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Abstract: In this research report, the benefits of channel-output feedback in the Gaussian interference channel (G-IC) are studied under the effect of additive Gaussian noise. Using a linear deterministic (LD) model, the signal to noise ratios (SNRs) in the feedback links beyond which feedback plays a significant role in terms of increasing the individual rates or the sum-rate are approximated. The relevance of this work lies on the fact that it identifies the feedback SNRs for which in any G-IC one of the following statements is true: (a) Feedback does not enlarge the capacity region; (b) Feedback enlarges the capacity region and the sum-rate is higher than the largest sum-rate without feedback; and (c) Feedback enlarges the capacity region but no significant improvement is observed in the sum-rate.

Key-words: Interference Channel, Noisy Channel-Output Feedback, Capacity Region.

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Quand est-ce que la rétro-alimentation améliore la region de capacité du canal à interférences?

Résumé : Dans ce rapport, l'impact du bruit additif sur les liens de rétro-alimentation dans le canal Gaussien à interférences est étudié en utilisant des approximations linéaires déterministes. Sous ces hypothèses, la valeur exacte du rapport signal à bruit (RSB) sur le lien de rétro-alimentation, au-delà de laquelle l'approximation linéaire déterministe de la région de capacité est améliorée, est caractérisée en fonction des RSB et des rapports interférences sur bruit (RIB). En général, trois scénarios peuvent être observés selon les valeurs exactes des RSB sur les liens directs et des RIBs: (a) L'utilisation de la rétro-alimentation est inutile pour améliorer la région de capacité; (b) L'utilisation de la rétro-alimentation améliore la région de capacité et la somme des taux de transmission; et (c) L'utilisation de la rétro-alimentation améliore la région de capacité mais la somme des taux de transmission n'est pas ameliorée.

Mots-clés : canal à interférences, rétroalimentation dégradée, Région de Capacité.

Contents

1 Introduction

The two-user Gaussian interference channel (G-IC) is the simplest channel model that captures the impairments brought by mutual interference into point-to-point communications subject to additive Gaussian noise. The interference channel (IC), in its most general form, was first proposed by Claude E. Shannon in [1]. The G-IC is a particular case that has been studied by several authors, see for instance [2–12] and references therein. However, despite this active research, the capacity region of the G-IC is characterized only in some special cases [3]. In general, the capacity region is not known exactly and only approximations to within a constant number of bits per channel-use per user are known [8].

On the other hand, channel-output feedback, which consists in letting a transmitter observe the channel-output at its intended receiver, was one of the first models for studying two-way point-to-point communications [13]. A G-IC with channel-output feedback is a model in which the backward direction (from receivers to transmitters) is exclusively used to let the transmitters observe the channel-output at the receivers with the goal of increasing the information rate or the reliability in the forward direction (from transmitters to receivers). Note that the backward direction may also be an IC since the point-to-point feedback links might be subject to mutual interference. There are several special cases of channel-output feedback in the G-IC. First, the case in which the observation of the channel-output from the intended receiver is noiseless corresponds to *perfect channel-output feedback* (POF) [14]. Second, the case in which such observation is noisy corresponds to *noisy channel-output feedback* (NOF) [15, 16]. Third, the case in which such observation is a linear combination of the channel-outputs from both receivers subject to additive noise corresponds to *wireless channel-output feedback* (WOF) [17]. The most general formulation is referred to as *general channel-output feedback* (GOF) [18–21]. Other types of feedback, including a channel-output processing, e.g., signal decoding, are known as *rate-limited feedback* (RLF) [22].

This work focuses in the case of G-IC with NOF (G-IC-NOF). One of the main motivations to focus on the G-IC-NOF stems from the recent findings regarding the impact of additive noise in the feedback links. In particular, in [15] and [16], it is shown that additive noise in the feedback links can dramatically change the number of generalized degrees of freedom (G-DoF) of the G-IC. In particular, one of the main benefits of feedback is that the number of G-DoF with perfect feedback increases monotonically with the interference to noise ratio (INR) in the very strong interference regime. However, in the presence of additive Gaussian noise in the feedback links, the number of G-DoF is bounded [15, 16].

From the discussion above a relevant question arises: *"When does channel-output feedback enlarge the capacity region of the G-IC?"* This paper provides the answer when feedback links are impaired by noise and free of mutual interference, i.e., G-IC-NOF. The desired answer is of the form: *"Implementing channel-output feedback in transmitter-receiver i enlarges the capacity region if the feedback SNR is bigger than* SNR_i^**"*, with $i \in \{1, 2\}$ and fixed SNRs and INRs in the forward G-IC. Note that the description of the capacity region of the G-IC-NOF in [16] does not provide an answer of the form mentioned above. An answer in the desired form requires some calculations that, despite the conceptual simplicity of this analysis, are long and tedious. More specifically, the value SNR_i^* is obtained by comparing the capacity region of the linear deterministic IC (LD-IC) in [8] and the capacity region of the LD-IC with noisy channel-output feedback (LD-IC-NOF) in [16] to identify the feedback parameters that ensure strict inclusion of the former into the latter. After, using the fact that the capacity region of the LD-IC-NOF approximates the capacity region of the G-IC-NOF, an approximation of SNR_i^* is obtained. Solving this problem leads to a handful of equally relevant byproducts to determine whether or not implementing feedback in one of the transmitter-receiver pairs increases any of the individual rates or the sum-

