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Abstract: Let X_n be a d dimensional Poisson point process of intensity n. We prove that the expected length of the Voronoi path between two points at distance 1 in the Delaunay triangulation associated with X_n is $\sqrt{\frac{2d}{\pi}} + O(d^{-\frac{1}{2}})$ for all $n \in \mathbb{N}$ and $d \to \infty$. In any dimension, we provide a precise interval containing the exact value, in 3D the expected length is between 1.4977 and 1.50007.

Key-words: Probabilistic analysis – Worst-case analysis – Walking algorithms

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Longueur moyenne de la marche de Vornoi dans une triangulation de Poisson-Delaunay en dimension d.

Résumé : Soit X_n un processus ponctuel de Poisson d'intensité n en dimension d. Nous démontrons que l'espérance de la longueur du chemin de Voronoi entre l'origine et un point à distance 1 dans la triangulation de Delaunay de X_n est $\sqrt{\frac{2d}{\pi}} + O(d^{-\frac{1}{2}})$ pour tout $n \in \mathbb{N}$ quand $d \to \infty$. Nous donnons des bornes inférieures et supérieures sur la bonne valeur en toute dimension, en 3D ces bornes sont 1.4977 et 1.50007.

Mots-clés : Analyse probabiliste – Analyse dans le cas le pire – Algorithmes de marche

1 Introduction

Finding paths in a Delaunay triangulation is a classical problem in computational geometry [7]. In the context of random points, several kind of paths have been studied in two dimensions such as straight walk [2, 8], cone walk [3], visibility walk [5], the shortest path [4], or the Voronoi path [1].

In this paper we take interest in the stretch ratio of a particular path in the Delaunay triangulation – the Voronoi path – and study its expected length in dimension d when the point set is a Poisson point process of density n with n going to infinity. The Voronoi path links the seeds of the Voronoi regions intersected by a line segment. An illustration for dimension 2 is given in Figure 1. The main result of this paper is the computation of upper and lower bound on the expected length of the Voronoi path. These bounds show that the asymptotic behavior of the length is $\sqrt{\frac{2d}{\pi}} + O(d^{-\frac{1}{2}})$. Table 1 provides the values of our bounds for small dimensions aside the exact value obtained from numerical integration.

Obviously, the length of the Voronoi path give an upper bound on the length of the shortest path and provide an upper bound on the expected stretch ratio of long walk in the Delaunay triangulation of a random point set.

Previous results provide values only for dimension two, where the expected length of the Voronoi path is $\frac{4}{\pi} \simeq 1.27$ and its variance is $O(n^{-\frac{1}{2}})$ [1, 6].

2 Voronoi Path

We define the Voronoi path VP_χ as the list of closest neighbors in a set of points χ of a point moving linearly from $s = (0, 0, 0)$ to $t = (1, 0, 0)$. This path is x-monotone, it starts at the closest neighbor of s and reaches the closest neighbor of t. It uses a sequence of edges of DT_χ the Delaunay triangulation of χ.

Notice that if somebody wants a path that goes actually from s to t, he can consider $VP_{\chi\cup\{s,t\}}$. This path differs from VP_χ only by few edges around s and around t.

An edge p_1p_2 belongs to VP_χ iff the unique ball $B_x(p_1, p_2)$ centered on the x axis passing through p_1 and p_2 is centered on the segment $[st]$ and does not contain any other points of χ (see Figure 1). Thus, denoting $M(p_1, p_2)$ the center of $B_x(p_1, p_2)$, we can write the length of the

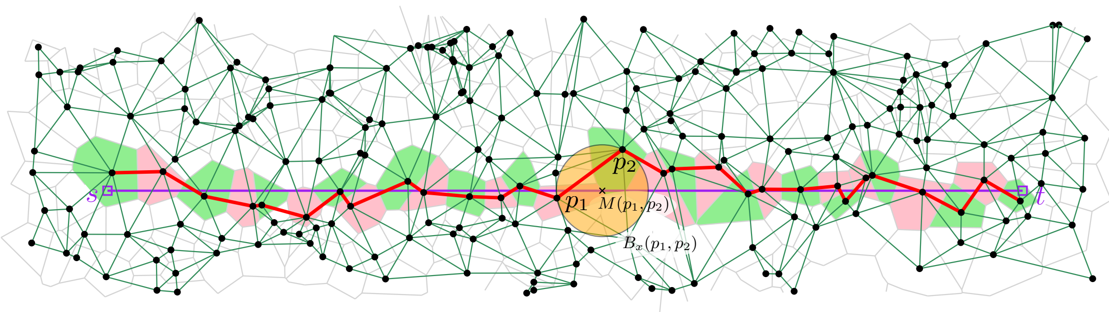


Figure 1: The Voronoi path.

dimension	1	2	3	4	5	6	7	8	$d \to \infty$
lower bound			1.497	1.682	1.835	1.918	2.077	2.224	$\sim \sqrt{\frac{2d}{\pi}}$
exact value	1	$\frac{4}{\pi} \simeq 1.27^{\dagger}$	$1.500^{\star}$	$1.698^{\star}$	$1.875^{\star}$	$2.04^{\star}$	$2.2^{\star}$	$2.3^{\star}$	$\sim \sqrt{\frac{2d}{\pi}}$
upper bound			1.5001	1.699	1.881	2.078	2.225	2.364	$\sim \sqrt{\frac{2d}{\pi}}$

† [1] ⋆ obtained from numerical integration

Table 1: Lower and upper bounds for the expected length of the Voronoi path.

Voronoi path as

$$\ell(VP_{\mathcal{X}}) = \frac{1}{2} \sum_{(p_1,p_2)\in\mathcal{X}^2} \mathbb{1}_{[B_x(p_1,p_2)\cap\mathcal{X}=\emptyset]} \mathbb{1}_{[M(p_1,p_2)\in[st]]} \|p_1p_2\|$$

where the $\frac{1}{2}$ arise because each edge is counted twice, once for each orientation. Now we turn our interest to the case where the point set is X_n a Poisson point process of intensity n. Using Slivnyak-Mecke formula [9, Theorem 3.3.5], we have:

$$\mathbb{E}\left[\ell(VP_{X_n})\right] = \frac{n^2}{2} \int_{(\mathbb{R}^3)^2} \mathbb{P}\left[B_x(p_1,p_2)\cap X_n = \emptyset\right] \mathbb{1}_{[M(p_1,p_2)\in[st]]} \|p_1p_2\| dp_1 dp_2. \qquad (1)$$

The path $VP_{X_n\cup\{s,t\}}$ has about the same length, actually $\mathbb{E}\left[\left|\ell(VP_{X_n}) - \ell(VP_{X_n\cup\{s,t\}})\right|\right] = O\left(n^{-\frac{1}{d}}\right)$ can be proven as an easy generalization from dimension two [6].

