Reconstruction of discontinuous parameters in a second order impedance boundary operator

Slim Chaabane, Bilel Charfi, Houssem Haddar

Reconstruction of discontinuous parameters in a second order impedance boundary operator

S. Chaabane[1], **B. Charfi**[1], **H. Haddar**[2]

[1]Laboratory of Applied Mathematics and Harmonic Analysis, LAMHA-LR 11ES52, Department of Mathematics, Faculty of Sciences of Sfax, Sfax University, BP 1171, Sfax 3000, Tunisia.

[2]INRIA, Ecole Polytechnique, Université Paris Saclay, Route de Saclay, 91128, Palaiseau Cedex, France.

E-mail: `slim.chaabane@fsm.rnu.tn`, `houssem.haddar@inria.fr`, `charfibilel@yahoo.fr`

February 2014

Abstract. We consider the inverse problem of retrieving the coefficients of a second order boundary operator from Cauchy data associated with the Laplace operator at a measurement curve. We study the identifiability and reconstruction in the case of piecewise continuous parameters. We prove in particular the differentiability of the Khon-Vogelius functional with respect to the discontinuity points and employ the result in a gradient type minimizing algorithm. We provide validating numerical results discussing in particular the case of unknown number of discontinuity points.

1. Introduction:

We study the inverse problem of retrieving the coefficients of an impedance operator from available Cauchy data on a given surface. This problem arises in a variety of applications related mainly to non destructive testing of corrugated surfaces or the identification of thin deposits [16, 14, 18, 11, 12]. The specificity of our work is to consider the case of a generalized boundary operator involving the Laplace-Beltrami operator with discontinuous coefficients. For the study of inverse problems with generalized impedance boundary conditions we refer to [3, 5, 2, 10]. We here employ a method that exploits the differentiability of the Kohn-Vogelius cost functional with respect to the discontinuity points. We show in particular that this differentiability holds although the state variable may not be differentiable (with respect to those discontinuity points). Indeed, from the numerical perspective, this method is attractive as it reduces the number of unknowns and therefore does not need sophisticated regularizations. This type of approach has been used in [7] for the case of impedance coefficients. The mathematical justification of the method in the case of generalized impedance boundary conditions is more delicate. Moreover, given the higher sensitivity of the inverse problem with respect to the Laplace-Beltrami coefficient, the interest of this type of approach is more relevant in this context.

We consider here the case of the Laplace operator similarly to [5, 4] where the case of regular coefficients is considered and a method based on surface integral formulation of the problem is employed. We also restrict ourselves to a two-dimensional setting of the problem. The directions of generalizations are therefore multiple and are part of ongoing efforts.

The outline of the paper is as follows. We first state the direct problem with a sketch of some useful regularity properties of the solutions. The impedance boundary conditions are assumed to hold on a known interior curve and the data for the inverse problem is formed by Cauchy data on an exterior boundary. We then discuss the identifiability issue in the case of piecewise continuous parameters. The case of general L^∞ coefficients is more complex (we refer to [8, 1] for a discussion of the inverse problem with L^∞ coefficients in the case of Robin type problem). We here show in particular that two sets of Cauchy data are needed to ensure the identifiability. We remove in particular the positivity assumption in [4] where a similar uniqueness result is proved. After summarizing some easy-expected differentiability results with respect to L^∞ perturbations of the parameters, we discuss the differentiability of the Khon-Vogelius function with respect to L^2 perturbations of the coefficients (see [17, 9, 15, 6] for more details concerning the Kohn-Vogelius method). This is done in the framework of piecewise continuous parameters. We also explain why the differentiability of the state is not guaranteed in that framework. In the last part of this article we exploit these results to design an inversion algorithm based on a gradient-descent procedure where the minimization is done alternatively on the coefficient values and the discontinuity points. We discuss the accuracy of this procedure and robustness with respect to noise. We show in particular that if the number of discontinuity points is not known a priori, one needs a regularization procedure that does not allow the appearance of Dirac-like singularities. An upper bound on the number of singularities can be automatically fixed in the algorithm.

2. The Direct and the inverse problem

Let Ω be a doubly connected bounded domain of $\mathbb{R}^2$ with $\mathcal{C}^{1,\alpha}$ boundary, for some $\alpha \in]0,1[$. We denote by Γ and Σ respectively the interior and exterior boundary of Ω. Let $H := \{u \in \mathcal{H}^1(\Omega) \; / u_{|\Gamma} \in \mathcal{H}^1(\Gamma)\}$ endowed with the following natural graph norm:

$$\|u\|_H^2 = \|u\|^2_{H^1(\Omega)} + \|u_{|\Gamma}\|^2_{H^1(\Gamma)}.$$

One can easily see that H is a Hilbert space. Let $\phi \in L^2(\Sigma)$ denotes the imposed current flux; $\phi \not\equiv 0$ and $q \in L^\infty(\Gamma)$ be a Robin parameter such that: $q \geq \gamma$; for some $\gamma > 0$. Let $\eta_* > 0$ and η_{ad} be the set of admissible parameters:

$$\eta_{ad} = \{\eta \in L^\infty(\Gamma) \text{ such that } \eta \geq \eta_* \text{ a.e. on } \Gamma\}.$$

For every $\eta \in \eta_{ad}$, we denote by $u_\eta \in H$ the solution of the following problem:

$$(\mathcal{N}) \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \phi & \text{on } \Sigma, \\ \dfrac{\partial u}{\partial n} + qu - \dfrac{\partial}{\partial \tau}\left(\eta \dfrac{\partial u}{\partial \tau}\right) = 0 & \text{on } \Gamma, \end{cases}$$

where $\dfrac{\partial u}{\partial \tau}$ denotes the tangential derivative of $u_{|\Gamma}$ and where n denotes the outward normal vector. The two last equations in $(\mathcal{N})$ hold in the sense of traces in $H^{-\frac{1}{2}}(\Sigma)$ and $H^{-\frac{1}{2}}(\Gamma)$ respectively. A function $u_\eta \in H$ satisfies $(\mathcal{N})$ if and only if it satisfies the following variational problem:

$$(V) \begin{cases} \text{Find } u \in H, \text{ such that :} \\ a_\eta(u, v) = L(v) \ \forall \ v \in H \end{cases}$$

with

$$a_\eta(u, v) := \int_\Omega \nabla u . \nabla v + \int_\Gamma quv + \int_\Gamma \eta \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau} \quad \text{and} \quad L(v) = \int_\Sigma \phi v. \tag{1}$$

One can prove by using the Lax-Milgram theorem, that the variational problem (V) admits only one solution u_η. Moreover, if $\phi \in L^2(\Sigma)$, then by standard elliptic regularities for PDE, $u_\eta \in H^{\frac{3}{2}}(\Omega)$ (we used here that $u_\eta \in H^1(\Gamma)$). Then, by trace results, we have $\dfrac{\partial u_\eta}{\partial n} \in L^2(\Gamma)$. Using the third equation in $(\mathcal{N})$, we then get that

$$\eta \frac{\partial u_\eta}{\partial \tau} \in H^1(\Gamma) \tag{2}$$

which shows in particular that $\eta \frac{\partial u_\eta}{\partial \tau} \in \mathcal{C}^0(\Gamma)$. These facts will be useful in the discussion of uniqueness for the following inverse problem:

$$(\mathcal{I}.\mathcal{P}) \begin{cases} \text{Given the prescribed flux } \phi \text{ together with the potential measurement } \ f := u_{\overline{\eta}_{|\Sigma}} \\ \text{recover the function } \overline{\eta} \in \eta_{ad}. \end{cases}$$

By considering the case of the annulus domain: $\Omega = \{(x, y) \in \mathbb{R}^2, \text{ such that } 1 < x^2 + y^2 < 4\}$, where $(\phi, f) = (1, 2\log(2) + 2))$ on Σ and $q = 1$ on Γ, we see that

$$u(x, y) = \log(x^2 + y^2) + 2$$

that satisfies

$$\frac{\partial u}{\partial \tau} = 0 \text{ on } \Gamma \tag{3}$$

is the unique solution of the problem $(\mathcal{N})$ for every parameter $\eta \in \eta_{ad}$. Consequently, only one measurement is not sufficient to determine the unknown parameter $\overline{\eta}$.

We hereafter establish the following two identifiability results.

Case 1. We first prove that if we avoid (3) by assuming that the set

$$\left\{x \in \Gamma \;/\; \overline{\eta}\frac{\partial u_{\overline{\eta}}}{\partial \tau}(x) = 0\right\} \text{ is of Lebesgue measure 0 in } \Gamma, \tag{4}$$

then only one measurement is sufficient to identify the unknown parameter $\overline{\eta}$.

Case 2: We exploit the result of Case 1 to prove that two different measurements of Cauchy pairs corresponding to two linearly independent fluxes ϕ and ψ are sufficient to identify the unknown parameter $\overline{\eta}$ in the class of piecewise continuous functions.