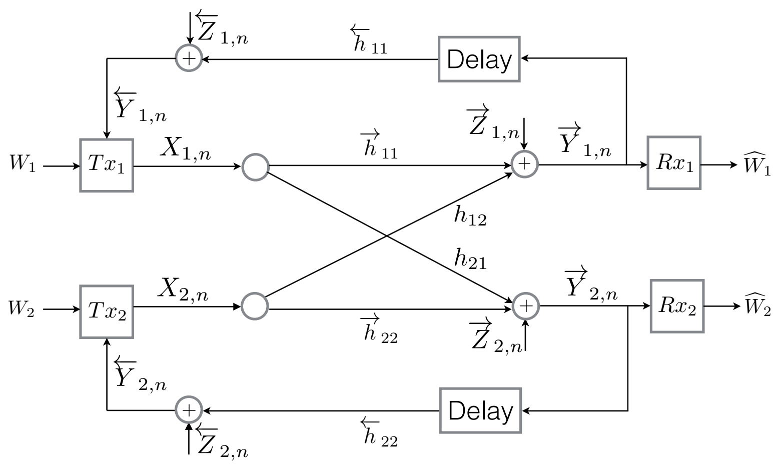

Figure 1: Gaussian interference channel with noisy channel-output feedback at channel use n.

rate. That is, answers to the following questions: *When does feedback in transmitter-receiver i allow achieving a rate R_1, such that for at least one R_2, all rate pairs (R_1', R_2) achievable without feedback satisfy $R_1 > R_1'$?*; *When does feedback in transmitter-receiver i allow achieving a rate R_2, such that for at least one R_1, all rate pairs (R_1, R_2') achievable without feedback satisfies $R_2 > R_2'$?*; or *When does feedback in transmitter-receiver i allow achieving a higher sum-rate than the maximum sum-rate achievable without feedback?*, with $i \in \{1, 2\}$ and fixed SNRs and INRs in the forward G-IC.

The answers to the questions above provide a lot of engineering insights about the benefits of feedback in the G-IC. For instance, all the cases in which feedback, even perfect channel-output feedback, is useless for increasing an individual rate or the sum-rate are identified. Similarly, this work provides guidelines for choosing in which of the point-to-point links feedback should be implemented for increasing either an individual rate or the sum-rate. Interestingly, in some cases, implementing feedback in only one of the transmitter-receiver pairs, despite the additive noise, turns out to be as beneficial as perfect channel-output feedback in both links.

2 Channel Models

2.1 Gaussian Interference Channels

Consider the two-user G-IC-NOF depicted in Figure 1. Transmitter i, with $i \in \{1, 2\}$, communicates with receiver i subject to the interference produced by transmitter j, with $j \in \{1, 2\} \backslash \{i\}$. There are two independent and uniformly distributed messages, $W_i \in \mathcal{W}_i$, with $\mathcal{W}_i = \{1, 2, \ldots, 2^{NR_i}\}$, where N denotes the fixed block-length in channel uses and R_i is the transmission rate in bits per channel use. At each block, transmitter i sends the codeword $\boldsymbol{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,N})^{\mathsf{T}} \in \mathcal{C}_i \subseteq \mathcal{X}_i^N$, where $\mathcal{X}_i$ and $\mathcal{C}_i$ are respectively the channel-input alphabet and the codebook of transmitter i.

The channel coefficient from transmitter i to receiver i is denoted by $\overrightarrow{h}_{ii}$, the channel coefficient from transmitter j to receiver i is denoted by h_{ij}; and the channel coefficient from channel-output i to transmitter i is denoted by $\overleftarrow{h}_{ii}$. All channel coefficients are assumed to be non-negative real numbers. At a given channel use $n \in \{1, 2, \ldots, N\}$, the channel output at receiver i is denoted by $\overrightarrow{Y}_{i,n}$. During channel use n, the input-output relation of the channel model is given by

$$\overrightarrow{Y}_{i,n}=\overrightarrow{h}_{ii}X_{i,n}+h_{ij}X_{j,n}+\overrightarrow{Z}_{i,n}, \tag{1}$$

where $\overrightarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver i. Let $d>0$ be the finite feedback delay measured in channel uses. At the end of channel use n, transmitter i observes $\overleftarrow{Y}_{i,n}$, which consists of a scaled and noisy version of $\overrightarrow{Y}_{i,n-d}$. More specifically,

$$\overleftarrow{Y}_{i,n}=\begin{cases}\overleftarrow{Z}_{i,n} & \text{for } n\in\{1,2,\ldots,d\}\\ \overleftarrow{h}_{ii}\overrightarrow{Y}_{i,n-d}+\overleftarrow{Z}_{i,n}, & \text{for } n\in\{d+1,d+2,\ldots,N\},\end{cases} \tag{2}$$

where $\overleftarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair i. The random variables $\overrightarrow{Z}_{i,n}$ and $\overleftarrow{Z}_{i,n}$ are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use, i.e., $d=1$. The encoder of transmitter i is defined by a set of deterministic functions $f_i^{(1)}, f_i^{(2)}, \ldots, f_i^{(N)}$, with $f_i^{(1)}: \mathcal{W}_i \rightarrow \mathcal{X}_i$ and for all $n\in\{2,3,\ldots,N\}$, $f_i^{(n)}: \mathcal{W}_i\times\mathbb{R}^{n-1}\rightarrow\mathcal{X}_i$, such that

$$X_{i,1}=f_i^{(1)}\left(W_i\right), \tag{3a}$$

and for all $n\in\{2,3,\ldots,N\}$,

$$X_{i,n}=f_i^{(n)}\left(W_i,\overleftarrow{Y}_{i,1},\overleftarrow{Y}_{i,2},\ldots,\overleftarrow{Y}_{i,n-1}\right). \tag{3b}$$