3 Voronoi Path in Dimension 3

We start by illustrating our method in three dimension.

Lemma 1. *In dimension 3*

$$\begin{aligned} 1.497706663 \simeq \tfrac{788984278470257640690697143}{745000536337515228912680960}\sqrt{2} \\ \leq \mathbb{E}\left[\ell(VP_{X_n})\right] \leq \tfrac{452337036471251065807696350 9}{4264485828690604413776035840}\sqrt{2} \simeq 1.500066356. \end{aligned}$$

Proof. The integral at Equation (1) is computed by substitution. The points p_1 and p_2 are defined by their sphere $B_x(p_1,p_2)$ and their spherical coordinates on that sphere. Let Φ be the function

$$\begin{array}{llll} \Phi: & \mathbb{R}\times\mathbb{R}_+\times([0,\pi)\times[0,2\pi))^2 & \longrightarrow & (\mathbb{R}^3)^2 \\ & (x,r,\alpha_1,\beta_1,\alpha_2,\beta_2) & \longmapsto & (p_1,p_2), \end{array}$$

with

$$p_i = \begin{pmatrix} x + r\cos\alpha_i \\ r\sin\alpha_i\cos\beta_i \\ r\sin\alpha_i\sin\beta_i \end{pmatrix} = (x,0,0) + ru_i \qquad \text{with} \quad u_i = \begin{pmatrix} \cos\alpha_i \\ \sin\alpha_i\cos\beta_i \\ \sin\alpha_i\sin\beta_i \end{pmatrix}.$$

Φ is a C^1-diffeomorphism up to a null set. Its Jacobian J_Φ has as determinant:

$$\begin{aligned} \det(J_\Phi) &= \begin{vmatrix} 1 & \cos\alpha_1 & -r\sin\alpha_1 & 0 & 0 & 0 \\ 0 & \sin\alpha_1\cos\beta_1 & r\cos\alpha_1\cos\beta_1 & -r\sin\alpha_1\sin\beta_1 & 0 & 0 \\ 0 & \sin\alpha_1\sin\beta_1 & r\cos\alpha_1\sin\beta_1 & r\sin\alpha_1\cos\beta_1 & 0 & 0 \\ 1 & \cos\alpha_2 & 0 & 0 & -r\sin\alpha_2 & 0 \\ 0 & \sin\alpha_2\cos\beta_2 & 0 & 0 & r\cos\alpha_2\cos\beta_2 & -r\sin\alpha_2\sin\beta_2 \\ 0 & \sin\alpha_2\sin\beta_2 & 0 & 0 & r\cos\alpha_2\sin\beta_2 & r\sin\alpha_2\cos\beta_2 \end{vmatrix} \\ &= r^4\sin\alpha_1\sin\alpha_2(\cos\alpha_1 - \cos\alpha_2) \end{aligned}$$

Then we substitute the new variables:[1]

$$
\begin{aligned}
\mathbb{E}\left[\ell(VP_{X_n})\right] &= \frac{n^2}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\mathbb{P}\left[B((x,0,0),r)\cap X_n=\emptyset\right]\mathbb{1}_{[(x,0,0)\in[st]]}\\
&\qquad\cdot r\|u_1u_2\|\,|\det(J_\Phi)|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}r\mathrm{d}x\\
&= \frac{n^2}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-n\frac{4}{3}\pi r^3}\mathbb{1}_{[x\in[0,1]]}\cdot r\|u_1u_2\|\\
&\qquad\cdot r^4\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}r\mathrm{d}x\\
&= \frac{n^2}{2}\left(\int_0^1 dx\right)\left(\int_0^{\infty}e^{-n\frac{4}{3}\pi r^3}r^5\mathrm{d}r\right)\\
&\qquad\cdot\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot\|u_1u_2\|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&= \frac{n^2}{2}\,\frac{3}{16n^2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot\|u_1u_2\|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \qquad (2)
\end{aligned}
$$

Eq.(7)

Using the trivial bound $\|u_1u_2\|\leq 2$ we get

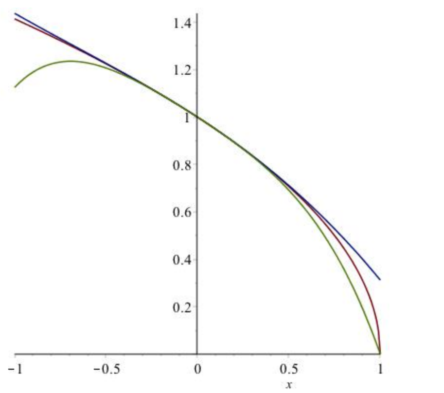
$$
\begin{aligned}
&\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot 2\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&= \left(2\int_0^{\pi}\int_0^{\alpha_2}\sin\alpha_1\sin\alpha_2\left(\cos\alpha_1-\cos\alpha_2\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right)\cdot\left(\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\beta_1\mathrm{d}\beta_2\right)\\
&= \frac{8}{3}\cdot 4\pi^2=\frac{32\pi^2}{3}.
\end{aligned}
$$

Eq.(9)

Plugging this bound in Equation (2) gives $\mathbb{E}\left[\ell(VP_{X_n})\right]\leq 2$.

We can improve on this result. Expressing $\|u_1u_2\|$ in terms of the spherical coordinates we have

$$
\begin{aligned}
\|u_1u_2\| &= \sqrt{(u_1-u_2)^2}=\sqrt{u_1^2+u_2^2-2u_1\cdot u_2}=\sqrt{2-2u_1\cdot u_2}\\
&= \sqrt{2}\sqrt{1-\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2\cos(\beta_1-\beta_2)} \qquad (3)\\
&= \sqrt{2}\sqrt{1-\delta} \qquad \text{with } \delta=\left(\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos(\beta_1-\beta_2)\right).
\end{aligned}
$$



We can use Taylor expansion to bound $\|u_1u_2\|$. If $T_k(y)$ is the Taylor expansion up to degree k of $\sqrt{1-y}$, for k odd and $y\in[-1,1]$ we have

$$T_k(y)-T_k(1)y^{k+1}\leq\sqrt{1-y}\leq T_k(y). \qquad (4)$$

The side figure illustrates this for $k=3$:

$$1-\frac{y}{2}-\frac{y^2}{8}-\frac{y^3}{16}-\frac{5y^4}{16}\leq\sqrt{1-y}\leq 1-\frac{y}{2}-\frac{y^2}{8}-\frac{y^3}{16}.$$

One can compute the following integrals (e.g., using Maple):

[1] Useful integrals on exponential or trigonometric functions are given in appendix.

$$2\int_0^{\pi}\int_0^{\alpha_2}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|T_{41}(\delta)\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2$$
$$=\tfrac{262994759490085880230232381}{232812667605473509035212280}\pi^2$$
$$2\int_0^{\pi}\int_0^{\alpha_2}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\left(T_{41}(\delta)-T_{41}(1)\delta^{42}\right)\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2$$
$$=\tfrac{150779012157083688602565450 3}{133265182146581387930501120}\pi^2.$$

Plugging these values in Equation (2) using Equation (3) and (4) yields the bounds announced in the lemma statement. □

A numerical evaluation of the integral at Equation (2) gives a value in that interval, pretty close to $\frac{3}{2}$ and we conjecture that $\frac{3}{2}$ is the correct value.