Theorem 2.1 *Let $\phi \in L^2(\Sigma)$; $\phi \not\equiv 0$ and $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$ such that 4 holds for $\overline{\eta} = \eta_1$. Then,*

$$u_{\eta_1|\Sigma} = u_{\eta_2|\Sigma} \Rightarrow \eta_1 = \eta_2.$$

Proof. One can see that the function $w = u_{\eta_1} - u_{\eta_2}$ verifies the following Cauchy problem:

$$\begin{cases} -\Delta w = 0 \text{ in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 \text{ and } w = 0 \quad \text{on } \Sigma. \end{cases}$$

From the unique continuation principle for the Laplace operator with Cauchy data, we deduce that $w \equiv 0$ in Ω. Consequently, $u_{\eta_1} = u_{\eta_2}$ in Ω and then we have

$$\frac{\partial}{\partial \tau}\left((\eta_1 - \eta_2)\frac{\partial u_{\eta_1}}{\partial \tau}\right) = 0 \text{ on } \Gamma.$$

Then, there exists $C \in \mathbb{R}$ such that

$$(\eta_1 - \eta_2)\frac{\partial u_{\eta_1}}{\partial \tau} = C \text{ on } \Gamma$$

and also we have

$$\frac{(\eta_1 - \eta_2)}{\eta_1}\left(\eta_1 \frac{\partial u_{\eta_1}}{\partial \tau}\right) = C \text{ on } \Gamma, \tag{5}$$

where $g = \eta_1 \dfrac{\partial u_{\eta_1}}{\partial \tau} \in \mathcal{C}^0(\Gamma)$ (as indicated in (2)). Let $x_0 \in \Gamma$ and $x_1 \in \Gamma$ such that:

$$u_{\eta_1}(x_0) = \min_{x \in \Gamma} u_{\eta_1} \quad \text{and} \quad u_{\eta_1}(x_1) = \max_{x \in \Gamma} u_{\eta_1}(x).$$

We prove that $C = 0$ by considering the two following cases.

First case: $u_{\eta_1}(x_0) = u_{\eta_1}(x_1)$. This implies that u_{η_1} is constant on Γ and therefore $C = 0$.

Second case: $u_{\eta_1}(x_0) \neq u_{\eta_1}(x_1)$. In this case the sign of the continuous function g must change on Γ. If not, for instance $g(x) = \eta_1 \frac{\partial u_{\eta_1}}{\partial \tau}(x) \geq 0$ for every $x \in \Gamma$, then from the condition $\eta_1 \geq \eta_* > 0$, we conclude that u_{η_1} is increasing along the connected curve in Γ joining x_1 to x_0 in the sense of increasing curvilinear abcissa. Consequently, $u_{\eta_1}(x_1) \leq u_{\eta_1}(x_0)$ which is of course a contradiction. We then conclude from the mean value theorem that there exists some point $a \in \Gamma$ such that $g(a) = 0$.

Let $\varepsilon > 0$ be a sufficiently small number and $\varphi : [0,1] \longmapsto \mathbb{R}^2$ be a $\mathcal{C}^{1,\alpha}$ parametrization of the curve Γ such that $\varphi'(t) \neq 0$ for every $t \in [0,1]$. Let $t_0 \in [0,1]$ such that $\varphi(t_0) = a$ and $\mathcal{V}_\varepsilon \subset \Gamma$ the arc joining a and $a_\varepsilon = \varphi(t_0 + \varepsilon)$. Integration (5) along $\mathcal{V}_\varepsilon$ implies

$$\frac{C}{\varepsilon}\int_{\mathcal{V}_\varepsilon} d\Gamma \leq \frac{1}{\varepsilon}\left\|\frac{(\eta_1-\eta_2)}{\eta_1}\right\|_{L^\infty(\Gamma)}\int_{\mathcal{V}_\varepsilon}|g|d\Gamma.$$

Consequently,

$$\frac{C}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\|\varphi'(t)\|dt \leq \frac{1}{\varepsilon}\left\|\frac{(\eta_1-\eta_2)}{\eta_1}\right\|_{L^\infty(\Gamma)}\int_{t_0}^{t_0+\varepsilon}|g|(\varphi(t))\|\varphi'(t)\|dt.$$

By letting $\varepsilon \to 0$, we obtain

$$C\|\varphi'(t_0)\| \leq \left\|\frac{(\eta_1-\eta_2)}{\eta_1}\right\|_{L^\infty(\Gamma)}\|\varphi'(t_0)\||g(\varphi(t_0))|$$

where $\|\varphi'(t_0)\| \neq 0$ and $g(\varphi(t_0)) = 0$. Consequently $C = 0$. Using again (5) we obtain

$$\frac{(\eta_1-\eta_2)}{\eta_1}\left(\eta_1\frac{\partial u_{\eta_1}}{\partial\tau}\right) = 0 \text{ on } \Gamma.$$

Using (4) we conclude that

$$\eta_1 = \eta_2 \text{ a.e. on } \Gamma.$$

$\square$

Theorem 2.2 *Let ϕ and $\psi \in L^2(\Sigma)$ be two linearly independent fluxes and $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$ be two piecewise continuous parameters. For $i \in \{1, 2\}$, we denote by $u^\phi_{\eta_i}$ and $u^\psi_{\eta_i}$ the unique solution of problem $(\mathcal{N})$ corresponding respectively to the fluxes ϕ and ψ for a parameter $\eta = \eta_i$. Set $f^i_\phi = u^\phi_{\eta_i|_\Sigma}$ and $f^i_\psi = u^\psi_{\eta_i|_\Sigma}$. Then we have the following implication*

$$(f^1_\phi, f^1_\psi) = (f^2_\phi, f^2_\psi) \ \Rightarrow\ \eta_1 = \eta_2.$$

Proof. Using the same arguments as in the previous Theorem, we deduce that:

$$\frac{(\eta_1-\eta_2)}{\eta_1}\left(\eta_1\frac{\partial u^\phi_{\eta_1}}{\partial\tau}\right) = 0 \text{ and } \frac{(\eta_1-\eta_2)}{\eta_1}\left(\eta_1\frac{\partial u^\psi_{\eta_1}}{\partial\tau}\right) = 0 \text{ on } \Gamma. \tag{6}$$

Consequently,

$$\left(\frac{(\eta_1-\eta_2)}{\eta_1}\right)^2\left(\left(\eta_1\frac{\partial u^\phi_{\eta_1}}{\partial\tau}\right)^2 + \left(\eta_1\frac{\partial u^\psi_{\eta_1}}{\partial\tau}\right)^2\right) = 0 \text{ on } \Gamma.$$

Assuming that we have $\eta_1 \neq \eta_2$, we conclude that there exists $x_0 \in \Gamma$ and an open connected subset I of Γ such that $x_0 \in I$ and $\eta_1(x) \neq \eta_2(x)$ for every $x \in I$. Consequently,

$$\frac{\partial u^\phi_{\eta_1}}{\partial\tau} = \frac{\partial u^\psi_{\eta_1}}{\partial\tau} = 0 \text{ on } I.$$

Then, there exists two constants α and β such that:

$$u_{\eta_1}^{\phi} = \alpha \text{ and } u_{\eta_1}^{\psi} = \beta \text{ on } I. \tag{7}$$

Consequently,

$$\frac{\partial u_{\eta_1}^{\phi}}{\partial n} + q\alpha = 0 \text{ and } \frac{\partial u_{\eta_1}^{\psi}}{\partial n} + q\beta = 0 \text{ on } I. \tag{8}$$

Equations (7) and (8) give

$$\begin{cases} \Delta(\beta u_{\eta_1}^{\phi} - \alpha u_{\eta_1}^{\psi}) = 0 \text{ in } \Omega \\ (\beta u_{\eta_1}^{\phi} - \alpha u_{\eta_1}^{\psi}) = 0 \text{ on } I, \\ \frac{\partial(\beta u_{\eta_1}^{\phi} - \alpha u_{\eta_1}^{\psi})}{\partial n} = 0 \text{ on } I. \end{cases}$$

By the unique continuation principle for the Laplace operator with surface homogeneous Cauchy data we deduce that $\beta u_{\eta_1}^{\phi} - \alpha u_{\eta_1}^{\psi} = 0$ in Ω. Therefore

$$\beta\phi - \alpha\psi = 0 \text{ on } \Sigma,$$

and then $\alpha = \beta = 0$. Using again equations (7) and (8), we deduce that:

$$\begin{cases} \Delta u_{\eta_1}^{\phi} = 0 \text{ in } \Omega \\ u_{\eta_1}^{\phi} = 0 \text{ and } \frac{\partial u_{\eta_1}^{\phi}}{\partial n} = 0 \text{ on } I. \end{cases}$$

Then, $u_{\eta_1}^{\phi} \equiv 0$ in Ω which is in contradiction with the fact that $\phi \not\equiv 0$. Consequently, $\eta_1 = \eta_2$ a.e. on Γ.

