1)\,\omega\big(\,\vec{n}\,\big)\,, \tag{2.15}$$

   ii) for any $p\in\{1,\dots,n\}$ and $\omega\in\Lambda^pT^*\Omega$, $\mathcal{K}^{(p)}_{\mathbf{t}}\omega$ is normal and for any local orthonormal frame $(E_1,\dots,E_n)$ on $U\subset\Omega$ such that $E_n\big|_{\partial\Omega}=\vec{n}$ (with $U\cap\partial\Omega\neq\emptyset$) and $X^T_1,\dots,X^T_p\in T\partial\Omega$, we have on $U\cap\partial\Omega$:
   $$\begin{aligned}&\big(\,\mathcal{K}^{(p)}_{\mathbf{t}}\omega\,\big)(\vec{n},X^T_1,\dots,X^T_{p-1})\\ &\qquad:=\;-\sum_{i=1}^{n-1}\Big(\big(\mathcal{K}_2(E_i,\cdot)\big)^{(p)}\omega\Big)(E_i,X^T_1,\dots,X^T_{p-1})\,,\end{aligned} \tag{2.16}$$

where $\big(\mathcal{K}_2(E_i,\cdot)\big)^{(p)} = \Big(\big(\mathcal{K}_2(E_i,\cdot)\big)^{(1)}\Big)^{(p)}$ and

$$\Big(\big(\mathcal{K}_2(E_i,\cdot)\big)^{(1)}\omega\Big)(X) \;=\; \omega\big(\,\mathcal{K}_2(E_i,X)\,\big)\,.$$

Note that the point 2.ii) is nothing but the statement of 2.i) when $p=1$.

2.2 Witten and weighted Laplacians

Using the following relations dealing with exterior and interior products (respectively denoted by $\wedge$ and $\mathbf{i}$), gradients, and Lie derivatives (denoted by $\mathcal{L}$),

$$(df\wedge)^* = \mathbf{i}_{\nabla f} \quad \text{as bounded operators in } L^2(\Omega,\Lambda^pT^*\Omega\,)\,, \tag{2.17}$$

$$d_f = d + df\,\wedge \quad \text{and} \quad d_f^* = d^* + \mathbf{i}_{\nabla f}\,, \tag{2.18}$$

$$\mathcal{L}_X = d\circ \mathbf{i}_X + \mathbf{i}_X\circ d \quad \text{and} \quad \mathcal{L}_X^* = d^*\circ(X^\flat\wedge\cdot) + X^\flat\wedge d^*\,, \tag{2.19}$$

the Witten Laplacian $(d_f+d_f^*)^2$ has the form

$$\Delta_f \;=\; (d+d^*)^2 + |\nabla f|^2 + \big(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}\big)\;. \tag{2.20}$$

When acting on 0-forms, the Witten Laplacian is then simply given by

$$\Delta_f^{(0)} \;=\; \Delta + |\nabla f|^2 + \Delta f$$

and is unitarily equivalent to the following operator acting on the weighted space $L^2(e^{-2f}d\mu)$, sometimes referred to as the weighted Laplacian (or Bakry-Émery Laplacian) in the literature (see e.g. [15]),

$$L_V^{(0)} \;:=\; \Delta + \nabla V\cdot\nabla \qquad \text{where} \qquad V \;:=\; 2f\,.$$

More precisely, it holds

$$\Delta_f^{(0)} \;=\; e^{-\frac{V}{2}}\,L_V^{(0)}\,e^{\frac{V}{2}}\,.$$

The operator $L_V^{(0)}$ has consequently a natural supersymmetric extension on the algebra of differential forms, acting in the weighted space $\Lambda L^2(e^{-2f}d\mu)$, which is simply defined for any $p\in\{0,\dots,n\}$ by the formula (1.14) that we recall here:

$$L_V^{(p)} \;:=\; e^f\,\Delta_f^{(p)}\,e^{-f} \qquad \text{where} \qquad V \;:=\; 2f\,.$$

To connect more precisely to the literature dealing with the Bakry-Émery theory of diffusion processes (see [5] for an overview), the operators $L_V^{(0)}$ and $L_V^{(1)}$ are related to the carré du champ operator of Bakry-Émery Γ and to its iteration Γ_2 via the relations

$$\int_\Omega \Gamma(\omega)\, e^{-2f} d\mu \;=\; \int_\Omega \big(L_V^{(0)}\,\omega\big)\,\omega\; e^{-2f} d\mu = \int_\Omega \langle d\omega, d\omega\rangle_{\Lambda^1}\;\; e^{-2f} d\mu \tag{2.21}$$

and

$$\int_\Omega \Gamma_2(\omega) e^{-2f} d\mu = \int_\Omega \big(L_V^{(0)}\,\omega\big)^2 e^{-2f} d\mu = \int_\Omega \langle L_V^{(1)}\, d\omega, d\omega\rangle_{\Lambda^1} e^{-2f} d\mu, \tag{2.22}$$

where ω is a smooth function supported in $\Omega \setminus \partial\Omega$ (see in particular [5] for many details and references about this notion).

Coming back to the Witten Laplacian, we have the following formula:

$$\Delta_f^{(p)} \;=\; (d+d^*)^2 \;+\; |\nabla f|^2 \;+\; 2\,\mathrm{Hess}^{(p)} f \;+\; \Delta f\,. \tag{2.23}$$

This relation is not very common in the literature dealing with semiclassical Witten Laplacians – i.e. where one studies $h^2 \Delta_{\frac{f}{h}}$ at the limit $h \to 0^+$ – which motivated this work, at least when Ω is not flat. We find generally there the formula (2.20) (see e.g. [11, 17] and references therein) and we thus give a proof below (see also [13] for another proof). Let us, before, specify the sense of (2.23). There, $\mathrm{Hess}^{(0)} f = 0$, $\mathrm{Hess}^{(1)} f$ is the element of $\mathcal{L}(\Lambda^1 T^*\Omega)$ canonically identified with $\mathrm{Hess}\, f$, and $\mathrm{Hess}^{(p)} f$ is the bundle symmetric endomorphism of $\Lambda^p T^*\Omega$ defined by

$$\mathrm{Hess}^{(p)} f \;:=\; \big(\mathrm{Hess}^{(1)} f\big)^{(p)} \quad \text{(see (2.10) for the meaning of } (A)^{(p)}\text{)}\,. \tag{2.24}$$