4 Voronoi Path in Dimension d

Theorem 2. *In dimension d, the expected length of the Voronoi path is bounded by:*

$$\frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1-\frac{d-1}{4d^2-1}\right)\sqrt{2}\leq\mathbb{E}\left[\ell(VP_{X_n})\right]\leq\frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1+\frac{1}{4d-2}\right)\sqrt{2}$$

and the behavior when $d\to\infty$ is:

$$\sqrt{\frac{2d}{\pi}}-\frac{1}{4\sqrt{2d\pi}}+O\left(d^{-\frac{3}{2}}\right)\leq\mathbb{E}\left[\ell(VP_{X_n})\right]\leq\sqrt{\frac{2d}{\pi}}+\frac{3}{4\sqrt{2d\pi}}+O\left(d^{-\frac{3}{2}}\right)$$

The rest of the section is devoted to the proof of Theorem 2.

As in dimension 3, we compute by substitution the integral in Equation (1) defining the points p_1 and p_2 by their sphere $B_x(p_1,p_2)$ and their spherical coordinates on that sphere. Let Φ be the function

$$\begin{array}{rccc}\Phi: & \mathbb{R}\times\mathbb{R}_+\times\left([0,\pi]^{d-1}\times[0,2\pi)\right)^2 & \longrightarrow & (\mathbb{R}^d)^2\\ & (x,r,\alpha_{1,1},\ldots,\alpha_{1,d-1},\alpha_{2,1},\ldots,\alpha_{2,d-1}) & \longmapsto & (p_1,p_2),\end{array}$$

with

$$p_i=\begin{pmatrix}x\\0\\0\\\vdots\\0\\\vdots\\0\\0\end{pmatrix}+ru_i\qquad\text{with}\quad u_i=\begin{pmatrix}\cos\alpha_{i,1}\\\sin\alpha_{i,1}\cos\alpha_{i,2}\\\sin\alpha_{i,1}\sin\alpha_{i,2}\cos\alpha_{i,3}\\\vdots\\\left(\prod_{l=1}^{j-1}\sin\alpha_{i,l}\right)\cos\alpha_{i,j}\\\vdots\\\left(\prod_{l=1}^{d-2}\sin\alpha_{i,l}\right)\cos\alpha_{i,d-1}\\\left(\prod_{l=1}^{d-2}\sin\alpha_{i,l}\right)\sin\alpha_{i,d-1}\end{pmatrix}.$$

Φ is a C^1-diffeomorphism up to a null set. Its Jacobian J_Φ has as determinant:

$$\det(J_\Phi)\quad=\quad r^{2(d-1)}\prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i})\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|.$$

$$
\begin{aligned}
\mathbb{E}\left[\ell(VP_{X_n})\right] &= \tfrac{n^2}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{(\mathbb{S}_{d-1})^2} \mathbb{P}\left[B((x,0,\ldots,0),r)\cap X_n=\emptyset\right]\mathbb{1}_{[(x,0,\ldots,0)\in[st]]}\\
&\qquad \cdot r\|u_1u_2\||\det(J_\Phi)|\mathrm{d}\alpha_{1,1}\ldots\mathrm{d}\alpha_{1,d-1}\mathrm{d}\alpha_{2,1}\ldots\mathrm{d}\alpha_{2,d-1}\mathrm{d}r\mathrm{d}x\\
&= \tfrac{n^2}{2}\int_0^1 \mathrm{d}x \times \int_0^{\infty} e^{-n\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}r^d}\ \boxed{\text{volume in } \mathbb{S}_{d-1}}\ r^{2d-1}\mathrm{d}r\\
&\quad \times \int_{(\mathbb{S}_{d-1})^2}\|u_1u_2\|\prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i})\\
&\qquad \cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1}\\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\ \boxed{\text{Eq.(7)}}\\
&\quad \times \int_{(\mathbb{S}_{d-1})^2}\|u_1u_2\|\prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i}) \qquad (5)\\
&\qquad \cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1}
\end{aligned}
$$

We use the Taylor expansion of Equation 4 with $k=1$ to bound $\|u_1u_2\|$:

$$\sqrt{2}\left(1-\frac{\delta}{2}-\frac{\delta^2}{2}\right)\le\|u_1u_2\|\le\sqrt{2}\left(1-\frac{\delta}{2}\right) \qquad (6)$$

with

$$\delta=u_1\cdot u_2=\sum_{i=1}^{d-2}\left(\prod_{j=1}^{i-1}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cos\alpha_{1,i}\cos\alpha_{2,i}+\left(\prod_{j=1}^{d-2}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cdot\cos(\alpha_{1,d-1}-\alpha_{2,d-1}).$$

We write δ as a sum of three terms $\delta=\delta_A+\delta_B+\delta_C$ with

$$
\begin{aligned}
\delta_A &= \left(\prod_{j=1}^{d-2}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cdot\cos(\alpha_{1,d-1}-\alpha_{2,d-1}),\\
\delta_B &= \sum_{i=2}^{d-2}\left(\prod_{j=1}^{i-1}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cos\alpha_{1,i}\cos\alpha_{2,i},\\
\delta_C &= \cos\alpha_{1,1}\cos\alpha_{2,1}.
\end{aligned}
$$

Replacing $\|u_1u_2\|$ by its bounds in Equation (5) yields the computation of the following integral for $k\in\{0,1,2\}$:

$$
\begin{aligned}
\mathcal{I}_k=\int_{(\mathbb{S}_{d-1})^2}\delta^k\prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i})\\
\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1}
\end{aligned}
$$