The components of the input vector $\boldsymbol{X}_i$ are real numbers subject to an average power constraint:

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left(X_{i,n}^2\right)\leq 1, \tag{4}$$

where the expectation is taken over the joint distribution of the message indices W_1, W_2, and the noise terms, i.e., $\overrightarrow{Z}_1$, $\overrightarrow{Z}_2$, $\overleftarrow{Z}_1$, and $\overleftarrow{Z}_2$. The dependence of $X_{i,n}$ on W_1, W_2, and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).

Hence, the decoder of receiver i is defined by a deterministic function $\psi_i: \mathbb{R}_i^N \rightarrow \mathcal{W}_i$. At the end of the communication, receiver i uses the vector $\left(\overrightarrow{Y}_{i,1}, \overrightarrow{Y}_{i,2}, \ldots, \overrightarrow{Y}_{i,N}\right)^{\mathsf{T}}$ to obtain an estimate of the message index:

$$\widehat{W}_i=\psi_i\left(\overrightarrow{Y}_{i,1},\overrightarrow{Y}_{i,2},\ldots,\overrightarrow{Y}_{i,N}\right), \tag{5}$$

where $\widehat{W}_i$ is an estimate of the message index. The decoding error probability in the two-user G-IC-NOF, denoted by $P_e(N)$, is given by

$$P_e(N)=\max\left(\Pr\left(\widehat{W_1}\neq W_1\right),\Pr\left(\widehat{W_2}\neq W_2\right)\right). \tag{6}$$

The definition of an achievable rate pair $(R_1,R_2)\in\mathbb{R}_+^2$ follows:

Definition 1 (Achievable Rate Pairs) A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks in $\mathcal{X}_1^N$ and in $\mathcal{X}_2^N$ with codewords of length N, the corresponding encoding functions $f_1^{(1)}, f_1^{(2)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, f_2^{(2)}, \ldots, f_2^{(N)}$, and the decoding functions ψ_1 and ψ_2, such that the decoding error probability can be made arbitrarily small by letting the block-length N grow to infinity.

The set of all achievable information rate pairs (R_1, R_2) is known as the information capacity region. The capacity region of a G-IC-NOF is described by six parameters: $\overrightarrow{\mathrm{SNR}}_i$, INR_{ij} and $\overleftarrow{\mathrm{SNR}}_i$, with $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$, which are defined as follows:

$$\overrightarrow{\mathrm{SNR}}_i = \overrightarrow{h}_{ii}^2, \tag{7}$$

$$\mathrm{INR}_{ij} = h_{ij}^2, \text{ and} \tag{8}$$

$$\overleftarrow{\mathrm{SNR}}_i = \overleftarrow{h}_{ii}^2 \left(\overrightarrow{h}_{ii}^2 + 2\overrightarrow{h}_{ii} h_{ij} + h_{ij}^2 + 1 \right). \tag{9}$$

Given fixed parameters $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$, the capacity region of the G-IC-NOF is approximated to within a constant number of bits by Theorem 4 in [16].

2.2 Linear Deterministic Interference Channels

Consider the two-user LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ depicted in Fig. 2. Parameter $\overrightarrow{n}_{ii}$ represents the number of bit-pipes between transmitter i and receiver i; parameter n_{ij} represents the number of bit-pipes between transmitter j and receiver i; and parameter $\overleftarrow{n}_{ii}$ represents the number of bit-pipes between receiver i and transmitter i (feedback).

At transmitter i, the channel-input $\boldsymbol{X}_{i,n}$ during channel use n, with $n \in \{1, 2, \ldots, N\}$, is a q-dimensional binary vector $\boldsymbol{X}_{i,n} = \left(X_{i,n}^{(1)}, X_{i,n}^{(2)}, \ldots, X_{i,n}^{(q)} \right)^{\mathsf{T}}$, where

$$q = \max\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21} \right), \tag{10}$$

and N is the block-length. At receiver i, the channel-output $\overrightarrow{\boldsymbol{Y}}_{i,n}$ during channel use n is also a q-dimensional binary vector $\overrightarrow{\boldsymbol{Y}}_{i,n} = \left(\overrightarrow{Y}_{i,n}^{(1)}, \overrightarrow{Y}_{i,n}^{(2)}, \ldots, \overrightarrow{Y}_{i,n}^{(q)} \right)^{\mathsf{T}}$. Let $\boldsymbol{S}$ be a $q \times q$ lower shift matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{11}$$

The input-output relation during channel use n is given by

$$\overrightarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{q-\overrightarrow{n}_{ii}} \boldsymbol{X}_{i,n} + \boldsymbol{S}^{q-n_{ij}} \boldsymbol{X}_{j,n}, \tag{12}$$

and the feedback signal $\overleftarrow{\boldsymbol{Y}}_{i,n}$ available at transmitter i at the end of channel use n satisfies

$$\overleftarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{\left(\max(\overrightarrow{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii} \right)^+} \overrightarrow{\boldsymbol{Y}}_{i,n-d}, \tag{13}$$

where d is a finite delay, additions and multiplications are defined over the Galois Field of two elements GF(2), and $(\cdot)^+$ is the positive part operator.