$\square$

3. The Kohn-Vogelius function

We present in this part a numerical method based on the Kohn-Vogelius cost function that allows us to determine the unknown piecewise constant parameter $\overline{\eta}$. For $\eta \in \eta_{ad}$, we denote by v_η the solution of the following problem:

$$(\mathcal{D}) \begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v = f & \text{on } \Sigma, \\ \dfrac{\partial v}{\partial n} + qv - \dfrac{\partial}{\partial \tau}\left(\eta \dfrac{\partial v}{\partial \tau}\right) = 0 & \text{on } \Gamma. \end{cases}$$

In the sequel, we denote by

$$H_0 = \{v \in H \text{ such that } v = 0 \text{ in } \Sigma\}.$$

The variational problem of $(\mathcal{D})$ is given by

$$(VD) \begin{cases} \text{Find } v \in H, \text{ such that } v_{|\Sigma} = f, \\ a_\eta(v, w) = 0 \; \forall \; w \in H_0. \end{cases}$$

where the bilinear form a_η is the same as in (1). We now consider the Kohn-Vogelius cost function J_ϕ corresponding to the flux ϕ, that measures the energy gap between u_η and v_η, defined as

$$\begin{array}{rcl} J_\phi: \ \eta_{ad} & \longrightarrow & \mathbb{R} \\ \eta & \longmapsto & J_\phi(\eta) = \displaystyle\int_\Omega |\nabla u_\eta - \nabla v_\eta|^2 + \int_\Gamma q|u_\eta - v_\eta|^2 + \int_\Gamma \eta \left|\frac{\partial u_\eta}{\partial \tau} - \frac{\partial v_\eta}{\partial \tau}\right|^2, \end{array}$$

where u_η is the solution of (V) and v_η is solution of $(\mathcal{D})$ with f being the potential measurements on Σ corresponding with the flux ϕ ($f := u_{\overline{\eta}_{|\Sigma}}$). As a consequence of Theorem 2.1, if (4) is satisfied, then the cost functional J_ϕ admits only one minimum which is the solution $\overline{\eta}$ of the inverse problem $(\mathcal{IP})$. If not than there may exist infinitly many solutions as attested by the counter example given at the beginning of the second section. Let $\eta'_{ad} = \eta_{ad} \cap \mathcal{C}_p$, where $\mathcal{C}_p$ denotes the all of piecewise continuous functions defined on Γ. If we assume that $\overline{\eta} \in \eta'_{ad}$ and we use two linear independent fluxes ϕ and ψ and a cost functional $J = J_\phi + J_\psi$, then by using Theorem 2.2, J has a unique minimizer on η'_{ad} given by the unique solution of $(\mathcal{I}.\mathcal{P})$.

3.1. Differentiability of the cost function J_ϕ

Let $\phi \in L^2(\Sigma)$ denotes the imposed current flux; $\phi \not\equiv 0$ and $q \in L^\infty(\Gamma)$ be a Robin parameter such that: $q \geq \gamma$; for some $\gamma > 0$.

Remark 3.1 *We can assume that we have only $q \geq 0$. Then, one needs to change the solution space to $\tilde{H} = \{v \in H, \displaystyle\int_\Sigma v = 0\}$ and impose that $\displaystyle\int_\Sigma \phi = 0$.*

We study in this part the differentiability of J_ϕ with respect to the parameter η. Let $\eta \in \eta_{ad}$ and $d \in L^\infty(\Gamma)$. For $h > 0$ small enough, we set by $\eta_h := \eta + hd$. One can prove that we have the two following expansions (the proof is simple and is left to reader).

Lemma 3.2 *There exist u^1_η and $\varepsilon(h)$ in H such that*

$$u_{\eta_h} = u_\eta + h u^1_\eta + h\varepsilon(h), \tag{9}$$

where $\lim_{h\to 0}\|\varepsilon(h)\|_H = 0$, and u^1_η is the solution of the following problem:

$$\left\{\begin{array}{l} \text{Find } u \in H \text{ such that} \\ \displaystyle\int_\Omega \nabla u . \nabla v + \int_\Gamma q u v + \int_\Gamma \eta \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} = -\int_\Gamma d \frac{\partial u_\eta}{\partial \tau}\frac{\partial v}{\partial \tau} \quad \forall\, v \in H. \end{array}\right. \tag{10}$$

Lemma 3.3 *There exist v^1_η and $\varepsilon(h)$ in H_0 such that*

$$v_{\eta_h} = v_\eta + h v^1_\eta + h\varepsilon(h) \tag{11}$$

where $\lim_{h\to 0}\|\varepsilon(h)\|_H = 0$, and v^1_η is the solution of the following problem:

$$\left\{\begin{array}{l} \text{Find } v \text{ in } H_0 \text{ such that} \\ \displaystyle\int_\Omega \nabla v \ \nabla w + \int_\Gamma q\ v\ w + \int_\Gamma \eta \frac{\partial v}{\partial \tau}\frac{\partial w}{\partial \tau} = -\int_\Gamma d\ \frac{\partial v_\eta}{\partial \tau}\frac{\partial w}{\partial \tau} \quad \forall\, w \in H_0. \end{array}\right.$$

As a consequence of the two previous lemmas we straightforwardly deduce that the function J_ϕ is Gateaux differentiable at every point $\eta \in \eta_{ad}$ and we have the following theorem.

Theorem 3.4 *The cost function J_ϕ has the following expansion (for sufficiently small h).*

$$J_\phi(\eta_h) = J_\phi(\eta) + h \int_\Gamma d. \left[\left(\frac{\partial v_\eta}{\partial \tau} \right)^2 - \left(\frac{\partial u_\eta}{\partial \tau} \right)^2 \right] \; + \; h\varepsilon(h).$$

where $\lim_{h \to 0} |\varepsilon(h)| = 0$.

Indeed the last theorem holds for small L^∞ perturbations hd. However this type of perturbations do not include the case of small perturbations of discontinuity points of η. This is what shall address now. We consider the case of piecewise constant parameter η and define the following set of admissible partitions of Γ where $\mathcal{T}$ denotes the set of nonempty connected open subsets of Γ:

$$\mathcal{V}_{ad} := \left\{ (\vartheta_i)_{i=1,\ldots,n};\; n > 1;\; \vartheta_i \in \mathcal{T};\; \vartheta_i \cap \vartheta_j = \emptyset \text{ if } i \neq j; \text{and} \bigcup_{i=1}^{n} \overline{\vartheta}_i = \Gamma \right\}$$

Let $\eta_* > 0$ be a given constant. The class of admissible parameters η is then redefined by

$$\eta_{ad} := \left\{ \eta = \sum_{i=1}^{n} c_i \chi_{\vartheta_i};\; (\vartheta_i)_{i=1,\ldots n} \in \mathcal{V}_{ad};\; c_i \geq \eta_*,\; i = 1, \ldots n \right\}.$$

By using Theorem 3.4 with a direction $d = \chi_{\vartheta_j}$, we obtain the following corollary:

Corollary 3.5 *Let $\eta = \sum_{i=1}^{n} c_i \chi_{\vartheta_i} \in \eta_{ad}$ and $j \in \{1, 2, \ldots, n\}$. Then, the mapping:*

$$\zeta_{j,\eta} : \mathbb{R}_+ \longrightarrow \mathbb{R} \quad t \longmapsto J_\phi(t\chi_{\vartheta_j} + \sum_{i \neq j} c_i \chi_{\vartheta_i})$$

is differentiable at c_j, and we have

$$\zeta'_{j,\eta}(c_j) \;=\; \int_{\vartheta_j} \left[\left(\frac{\partial v_\eta}{\partial \tau} \right)^2 - \left(\frac{\partial u_\eta}{\partial \tau} \right)^2 \right].$$

3.2. Derivative of J_ϕ with respect to the discontinuity points of η

In this section, we shall discuss the derivative of the cost function J_ϕ with respect to the (possible) discontinuity points of η. The major difficulty is that the solutions u_η and v_η are not differentiable with respect to discontinuity points. First, we begin by proving the two following lemmas on the dependance of the state u_η with respect to η.

Lemma 3.6 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$; $q \geq \gamma$; for some $\gamma > 0$. Then, there exists a constant $C > 0$, such that for every $\eta \in \eta_{ad}$, the functions $\eta \frac{\partial u_\eta}{\partial \tau}$ and $\eta \frac{\partial v_\eta}{\partial \tau}$ are in $H^1(\Gamma)$, and verify the following estimates:*

$$\left\| \eta \frac{\partial u_\eta}{\partial \tau} \right\|_{H^1(\Gamma)} \leq C \|u_\eta\|_{H^1(\Gamma)} \quad \textit{and} \quad \left\| \eta \frac{\partial v_\eta}{\partial \tau} \right\|_{H^1(\Gamma)} \leq C \|v_\eta\|_{H^1(\Gamma)}.$$

Proof. Let ς be the Neumann trace mapping:

$$\begin{array}{rccc} \varsigma : & H^{\frac{3}{2}}(\Omega) & \longrightarrow & L^2(\Gamma) \\ & v & \longmapsto & \frac{\partial v}{\partial n} \end{array}$$

and Ψ the linear continuous mapping:

$$\begin{array}{rccc} \Psi : & H^1(\Gamma) & \longrightarrow & H^{\frac{3}{2}}(\Omega) \\ & v & \longmapsto & \tilde{v} \end{array}$$

where $\tilde{v}$ denotes the harmonic extension of v defined as the unique solution of the following boundary value problem:

$$\begin{cases} -\Delta \tilde{v} = 0 & \text{in } \Omega, \\ \dfrac{\partial \tilde{v}}{\partial n} = \phi & \text{on } \Sigma, \\ \tilde{v} = v & \text{on } \Gamma. \end{cases}$$

We set $\alpha := \|\Psi\|_{\mathcal{L}(H^1(\Gamma), H^{\frac{3}{2}}(\Omega))}$ and $\beta := \|\varsigma\|_{\mathcal{L}(H^{\frac{3}{2}}(\Omega), L^2(\Gamma))}$. Let $\eta \in \eta_{ad}$, then the function $g = u_{\eta|_\Gamma} \in H^1(\Gamma)$ and we have

$$\|u_\eta\|_{H^{\frac{3}{2}}(\Omega)} \leq \alpha \|g\|_{H^1(\Gamma)}.$$

Consequently, $\dfrac{\partial u_\eta}{\partial n} \in L^2(\Gamma)$ and verifies the following estimate

$$\left\| \frac{\partial u_\eta}{\partial n} \right\|_{L^2(\Gamma)} \leq \alpha\beta \|u_\eta\|_{H^1(\Gamma)}.$$

Moreover, from the boundary conditions $\dfrac{\partial}{\partial \tau}\left(\eta \dfrac{\partial u_\eta}{\partial \tau}\right) = \dfrac{\partial u_\eta}{\partial n} + q u_\eta$, with $q u_\eta \in L^2(\Gamma)$, we deduce that $\eta \frac{\partial u_\eta}{\partial \tau} \in H^1(\Gamma)$ and verifies the following estimate:

$$\left\| \eta \frac{\partial u_\eta}{\partial \tau} \right\|_{H^1(\Gamma)} \leq C \|u_\eta\|_{H^1(\Gamma)}$$

for some constant $C > 0$ depending only on α, β, ϕ, q and Γ. The proof for v_η can be done in a similar way.