4.1 Computation of $\mathcal{I}_0$

$$
\begin{aligned}
\mathcal{I}_0 &= \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\alpha_{1,d-1}\mathrm{d}\alpha_{2,d-1} \times \\
&\quad \prod_{i=2}^{d-2}\int_0^{\pi} \sin^{d-i-1}(\alpha_{1,i})\mathrm{d}\alpha_{1,i} \times \prod_{i=2}^{d-2}\int_0^{\pi} \sin^{d-i-1}(\alpha_{2,i})\mathrm{d}\alpha_{2,i} \times \\
&\quad \int_0^{\pi}\int_0^{\pi} \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1} \\
&= 4\pi^2 \times \left(\prod_{i=2}^{d-2}\frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)}\right)^2 \pi^{d-3} \quad \text{[Eq.(8)]} \\
&\quad \times \int_0^{\pi}\int_0^{\pi} \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1} \\
&= 4\pi^2 \times \frac{1}{\Gamma\left(\frac{d-1}{2}\right)^2} \ \text{[telescoping]} \times \pi^{d-3} \times \frac{2^{2d}((d-2)!)^2}{(2d-2))!} \ \text{[Eq.(9)]} \quad = \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2(2d-2))!}
\end{aligned}
$$

4.2 Computation of $\mathcal{I}_1$

Decomposing on the different terms of δ, we can write $\mathcal{I}_1 = \mathcal{I}_1^A + \mathcal{I}_1^B + \mathcal{I}_1^C$.

$$
\begin{aligned}
\mathcal{I}_1^A &= \int_{(\mathbb{S}_{d-1})^2} \left(\prod_{j=1}^{d-2}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cdot\cos(\alpha_{1,d-1}-\alpha_{2,d-1})\left(\prod_{j=2}^{d-2}\sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right) \\
&\quad \cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1} \\
\mathcal{I}_1^B &= \sum_{i=2}^{d-2}\int_{(\mathbb{S}_{d-1})^2}\left(\prod_{j=1}^{i-1}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cos\alpha_{1,i}\cos\alpha_{2,i}\left(\prod_{j=2}^{d-2}\sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right) \\
&\quad \cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1} \\
\mathcal{I}_1^C &= \int_{(\mathbb{S}_{d-1})^2}\cos\alpha_{1,1}\cos\alpha_{2,1}\left(\prod_{j=2}^{d-2}\sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right) \\
&\quad \cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1}
\end{aligned}
$$

We have $\mathcal{I}_1^A = 0$ since integrating over $\alpha_{1,d-1}$ and $\alpha_{2,d-1}$ create a null factor according to Equation (10). We also have $\mathcal{I}_1^B = 0$ since integrating over $\alpha_{1,i}$ create a null factor in each term of the sum using Equation (11).

To compute $\mathcal{I}_1^C$, we can integrate all variables different from $\cos\alpha_{1,1}\cos\alpha_{2,1}$ in the same way as we have done for computing $\mathcal{I}_0$ and get

$$
\begin{aligned}
\mathcal{I}_1 = \mathcal{I}_1^C &= \frac{4\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)^2} \\
&\quad \times \int_0^{\pi}\int_0^{\pi}\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cos(\alpha_{1,1})\cos(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1}
\end{aligned}
$$

Then using Equation (12) one can finally compute $\mathcal{I}_1$. Observing that the result of Equation (12) is the same as Equation (9) up to a factor $-(2d-1)$ we get

$$\mathcal{I}_1 = -\frac{1}{2d-1}\mathcal{I}_0$$

4.3 The Upper Bound

Using the values of $\mathcal{I}_0$ and $\mathcal{I}_1$ in Equation (5), we get

$$\begin{aligned}
\mathbb{E}\left[\ell(VP_{X_n})\right] &\leq \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\sqrt{2}\left(\mathcal{I}_0 - \tfrac{1}{2}\mathcal{I}_1\right) \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\mathcal{I}_0\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\frac{4\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)^2}\frac{2^{2d}((d-2)!)^2}{(2d-2))!}\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \sqrt{\frac{2d}{\pi}} + \frac{3}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right) \qquad \text{when } d\to\infty
\end{aligned}$$

4.4 Computation of $\mathcal{I}_2$

We can easily get an upper bound of integral $\mathcal{I}_2$ by noticing that $\delta^2 \leq 1$ and thus $\mathcal{I}_2 \leq \mathcal{I}_0$, but this yields an unsatisfactory lower bound for the length of the shortest path. So we compute $\mathcal{I}_2$ exactly.

The integral $\mathcal{I}_2$ can be split in 6 terms according to the development of δ^2:

$$\delta^2 = \delta_A^2 + \delta_B^2 + \delta_C^2 + 2\delta_A\delta_B + 2\delta_A\delta_C + 2\delta_B\delta_C,$$

as for the computation of $\mathcal{I}_1$, because of Equations 10 and 11, the three terms corresponding to $\delta_A\delta_B$, $\delta_A\delta_C$, and $\delta_B\delta_C$ yields null integrals. Thus $\mathcal{I}_2 = \mathcal{I}_2^A + \mathcal{I}_2^B + \mathcal{I}_2^C$, where