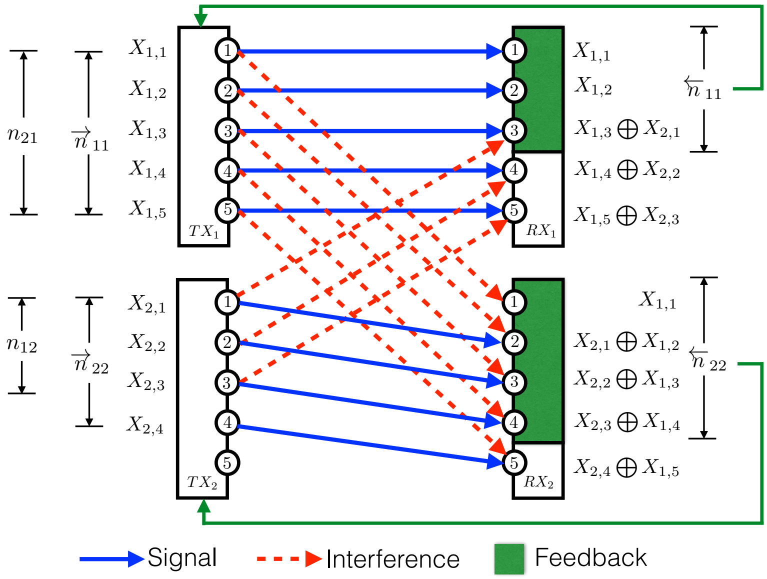

Figure 2: Two-user linear deterministic interference channel with noisy channel-output feedback. The bit-pipe line number 1 represents the most significant bit.

Without any loss of generality, the feedback delay is assumed to be equal to one channel use. Let $\mathcal{W}_i$ be the set of message indices of transmitter i. Transmitter i sends the message index $W_i \in \mathcal{W}_i$ by sending the codeword $\boldsymbol{X}_i = (\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \ldots, \boldsymbol{X}_{i,N})$, which is a binary $q \times N$ matrix. The encoder of transmitter i can be modeled as a set of deterministic mappings $f_i^{(1)}$, $f_i^{(2)}, \ldots, f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \rightarrow \{0,1\}^q$ and for all $n \in \{2, 3, \ldots, N\}$, $f_i^{(n)} : \mathcal{W}_i \times \{0,1\}^{q \times (n-1)} \rightarrow \{0,1\}^q$, such that

$$\boldsymbol{X}_{i,1} = f_i^{(1)}(W_i) \tag{14a}$$

and for all $n \in \{2, 3, \ldots, N\}$,

$$\boldsymbol{X}_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{\boldsymbol{Y}}_{i,1}, \overleftarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overleftarrow{\boldsymbol{Y}}_{i,n-1}\right). \tag{14b}$$

The decoder of receiver i is defined by a deterministic function $\psi_i : \{0,1\}^{q \times N} \rightarrow \mathcal{W}_i$. At the end of the communication, receiver i uses the sequence $\left(\overrightarrow{\boldsymbol{Y}}_{i,1}, \overrightarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overrightarrow{\boldsymbol{Y}}_{i,N}\right)$ to obtain an estimate $\widehat{W}_i$ of the message index W_i. The decoding error probability in the two-user LD-IC-NOF, denoted by $P_e(N)$, is given by (6).

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies Definition 1. The set of all achievable information rate pairs (R_1, R_2) is known as the information capacity region and it is characterized by Theorem 1 in [16].

2.3 Connections between Linear Deterministic and Gaussian Interference Channels

The capacity region of the G-IC-NOF with parameters $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$ can be approximated by the capacity region of an LD-IC-NOF with parameters $\overrightarrow{n}_{ii} =$

$\lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{SNR}}_i)\rfloor$; $n_{ij} = \lfloor \frac{1}{2}\log_2(\mathrm{INR}_{ij})\rfloor$; $\overleftarrow{n}_{ii} = \lfloor \frac{1}{2}\log_2(\overleftarrow{\mathrm{SNR}}_i)\rfloor$, with $i \in \{1,2\}$ and $j \in \{1,2\}\setminus\{i\}$. For instance, in the case without feedback, the capacity region of any G-IC with parameters $\overrightarrow{\mathrm{SNR}}_1 > 1$, $\overrightarrow{\mathrm{SNR}}_2 > 1$, $\mathrm{INR}_{12} > 1$ and $\mathrm{INR}_{21} > 1$ is within 18.6 bits per channel use per user of the capacity of an LD-IC with parameters $\overrightarrow{n}_{11} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{SNR}}_1)\rfloor$, $\overrightarrow{n}_{22} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{SNR}}_2)\rfloor$, $\overrightarrow{n}_{12} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{INR}}_{21})\rfloor$, and $\overrightarrow{n}_{21} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{INR}}_{21})\rfloor$ (Theorem 2 in [23]). More specifically, if the capacity region of the G-IC and LD-IC without feedback are denoted by $\mathcal{C}_{\mathrm{G}}$ and $\mathcal{C}_{\mathrm{LD}}$, respectively, the following holds:

$$\mathcal{C}_{\mathrm{LD}} \subseteq \mathcal{C}_{\mathrm{G}} + (5,5), \text{ and} \tag{15a}$$

$$\mathcal{C}_{\mathrm{G}} \subseteq \mathcal{C}_{\mathrm{LD}} + (13.6, 13.6). \tag{15b}$$