Denoting also by $\mathrm{Hess} f$ the bundle symmetric endomorphism of $T\Omega$ defined by $\langle \mathrm{Hess} f\, X, Y\rangle := \mathrm{Hess} f(X,Y)$ (i.e. by $\mathrm{Hess} f\, X = \nabla_X \nabla f$), remark that we have for any $p \in \{1, \dots, n\}$ and $\omega \in \Lambda^p T^*\Omega$:

$$\mathrm{Hess}^{(p)} f\, \omega(X_1, \dots, X_p) \;=\; \sum_{i=1}^{p} \omega(X_1, \dots, \mathrm{Hess} f\, X_i, \dots, X_p)\,. \tag{2.25}$$

Proof of formula (2.23)*:* Let us first recall that the covariant derivative ∇_X on $\Lambda^p T^*\Omega$ induced by the Levi-Civita connection is defined by

$$\begin{aligned}(\nabla_X \omega)(Y_1, \dots, Y_p) \;:=\; &\nabla_X\big(\omega(Y_1, \dots, Y_p)\big)\\ &- \sum_{k=1}^{p} \omega(Y_1, \dots, \nabla_X Y_k, \dots, Y_p)\end{aligned} \tag{2.26}$$

and satisfies in particular the relations

$$\nabla_X\big(\langle\omega,\eta\rangle_{\Lambda^p}\big)=\langle\nabla_X\omega,\eta\rangle_{\Lambda^p}+\langle\omega,\nabla_X\eta\rangle_{\Lambda^p} \tag{2.27}$$

and

$$\nabla_X\big(\omega_1\wedge\omega_2\big)=(\nabla_X\omega_1)\wedge\omega_2+\omega_1\wedge(\nabla_X\omega_2)\,. \tag{2.28}$$

The differential d and ∇ are moreover related by the relation

$$d\omega(X_0,\dots,X_p) \;=\; \sum_{k=0}^{p}(-1)^k(\nabla_{X_k}\omega)(X_0,\dots,\dot{X}_k,\dots,X_p)\,, \tag{2.29}$$

where $\omega\in\Lambda^pT^*\Omega$ and the notation $\dot{X}_k$ means that X_k has been removed from the parenthesis. Furthermore, if $(E_1,\dots,E_n)$ is a local orthonormal frame on $U\subset\Omega$, the codifferential d^* is given there by

$$d^* \;=\; -\sum_{i=1}^{n}\mathbf{i}_{E_i}\nabla_{E_i}\,. \tag{2.30}$$

Hence, we deduce from (2.25) and from the relation relating $\mathcal{L}_X$ and ∇_X,

$$(\mathcal{L}_X^{(p)}\omega)(X_1,\dots,X_p) \;=\; (\nabla_X\omega)(X_1,\dots,X_p)+\sum_{i=1}^{p}\omega(X_1,\dots,\nabla_{X_i}X,\dots,X_p)$$

which arises from (2.26), (2.29), and (2.19), the following equality:

$$\mathcal{L}_{\nabla f}^{(p)} \;=\; \nabla_{\nabla f}\;+\;\mathrm{Hess}^{(p)}f\,. \tag{2.31}$$

Taking now a local orthonormal frame $(E_1,\dots,E_n)$ on an open set $U\subset\Omega$, we deduce from (2.28), (2.30), and (2.19) the following relations (on U):

$$\begin{aligned}\mathcal{L}_{\nabla f}^{*,(p)}\omega \;&=\; \sum_{i=1}^{n}\Big((\nabla_{E_i}df)\wedge\mathbf{i}_{E_i}\omega-df(E_i)\nabla_{E_i}\omega-\big(\nabla_{E_i}df(E_i)\big)\Big)\omega\\ &=\; -\nabla_{\nabla f}\,\omega+(\Delta f)\omega+\sum_{i=1}^{n}(\nabla_{E_i}df)\wedge\mathbf{i}_{E_i}\omega\,.\end{aligned} \tag{2.32}$$

Lasty, we have

$$\begin{aligned}&\sum_{i=1}^{n}\big((\nabla_{E_i}df)\wedge\mathbf{i}_{E_i}\omega\big)(X_1,\dots,X_p)\\ &\qquad=\;\sum_{i=1}^{n}\sum_{k=1}^{p}(-1)^{k+1}(\nabla_{E_i}df)(X_k)(\mathbf{i}_{E_i}\omega)(X_1,\dots,\dot{X}_k,\dots,X_p)\\ &\qquad=\;\sum_{k=1}^{p}(-1)^{k+1}\omega(\mathrm{Hess}\,f\,X_k,X_1,\dots,\dot{X}_k,\dots,X_p)\\ &\qquad=\;\sum_{k=1}^{p}\omega(X_1,\dots,\mathrm{Hess}\,f\,X_k,\dots,X_p)\end{aligned} \tag{2.33}$$

and formula (2.23) for the Witten Laplacian then follows from (2.20) and (2.31)–(2.33). ∎

3 Self-adjoint realizations of the Witten Laplacian

In the sequel, we will use for any $(\omega,\eta)\in\left(\Lambda^p H^1\right)^2$ the more compact notation

$$\mathcal{D}^{(p)}_f(\omega,\eta) \;:=\; \langle d_f\omega, d_f\eta\rangle_{\Lambda^{p+1}L^2} + \langle d^*_f\omega, d^*_f\eta\rangle_{\Lambda^{p-1}L^2}$$

as well as

$$\mathcal{D}^{(p)}_f(\omega) \;:=\; \mathcal{D}^{(p)}_f(\omega,\omega)$$

and

$$\mathcal{D}^{(p)}(\omega,\eta) \;:=\; \mathcal{D}^{(p)}_0(\omega,\eta) \quad \text{and} \quad \mathcal{D}^{(p)}(\omega) := \mathcal{D}^{(p)}(\omega,\omega)\,.$$