$$
\begin{aligned}
\mathcal{I}_2^A &= \int_{(\mathbb{S}_{d-1})^2} \left(\left(\prod_{j=1}^{d-2} \sin\alpha_{1,j} \sin\alpha_{2,j} \right) \cdot \cos(\alpha_{1,d-1} - \alpha_{2,d-1}) \right)^2 \left(\prod_{j=2}^{d-2} \sin^{d-j-1}(\alpha_{1,j}) \sin^{d-j-1}(\alpha_{2,j}) \right) \\
&\qquad \cdot \sin^{d-2}(\alpha_{1,1}) \sin^{d-2}(\alpha_{2,1}) \cdot |\cos\alpha_{1,1} - \cos\alpha_{2,1}| \mathrm{d}\alpha_{1:2,1:d-1} \\
&= \int_{(\mathbb{S}_{d-1})^2} \left(\prod_{i=2}^{d-2} \sin^{d-i+1}(\alpha_{1,i}) \sin^{d-i+1}(\alpha_{2,i}) \right) \cos^2(\alpha_{1,d-1} - \alpha_{2,d-1}) \\
&\qquad \cdot \left(\sin^d(\alpha_{1,1}) \sin^d(\alpha_{2,1}) |\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})| \right) \mathrm{d}\alpha_{1:2,1:d-1} \\
&= \left(\prod_{i=2}^{d-2} \int_0^\pi \sin^{d-i+1}(\alpha_{1,i}) \mathrm{d}\alpha_{1,i} \right)^2 \\
&\qquad \cdot \int_0^{2\pi} \int_0^{2\pi} \cos^2(\alpha_{1,d-1} - \alpha_{2,d-1}) \mathrm{d}\alpha_{2,d-1} \mathrm{d}\alpha_{1,d-1} \\
&\qquad \cdot \int_0^\pi \int_0^\pi \sin^d(\alpha_{1,1}) \sin^d(\alpha_{2,1}) |\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})| \mathrm{d}\alpha_{2,1} \mathrm{d}\alpha_{1,1} \\
&= \left(\prod_{i=2}^{d-2} \frac{\Gamma\left(\frac{d-i}{2}+1\right)}{\Gamma\left(\frac{d-i+1}{2}+1\right)} \right)^2 \pi^{d-3} \underbrace{}_{\text{Eq.(8)}} \cdot 2\pi^2 \underbrace{}_{\text{Eq.(13)}} \cdot \frac{2^{2d+4} d!^2}{(2d+2)!} \underbrace{}_{\text{Eq.(9)}} \\
&= \frac{2\pi^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2} \underbrace{}_{\text{telescoping}} \cdot \frac{2^{2d+4}(d!)^2}{(2d+2)!} = \frac{2^{2d+5}\pi^{d-1}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{2}^{B}. \quad &= \int_{(\mathbb{S}_{d-1})^2} \left(\sum_{i=2}^{d-2} \left(\prod_{j=1}^{i-1} \sin\alpha_{1,j} \sin\alpha_{2,j} \right) \cos\alpha_{1,i} \cos\alpha_{2,i} \right)^2 \left(\prod_{j=2}^{d-2} \sin^{d-j-1}(\alpha_{1,j}) \sin^{d-j-1}(\alpha_{2,j}) \right) \\
&\quad \cdot \sin^{d-2}(\alpha_{1,1}) \sin^{d-2}(\alpha_{2,1}) \cdot |\cos\alpha_{1,1} - \cos\alpha_{2,1}| \mathrm{d}\alpha_{1:2,1:d-1} \\
&= \int_{(\mathbb{S}_{d-1})^2} \left(\prod_{i=2}^{d-2} \sin^{d-i-1}(\alpha_{1,i}) \sin^{d-i-1}(\alpha_{2,i}) \times \sum_{i=2}^{d-2} \left(\prod_{j=2}^{i-1} \sin^2(\alpha_{1,j}) \sin^2(\alpha_{2,j}) \right) \cos^2(\alpha_{1,i}) \cos^2(\alpha_{2,i}) \right) \\
&\quad \times \left(\sin^d(\alpha_{1,1}) \sin^d(\alpha_{2,1}) |\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})| \right) \mathrm{d}\alpha_{1:2,1:d-1} \\
&\quad \text{since all terms where } \cos\alpha_{1,i} \text{ is not squared have null integral by Equation (11)} \\
&= \left(\sum_{i=2}^{d-2} \left(\prod_{j=2}^{i-1} \int_0^\pi \sin^{d-j+1}(\alpha_{1,j}) \mathrm{d}\alpha_{1,j} \right)^2 \right. \\
&\qquad \left. \left(\int_0^\pi \sin^{d-i-1}(\alpha_{1,i}) \cos^2(\alpha_{1,i}) \mathrm{d}\alpha_{1,i} \right)^2 \left(\prod_{j=i+1}^{d-2} \int_0^\pi \sin^{d-j-1}(\alpha_{1,j}) \mathrm{d}\alpha_{1,j} \right)^2 \right) \\
&\quad \times \int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\alpha_{2,d-1} \mathrm{d}\alpha_{1,d-1} \times \int_0^\pi \int_0^\pi \sin^d(\alpha_{1,1}) \sin^d(\alpha_{2,1}) |\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})| \mathrm{d}\alpha_{2,1} \mathrm{d}\alpha_{1,1} \\
&= \left(\sum_{i=2}^{d-2} \left(\prod_{j=2}^{i-1} \frac{\Gamma\left(\frac{d-j+2}{2}\right)}{\Gamma\left(\frac{d-j+3}{2}\right)} \right)^2 \underset{\text{Eq.(8)}}{} \left(\frac{1}{d-i+1} \cdot \frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)} \right)^2 \underset{\text{Eq.(14)}}{} \right. \\
&\qquad \left. \left(\prod_{j=i+1}^{d-2} \frac{\Gamma\left(\frac{d-j}{2}\right)}{\Gamma\left(\frac{d-j+1}{2}\right)} \right)^2 \underset{\text{Eq.(8)}}{} \right) \pi^{d-3} \times 4\pi^2 \times \underset{\text{Eq.(12)}}{\frac{2^{2d+4} d!^2}{(2d+2)!}} \\
&= \left(\frac{1}{\Gamma\left(\frac{d+1}{2}\right)^2} \sum_{i=3}^{d-1} \left(\frac{\Gamma\left(1+\frac{i}{2}\right)}{i\Gamma\left(\frac{i}{2}\right)} \right)^2 \right) \underset{\text{telescoping}}{} \times 4\pi^{d-1} \frac{2^{2d+4} d!^2}{(2d+2)!} \\
&= \left(\frac{1}{\Gamma\left(\frac{d+1}{2}\right)^2} \sum_{i=3}^{d-1} \left(\frac{1}{2} \right)^2 \right) \underset{\Gamma \text{ properties}}{} \times 4\pi^{d-1} \frac{2^{2d+4} d!^2}{(2d+2)!} \\
&= \left(\frac{(d-3)/4}{\Gamma\left(\frac{d+1}{2}\right)^2} \right) \times 4\pi^{d-1} \frac{2^{2d+4} d!^2}{(2d+2)!} = \frac{\pi^{d-1}(d-3) 2^{2d+4} (d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_2^C &= \int_{(\mathbb{S}_{d-1})^2} (\cos\alpha_{1,1}\cos\alpha_{2,1})^2 \left(\prod_{i=2}^{d-2} \sin^{d-i-1}(\alpha_{1,j}) \sin^{d-i-1}(\alpha_{2,i})\right) \\
&\qquad \cdot \sin^{d-2}(\alpha_{1,1}) \sin^{d-2}(\alpha_{2,1}) \cdot |\cos\alpha_{1,1} - \cos\alpha_{2,1}| \mathrm{d}\alpha_{1:2,1:d-1} \\
&= \left(\prod_{i=2}^{d-2} \int_0^\pi \sin^{d-i-1}\alpha_{1,i} \mathrm{d}\alpha_{1,i}\right)^2 \times \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\alpha_{2,d-1}\mathrm{d}\alpha_{1,d-1} \\
&\quad \times \int_0^\pi \int_0^\pi \sin^{d-2}(\alpha_{1,1}) \sin^{d-2}(\alpha_{2,1}) \cos^2(\alpha_{1,1}) \cos^2(\alpha_{2,1}) |\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})| \mathrm{d}\alpha_{2,1}\mathrm{d}\alpha_{1,1} \\
&= \left(\prod_{i=2}^{d-2} \frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)}\right)^2 \pi^{d-3} \times 4\pi^2 \times \frac{2^{2d+2}(7d-1)d!^2}{(d-1)^2 d(2d+2)!} \\
&= \frac{4\pi^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2} \cdot \frac{2^{2d+4}(d!)^2}{(2d+2)!} \cdot \frac{7d-1}{16d} = \frac{2^{2d+2}\pi^{d-1}(d!)^2(7d-1)}{d\,\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!}.
\end{aligned}
$$