In a more general setting, for instance in the case with noisy channel-output feedback, the LD-IC is known to be a close approximation of the G-IC [16]. In Section 5, this approximation is used to simplify the identification of the cases in which channel-output feedback, even subject to additive noise, enlarges the capacity region of the G-IC.

3 Main Results

3.1 Preliminaries

Let $\alpha_i \in \mathbb{Q}$, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$ be defined as

$$\alpha_i = \frac{n_{ij}}{\overrightarrow{n}_{ii}}. \tag{16}$$

For each transmitter-receiver pair i, there exist five possible interference regimes (IRs), as suggested in [8]: the very weak IR (VWIR), i.e., $\alpha_i \leqslant \frac{1}{2}$, the weak IR (WIR), i.e., $\frac{1}{2} < \alpha_i \leqslant \frac{2}{3}$, the moderate IR (MIR), i.e., $\frac{2}{3} < \alpha_i < 1$, the strong IR (SIR), i.e., $1 \leqslant \alpha_i \leqslant 2$ and the very strong IR (VSIR), i.e., $\alpha_i > 2$. The scenarios in which the desired signal is stronger than the interference ($\alpha_i < 1$), namely the VWIR, the WIR, and the MIR, are referred to as the low-interference regimes (LIRs). Conversely, the scenarios in which the desired signal is weaker than or equal to the interference ($\alpha_i \geqslant 1$), namely the SIR and the VSIR, are referred to as the high-interference regimes (HIRs).

The main results of this paper are presented using a set of events (Boolean variables) that are determined by the parameters $\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}$, and n_{21}. Given a fixed tuple ($\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}), the events are defined below:

$$E_1: \quad \alpha_1 < 1 \wedge \alpha_2 < 1, \tag{17}$$

$$E_{2,i}: \quad \alpha_i \leqslant \frac{1}{2} \wedge 1 \leqslant \alpha_j \leqslant 2, \tag{18}$$

$$E_{3,i}: \quad \alpha_i \leqslant \frac{1}{2} \wedge \alpha_j > 2, \tag{19}$$

$$E_{4,i}: \quad \frac{1}{2} < \alpha_i \leqslant \frac{2}{3} \wedge \alpha_j \geqslant 1, \tag{20}$$

$$E_{5,i}: \quad \frac{2}{3} < \alpha_i < 1 \wedge \alpha_j \geqslant 1, \tag{21}$$

$$E_{6,i}: \quad \frac{1}{2} < \alpha_i \leqslant 1 \wedge \alpha_j > 1, \tag{22}$$

$$E_{7,i}: \quad \alpha_i \geqslant 1 \wedge \alpha_j \leqslant 1, \tag{23}$$

$$E_{8,i}: \quad \overrightarrow{n}_{ii} > n_{ji}, \tag{24}$$

$$E_9: \quad \overrightarrow{n}_{11} + \overrightarrow{n}_{22} > n_{12} + n_{21}, \tag{25}$$

$$E_{10,i}: \quad \overrightarrow{n}_{ii} + \overrightarrow{n}_{jj} > n_{ij} + 2n_{ji}, \tag{26}$$
$$E_{11,i}: \quad \overrightarrow{n}_{ii} + \overrightarrow{n}_{jj} < n_{ij}. \tag{27}$$

In the following, given an event, e.g. $E_{8,i}$: $\overrightarrow{n}_{ii} > n_{ji}$, the notation $\widetilde{E}_{8,i}$ indicates $\overrightarrow{n}_{ii} < n_{ji}$; the notation $\overline{E}_{8,i}$ indicates $\overrightarrow{n}_{ii} \leqslant n_{ji}$ (logical complement); and the notation $\check{E}_{8,i}$ indicates $\overrightarrow{n}_{ii} \geqslant n_{ji}$.