Let us also recall, for $\mathbf{b}\in\{\mathbf{n},\mathbf{t}\}$, the definition of $\Lambda^p H^1_{\mathbf{b}}$ given in (1.11):

$$\Lambda^p H^1_{\mathbf{b}} \;=\; \left\{\omega\in\Lambda^p H^1\,,\ \mathbf{b}\omega=0 \text{ on } \partial\Omega\right\}\,.$$

In particular, $\Lambda^0 H^1_{\mathbf{n}} = H^1_{\mathbf{n}}$ is simply $H^1(\Omega)$ while $H^1_{\mathbf{t}} = H^1_0(\Omega)$. Moreover, since the boundary $\partial\Omega$ is smooth, the space

$$\Lambda^p\mathcal{C}^\infty_{\mathbf{b}} \;:=\; \{\omega\in\Lambda^p\mathcal{C}^\infty\,,\ \mathbf{b}\omega=0 \text{ on } \partial\Omega\}$$

is dense in $\left(\Lambda^p H^1_{\mathbf{b}}\,,\ \|\cdot\|_{\Lambda^p H^1}\right)$.

The following lemma states two Green's identities comparing $\mathcal{D}^{(p)}(\cdot)$ and $\mathcal{D}^{(p)}_f(\cdot)$ on the space of tangential or normal p-forms. We refer to [11, Section 2.3] and [17, Section 2.2] for a proof.

Lemma 3.1. *We have the two following identities:*

i) for any $\omega\in\Lambda^p H^1_{\mathbf{t}}$,

$$\mathcal{D}^{(p)}_f(\omega) = \mathcal{D}^{(p)}(\omega) + \|\,|\nabla f|\,\omega\|^2_{\Lambda^p L^2} + \langle(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f})\omega,\omega\rangle_{\Lambda^p L^2} + \int_{\partial\Omega}\langle\omega,\omega\rangle_{\Lambda^p}\,\partial_n f\ d\mu_{\partial\Omega}\,, \quad (3.1)$$

ii) for any $\omega\in\Lambda^p H^1_{\mathbf{n}}$,

$$\mathcal{D}^{(p)}_f(\omega) = \mathcal{D}^{(p)}(\omega) + \|\,|\nabla f|\,\omega\|^2_{\Lambda^p L^2} + \langle(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f})\omega,\omega\rangle_{\Lambda^p L^2} - \int_{\partial\Omega}\langle\omega,\omega\rangle_{\Lambda^p}\,\partial_n f\ d\mu_{\partial\Omega}\,. \quad (3.2)$$

We now compile in the following proposition basic facts about Witten Laplacians on manifolds with boundary proven in [11, Section 2.4] and in [17, Section 2.3].

Proposition 3.2. *i) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the nonnegative quadratic form $\omega \to \mathcal{D}^{(p)}_{f,h}(\omega)$ is closed on $\Lambda^p H^1_{\mathbf{b}}$. Its associated self-adjoint Friedrichs extension is denoted by $\big(\Delta_f^{\mathbf{b},(p)}\,,\, D(\Delta_f^{\mathbf{b},(p)})\big)$.*

ii) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the domain of $\Delta_f^{\mathbf{b},(p)}$ is given by

$$D(\Delta_f^{\mathbf{b},(p)}) \;=\; \big\{u \in \Lambda^p H^2,\ \mathbf{b}\omega = 0\,,\ \mathbf{b}d_f^*\omega = 0 \text{ and } \mathbf{b}d_f\omega = 0 \text{ on } \partial\Omega\big\}.$$

We have moreover:

$$\forall\, \omega \,\in\, D(\Delta_f^{\mathbf{b},(p)})\,, \quad \Delta_f^{\mathbf{b},(p)}\omega \;=\; \Delta_f^{(p)}\omega \quad \text{in} \quad \Omega$$

and the equalities $\mathbf{n}d_f^*\omega = 0$ and $\mathbf{t}d_f\omega = 0$ are actually satisfied for any $\omega \in \Lambda^p H^2 \cap \Lambda^p H^1_{\mathbf{b}}$.

iii) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, $\Delta_f^{\mathbf{b},(p)}$ has a compact resolvent.

iv) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the following commutation relations hold for any $v \in \Lambda^p H^1_{\mathbf{b}}$:
– for every $z \in \varrho(\Delta_f^{\mathbf{b},(p)}) \cap \varrho(\Delta_f^{\mathbf{b},(p+1)})$,

$$(z - \Delta_f^{\mathbf{b},(p+1)})^{-1}\, d_f^{(p)}\, v \;=\; d_f^{(p)}\, (z - \Delta_f^{\mathbf{b},(p)})^{-1}\, v$$

– and for every $z \in \varrho(\Delta_f^{\mathbf{b},(p)}) \cap \varrho(\Delta_f^{\mathbf{b},(p-1)})$,

$$(z - \Delta_f^{\mathbf{b},(p-1)})^{-1}\, d_f^{(p-1),*}\, v \;=\; d_f^{(p-1),*}\, (z - \Delta_f^{\mathbf{b},(p)})^{-1}\, v\,.$$

In the spirit of the above point iv), we have also the following Witten-Hodge-decomposition which will be useful when proving Corollary 1.4:

Proposition 3.3. *For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, it holds*

$$\begin{aligned} \Lambda^p L^2 \;&=\; \operatorname{Ker} \Delta_f^{\mathbf{b},(p)} \oplus^\perp \operatorname{Ran}\big(d_f\big|_{\Lambda^{p-1}H^1_{\mathbf{b}}}\big) \oplus^\perp \operatorname{Ran}\big(d_f^*\big|_{\Lambda^{p+1}H^1_{\mathbf{b}}}\big) \\ &=:\; K^{\mathbf{b},(p)} \oplus^\perp R^{\mathbf{b},(p)} \oplus^\perp R^{*,\mathbf{b},(p)}\,, \end{aligned} \tag{3.3}$$

the spaces $R^{\mathbf{b},(p)}$ and $R^{*,\mathbf{b},(p)}$ being consequently closed in $\Lambda^p L^2$. Denoting moreover by $\pi_{R^{\mathbf{b},(p)}}$ and $\pi_{R^{*,\mathbf{b},(p)}}$ the orthogonal projectors on these respective spaces, the following relations hold in the sense of unbounded operators:

$$\pi_{R^{\mathbf{b},(p)}}\, \Delta_f^{\mathbf{b},(p)} \;\subset\; \Delta_f^{\mathbf{b},(p)}\, \pi_{R^{\mathbf{b},(p)}} \text{ and } \pi_{R^{*,\mathbf{b},(p)}}\, \Delta_f^{\mathbf{b},(p)} \;\subset\; \Delta_f^{\mathbf{b},(p)}\, \pi_{R^{*,\mathbf{b},(p)}}\,. \tag{3.4}$$