Eq.(8) Eq.(15) telescoping

Summing the three terms together and simplifying gives:

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{2^{2d+5}\pi^{d-1}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!} + \frac{\pi^{d-1}(d-3)2^{2d+4}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!} + \frac{2^{2d+2}\pi^{d-1}(d!)^2(7d-1)}{d\,\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!} \\
&= \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2 (2d-2))!} \cdot \frac{d^2(d-1)^2}{\left(\frac{d-1}{2}\right)^2 (2d+2)(2d+1)2d(2d-1)} \left(8 + 4(d-3) + \frac{7d-1}{d}\right) \\
&= \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2 (2d-2))!} \cdot \frac{4d^2}{(2d+2)(2d+1)2d(2d-1)} \cdot \frac{(d+1)(4d-1)}{d} \\
&= \mathcal{I}_0 \cdot \frac{4d-1}{4d^2-1}
\end{aligned}
$$

4.5 The Lower Bound

Using the values of $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$ in Equation (5), we get

$$
\begin{aligned}
\mathbb{E}\left[\ell(VP_{X_n})\right] &\geq \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\sqrt{2}\left(\mathcal{I}_0 - \tfrac{1}{2}\mathcal{I}_1 - \tfrac{1}{2}\mathcal{I}_2\right) \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\mathcal{I}_0\left(1 + \frac{1}{4d-2} - \frac{4d-1}{4d^2-1}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1 - \frac{d-1}{4d^2-1}\right)\sqrt{2} \\
&= \sqrt{\frac{2d}{\pi}} - \frac{1}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right) \qquad \text{when } d \to \infty
\end{aligned}
$$

d	k	lower bound	$\simeq$	exact value	upper bound	$\simeq$
3	41	$\frac{788984278470257640690697143}{745000536337515228912680960}\sqrt{2}$	1.49770	1.500	$\frac{4523370364712510658076963509}{4264485828690604413776035840}\sqrt{2}$	1.50007
4	7	$\frac{102494570}{8729721}\frac{\sqrt{2}}{\pi^2}$	1.6823	1.698	$\frac{121774997}{10270260}\frac{\sqrt{2}}{\pi^2}$	1.6990
5	3	$\frac{135}{104}\sqrt{2}$	1.8357	1.875	$\frac{21305}{16016}\sqrt{2}$	1.8812
6	1	$\frac{3014656}{225225}\frac{\sqrt{2}}{\pi^2}$	1.9179	2.04	$\frac{753664}{51975}\frac{\sqrt{2}}{\pi^2}$	2.0778
7	1	$\frac{210}{143}\sqrt{2}$	2.0768	2.2	$\frac{225}{143}\sqrt{2}$	2.2252
8	1	$\frac{2080374784}{134008875}\frac{\sqrt{2}}{\pi^2}$	2.2244	2.3	$\frac{130023424}{7882875}\frac{\sqrt{2}}{\pi^2}$	2.3635

Table 2: Lower and upper bounds for the expected length of the Voronoi path using Taylor expansion of order k and exact values from numerical integration.

5 Small Dimensions

For d small and similarly to what we have done in dimension 3, we can use symbolic computation to get better bounds using higher order Taylor expansions, and to get idea of the exact value using numerical integration. Results are in Table 2, Maple code is available with this report.

A Useful Integrals

Proofs are below.

$$\int_0^\infty e^{-cr^d} r^{2d-1}\mathrm{d}r = \frac{1}{d\cdot c^2}. \tag{7}$$

$$\int_0^\pi \sin^d\alpha\,\mathrm{d}\alpha = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1+\frac{d}{2}\right)}\cdot\sqrt{\pi}. \tag{8}$$

$$\int_0^\pi\int_0^\pi \sin^d\alpha_1\sin^d\alpha_2|\cos\alpha_1-\cos\alpha_2|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = \frac{2^{2d+4}d!^2}{(2d+2)!}. \tag{9}$$

$$\int_0^{2\pi}\int_0^{2\pi}\cos\left(\alpha_1-\alpha_2\right)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 = 0. \tag{10}$$

$$\int_0^\pi \sin^d\alpha\cos\alpha\mathrm{d}\alpha = 0. \tag{11}$$

$$\int_0^\pi\int_0^\pi \sin^d\alpha_1\sin^d\alpha_2|\cos\alpha_1-\cos\alpha_2|\cos\alpha_1\cos\alpha_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = -\frac{2^{2d+4}d!^2}{(2d+3)!}. \tag{12}$$

$$\int_0^{2\pi}\int_0^{2\pi}\cos^2(\alpha_1-\alpha_2)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 = 2\pi^2. \tag{13}$$

$$\int_0^\pi \sin^d(\alpha)\cos^2(\alpha)\mathrm{d}\alpha = \left(\frac{1}{d+2}\right)\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1+\frac{d}{2}\right)}\cdot\sqrt{\pi}. \tag{14}$$

$$\int_0^\pi\int_0^\pi \sin^d\alpha_1\sin^d\alpha_2\cos^2\alpha_1\cos^2\alpha_2|\cos\alpha_1-\cos\alpha_2|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{15}$$

$$= \frac{2^{2d+6}(7d+13)(d+2)!^2}{(d+1)^2(d+2)(2d+6)!}.$$

$$\int_a^b \sin^d \alpha \,\mathrm{d}\alpha = \begin{cases} -\sum\limits_{\substack{1\le i\le d\\ i \text{ odd}}} \frac{1}{i}\left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})}\right)\sin^{i-1}(\alpha)\cos(\alpha)\Big|_a^b, & \text{if } d \text{ is odd,}\\ -\sum\limits_{\substack{2\le i\le d\\ i \text{ even}}} \frac{1}{i}\left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})}\right)\sin^{i-1}(\alpha)\cos(\alpha)\Big|_a^b + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}\frac{\alpha}{\sqrt{\pi}}\Big|_a^b, & \text{if } d \text{ is even.}\end{cases} \tag{16}$$

$$\int_0^\pi \alpha \sin^d \alpha \cos\alpha \,\mathrm{d}\alpha = -\left(\frac{1}{d+1}\right)\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(1+\frac{d+1}{2}\right)}\cdot\sqrt{\pi}. \tag{17}$$