Combining the events (17)-(27), five main scenarios are identified:

$$S_{1,i}: \quad (E_1 \wedge E_{8,i}) \vee (E_{2,i} \wedge E_{8,i}) \vee (E_{3,i} \wedge E_{8,i} \wedge E_9) \vee (E_{4,i} \wedge E_{8,i} \wedge E_9) \vee (E_{5,i} \wedge E_{8,i} \wedge E_9), \tag{28}$$
$$S_{2,i}: \quad \left(E_{3,i} \wedge \widetilde{E}_{8,j} \wedge \overline{E}_9\right) \vee \left(E_{6,i} \wedge \widetilde{E}_{8,j} \wedge \overline{E}_9\right) \vee \left(\widetilde{E}_1 \wedge \widetilde{E}_{8,j}\right), \tag{29}$$
$$S_{3,i}: \quad \left(E_1 \wedge \overline{E}_{8,i}\right) \vee \left(E_{2,i} \wedge \overline{E}_{8,i}\right) \vee \left(E_{3,i} \wedge \check{E}_{8,j} \wedge \overline{E}_{8,i}\right) \vee \left(E_{4,i} \wedge \check{E}_{8,j} \wedge \overline{E}_{8,i}\right)$$
$$\vee \left(E_{5,i} \wedge \check{E}_{8,j} \wedge \overline{E}_{8,i}\right) \vee \left(\overline{E}_1 \wedge \check{E}_{8,j}\right) \vee \left(E_{7,i}\right), \tag{30}$$
$$S_4: \quad E_1 \wedge E_{8,1} \wedge E_{8,2} \wedge E_{10,1} \wedge E_{10,2}, \tag{31}$$
$$S_5: \quad \overline{E}_1 \wedge E_{11,1} \wedge E_{11,2}. \tag{32}$$

For all $i \in \{1, 2\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, S_4 and S_5 exhibit the properties stated by the following corollaries.

Corollary 1 For all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, given a fixed $i \in \{1, 2\}$, only one of the events $S_{1,i}$, $S_{2,i}$ and $S_{3,i}$ is true.

Corollary 2 For all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, when one of the events S_4 or S_5 holds true, then the other necessarily holds false.

Note that Corollary 2 does not exclude the case in which both S_4 and S_5 are simultaneously false.

Corollary 3 For all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, when S_4 holds true, then both $S_{1,1}$ and $S_{1,2}$ hold true; and when S_5 holds true, then both $S_{2,1}$ and $S_{2,2}$ hold true.

3.2 Rate Improvement Metrics

Given a fixed tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, let $\mathcal{C}(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ be the capacity region of an LD-IC with noisy channel-output feedback with parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. The maximum improvement of the individual rates R_1 and R_2, denoted by $\Delta_1(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and $\Delta_2(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})$, due to the effect of channel-output feedback with respect to the case without feedback is

$$\Delta_1(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \max_{0 < R_2 < R_2^*} \left\{ \sup_{(R_1, R_2) \in \mathcal{C}(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})} R_1 - \sup_{(R_1^\dagger, R_2) \in \mathcal{C}(0,0)} R_1^\dagger \right\} \text{ and} \tag{33}$$
$$\Delta_2(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \max_{0 < R_1 < R_1^*} \left\{ \sup_{(R_1, R_2) \in \mathcal{C}(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})} R_2 - \sup_{(R_1, R_2^\dagger) \in \mathcal{C}(0,0)} R_2^\dagger \right\}, \tag{34}$$

with

$$R_1^* = \sup_{(r_1, r_2) \in \mathcal{C}(0,0)} r_1 \text{ and} \tag{35}$$
$$R_2^* = \sup_{(r_1, r_2) \in \mathcal{C}(0,0)} r_2. \tag{36}$$

Note that for a fixed $i \in \{1,2\}$, $\Delta_i(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) > 0$ if and only if it is possible to achieve a rate pair (R_1, R_2) with channel-output feedback such that R_i is higher than the maximum rate achievable by transmitter-receiver i without feedback when the rate of transmitter-receiver pair j is fixed at R_j. In the following, given fixed parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, the statement *"the rate R_i is improved by using feedback"* is used to indicate that $\Delta_i(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) > 0$.

Alternatively, the maximum improvement of the sum-rate $\Sigma(\overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ with respect to the case without feedback is

$$\Sigma(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \sup_{(R_1,R_2)\in\mathcal{C}(\overleftarrow{n}_{11},\overleftarrow{n}_{22})} \left\{ R_1 + R_2 \right\} - \sup_{(R_1^{\dagger},R_2^{\dagger})\in\mathcal{C}(0,0)} \left\{ R_1^{\dagger} + R_2^{\dagger} \right\}. \tag{37}$$

Note that $\Sigma(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) > 0$ if and only if there exists a rate pair with feedback whose sum is higher than the maximum sum-rate achievable without feedback. In the following, given fixed parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, the statement *"the sum-rate is improved by using feedback"* is used to imply that $\Sigma(\overleftarrow{n}_{11}, \overleftarrow{n}_{22}) > 0$.

In the following, when feedback is exclusively used by transmitter-receiver pair i, i.e., $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, then the maximum improvement of the individual rate of transmitter-receiver k, with $k \in \{1,2\}$, and the maximum improvement of the sum-rate are denoted by $\Delta_k(\overleftarrow{n}_{ii})$ and $\Sigma(\overleftarrow{n}_{ii})$, respectively. Hence, this notation $\Delta_k(\overleftarrow{n}_{ii})$ replaces either $\Delta_k(\overleftarrow{n}_{11}, 0)$ or $\Delta_k(0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. The same holds for the notation $\Sigma(\overleftarrow{n}_{ii})$ that replaces $\Sigma(\overleftarrow{n}_{11}, 0)$ or $\Sigma(0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively.