□

Remark 3.7 *One can prove using similar arguments as in the previous Lemma that there exists $C > 0$, such that for every $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$, we have:*

$$\left\| \eta_1 \frac{\partial u_{\eta_1}}{\partial \tau} - \eta_2 \frac{\partial u_{\eta_2}}{\partial \tau} \right\|_{H^1(\Gamma)} \leq C \| u_{\eta_1} - u_{\eta_2} \|_{H^1(\Gamma)} \tag{12}$$

and,

$$\left\| \eta_1 \frac{\partial v_{\eta_1}}{\partial \tau} - \eta_2 \frac{\partial v_{\eta_2}}{\partial \tau} \right\|_{H^1(\Gamma)} \leq C \| v_{\eta_1} - v_{\eta_2} \|_{H^1(\Gamma)}. \tag{13}$$

The following Lemma proves that the two mappings $\eta \longmapsto u_\eta$ and $\eta \longmapsto v_\eta$ are Lipschitz from $L^2(\Gamma)$ into H.

Lemma 3.8 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$; $q \geq \gamma$; for some $\gamma > 0$. Then, there exists a constant $C > 0$, such that for every $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$, we have:*

$$\| u_{\eta_1} - u_{\eta_2} \|_H \leq C \| \eta_1 - \eta_2 \|_{L^2(\Gamma)} \quad \textit{and} \quad \| v_{\eta_1} - v_{\eta_2} \|_H \leq C \| \eta_1 - \eta_2 \|_{L^2(\Gamma)}.$$

Proof. First, we can see from the variational formulation (V) that for every $\eta \in \eta_{ad}$, we have:

$$min\{1, \eta_*, \gamma\} \| u_\eta \|_H^2 \leq \| \phi \|_{L^2(\Sigma)} \| u_\eta \|_{L^2(\Sigma)}.$$

Let δ be the norm of the trace mapping from H to $L^2(\Sigma)$, then we have:

$$\| u_\eta \|_H \leq \frac{\delta \| \phi \|_{L^2(\Sigma)}}{min\{1, \eta_*, \gamma\}} \tag{14}$$

Let us consider now two admissible parameters $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$ and $e = u_{\eta_1} - u_{\eta_2}$, then we have:

$$\int_\Omega \nabla e \nabla v + \int_\Gamma qev + \int_\Gamma \eta_1 \frac{\partial e}{\partial \tau} \frac{\partial v}{\partial \tau} = - \int_\Gamma (\eta_1 - \eta_2) \frac{\partial u_{\eta_2}}{\partial \tau} \frac{\partial v}{\partial \tau} \text{ for every } v \in H.$$

Therefore

$$min\{1, \eta_*, \gamma\} \| e \|_H \leq \left\| (\eta_1 - \eta_2) \frac{\partial u_{\eta_2}}{\partial \tau} \right\|_{L^2(\Gamma)}.$$

By using the previous Lemma, we deduce that $\eta_2 \frac{\partial u_{\eta_2}}{\partial \tau} \in L^\infty(\Gamma)$, and from the condition $\eta_2 \geq \eta_*$, we get

$$\| e \|_H \leq \frac{\left\| \eta_2 \frac{\partial u_{\eta_2}}{\partial \tau} \right\|_{L^\infty(\Gamma)}}{\eta_* \, min\{1, \eta_*, \gamma\}} \| \eta_1 - \eta_2 \|_{L^2(\Gamma)}.$$

Using again the previous Lemma together with the continuous embedding of $H^1(\Gamma)$ into $L^\infty(\Gamma)$, we deduce from equation (14) that there exists a constant $C > 0$ such that

$$\| e \|_H \leq C \| \eta_1 - \eta_2 \|_{L^2(\Gamma)}. \tag{15}$$

To prove the same result for the Dirichlet problem $(\mathcal{D})$ we first prove the uniform boundedness of v_η with respect to η. Let v_0 be the solution of the following Dirichlet problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v = f & \text{sur } \Sigma, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

Taking $w = v_\eta - v_0$ as a test function in the variational formulation for v_η, we simply get:

$$\int_\Omega |\nabla v_\eta|^2 + \int_\Gamma q|v_\eta|^2 + \int_\Gamma \eta \left(\frac{\partial v_\eta}{\partial \tau}\right)^2 = \int_\Omega \nabla v_\eta \nabla v_0.$$

Consequently,

$$\|v_\eta\|_H \leq \frac{|v_0|_{1,\Omega}}{min\{1, \eta_*, \gamma\}}.$$

Let us consider now two admissible parameters $(\eta_1, \eta_2) \in \eta_{ad} \times \eta_{ad}$ and $e' = v_{\eta_1} - v_{\eta_2}$, then we have

$$\int_\Omega \nabla e' \nabla w + \int_\Gamma q e' w + \int_\Gamma \eta_1 \frac{\partial e'}{\partial \tau}\frac{\partial w}{\partial \tau} = -\int_\Gamma (\eta_1 - \eta_2)\frac{\partial v_{\eta_2}}{\partial \tau}\frac{\partial w}{\partial \tau} \text{ for every } w \in H_0.$$

Using exactly the same proof as for the first estimate (15), we obtain

$$\|e'\|_H \leq C\|\eta_1 - \eta_2\|_{L^2(\Gamma)}$$

for some constant $C > 0$ only depending on α, β, ϕ, q, η_* and Γ.

$\square$

In the sequel, we suppose that the curve Γ is parameterized by a $\mathcal{C}^{1,\alpha}$ function $\varphi : [0,1] \longmapsto \mathbb{R}^2$. Consider a parameter $\eta \in \eta_{ad}$. There exists some constants $c_1, ..., c_n \in \mathbb{R}_+$; $n \geq 2$ and a strictly increasing subdivision $0 = \alpha_0 < \alpha_1 < ... < \alpha_n = 1$ such that $\eta = \sum_{i=1}^{n} c_i \, \chi_{\vartheta_i}$ with $\vartheta_i = \varphi([\,\alpha_i, \alpha_{i+1}[)$.

Let $h > 0$ be a small enough parameter and $c_0 = c_n$. We denote by $m_i = \varphi(\alpha_i)$, $m_{i,h} = \varphi(\alpha_i + h)$ and by $\vartheta_{i,h} = \varphi([\alpha_i, \alpha_i + h])$ (see Figure 1).

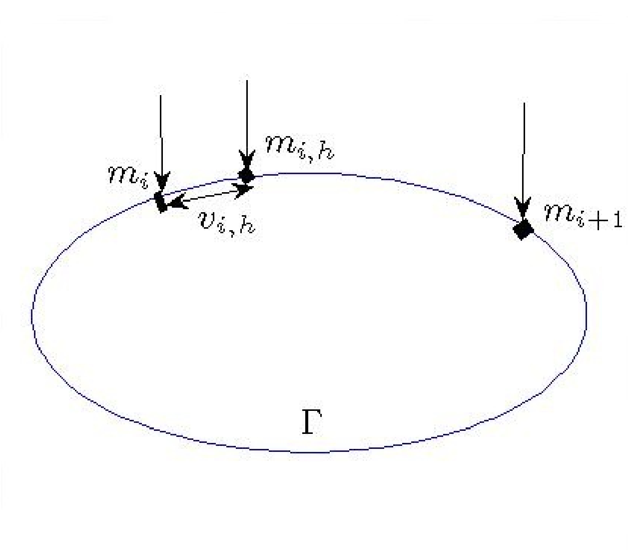

Figure 1. The set $\vartheta_{i,h}$

Let us denote by η_h the parameter defined by:

$$\eta_h(x) \begin{cases} c_{i-1}, & \text{if} \quad x \in \vartheta_{i,h} \\ \eta(x) & \text{else.} \end{cases}$$

To compute the derivative of the cost function J_ϕ with respect to m_i we need the following lemma.