In particular, for $A \in \{(K^{\mathbf{b},(p)})^{\perp}, R^{\mathbf{b},(p)}, R^{*,\mathbf{b},(p)}\}$, the unbounded operator $\Delta_f^{\mathbf{b},(p)}\big|_A$ with domain $D(\Delta_f^{\mathbf{b},(p)}) \cap A$ is well defined, self-adjoint, invertible on A, and it holds for every $v \in \Lambda^p H^1_{\mathbf{b}} \cap (K^{\mathbf{b},(p)})^{\perp}$:

$$\begin{aligned}\left(\Delta_f^{\mathbf{b},(p+1)}\big|_{(K^{\mathbf{b},(p+1)})^{\perp}}\right)^{-1} d_f\, v \;&=\; \left(\Delta_f^{\mathbf{b},(p+1)}\big|_{R^{\mathbf{b},(p+1)}}\right)^{-1} d_f\, v \\ &=\; d_f \left(\Delta_f^{\mathbf{b},(p)}\big|_{(K^{\mathbf{b},(p)})^{\perp}}\right)^{-1} v \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\left(\Delta_f^{\mathbf{b},(p-1)}\big|_{(K^{\mathbf{b},(p-1)})^{\perp}}\right)^{-1} d_f^*\, v \;&=\; \left(\Delta_f^{\mathbf{b},(p-1)}\big|_{R^{*,\mathbf{b},(p-1)}}\right)^{-1} d_f^*\, v \\ &=\; d_f^* \left(\Delta_f^{\mathbf{b},(p)}\big|_{(K^{\mathbf{b},(p)})^{\perp}}\right)^{-1} v\,. \end{aligned} \tag{3.6}$$

Proof. The orthogonality of the sum appearing in the r.h.s. of (3.3) follows easily from the distorted Green's formula valid for any $(\omega, \eta) \in \Lambda^{p-1}H^1 \times \Lambda^p H^1$,

$$\langle d_f\omega, \eta\rangle_{\Lambda^p L^2} \;=\; \langle \omega, d_f^*\eta\rangle_{\Lambda^{p-1}L^2} + \int_{\partial\Omega} \langle \omega, \mathbf{i}_{\vec{n}}\eta\rangle_{\Lambda^p}\, d\mu_{\partial\Omega}\,, \tag{3.7}$$

which is a straightforward consequence of (2.17)–(2.18) and of the usual Green's formula:

$$\langle d\omega, \eta\rangle_{\Lambda^p L^2} \;=\; \langle \omega, d^*\eta\rangle_{\Lambda^{p-1}L^2} + \int_{\partial\Omega} \langle \omega, \mathbf{i}_{\vec{n}}\eta\rangle_{\Lambda^p}\, d\mu_{\partial\Omega}\,. \tag{3.8}$$

Moreover, since $\Delta_f^{\mathbf{b},(p)}$ has a compact resolvent, the self-adjoint operator $\tilde{\Delta}_f^{\mathbf{b},(p)}$ on $(K^{\mathbf{b},(p)})^{\perp} := \mathrm{Ker}\,(\Delta_f^{\mathbf{b},(p)})^{\perp}$,

$$\tilde{\Delta}_f^{\mathbf{b},(p)} \;:=\; \Delta_f^{\mathbf{b},(p)}\big|_{(K^{\mathbf{b},(p)})^{\perp}} :\; D(\Delta_f^{\mathbf{b},(p)}) \cap \mathrm{Ker}\,(\Delta_f^{\mathbf{b},(p)})^{\perp} \longrightarrow \mathrm{Ker}\,(\Delta_f^{\mathbf{b},(p)})^{\perp}\,,$$

is invertible and hence any $u \in \Lambda^p L^2$ has the form

$$u \;=\; \pi_{f,\mathbf{b}}u \,+\, \Delta_f^{\mathbf{b},(p)}v \;=\; \pi_{f,\mathbf{b}}u \,+\, d_f\big(d_f^*v\big) \,+\, d_f^*\big(d_f v\big)\,, \tag{3.9}$$

for some uniquely determined $v \in D(\Delta_f^{\mathbf{b},(p)}) \cap (K^{\mathbf{b},(p)})^{\perp}$, denoting by $\pi_{f,\mathbf{b}} = \pi_{f,h,\mathbf{b}}^{(p)}$ the orthogonal projection on $(K^{\mathbf{b},(p)})^{\perp}$. This implies (3.3).
Let us now prove (3.4) and take then $u \in D(\Delta_f^{\mathbf{b},(p)})$. It holds

$$\pi_{R^{\mathbf{b},(p)}}\, \Delta_f^{\mathbf{b},(p)}\, u \;=\; d_f\big(d_f^*u\big) \quad\text{and}\quad \pi_{R^{*,\mathbf{b},(p)}}\, \Delta_f^{\mathbf{b},(p)}\, u \;=\; d_f^*\big(d_f u\big) \tag{3.10}$$

and, according to (3.9), we have moreover

$$\pi_{R^{\mathbf{b},(p)}}u \;=\; d_f\big(d_f^*v\big) \quad\text{and}\quad \pi_{R^{*,\mathbf{b},(p)}}u \;=\; d_f^*\big(d_f v\big) \tag{3.11}$$

where $v = \big(\tilde{\Delta}_f^{\mathbf{b},(p)}\big)^{-1}(u - \pi_{f,\mathbf{b}}u) \in D(\Delta_f^{\mathbf{b},(p)}) \cap (K^{\mathbf{b},(p)})^{\perp}$. Using now iv) of Proposition 3.2, we have for every $z \in \mathbf{R}$, $z < 0$,

$$
\begin{aligned}
d_f\, d_f^*\big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1}(u - \pi_{f,\mathbf{b}}u) &= \big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1} d_f\, d_f^*\,(u - \pi_{f,\mathbf{b}}u) \\
&\underset{z\to 0^-}{\longrightarrow} \big(\tilde{\Delta}_f^{\mathbf{b},(p)}\big)^{-1} d_f\, d_f^*(u - \pi_{f,\mathbf{b}}u).
\end{aligned} \tag{3.12}
$$