3.3 Enlargement of the Capacity Region

Given fixed parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, $i \in \{1,2\}$, and $j \in \{1,2\} \setminus \{i\}$, the capacity region of a two-user LD-IC, when feedback is available only at transmitter-receiver pair i, i.e., $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, is denoted by $\mathcal{C}(\overleftarrow{n}_{ii})$ instead of $\mathcal{C}(\overleftarrow{n}_{11}, 0)$ or $\mathcal{C}(0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. Following this notation, Theorem 1 identifies the exact values of $\overleftarrow{n}_{ii}$ for which the strict inclusion $\mathcal{C}(0,0) \subset \mathcal{C}(\overleftarrow{n}_{ii})$ holds for $i \in \{1,2\}$.

Theorem 1 Let $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ be a fixed tuple. Let also $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ and $\overleftarrow{n}_{ii}^{*} \in \mathbb{N}$ be fixed integers, with

$$\overleftarrow{n}_{ii}^{*} = \begin{cases} \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right) & \text{if } S_{1,i} = \text{True} \\ \overrightarrow{n}_{jj} + (\overrightarrow{n}_{ii} - n_{ij})^{+} & \text{if } S_{2,i} = \text{True}. \end{cases} \tag{38}$$

Assume that $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\mathcal{C}\Big(0,0\Big) = \mathcal{C}\Big(\overleftarrow{n}_{ii}\Big)$. Assume that either $S_{1,i} = \text{True}$ or $S_{2,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}_{ii}^{*}$, $\mathcal{C}\Big(0,0\Big) = \mathcal{C}\Big(\overleftarrow{n}_{ii}\Big)$ and for all $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{*}$, $\mathcal{C}\Big(0,0\Big) \subset \mathcal{C}\Big(\overleftarrow{n}_{ii}\Big)$.

Proof: The proof of Theorem 1 is presented in Appendix A. ■

Theorem 1 shows that under event $S_{3,i}$ in (30), implementing feedback in transmitter-receiver pair i, with any $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, does not enlarge the capacity region. Note that when both $E_{8,i}$ and $\widetilde{E}_{8,j}$ hold false, then both $S_{1,i}$ and $S_{2,i}$ hold false, which implies that $S_{3,i}$ holds true (Corollary 1). The following remark is a consequence of this observation.

Remark 1: *A necessary but not sufficient condition for enlarging the capacity region by using feedback in transmitter-receiver pair i is: there exists at least one transmitter able to send more information bits to receiver i than to receiver j, i.e., $\overrightarrow{n}_{ii} > n_{ji}$ (Event $E_{8,i}$) or $n_{ij} > \overrightarrow{n}_{jj}$ (Event $\widetilde{E}_{8,j}$).*

Alternatively, under events $S_{1,i}$ in (28) and $S_{2,i}$ in (29), the capacity region can be enlarged when $\overleftarrow{n}_{ii} > \overleftarrow{n}^{*}_{ii}$. It is important to highlight that in the cases in which feedback enlarges the capacity region of the two-user LD-IC-NOF, that is, in events $S_{1,1}$, $S_{2,1}$, $S_{1,2}$ or $S_{2,2}$, for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, the following is always true :

$$\overleftarrow{n}^{*}_{ii} > (\overrightarrow{n}_{ii} - n_{ij})^{+}. \tag{39}$$

Essentially, the inequality in (39) unveils a necessary but not sufficient condition to enlarge the capacity region using channel-output feedback. This condition is that for at least one $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$, transmitter i decodes a subset of the information bits sent by transmitter j at each channel use.

Another interesting observation is that the threshold $\overleftarrow{n}^{*}_{ii}$ beyond which feedback is useful is different under event $S_{1,i}$ in (28) and event $S_{2,i}$ in (29). In general when $S_{1,i}$ holds true, the enlargement of the capacity region is due to the fact that feedback allows using *interference as side information* [25]. Alternatively, when $S_{2,i}$ in (29) holds true, the enlargement of the capacity region occurs as a consequence of the fact that some of the bits that cannot be transmitted directly from transmitter j to receiver j, can arrive to receiver j via an alternative path: transmitter j - receiver i - transmitter i - receiver j. Both scenarios, interference as side information and alternative path, are extensively discussed in [14], [15], and [16].

3.4 Improvement of the Individual Rate R_i by Using Feedback in Link i

Given fixed parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, and $i \in \{1,2\}$, implementing channel-output feedback in transmitter-receiver pair i increases the individual rate R_i, i.e., $\Delta_i(\overleftarrow{n}_{ii}) > 0$ for some values of $\overleftarrow{n}_{ii}$. Theorem 2 identifies the exact values of $\overleftarrow{n}_{ii}$ for which $\Delta_i(\overleftarrow{n}_{ii}) > 0$.

Theorem 2 Let $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ be a fixed tuple. Let also $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ and $\overleftarrow{n}^{\dagger}_{ii} \in \mathbb{N}$ be fixed integers, with

$$\overleftarrow{n}^{\dagger}_{ii} = \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right). \tag{40}$$

Assume that either $S_{2,i}$ = True or $S_{3,i}$ = True. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_i(\overleftarrow{n}_{ii}) = 0$. Assume that $S_{1,i}$ = True. Then, when $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}^{\dagger}_{ii}$, it holds that $\Delta_i(\overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}^{\dagger}_{ii}$, it holds that $\Delta_i(\overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 2 is presented in Appendix B. ■

Theorem 2 highlights that under events $S_{2,i}$ in (29) and $S_{3,i}$ in (30), the individual rate R_i cannot be improved by using feedback in transmitter-receiver pair i, i.e., $\Delta_i(\overleftarrow{n}_{ii}) = 0$. Alternatively, under event $S_{1,i}$ in (28), the individual rate R_i can be improved, i.e., $\Delta_i\left(\overleftarrow{n}_{ii}\right) > 0$, whenever $\overleftarrow{n}_{ii} > \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right)$. Hence, given the definition of $S_{1,i}$, the following remark is relevant.