Proofs

Proof of Equation (7). Classic. □

Proof of Equation (16). Integration by parts gives

$$\int_a^b \sin^d \alpha \,\mathrm{d}\alpha = -\cos(\alpha)\sin^{d-1}(\alpha)\Big|_a^b + \int_a^b \cos^2\alpha(d-1)\sin^{d-2}(\alpha)\,\mathrm{d}\alpha.$$

Using the identity $\cos^2(\alpha) = 1-\sin^2(\alpha)$ and simplifying gives

$$\int_a^b \sin^d \alpha \,\mathrm{d}\alpha = -\left(\frac{1}{d}\right)\cos(\alpha)\sin^{d-1}(\alpha)\Big|_a^b + \left(\frac{d-1}{d}\right)\int_a^b \sin^{d-2}(\alpha)\,\mathrm{d}\alpha.$$

Now, using $\int_a^b \sin^0\alpha\,\mathrm{d}\alpha = \alpha\Big|_a^b$ and the identity

$$\prod_{k=1}^{k}\frac{2i-1}{2i} = \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(k+1)\sqrt{\pi}},$$

by induction on d, we get that, for d even,

$$\int_a^b \sin^d \alpha \,\mathrm{d}\alpha = -\sum_{\substack{2\le i\le d\\ i \text{ even}}}\frac{1}{i}\left(\frac{\Gamma(\frac{d+1}{2})/\left(\Gamma(1+\frac{d}{2})\sqrt{\pi}\right)}{\Gamma(\frac{i+1}{2})/\left(\Gamma(1+\frac{i}{2})\sqrt{\pi}\right)}\right)\sin^{i-1}(\alpha)\cos(\alpha)\Big|_a^b + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}\frac{\alpha}{\sqrt{\pi}}\Big|_a^b.$$

Analogously, for d odd, we use $\int_a^b \sin^1\alpha\,\mathrm{d}\alpha = -\cos(\alpha)\Big|_a^b$ and the identity

$$\prod_{k=1}^{k}\frac{2i}{2i+1} = \frac{\Gamma(d+1)\sqrt{\pi}}{\Gamma(d+\frac{3}{2})},$$

to get by induction

$$\int_a^b \sin^d \alpha \,\mathrm{d}\alpha = -\sum_{\substack{1\le i\le d\\ i \text{ odd}}}\frac{1}{i}\left(\frac{\left(\Gamma(\frac{d+1}{2})\sqrt{\pi}\right)/\Gamma(1+\frac{d}{2})}{\left(\Gamma(\frac{i+1}{2})\sqrt{\pi}\right)/\Gamma(1+\frac{i}{2})}\right)\sin^{i-1}(\alpha)\cos(\alpha)\Big|_a^b.$$

Grouping the two expressions above completes the proof. □

Proof of Equation (8). It is a direct corollary of Equation 16. □

Proof of Equation (17). Integration by parts gives

$$\begin{aligned}\int_0^\pi \alpha \sin^d \alpha \cos \alpha \, \mathrm{d}\alpha &= \frac{1}{d+1}\alpha \sin^{d+1}(\alpha)\Big|_0^\pi - \frac{1}{d+1}\int_0^\pi \sin^{d+1}\alpha \, \mathrm{d}\alpha \\ &= -\frac{1}{d+1}\int_0^\pi \sin^{d+1}\alpha \, \mathrm{d}\alpha \\ &\overset{\text{Eq.(8)}}{=} -\left(\frac{1}{d+1}\right)\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(1+\frac{d+1}{2}\right)} \cdot \sqrt{\pi}\end{aligned}$$

□

Proof of Equation (9). First, we use the symmetry with respect to the line $\alpha_1 = \alpha_2$ to split the integral in two equal parts without absolute values.

$$\begin{aligned}\text{integral} &= 2\iint_{\substack{\alpha_1,\alpha_2\in[0,\pi]^2\\ \alpha_1<\alpha_2}} \sin^d \alpha_1 \sin^d \alpha_2 (\cos\alpha_1 - \cos\alpha_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &= 2\left(\int_0^\pi \sin^d\alpha_1\cos\alpha_1\int_{\alpha_1}^\pi \sin^d\alpha_2\mathrm{d}\alpha_2\mathrm{d}\alpha_1 - \int_0^\pi \sin^d\alpha_2\cos\alpha_2\int_0^{\alpha_2}\sin^d\alpha_1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right) \\ &= 2\int_0^\pi \sin^d x\cos x\left(\int_x^\pi \sin^d y\mathrm{d}y - \int_0^x \sin^d y\mathrm{d}y\right).\end{aligned}$$

Let $f_d(x) = \left(\int_x^\pi \sin^d y\mathrm{d}y - \int_0^x \sin^d y\mathrm{d}y\right)$. Then, by using Equation 16, we get

$$f_d(x) = \begin{cases} 2\displaystyle\sum_{\substack{1\le i\le d\\ i \text{ odd}}} \frac{1}{i}\left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})}\right)\sin^{i-1}x\cos x, & \text{if } d \text{ is odd,}\\ 2\displaystyle\sum_{\substack{2\le i\le d\\ i \text{ even}}} \frac{1}{i}\left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})}\right)\sin^{i-1}x\cos x + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}\sqrt{\pi}\left(1-\frac{2x}{\pi}\right), & \text{if } d \text{ is even.}\end{cases}$$

First, suppose d is odd. Then, replacing the equation above in the simplified integral gives

$$2\int_0^\pi \sin^d x\cos x\left(2\sum_{\substack{1\le i\le d\\ i \text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x\right)\mathrm{d}x,$$

which simplifies to

$$4\sum_{\substack{1\le i\le d\\ i \text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\int_0^\pi \sin^{d+i-1}x\cos^2 x\mathrm{d}x.$$

Using Equation 14 gives

$$4\sum_{\substack{1\le i\le d\\ i \text{ odd}}}\frac{1}{i}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\left(\frac{1}{d+i+1}\right)\frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)}\sqrt{\pi}.$$

Taking $d = 2k-1$, $i = 2j-1$, replacing in the equation above and summing, gives

$$4\frac{(4k+1)\sqrt{\pi}\Gamma(k)\Gamma(2k)}{\Gamma(2k+\frac{3}{2})\Gamma(1+k)}.$$

Finally, replacing k by $\frac{d+1}{2}$ and simplifying gives

$$2\sqrt{\pi}\frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})}.$$

Now, suppose d is even. Then, again, replacing f_d in the simplified integral gives

$$2\int_0^\pi \sin^d x\cos x\left(\underbrace{2\sum_{\substack{2\le i\le d\\ i \text{ even}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x}_{A}+\underbrace{\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}\sqrt{\pi}\Big(1-\frac{2x}{\pi}\Big)}_{B}\right)\mathrm{d}x,$$