Since moreover $\big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1}(u - \pi_{f,\mathbf{b}}u) = v + z\,\big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1} v$, it also holds

$$
\begin{aligned}
d_f\, d_f^*\big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1}(u - \pi_{f,\mathbf{b}}u) &= d_f\, d_f^*\, v + z\,\big(\Delta_f^{\mathbf{b},(p)} - z\big)^{-1} d_f\, d_f^*\, v \\
&\underset{z\to 0^-}{\longrightarrow} d_f\big(d_f^*\, v\big)
\end{aligned} \tag{3.13}
$$

and we deduce from (3.12) and (3.13) that

$$
d_f\, d_f^*\, v \in D(\Delta_f^{\mathbf{b},(p)}) \quad \text{and} \quad \Delta_f^{\mathbf{b},(p)}\, d_f\, d_f^*\, v \;=\; d_f\, d_f^*(u - \pi_{f,\mathbf{b}}u) \;=\; d_f\big(d_f^*\, u\big)\,,
$$

which proves the first equality of (3.4) according to (3.10) and (3.11). The second equality of (3.4) is proven similarly after establishing the analogous versions of (3.12) and (3.13) with $d_f\, d_f^*$ replaced by $d_f^*\, d_f$. The last part of Proposition 3.3 then follows easily, using again iv) of Proposition 3.2 as in (3.12) and (3.13) to obtain (3.5) and (3.6). ∎

4 Proofs of the main results

4.1 Proof of Theorem 1.1

We first prove Theorem 1.1 in the case $f = 0$. As shown in [23], it implies in particular Gaffney's inequalities which state the equivalence between the norms $\|\cdot\|_{\Lambda^p H^1}$ and $\sqrt{\mathcal{D}^{(p)}(\cdot) + \|\cdot\|^2_{\Lambda^p L^2}}$ for tangential or normal p-forms.

Theorem 4.1. *Let $\omega \in \Lambda^p H^1_{\mathbf{b}}$ with $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$. We have then the identity*

$$
\|\omega\|^2_{\dot{H}^1} \;=\; \mathcal{D}^{(p)}(\omega) - \langle \mathrm{Ric}^{(p)}\omega, \omega\rangle_{L^2} - \int_{\partial\Omega} \langle \mathcal{K}^{(p)}_{\mathbf{b}}\omega, \omega\rangle_{\Lambda^p}\, d\mu_{\partial\Omega}\,,
$$

where $\mathrm{Ric}^{(p)} \in \mathcal{L}(\Lambda^p T^*\Omega)$ and $\mathcal{K}^{(p)}_{\mathbf{b}} \in \mathcal{L}(\Lambda^p T^*\Omega\big|_{\partial\Omega})$ have been respectively defined in (2.9) *and in* (2.13)*–*(2.16)*, and*

$$
\|\cdot\|^2_{\dot{H}^1} \;:=\; \|\cdot\|^2_{H^1} - \|\cdot\|^2_{L^2}\,.
$$

The statement of Theorem 4.1 is essentially the statement of [23, Theorem 2.1.5] and the content of its proof in the case $\mathbf{t}\omega = 0$, and is closely related to the statement of [23, Theorem 2.1.7] in the case $\mathbf{n}\omega = 0$. We have nevertheless to compute the exact form of $\mathcal{K}_{\mathbf{n}}^{(p)}$ and especially of $\mathcal{K}_{\mathbf{n}}^{(1)}$ in the latter case. We also give a complete proof in the case $\mathbf{t}\omega = 0$ for the sake of clarity.

Proof. By density of $\Lambda^p\mathcal{C}_{\mathbf{b}}^\infty$ in $\Lambda^p H_{\mathbf{b}}^1$ for $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, it is sufficient to prove Theorem 4.1 for $\omega \in \Lambda^p\mathcal{C}_{\mathbf{b}}^\infty$. Moreover, it follows from the Weitzenböck formula (2.6) and from the Green's formulas for the Hodge and Bochner Laplacians,

$$\mathcal{D}^{(p)}(\omega) \;=\; \langle \Delta\omega, \omega\rangle_{\Lambda^p L^2} + \int_{\partial\Omega} \big(\langle \mathbf{i}_{\vec{n}} d\omega, \omega\rangle_{\Lambda^p} \;-\; \langle d^*\omega, \mathbf{i}_{\vec{n}}\omega\rangle_{\Lambda^{p-1}}\big)\, d\mu_{\partial\Omega} \quad (4.1)$$

and

$$\|\omega\|^2_{\Lambda^p \dot{H}^1} \;=\; \langle \Delta_B\omega, \omega\rangle_{\Lambda^p L^2} + \int_{\partial\Omega} \langle \nabla_{\vec{n}}\omega, \omega\rangle_{\Lambda^p} d\mu_{\partial\Omega}\,, \quad (4.2)$$

that for any $\omega \in \Lambda^p\mathcal{C}^\infty$, the expression

$$\|\omega\|^2_{\Lambda^p \dot{H}^1} - \mathcal{D}^{(p)}(\omega) + \langle \mathrm{Ric}^{(p)}\omega, \omega\rangle_{\Lambda^p L^2}$$

reduces to the boundary integral

$$\int_{\partial\Omega} \big(\langle \nabla_{\vec{n}}\omega, \omega\rangle_{\Lambda^p} - \langle \mathbf{i}_{\vec{n}} d\omega, \omega\rangle_{\Lambda^p} + \langle d^*\omega, \mathbf{i}_{\vec{n}}\omega\rangle_{\Lambda^{p-1}}\big)\, d\mu_{\partial\Omega}$$

and we have then just to check that for any ω in $\Lambda^p\mathcal{C}_{\mathbf{b}}^\infty$, it holds

$$\langle \mathcal{K}_{\mathbf{b}}^{(p)}\omega, \omega\rangle_{\Lambda^p} \;=\; \langle \mathbf{i}_{\vec{n}} d\omega, \omega\rangle_{\Lambda^p} - \langle d^*\omega, \mathbf{i}_{\vec{n}}\omega\rangle_{\Lambda^{p-1}} - \langle \nabla_{\vec{n}}\omega, \omega\rangle_{\Lambda^p}\,, \quad (4.3)$$

where $\mathcal{K}_{\mathbf{b}}^{(p)} \in \mathcal{L}(\Lambda^p T^*\Omega\big|_{\partial\Omega})$ has been defined in (2.13)–(2.16).