Lemma 3.9 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$; $q \geq \gamma$; for some $\gamma > 0$. Then, there exists $C > 0$ such that for every $h > 0$ small enough, we have:*

$$\int_{\vartheta_{i,h}} \left| \left(\eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau} \right) \eta \frac{\partial u_\eta}{\partial \tau} \right| \leq C h^{\frac{3}{2}},$$

$$\int_{\vartheta_{i,h}} \left| \left(\eta_h \frac{\partial v_{\eta_h}}{\partial \tau} - \eta \frac{\partial v_\eta}{\partial \tau} \right) \eta \frac{\partial v_\eta}{\partial \tau} \right| \leq C h^{\frac{3}{2}}.$$

Proof. We first observe that

$$\int_{\vartheta_{i,h}} \left| \left(\eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau} \right) \eta \frac{\partial u_\eta}{\partial \tau} \right| \leq \left[\left\| \eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau} \right\|_{L^\infty(\Gamma)} \left\| \eta \frac{\partial u_\eta}{\partial \tau} \right\|_{L^\infty(\Gamma)} \|\varphi'\|_{L^\infty([0,1])} \right] h (16)$$

From Remark 3.7 and the continuous embedding of $H^1(\Gamma)$ into $L^\infty(\Gamma)$, we deduce that there exists a constant $C_1 > 0$ such that:

$$\left\| \eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau} \right\|_{L^\infty(\Gamma)} \leq C_1 \|u_{\eta_h} - u_\eta\|_{H^1(\Gamma)} \tag{17}$$

and by Lemma 3.8, we can deduce that there exists a constant $C_2 > 0$ such that:

$$\|u_{\eta_h} - u_\eta\|_{H^1(\Gamma)} \leq C_2 \|\eta_h - \eta\|_{L^2(\Gamma)}. \tag{18}$$

From the definition of η_h, we get for some $c > 0$ the following inequality

$$\|\eta_h - \eta\|_{L^2(\Gamma)} \leq c h^{\frac{1}{2}} \tag{19}$$

for $h > 0$ small enough. From (16), (17), (18) and (19), we infer the existence of a constant $C > 0$, such that, for $h > 0$ small enough, we have

$$\int_{\vartheta_{i,h}} \left| \left(\eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau} \right) \eta \frac{\partial u_\eta}{\partial \tau} \right| \leq C h^{\frac{3}{2}}. \tag{20}$$

Using similar arguments, we also get:

$$\int_{\vartheta_{i,h}} \left| \left(\eta_h \frac{\partial v_{\eta_h}}{\partial \tau} - \eta \frac{\partial v_\eta}{\partial \tau} \right) \eta \frac{\partial v_\eta}{\partial \tau} \right| \leq C h^{\frac{3}{2}}.$$

□

Let us define

$$J^1_\phi(\eta) := \int_\Omega |\nabla u_\eta|^2 + \int_\Gamma q |u_\eta|^2 + \int_\Gamma \eta \left| \frac{\partial u_\eta}{\partial \tau} \right|^2.$$

Then we have the following theorem.

Theorem 3.10 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$, $q \geq \gamma$ for some $\gamma > 0$. Then:*

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = -\left[\frac{1}{\eta}\right]\left(\eta\frac{\partial u_\eta}{\partial \tau}(m_i)\right)^2 \|\varphi'(\alpha_i)\|h + h\varepsilon(h),$$

where $\left[\frac{1}{\eta}\right] := \frac{1}{c_i} - \frac{1}{c_{i-1}}$ and $\varepsilon(h) \to 0$ as $h \to 0$.

Proof. Taking u_η (respectively u_{η_h}) as a test function in the variational formulation for u_η (respectively u_{η_h}), we simply get:

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = \int_\Sigma \phi(u_{\eta_h} - u_\eta).$$

Using u_η as a test function in the variational formulation for u_{η_h}, we get:

$$\int_\Omega \nabla u_{\eta_h} \nabla u_\eta + \int_\Gamma q u_{\eta_h} u_\eta + \int_\Gamma \eta_h \frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial u_\eta}{\partial \tau} = \int_\Sigma \phi u_\eta$$

and in a symmetric way, we get

$$\int_\Omega \nabla u_{\eta_h} \nabla u_\eta + \int_\Gamma q u_{\eta_h} u_\eta + \int_\Gamma \eta \frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial u_\eta}{\partial \tau} = \int_\Sigma \phi u_{\eta_h}.$$

Consequently,

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = \int_\Gamma (\eta - \eta_h)\frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial u_\eta}{\partial \tau}.$$

Using the definition of η_h we obtain:

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = \int_{\vartheta_{i,h}} \frac{(c_i - c_{i-1})}{c_i c_{i-1}}\left(c_{i-1}\frac{\partial u_{\eta_h}}{\partial \tau}\right)\left(c_i\frac{\partial u_\eta}{\partial \tau}\right) = -\left[\frac{1}{\eta}\right]\int_{\vartheta_{i,h}}\left(\eta_h\frac{\partial u_{\eta_h}}{\partial \tau}\right)\left(\eta\frac{\partial u_\eta}{\partial \tau}\right).$$

Rearranging the terms yields

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = -\left[\frac{1}{\eta}\right]\int_{\vartheta_{i,h}}\left(\eta\frac{\partial u_\eta}{\partial \tau}\right)^2 - \left[\frac{1}{\eta}\right]\int_{\vartheta_{i,h}}\left(\eta_h\frac{\partial u_{\eta_h}}{\partial \tau} - \eta\frac{\partial u_\eta}{\partial \tau}\right)\left(\eta\frac{\partial u_\eta}{\partial \tau}\right). \quad (21)$$

From Lemma 3.6, the function $\eta\frac{\partial u_\eta}{\partial \tau} \in \mathcal{C}^0(\Gamma)$, and therefore

$$\int_{\vartheta_{i,h}}\left(\eta\frac{\partial u_\eta}{\partial \tau}\right)^2 = \|\varphi'(\alpha_i)\|\left(\eta\frac{\partial u_\eta}{\partial \tau}(m_i)\right)^2 h + h\varepsilon(h). \quad (22)$$

By (21), (22) and Lemma 3.9, we obtain:

$$J^1_\phi(\eta_h) - J^1_\phi(\eta) = -\|\varphi'(\alpha_i)\|\left[\frac{1}{\eta}\right]\left(\eta\frac{\partial u_\eta}{\partial \tau}(m_i)\right)^2 h + h\varepsilon(h).$$

□

Now consider

$$J^2_\phi(\eta) := \int_\Omega |\nabla v_\eta|^2 + \int_\Gamma q|v_\eta|^2 + \int_\Gamma \eta\left|\frac{\partial v_\eta}{\partial \tau}\right|^2.$$

Then we can establish similarly as in the proof of Theorem 3.10 the following result.

Theorem 3.11 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$; $q \geq \gamma$; for some $\gamma > 0$. Then we have:*

$$J_\phi^2(\eta_h) - J_\phi^2(\eta) = \left[\frac{1}{\eta}\right]\left(\eta\frac{\partial v_\eta}{\partial \tau}(m_i)\right)^2 \|\varphi'(\alpha_i)\|h + h\varepsilon(h),$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

Proof. Taking $w = v_{\eta_h} - v_\eta \in H_0$ as a test function in the variational formulation of v_η, we simply get the following equation:

$$\int_\Omega |\nabla v_\eta|^2 + \int_\Gamma q|v_\eta|^2 + \int_\Gamma \eta\left(\frac{\partial v_\eta}{\partial \tau}\right)^2 = \int_\Omega \nabla v_\eta \nabla v_{\eta_h} + \int_\Gamma q v_\eta v_{\eta_h} + \int_\Gamma \eta \frac{\partial v_\eta}{\partial \tau}\frac{\partial v_{\eta_h}}{\partial \tau}$$

and in a symmetric way, we get

$$\int_\Omega |\nabla v_{\eta_h}|^2 + \int_\Gamma q|v_{\eta_h}|^2 + \int_\Gamma \eta_h\left(\frac{\partial u_{\eta_h}}{\partial \tau}\right)^2 = \int_\Omega \nabla v_{\eta_h} \nabla v_\eta + \int_\Gamma q v_{\eta_h} v_\eta + \int_\Gamma \eta_h \frac{\partial v_{\eta_h}}{\partial \tau}\frac{\partial v_\eta}{\partial \tau}.$$

Consequently,

$$J_\phi^2(\eta_h) - J_\phi^2(\eta) = -\left[\int_\Gamma (\eta - \eta_h)\frac{\partial v_{\eta_h}}{\partial \tau}\frac{\partial v_\eta}{\partial \tau}\right].$$

The remaining of the proof follows exactly the same line as the proof of Theorem 3.10.

□

Using the fact that:

$$\int_\Omega \nabla u_{\eta_h}\nabla v_{\eta_h} + \int_\Gamma q u_{\eta_h} v_{\eta_h} + \int_\Gamma \eta_h \frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial v_{\eta_h}}{\partial \tau} = \int_\Sigma \phi f,$$

then,

$$J_\phi(\eta_h) - J_\phi(\eta) = \left(J_\phi^1(\eta_h) - J_\phi^1(\eta)\right) + \left(J_\phi^2(\eta_h) - J_\phi^2(\eta)\right).$$

Therefore, we obtain the following straightforward corollary of Theorems 3.10 and 3.11.

Theorem 3.12 *Let $\phi \in L^2(\Sigma)$ and $q \in L^\infty(\Gamma)$; $q \geq \gamma$; for some $\gamma > 0$. Then, the Kohn-Vogelius function J_ϕ is differentiable with respect to the discontinuity points of η and we have:*

$$J_\phi(\eta_h) - J_\phi(\eta) = \left[\frac{1}{\eta}\right]\left(\left(\eta\frac{\partial v_\eta}{\partial \tau}(m_i)\right)^2 - \left(\eta\frac{\partial u_\eta}{\partial \tau}(m_i)\right)^2\right)\|\varphi'(\alpha_i)\|h + h\varepsilon(h).$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

Remark 3.13 *We have established in the previous theorem that the cost function J_ϕ is differentiable with respect to the discontinuity points of η which allows us to use an algorithm of gradient type to determine the unknown parameter $\overline{\eta}$. The following Lemma allows us to prove that the state u_η is not differentiable with respect to the discontinuity points of η and then the theorem 3.12 cannot be established as a simple consequence of the state derivatives.*

Lemma 3.14 *Let $v \in \mathcal{C}^1(\overline{\Omega})$ and $e_h = u_{\eta_h} - u_\eta$, then we have:*

$$\int_\Omega \nabla e_h \nabla v + \int_\Gamma q e_h v + \int_\Gamma \eta \frac{\partial e_h}{\partial \tau}\frac{\partial v}{\partial \tau} = \|\varphi'(\alpha_i)\|(\frac{c_i - c_{i-1}}{c_{i-1}}) \left(\eta \frac{\partial u_\eta}{\partial \tau}\right)(m_i)\frac{\partial v}{\partial \tau}(m_i)h + h\varepsilon(h)$$

with $\varepsilon(h) \to 0$ as $h \to 0$.