We handle the A term first, which is pretty similar to the d odd case. Again, for the A term, expanding the sum and using Equation 14 gives

$$4\sum_{\substack{2\le i\le d\\ i \text{ even}}}\frac{1}{i}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\left(\frac{1}{d+i+1}\right)\frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)}\sqrt{\pi}.$$

Taking $d = 2k$, $i = 2j$, replacing in the equation above and summing, gives

$$4\left(\frac{\sqrt{\pi}\Gamma(2k+1)}{(2k+1)\Gamma(2k+\frac{3}{2})}-\frac{2}{(2k+1)^2}\right),$$

Finally, replacing k by $d/2$ and simplifying gives

$$2\sqrt{\pi}\frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})}-2\frac{1}{\left(\frac{d+1}{2}\right)^2}.$$

Now, we handle the B term, which is

$$2\int_0^\pi \sin^d x\cos x\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x.$$

Expanding it, gives

$$2\int_0^\pi \sin^d x\cos x\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\mathrm{d}x-2\cdot\frac{2}{\pi}\cdot\int_0^\pi x\sin^d x\cos x\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\mathrm{d}x.$$

The left term is null because of Equation 11. And plugging Equation 17 in the right term above gives

$$2\cdot\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+d/2)}\sqrt{\pi}\cdot\left(\frac{1}{d+1}\right)\cdot\frac{\Gamma(1+d/2)}{\Gamma(1+\frac{d+1}{2})}\cdot\sqrt{\pi}\cdot\frac{2}{\pi}=2\frac{1}{\left(\frac{d+1}{2}\right)^2}.$$

Summing A term with B term gives

$$2\sqrt{\pi}\frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})}.$$

Therefore, the expression above holds for both odd and even d. Moreover, it can be simplified to the following nicer expression

$$\frac{2^{2d+4}(d!)^2}{(2d+2)!}.$$

□

Proof of Equation (10). Symmetry on the circle gives 0. □

Proof of Equation (11). Symmetry with respect to $\pi/2$ gives 0. □

Proof of Equation (12). As in the proof of Equation 9, we simplify this integral to get rid of the absolute value and isolate each of its variable.

$$\begin{aligned}
\text{integral} &= 2\iint_{\substack{\alpha_1,\alpha_2\in[0,\pi]^2\\ \alpha_1<\alpha_2}} \sin^d\alpha_1 \sin^d\alpha_2(\cos\alpha_1-\cos\alpha_2)\cos\alpha_1\cos\alpha_2 \mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
&= 2\left(-\int_0^\pi \sin^d\alpha_1\cos\alpha_1\int_{\alpha_1}^\pi \sin^d\alpha_2\cos^2\alpha_2\mathrm{d}\alpha_2\mathrm{d}\alpha_1+\int_0^\pi\sin^d\alpha_2\cos\alpha_2\int_0^{\alpha_2}\sin^d\alpha_1\cos^2\alpha_1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right) \\
&= 2\int_0^\pi \sin^d x\cos x\left(-\int_x^\pi \sin^d y\cos^2 y\mathrm{d}y+\int_0^x\sin^d y\cos^2 y\mathrm{d}y\right). \\
&= -2\int_0^\pi \sin^d x\cos x f_d(x)\mathrm{d}x+2\int_0^\pi\sin^d x\cos x f_{d+2}(x)\mathrm{d}x
\end{aligned}$$

where f_d is defined in the proof of Equation 9. The first integral is exactly the opposite as the one in Equation 9, the second integral is quite similar, for d odd it gives:

$$\begin{aligned}
&2\int_0^\pi \sin^d x\cos x\left(2\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x\right)\mathrm{d}x \\
&= 4\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\int_0^\pi \sin^{d+i-1}x\cos^2 x \\
&= 4\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\left(\frac{1}{d+i+1}\right)\frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)}\sqrt{\pi}=\frac{\Gamma(d-1)}{\Gamma\left(d+\frac{1}{2}\right)}.
\end{aligned}$$

Substracting the two integrals yields the claimed result. The case with d even can be solved similarly. □

Proof of Equation (13). Simple integration. □

Proof of Equation (14). Use $\cos^2 x = 1-\sin^2 x$ and Equation 8, then simplify. □

Proof of Equation (15). As in the proof of Equations 9 and 12, we simplify this integral to get rid of the absolute value and isolate each of its variable.

$$\begin{aligned}
\text{integral} &= 2\iint_{\substack{\alpha_1,\alpha_2\in[0,\pi]^2\\ \alpha_1<\alpha_2}} \sin^d\alpha_1\sin^d\alpha_2(\cos\alpha_1-\cos\alpha_2)\cos^2\alpha_1\cos^2\alpha_2 \mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&= 2\left(\int_0^\pi \sin^d\alpha_1\cos^3\alpha_1\int_{\alpha_1}^\pi \sin^d\alpha_2\cos^2\alpha_2\mathrm{d}\alpha_2\mathrm{d}\alpha_1 - \int_0^\pi \sin^d\alpha_2\cos^3\alpha_2\int_0^{\alpha_2}\sin^d\alpha_1\cos^2\alpha_1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right)\\
&= 2\int_0^\pi \sin^d x\cos x(1-\sin^2 x)\left(\int_x^\pi \sin^d y(1-\sin^2 y)\mathrm{d}y - \int_0^x \sin^d y(1-\sin^2 y)\mathrm{d}y\right).\\
&= 2\int_0^\pi \sin^d x\cos x f_d(x)\mathrm{d}x - 2\int_0^\pi \sin^d x\cos x f_{d+2}(x)\mathrm{d}x\\
&\qquad -2\int_0^\pi \sin^{d+2} x\cos x f_d(x)\mathrm{d}x + 2\int_0^\pi \sin^{d+2}x\cos x f_{d+2}(x)\mathrm{d}x
\end{aligned}$$

where f_d is defined in the proof of Equation 9. The first and fourth integrals are the same as the one in Equation 9 (with d replaced by $d+2$ in the fourth integral), the second integral is the same as the one in Equation 12, the third integral is quite similar, for d odd it gives:

$$\begin{aligned}
&2\int_0^\pi \sin^{d+2}x\cos x\left(2\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x\right)\mathrm{d}x\\
&= 4\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\int_0^\pi \sin^{d+i+1}x\cos^2 x\\
&= 4\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\left(\frac{1}{d+i+3}\right)\frac{\Gamma\left(\frac{d+i+2}{2}\right)}{\Gamma\left(\frac{d+i+3}{2}\right)}\sqrt{\pi}\\
&= \frac{4\,\Gamma(d+3)}{(d+3)\Gamma\left(d+\frac{7}{2}\right)}.
\end{aligned}$$

Adding the four integrals yields the claimed result. The case with d even can be solved similarly. □

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