Case $\mathbf{n}\omega = 0$:
We have then $\langle d^*\omega, \mathbf{i}_{\vec{n}}\omega\rangle_{\Lambda^p} = 0$ and

$$\langle \mathbf{i}_{\vec{n}} d\omega, \omega\rangle_{\Lambda^p} - \langle \nabla_{\vec{n}}\omega, \omega\rangle_{\Lambda^p} \;=\; \langle \mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}}\omega, \omega\rangle_{\Lambda^p} \;=\; \langle \mathbf{t}\big(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}}\omega\big), \omega\rangle_{\Lambda^p}\,,$$

the last equality following again from $\mathbf{n}\omega = 0$. It is then sufficient to show that for any ω in $\Lambda^p\mathcal{C}_{\mathbf{n}}^\infty$, it holds

$$\mathcal{K}_{\mathbf{n}}^{(p)}\omega \;=\; \mathbf{t}\big(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}}\omega\big)\,. \quad (4.4)$$

Taking now p tangential vector fields $X_1, \dots, X_p$ and denoting for simplicity $\vec{n}$ by X_0, we deduce from (2.29) that:

$$
\begin{aligned}
(\mathbf{i}_{X_0} d\omega - \nabla_{X_0}\omega)(X_1, \dots, X_p) &= d\omega(X_0, X_1, \dots, X_p) - (\nabla_{X_0}\omega)(X_1, \dots, X_p) \\
&= \sum_{k=1}^{p} (-1)^k (\nabla_{X_k}\omega)(X_0, \dots, \dot{X}_k, \dots, X_p)\,.
\end{aligned}
$$

Moreover, using (2.26), the tangentiality of $X_1, \dots, X_p$, and $\mathbf{n}\omega = 0$, we have for any $k \in \{1, \dots, p\}$:

$$
\begin{aligned}
(\nabla_{X_k}\omega)(X_0, \dots, \dot{X}_k, \dots, X_p) \; &= \nabla_{X_k}\big(\omega(X_0, \dots, \dot{X}_k, \dots, X_p)\big) \\
&\quad - \sum_{k \neq \ell = 0, \dots, p} \omega(X_0, \dots, \nabla_{X_k} X_\ell, \dots, \dot{X}_k, \dots, X_p) \\
&= \; -\,\omega(\nabla_{X_k} X_0, \dots, \dot{X}_k, \dots, X_p) \\
&= (-1)^k \omega(X_1, \dots, \nabla_{X_k} X_0, \dots, X_p)\,.
\end{aligned}
$$

Hence, it holds for any $\omega \in \Lambda^p \mathcal{C}^\infty_{\mathbf{n}}$ and p tangential vector fields $X_1, \dots, X_p$,

$$
(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}}\omega)(X_1, \dots, X_p) \; = \; \sum_{k=1}^{p} \omega(X_1, \dots, \nabla_{X_k}\vec{n}, \dots, X_p)\,,
$$

the r.h.s. being nothing but $\big(\mathcal{K}^{(p)}_{\mathbf{n}}\omega\big)(X_1, \dots, X_p)$ according to (2.13) and (2.14). This proves (4.4) and then concludes the proof in the case $\mathbf{n}\omega = 0$.

Case $\mathbf{t}\omega = 0$:
We have here $\langle \mathbf{i}_{\vec{n}} d\omega, \omega \rangle_{\Lambda^p} = 0$ and from (4.3), we are then led to compute more precisely

$$
\begin{aligned}
\langle d^*\omega, \mathbf{i}_{\vec{n}}\omega \rangle_{\Lambda^{p-1}} + \langle \nabla_{\vec{n}}\omega, \omega \rangle_{\Lambda^p} \; &= \; \langle \vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega, \omega \rangle_{\Lambda^p} \\
&= \; \langle \mathbf{n}\big(\vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega\big), \omega \rangle_{\Lambda^p} \\
&= \; \langle \vec{n}^\flat \wedge (d^*\omega + \mathbf{i}_{\vec{n}} \nabla_{\vec{n}}\omega), \omega \rangle_{\Lambda^p}\,,
\end{aligned}
$$

the second to last equality following from $\mathbf{t}\omega = 0$ and the last one from $\mathbf{n}\omega = \vec{n}^\flat \wedge (\mathbf{i}_{\vec{n}}\omega)$. To conclude, it then remains to show that for any ω in $\Lambda^p \mathcal{C}^\infty_{\mathbf{t}}$, it holds

$$
\mathcal{K}^{(p)}_{\mathbf{t}}\omega \; = \; -\mathbf{n}\big(\vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega\big) \; = \; -\vec{n}^\flat \wedge (d^*\omega + \mathbf{i}_{\vec{n}} \nabla_{\vec{n}}\omega)\,. \tag{4.5}
$$

Denoting by $(E_1, \dots, E_n)$ a local orthonormal frame such that $E_n = \vec{n}$ on $\partial\Omega$ and using (2.30), we get

$$
-\mathbf{n}\big(\vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega\big) \; = \; \vec{n}^\flat \wedge \Big(\sum_{i=1}^{n-1} \mathbf{i}_{E_i} \nabla_{E_i}\omega \Big)\,.
$$