Proof. By using the variational formulations of u_η and u_{η_h}, we simply get:

$$\int_\Omega \nabla e_h \nabla v + \int_\Gamma q e_h v + \int_\Gamma \eta \frac{\partial e_h}{\partial \tau}\frac{\partial v}{\partial \tau} = -\int_{\vartheta_{i,h}} (\eta_h - \eta)\frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial v}{\partial \tau}.$$

Then we have

$$\int_\Omega \nabla e_h \nabla v + \int_\Gamma q e_h v + \int_\Gamma \eta \frac{\partial e_h}{\partial \tau}\frac{\partial v}{\partial \tau} = (\frac{c_i - c_{i-1}}{c_{i-1}})\int_{\vartheta_{i,h}} \eta_h \frac{\partial u_{\eta_h}}{\partial \tau}\frac{\partial v}{\partial \tau}.$$

Consequently,

$$\int_\Omega \nabla e_h \nabla v + \int_\Gamma q e_h v + \int_\Gamma \eta \frac{\partial e_h}{\partial \tau}\frac{\partial v}{\partial \tau} = (\frac{c_i - c_{i-1}}{c_{i-1}})\int_{\vartheta_{i,h}} \eta \frac{\partial u_\eta}{\partial \tau}\frac{\partial v}{\partial \tau} + (\frac{c_i - c_{i-1}}{c_{i-1}})\int_{\vartheta_{i,h}} (\eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta\frac{\partial u_\eta}{\partial \tau})\frac{\partial v}{\partial \tau}$$

where the function: $\eta \dfrac{\partial u_\eta}{\partial \tau}\dfrac{\partial v}{\partial \tau} \in \mathcal{C}^0(\Gamma)$. Then, we have:

$$\int_{\vartheta_{i,h}} \eta \frac{\partial u_\eta}{\partial \tau}\frac{\partial v}{\partial \tau} = \|\varphi'(\alpha_i)\| \left(\eta\frac{\partial u_\eta}{\partial \tau}\right)(m_i)\frac{\partial v}{\partial \tau}(m_i)h + h\varepsilon(h).$$

Moreover, one can prove like in (20) the existence of a constant $C > 0$ such that

$$\left|\int_{\vartheta_{i,h}} (\eta_h \frac{\partial u_{\eta_h}}{\partial \tau} - \eta \frac{\partial u_\eta}{\partial \tau})\frac{\partial v}{\partial \tau}\right| \le C h^{\frac{3}{2}}.$$

Consequently,

$$\int_\Omega \nabla e_h \nabla v + \int_\Gamma q e_h v + \int_\Gamma \eta \frac{\partial e_h}{\partial \tau}\frac{\partial v}{\partial \tau} = \|\varphi'(\alpha_i)\|(\frac{c_i - c_{i-1}}{c_{i-1}}) \left(\eta \frac{\partial u_\eta}{\partial \tau}\right)(m_i)\frac{\partial v}{\partial \tau}(m_i)h + h\varepsilon(h).$$

□

Theorem 3.15 *If $[\eta](m_i) \neq 0$ and $\eta \frac{\partial u_\eta}{\partial \tau}(m_i) \neq 0$, then we cannot find a function $u^1 \in H$ such that $\frac{u_{\eta_h} - u_\eta}{h} \rightharpoonup u^1$ weakly in H (i.e the state u_η is not weakly differentiable with respect to m_i).*

Proof. Clearly the linear mapping

$$\Psi : \mathcal{C}^1(\overline{\Omega}) \longrightarrow \mathbb{R}, \quad v \longmapsto \frac{\partial v}{\partial \tau}(m_i)$$

cannot be extended by density to a continuous mapping from H to $\mathbb{R}$. Then using Lemma 3.14 we can conclude that if $[\eta](m_i) \neq 0$ and $\eta \frac{\partial u_\eta}{\partial \tau}(m_i) \neq 0$, the state u_η is not weakly differentiable with respect to the discontinuity point m_i.

□

4. Numerical algorithm and results

In this section, we present some validating numerical results using a minimization algorithm of gradient type. The numerical examples are based on synthetic data numerically simulated using the FreeFem++ software [13]. We also use the same software in solving the inverse problem and make a special attention to avoid "inverse crimes" by using different meshes and adding random noise to the synthetic data.

Description of the numerical algorithm From the identifiability study presented in section 2 of the paper, we already see that at least two different measurements are needed to guarantee unique determination of the parameter $\overline{\eta}$. Since our algorithm is based on minimizing the cost functional J_ϕ, one also has to avoid (as much as possible) the presence of local minima. We numerically observed that this can be done by increasing the number of used fluxes ϕ. More specifically, given N linearly independent fluxes $\phi_1,\ \phi_2,\ ...,\phi_N$, the cost functional $J(\eta)$ that we shall consider is

$$J(\eta) := \sum_{j=1}^{N} J_{\phi_j}(\eta).$$

Let us denote by $\{\varphi(\alpha) = (x(\alpha), y(\alpha)); \alpha \in [-\pi, \pi[\}$ a parametrization of the curve Γ. The impedance function η is then sought as a piecewise constant function of $\alpha \in [-\pi, \pi)$ parametrized by the number of discontinuity points n, the points of discontinuities $m_i = (x(\alpha_i), y(\alpha_i))$ and the values $c_i > 0$ of η on $]\alpha_{i-1}, \alpha_i[$ for $i = 1, \ldots, n$ where we have set $\alpha_0 = \alpha_n$ and assumed that $\alpha_i < \alpha_{i+1}$. The cost functional J can then be seen as a function of α_i and c_i, $i = 1, \ldots n$. The derivative of J with respect to these parameters can be written as

$$\frac{\partial J}{\partial c_i} = \sum_{j=1}^{N} \frac{\partial J_{\phi_j}}{\partial c_i}, \quad \frac{\partial J}{\partial \alpha_i} = \sum_{j=1}^{N} \frac{\partial J_{\phi_j}}{\partial m_i},$$

where $\frac{\partial J_{\phi_j}}{\partial c_i}$ is given by the Corollary 3.5 and $\frac{\partial J_{\phi_j}}{\partial m_i}$ by the Theorem 3.12. Our algorithm alternates iterations on c_i and α_i and can be synthetically written as

$$c_i^{k+1} = c_i^k - \rho_c^k \frac{\partial J}{\partial c_i}(c_1^k, \ldots, c_n^k, \alpha_1^k, \ldots, \alpha_n^k),$$
$$\alpha_i^{k+1} = \alpha_i^k - \rho_\eta^k \frac{\partial J}{\partial \alpha_i}(c_1^{k+1}, \ldots, c_n^{k+1}, \alpha_1^k, \ldots, \alpha_n^k).$$

At each iteration, the steps ρ_c^k and ρ_η^k are determined so that the cost functional decreases. This is done by reducing the step size by a factor $\gamma < 1$ if the cost functional does not decrease till a small tolerances ϵ_c and ϵ_η. The algorithm stops if

$$\left|\rho_c^k \frac{\partial J}{\partial c_i}(c_1^k, \ldots, c_n^k, \alpha_1^k, \ldots, \alpha_n^k)\right| < \epsilon_c \text{ and } \left|\rho_\eta^k \frac{\partial J}{\partial \alpha_i}(c_1^{k+1}, \ldots, c_n^{k+1}, \alpha_1^k, \ldots, \alpha_n^k)\right| < \epsilon_\eta.$$

In the case on unknown number of singularities, an additional constrain is used in our algorithm to prevent instabilities coming from two identical singularity points. We shall discuss two strategies. In the first one we enforce at each iteration step

$$|\alpha_{i+1} - \alpha_i| > \alpha_* > 0 \tag{23}$$

where α_* is a fixed parameter. In the second strategy, if (23) is not satisfied, then the two discontinuity points are merged together.

Numerical experiments For our numerical validating examples we choose the fluxes ϕ_j defined on Σ as

$$\phi_j(\vec{x}) = \frac{(\vec{x} - \vec{x}_j) \cdot \vec{n}}{\|\vec{x} - \vec{x}_j\|^2} \quad \vec{x} \in \Sigma$$

where $\vec{n}$ is the outer normal on Σ and where the "point source" $\vec{x}_j$ is chosen as

$$\vec{x}_j = R(\cos(\theta_j), \sin(\theta_j)) \quad \theta_j = 2\pi \frac{j-1}{N}.$$

In the following examples, Σ is chosen to be the unit circle, $R = 1.3$ and the impedance function $q(x, y) = (6x + 7)/(x^3 - 3x + 3)$. The parameters for the inversion algorithm are $\epsilon_c = 10^{-3}$ and $\epsilon_\eta = 10^{-3}$. The synthetic data f_j, $j = 1, \ldots, N$ are numerically computed by solving (V). In order to avoid an inverse crime, we used a different mesh than the one used in the inversion algorithm and we corrupt the data with random noise as $f_j^\delta = f_j(1 + \delta r_j)$ where r_j is a vector of uniformly randomly distributed values between -1 and 1 and where the positive number $\delta < 1$ indicates the noise level

The case of a mildly non convex kite We first consider the case of Γ being a kite parametrized as

$$\begin{cases} x(\alpha) = 0.6\cos(\alpha) + 0.2\cos(2\alpha) - 0.1, \\ y(\alpha) = 0.5\sin(\alpha), \end{cases}$$

and four discontinuity points at $\alpha_1 = -\frac{3\pi}{4}$, $\alpha_2 = 0$, $\alpha_3 = \frac{\pi}{4}$ and $\alpha_4 = \frac{3\pi}{4}$ (See Figure 2).

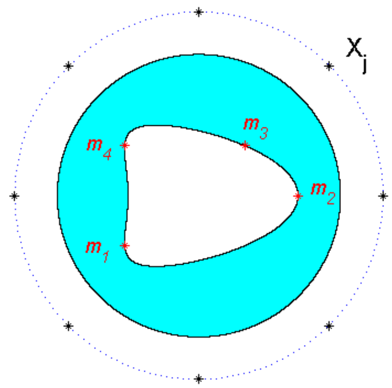

Figure 2. Description of the geometry for the first example

Figure 3 indicates the two different meshes used for simulating the synthetic data and in the inversion algorithm (where problems $\mathcal{N}$ and $\mathcal{D}$ are solved at each iteration).

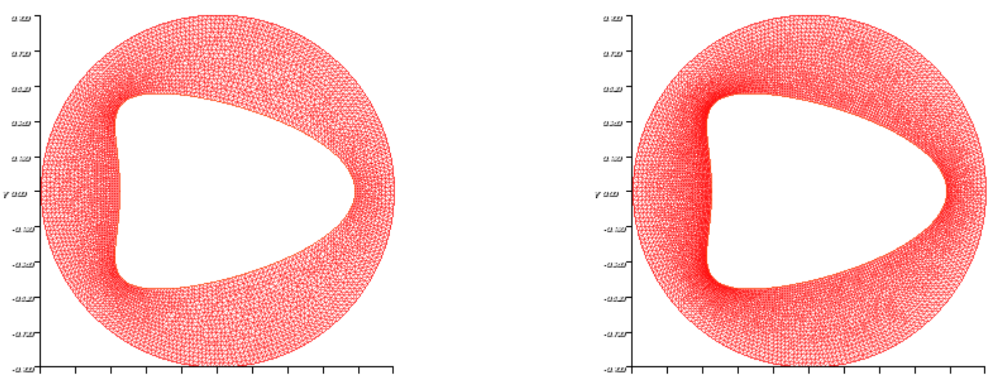

Figure 3. The mesh used to compute the synthetic data (left) and the (refined) mesh used for the inversion algorithm (right).

a. The case of known c_i and n

We first discuss the case where only the discontinuity points m_i are unknown. The obtained results for a noise level $\delta = 5\%$ and different values of the number of used fluxes N are depicted in Figures 4 and 5. We remark that the precision is indeed increased by increasing the number of fluxes. If only one flux is used, we observe that a different local minima is found. We observed that in the case of known c_i and n the number of local minima is usually drastically reduced as long as more than one flux is used. This is attested by the quality of the reconstructions observed in Figures 4 and 5. As a general conclusion of similar experiments conducted but not reported here, we can reasonably say that the algorithm is efficient and stable in this type of configurations, i.e. when c_i and n are known.

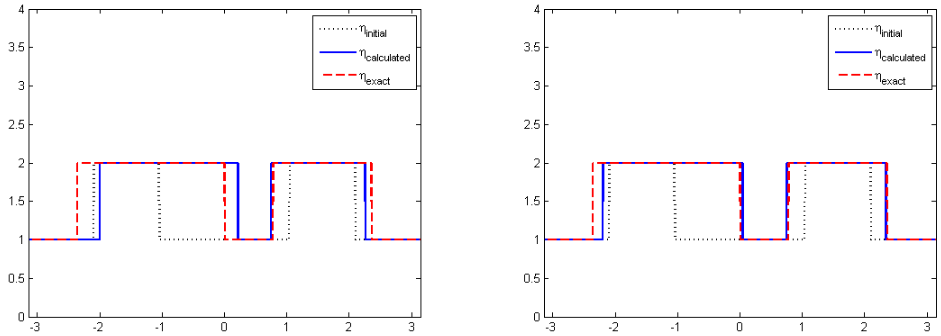

Figure 4. Values of η versus $\alpha \in [-\pi, \pi[$ for different number N of used fluxes: $N = 1$ left and $N = 2$ right. Exact η: dashed line. Initial guess: dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 2 for known values of c_i and known n. The noise level $\delta = 5\%$.

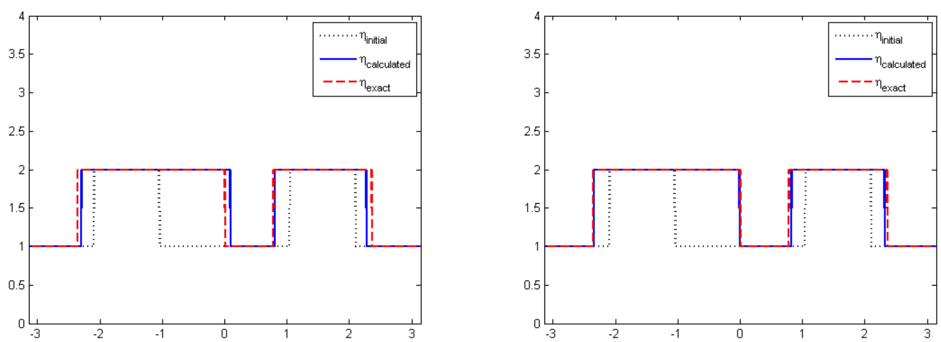

Figure 5. Values of η versus $\alpha \in [-\pi, \pi[$ for different number N of used fluxes: $N = 4$ left and $N = 8$ right. Exact η: dashed line. Initial guess: dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 2 for known values of c_i and known n. The noise level $\delta = 5\%$.

Figure 6 indicates the evolution of the cost functional during iterations. One observes that only few number of iterations is needed to obtain a small residual. The number of needed iterations in fact increases if the number of fluxes is reduced.

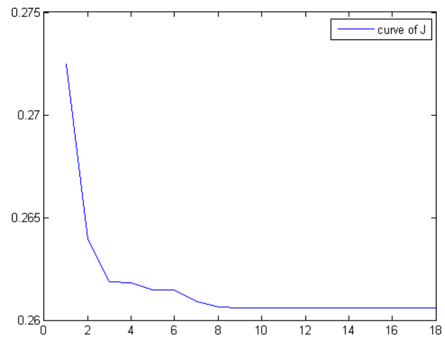

Figure 6. The cost functional J versus the number of iterations for the experiment of Figure 5 right (8 fluxes).

b. The case of unknown c_i but known n

We here consider the same experiment as the one of Figure 5 but we assume that the values of c_i, $i = 1, \ldots, 4$ are not known. We choose as initial guess a constant value and initiate the discontinuity points at $\alpha_1^{initial} = -\frac{2\pi}{3}, \alpha_2^{initial} = -\frac{\pi}{3}, \alpha_3^{initial} = \frac{\pi}{3}$ and $\alpha_4^{initial} = \frac{2\pi}{3}$. In the case of 1 or 2 fluxes the algorithm converges to a local minimum that is far from the exact solution. We observe that reasonable accuracy is obtained for the case of 4 and 8 fluxes. We also observed that the convergence for the discontinuity points is faster than the one for the discontinuity values. This means in particular that the direct problem is more sensitive to m_i than to c_i.

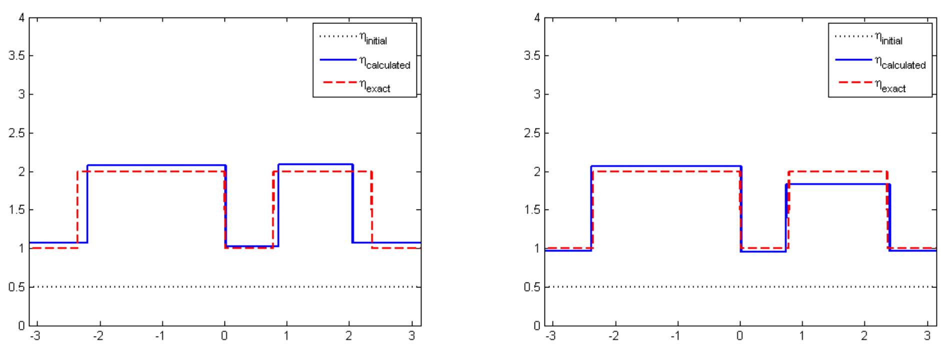

Figure 7. Values of η versus $\alpha \in [-\pi, \pi[$ for different number N of used fluxes: $N = 4$ left and $N = 8$ right. Exact η: dashed line. Initial guess: dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 2 for unknown values of c_i but known n. The noise level $\delta = 5\%$.

c. The case of unknown c_i and unknown n.

We still consider the same experiment as the one of Figure 5 or Figure 7. We now treat the case where one overestimates the number of discontinuity points n. We indicate in Figure 8 the obtained reconstructions using the first strategy, i.e. enforcing (23) to hold at each iteration, for two different values of the parameter α_* (0.2 and 0.5). For this example we initialize η with 6 discontinuity points that are indicated in Figure 8. In general, better reconstructions are obtained if we increase the value of α_*.