Taking now $p-1$ tangential vector fields $X_1, \dots, X_{p-1}$, we then have:

$$-\mathbf{n}\big(\vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega\big)(\vec{n}, X_1, \dots, X_{p-1}) \;=\; \sum_{i=1}^{n-1} (\nabla_{E_i}\omega)(E_i, X_1, \dots, X_{p-1})\,,$$

where, for any $i \in \{1, \dots, n-1\}$, using (2.26), the tangentiality of the vector fields $X_1, \dots, X_{p-1}$, $\mathbf{t}\omega = 0$, and denoting for simplicity E_i by X_0,

$$\begin{aligned}
(\nabla_{X_0}\omega)(X_0, X_1, \dots, X_{p-1}) \;&=\; \nabla_{X_0}\big(\omega(X_0, X_1, \dots, X_{p-1})\big)\\
&\quad - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, \nabla_{X_0}X_\ell, \dots, X_{p-1})\\
&=\; - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, (\nabla_{X_0}X_\ell)^\perp, \dots, X_{p-1})\\
&=\; - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, \mathcal{K}_2(X_0, X_\ell), \dots, X_{p-1})\\
&=\; -\Big(\big(\mathcal{K}_2(X_0, \cdot)\big)^{(p)}\omega\Big)(X_0, \dots, X_{p-1})\,,
\end{aligned}$$

where the notation $\big(\mathcal{K}_2(X_0, \cdot)\big)^{(p)}\omega$ has been defined at the line following (2.16). Consequently, it holds for any $\omega \in \Lambda^p\mathcal{C}^\infty_{\mathbf{t}}$ and $p-1$ tangential vector fields $X_1, \dots, X_{p-1}$,

$$\begin{aligned}
&-\mathbf{n}\big(\vec{n}^\flat \wedge d^*\omega + \nabla_{\vec{n}}\omega\big)(\vec{n}, X_1, \dots, X_{p-1})\\
&\qquad=\; - \sum_{i=1}^{n-1} \Big(\big(\mathcal{K}_2(E_i, \cdot)\big)^{(p)}\omega\Big)(E_i, X_1, \dots, X_{p-1})\\
&\qquad=\; \big(\mathcal{K}_{\mathbf{t}}^{(p)}\omega\big)(\vec{n}, X_1, \dots, X_{p-1})\,,
\end{aligned}$$

which proves (4.5) and concludes the proof of Theorem 4.1. ∎

We end up this subsection with the proof of Theorem 1.1.

Proof of Theorem 1.1. According to (2.23), (2.20), Lemma 3.1, and to Theorem 4.1, we have just to show the identity

$$\begin{aligned}
\|e^f\omega\|^2_{\Lambda^p\dot{H}^1(e^{-2f}d\mu)} = \|\omega\|^2_{\Lambda^p\dot{H}^1} + \langle(|\nabla f|^2 + \Delta f)\omega, \omega\rangle_{\Lambda^p L^2}&\\
+ \int_{\partial\Omega} \langle\omega, \omega\rangle_{\Lambda^p}\, \partial_n f\; d\mu_{\partial\Omega}\,.& \quad (4.6)
\end{aligned}$$

Let now $(U_j)_{j\in\{1,\dots,K\}}$ be any nice cover of Ω with associated local orthonormal frames $(E_1, \dots, E_n)$ (we drop the dependence on j to lighten the notation) and a subordinated partition of unity $(\rho_j)_{j\in\{1,\dots,K\}}$. We have then the relation (see [23, p. 31] for more details about the H^1 norm):

$$\|e^f\omega\|^2_{\Lambda^p\dot{H}^1(e^{-2f}d\mu)} \;=\; \sum_{j=1}^{K}\sum_{i=1}^{n}\int_{U_j}\rho_j\left\|e^{-f}\nabla_{E_i}(e^f\omega)\right\|^2_{\Lambda^p}d\mu\,.$$

Moreover, the relation $e^{-f}\nabla_{E_i}(\,e^f\cdot\,) = (\nabla_{E_i}f)\,\cdot\, + \nabla_{E_i}(\,\cdot\,)$ implies

$$\left\|e^{-f}\nabla_{E_i}(e^f\omega)\right\|^2_{\Lambda^p} = \left\|\nabla_{E_i}\omega\right\|^2_{\Lambda^p} + \left\|(\nabla_{E_i}f)\omega\right\|^2_{\Lambda^p} + 2\langle\nabla_{E_i}\omega, (\nabla_{E_i}f)\omega\rangle_{\Lambda^p}$$

so according to (4.6), we are simply led to prove that

$$2\sum_{j=1}^{K}\sum_{i=1}^{n}\rho_j\int_{U_j}\langle\nabla_{E_i}\omega,(\nabla_{E_i}f)\omega\rangle_{\Lambda^p}d\mu = \langle(\Delta f)\omega,\omega\rangle_{L^2} + \int_{\partial\Omega}\|\omega\|^2_{\Lambda^p}\,\partial_n f\;d\mu_{\partial\Omega}\,.$$

To conclude, we use the Green's formula (3.8) and the formula (2.30) for the codifferential which give

$$\begin{aligned}\int_{\partial\Omega}\|\omega\|^2_{\Lambda^p}\,\partial_n f\;d\mu_{\partial\Omega} &= -\int_{\Omega}d^*\big(\|\omega\|^2_{\Lambda^p}\,df\big)d\mu\\ &= \sum_{j=1}^{K}\sum_{i=1}^{n}\int_{U_j}\rho_j\,\mathbf{i}_{E_i}\nabla_{E_i}\big(\|\omega\|^2_{\Lambda^p}\,df\big)d\mu\\ &= \sum_{j=1}^{K}\sum_{i=1}^{n}\int_{U_j}\rho_j\Big(2\langle\nabla_{E_i}\omega,\omega\rangle_{\Lambda^p}df(E_i)\\ &\qquad\qquad + \|\omega\|^2_{\Lambda^p}(\nabla_{E_i}df)(E_i)\Big)d\mu\\ &= 2\sum_{j=1}^{K}\sum_{i=1}^{n}\int_{U_j}\rho_j\langle\nabla_{E_i}\omega,(\nabla_{E_i}f)\omega\rangle d\mu - \langle(\Delta f)\omega,\omega\rangle_{L^2}.\end{aligned}$$

This implies (4.6) and then concludes the proof of Theorem 1.1. ∎

4.2 Proof of Theorem 1.2

We first prove 1.i) and then consider $p>0$ and $\omega\in\Lambda^{p-1}H^1_{\mathbf{n}}(d\nu)$ such that $d^*_V\omega = 0$. Let us also consider the corresponding form on the flat space:

$$\eta \;:=\; e^{-f}\,\omega \quad\text{where}\quad f \;:=\; \frac{V}{2}\,.$$

We have then in particular $\eta\,\in\,\Lambda^{p-1}H^1_{\mathbf{n}}$ and $d^*_f\eta = 0$. Note also that $\mathcal{K}^{(p)}_{\mathbf{n}}\geq 0$ and $\mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)}f > 0$ together with Theorem 1.1 imply that

$$\Delta_f^{\mathbf{n},(p)} \;\geq\; \mathrm{Ric}^{(p)} + 2\,\mathrm{Hess}^{(p)}f \;>\; 0 \tag{4.7}$$

and therefore that $0 \in \varrho(\Delta_f^{\mathbf{n},(p)})$. As already explained in the introduction (see (1.2) there), the trick is to use now the following relation which results easily from $d_f^*\eta = 0$, (3.7), (3.5), and (3.6):

$$\begin{aligned} \| \eta - \pi_{f,\mathbf{n}}\eta \|_{L^2}^2 &= \langle (\Delta_f^{\mathbf{n},(p)})^{-1} d_f(\eta - \pi_{f,\mathbf{n}}\eta) , d_f(\eta - \pi_{f,\mathbf{n}}\eta) \rangle_{L^2} \\ &= \langle (\Delta_f^{\mathbf{n},(p)})^{-1} d_f\eta , d_f\eta \rangle_{L^2} , \end{aligned} \tag{4.8}$$

where $\pi_{f,\mathbf{n}} = \pi_{f,\mathbf{n}}^{(p)}$ denotes the orthogonal projection on $\mathrm{Ker}\,(\Delta_f^{\mathbf{n},(p)})$. The estimate to prove involving $\omega = e^f\,\eta$ is then a simple consequence of (4.7) and (4.8) according to the unitary equivalence

$$L_V^{\mathbf{n},(p)} = e^f\,\Delta_f^{\mathbf{n},(p)}\,e^{-f} \quad \text{where} \quad f = \frac{V}{2}\,.$$

The proof of 1.ii) is completely similar as well as the proofs of 2.i) and 2.ii).

4.3 Proof of Corollary 1.4

This proof is similar to the previous one and we only prove it in the normal case, the tangential case being completely analogous. In order to improve the latter result, we want to derive an estimate of the type (4.7) with $\Delta_f^{\mathbf{n},(1)}$ replaced by the self-adjoint unbounded operator $\Delta_f^{\mathbf{n},(1)}\big|_{\mathrm{Ran}\,d_f}$ defined in Proposition 3.3. To do so, we will in particular make use of the nonnegative term $\|e^f\cdot\|^2_{\dot{H}^1(e^{-2f}\,d\mu)}$ of the integration by part formula (1.13) stated in Theorem 1.1.

Let us then consider $\omega \in D(L_V^{\mathbf{n},(0)}) = \{u \in H^2 \cap H^1_{\mathbf{n}}(d\nu) \text{ s.t. } \mathbf{n}du = 0 \text{ on } \partial\Omega\}$ and its corresponding form on the flat space

$$\eta := e^{-f}\omega \quad \text{where} \quad f := \frac{V}{2}$$

which consequently belongs to $D(\Delta_f^{\mathbf{n},(0)})$. Denoting by $(E_1,\dots,E_n)$ a local orthonormal frame on $U \subset \Omega$, we deduce from the Cauchy-Schwarz inequality the following relations satisfied by the integrand of $\|e^f d_f\eta\|^2_{\dot{H}^1(e^{-2f}\,d\mu)}$ a.e. on U and for every N such that $\frac{1}{N} \in [-\infty, \frac{1}{n})$ or $N = n$ if f is constant:

$$\begin{aligned} \sum_{i=1}^n \left\| e^{-f}\nabla_{E_i}(e^f\,d_f\eta)\right\|^2_{\Lambda^1} &\geq \frac{1}{n}\big(e^{-f}\,\Delta^{(0)}\,e^f\eta\big)^2 \\ &= \frac{1}{n}\big(\Delta_f^{(0)}\eta - 2\,\langle df, d_f\eta\rangle_{\Lambda^1}\big)^2 \\ &\geq \frac{1}{N}\big(\Delta_f^{(0)}\eta\big)^2 - \frac{4}{N-n}\,df\otimes df(d_f\eta, d_f\eta)\,. \end{aligned}$$

This implies, after integration on Ω:

$$\|e^f d_f\eta\|^2_{\dot{H}^1(e^{-2f}d\mu)} + \frac{4}{N-n}\int_\Omega df\otimes df(d_f\eta, d_f\eta)\; d\mu$$
$$\geq\; \frac{1}{N}\|\Delta_f^{(0)}\eta\|^2_{L^2} \;=\; \frac{1}{N}\mathcal{D}_f^{(1)}(d_f\eta)\,. \tag{4.9}$$

Moreover, $\mathcal{K}_{\mathbf{n}}^{(1)} \geq 0$ and

$$\mathrm{Ric}_{2f,N} \;:=\; \mathrm{Ric} \,+\, 2\,\mathrm{Hess}\, f \,-\, \frac{4}{N-n}\, df\otimes df \;>\; 0$$

together with Theorem 1.1 and (4.9) imply that

$$(1-\frac{1}{N})\Delta_f^{\mathbf{n},(1)}\Big|_{\mathrm{Ran}\, d_f} \;\geq\; \mathrm{Ric}_{2f,N} \;>\; 0\,. \tag{4.10}$$

The estimate to prove is then a simple consequence of (4.10) and of the relation

$$\|\,\eta-\pi_{f,\mathbf{n}}\eta\,\|^2_{L^2} = \langle\,(\Delta_f^{\mathbf{n},(1)}\Big|_{\mathrm{Ran}\, d_f})^{-1}\, d_f(\eta-\pi_{f,\mathbf{n}}\eta)\,,\, d_f(\eta-\pi_{f,\mathbf{n}}\eta)\,\rangle_{L^2} \tag{4.11}$$

valid for any $\eta\in\Lambda^0H^1_{\mathbf{n}} = H^1(\Omega)$ and resulting from (3.7), (3.5), and (3.6).

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