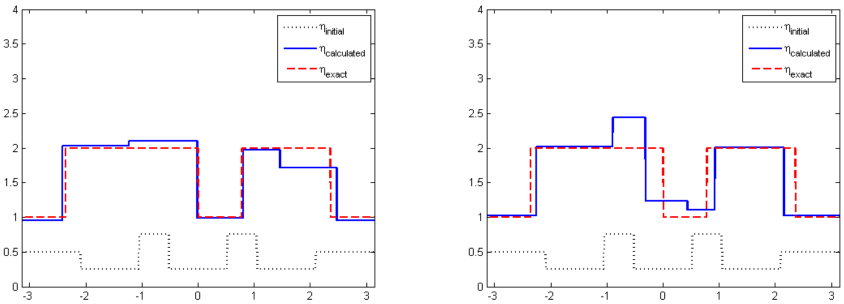

Figure 8. Values of η versus $\alpha \in [-\pi, \pi[$ for different values of α_* that controls the minimal distance between two discontinuity points and for a number of fluxes $N = 8$. Left: $\alpha_* = 0.2$. Right: $\alpha_* = 0.4$. Exact η: dashed line. Initial guess: dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 2 for unknown values of c_i and an over estimated initial number of discontinuity points $n = 6$. The noise level $\delta = 5\%$.

The second strategy, consisting in merging together two discontinuity points if the distance is less that α_* is tested in Figure 9. We choose $\alpha_* = 0.001$ and indicate the two steps at which this tolerance is reached. The value of c is then taken as the average value between the two intervals that have been merged. We clearly see that this procedure lead to better reconstructions.

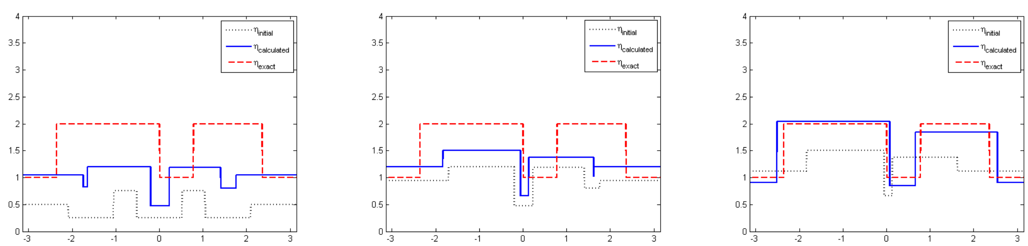

Figure 9. Values of η versus $\alpha \in [-\pi, \pi[$ at the iteration steps where a distance between two consecutive discontinuity points is less than $\alpha_* = 0.001$. Final reconstruction is on the right. Number of fluxes $N = 8$. Exact η: dashed line. Initial guess (at the current iteration): dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 2 for unknown values of c_i and an over estimated initial number of discontinuity points $n = 6$. The noise level $\delta = 5\%$.

A kite with stronger "non convexity" We end our discussion by reproducing the experiment of Figure 7 in the case where the geometry of the kite is modified as

$$\begin{cases} x(\alpha) = 0.6\cos(\alpha) + 0.3\cos(2\alpha) - 0.15, \\ y(\alpha) = 0.4\sin(\alpha). \end{cases}$$

The location of the discontinuity points is indicated in Figure 10. Let us emphasize

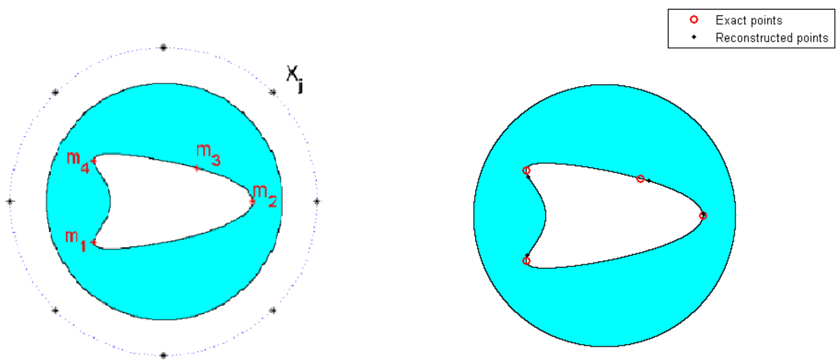

Figure 10. Description of geometry (left) and the reconstructed discontinuity points (right).

that in the case where the values of c_i are known, there is no notable difference with the previous case in terms of accuracy and stability of the reconstructions of the discontinuity points. However, when the values of c_i are not known, the algorithm becomes much more sensitive to the initial guess as the number of local minima significantly increases. This is particularly the case when discontinuity points are located in the non convex regions. This is what we consider in Figure 10. It appears that a possible path to avoid as much as possible local minima is to increase gradually the number of used fluxes. This is what we illustrate in Figure 11 where we start with two fluxes and use the final result as an initial guess for the case where we multiply the number of fluxes by 2 till reaching the case of 8 fluxes. While starting with 8 fluxes does not give satisfactory results, the current procedure provide a better reconstruction as attested by Figure 11-right. Let us notice that the accuracy of the reconstruction of the discontinuity points is better represented by Figure 10-right since our parametrization is not given in terms of the curvilinear abscissa.

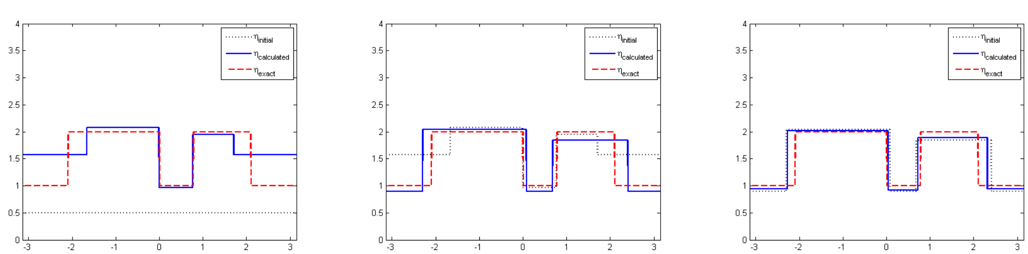

Figure 11. Values of η versus $\alpha \in [-\pi, \pi[$ when gradually increasing the number N of used fluxes: $N = 2$ left, $N = 4$ right and $N = 8$ right. Exact η: dashed line. Initial guess: dotted line. reconstructed η: solid line. This experiment is associated with the geometry of Figure 10-left for unknown values of c_i but known n. The noise level $\delta = 3\%$.

References

[1] Baratchart L., Bourgeois L. and Leblond J. Uniqueness results for inverse Robin problems with bounded coefficient preprint.

[2] 2012 Bourgeois L., Chaulet N. and Haddar H. 2012 On simultaneous identification of the shape and generalized impedance boundary condition in obstacle scattering *SIAM J. Sci. Comput.* **34(3)**.

[3] Bourgeois L. and Haddar H. 2010 Identification of generalized impedance boundary conditions in inverse scattering problems *Inverse Problems and Imaging* **4(1)** 19–38.

[4] Cakoni F., Hu Y. and Kress R. 2014 Simultaneous reconstruction of shape and generalized impedance functions in electrostatic imaging *Inverse Problems* **30** 105010.

[5] Cakoni F. and Kress R. 2013 Integral equation methods for the inverse obstacle problem with generalized impedance boundary condition *Inverse Problems* **29(1)** 015005.

[6] Chaabane S., Elhechmi C. and Jaoua M. 2004 A stable recovery algorithm for the Robin inverse problem *Math. Comput. Simul.* **66** 367-83.

[7] Chaabane S., Feki I. and Mars N. 2012 Numerical reconstruction of a piecewise constant Robin parameter in the two- or three-dimensional case *Inverse Problems* **28** 065016.

[8] Chaabane S., Ferchichi J. and Kunisch K. 2003 Differentiability properties of the L1-tracking functional and application to the Robin inverse problem *Inverse Problems* **20** 1083-97.

[9] Chaabane S. and Jaoua M. 1999 Identification of Robin coefficients by the means of boundary measurements *Inverse Problems* **15** 1425-1438.

[10] Chamaillard M., Chaulet N. and Haddar H. 2013 Analysis of the factorization method for a general class of boundary conditions *Journal of Inverse and Ill-posed Problems.*

[11] Duruflé M., Haddar H. and Joly P. 2006 Higher Order Generalized Impedance Boundary Conditions in Electromagnetic Scattering Problems *C.R. Physique* **7(5)** 533-542.

[12] Haddar, H. and Joly, P. and Nguyen, H.-M. 2005 Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case *Math. Models Methods Appl. Sci.* **15(8)** 1273–1300.

[13] Hecht F. 2012 New development in freefem++ *Journal of Numerical Mathmatics* **20** 251-265.

[14] Inglese G. 1997 An inverse problem in corrosion detection *Inverse Problems* **13** 977-94.

[15] Jin B. 2007 Conjugate gradient method for the Robin inverse problem associated with the Laplace equation *J. Numer. Methods Eng.* **71** 433-53.

[16] Kaup P., Santosa F. and Vogelius M. 1996 Method for imaging corrosion damage int hin plates frome lectrostatic data *Inverse Problems* **12** 279-93.

[17] Kohn R.V. and Vogelius M. 1987 Relaxation of a variational method for impedance computed tomography *Commun. Pure Appl. Math.* **40** 745-77.

[18] Senior T.B.A. and Volakis J.L. 1995 Approximate boundary conditions in electromagnetics *IEE Electromagnetic waves series* **41**.