Remark 2: *A necessary but not sufficient condition for $\Delta_i\left(\overleftarrow{n}_{ii}\right) > 0$ is: the number of bit-pipes from transmitter i to receiver i is higher than the number of bit-pipes from transmitter i to receiver j, i.e., $\overrightarrow{n}_{ii} > n_{ji}$ (Event $E_{8,i}$)*

3.5 Improvement of the Individual Rate R_j by Using Feedback in Link i

Given fixed parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, $i \in \{1,2\}$, and $j \in \{1,2\} \setminus \{i\}$, implementing channel-output feedback in transmitter-receiver pair i increases the individual rate R_j, i.e.,

$\Delta_j(\overleftarrow{n}_{ii}) > 0$ for some values of $\overleftarrow{n}_{ii}$. Theorem 3 identifies the exact values of $\overleftarrow{n}_{ii}$ for which $\Delta_j(\overleftarrow{n}_{ii}) > 0$.

Theorem 3 Let $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ be a fixed tuple. Let also $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ and $\overleftarrow{n}^*_{ii} \in \mathbb{N}$ given in (38), be fixed integers. Assume that $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_j(\overleftarrow{n}_{ii}) = 0$. Assume that either $S_{1,i} = \text{True}$ or $S_{2,i} = \text{True}$. Then, when $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}^*_{ii}$, it holds that $\Delta_j(\overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}^*_{ii}$, it holds that $\Delta_j(\overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 3 follows along the same lines of the proof of Theorem 2 in Appendix B. ■

Theorem 3 shows that under event $S_{3,i}$ in (30), implementing feedback in transmitter-receiver pair i does not bring any improvement on the rate R_j. This is in line with the results of Theorem 1. In contrast, under events $S_{1,i}$ in (28) and $S_{2,i}$ in (29), the individual rate R_j can be improved, i.e., $\Delta_j(\overleftarrow{n}_{ii}) > 0$ for all $\overleftarrow{n}_{ii} > \overleftarrow{n}^*_{ii}$. From the definition of events $S_{1,i}$ and $S_{2,i}$, the following remark holds:

Remark 3: *A necessary but not sufficient condition for* $\Delta_j\left(\overleftarrow{n}_{ii}\right) > 0$ *is: there exists at least one transmitter able to send more information bits to receiver i than to receiver j, i.e.,* $\overrightarrow{n}_{ii} > n_{ji}$ *(Event* $E_{8,i}$*) or* $n_{ij} > \overrightarrow{n}_{jj}$ *(Event* $\widetilde{E}_{8,j}$*).*

It is important to highlight that under event $S_{1,i}$, the threshold on $\overleftarrow{n}_{ii}$ for increasing the individual rate R_i i.e., $\overleftarrow{n}^\dagger_{ii}$, and R_j i.e., $\overleftarrow{n}^*_{ii}$, are identical, see Theorem 2 and Theorem 3. This implies that in this case, the use of feedback in transmitter-receiver pair i, with $\overleftarrow{n}_{ii} > \overleftarrow{n}^\dagger_{ii} = \overleftarrow{n}^*_{ii}$, benefits both transmitter-receiver pairs, i.e., $\Delta_i(\overleftarrow{n}_{ii}) > 0$ and $\Delta_j(\overleftarrow{n}_{ii}) > 0$. Under event $S_{2,i}$, using feedback in transmitter-receiver pair i, with $\overleftarrow{n}_{ii} > \overleftarrow{n}^*_{ii}$, exclusively benefits transmitter-receiver pair j, i.e., $\Delta_i(\overleftarrow{n}_{ii}) = 0$ and $\Delta_j(\overleftarrow{n}_{ii}) > 0$.

3.6 Improvement of the Sum-Rate

Given fixed parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, and $i \in \{1,2\}$, implementing channel-output feedback in transmitter-receiver pair i increases the sum-rate, i.e., $\Sigma(\overleftarrow{n}_{ii}) > 0$ for some values of $\overleftarrow{n}_{ii}$. Theorem 4 identifies the exact values of $\overleftarrow{n}_{ii}$ for which $\Sigma(\overleftarrow{n}_{ii}) > 0$.

Theorem 4 Let $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ be a fixed tuple. Let also $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ and $\overleftarrow{n}^+_{ii} \in \mathbb{N}$ be fixed integers, with

$$\overleftarrow{n}^+_{ii} = \begin{cases} \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^+\right) & \text{if } S_4 = \text{True} \\ \overrightarrow{n}_{jj} + (\overrightarrow{n}_{ii} - n_{ij})^+ & \text{if } S_5 = \text{True}. \end{cases} \tag{41}$$

Assume that $S_4 = \text{False}$ and $S_5 = \text{False}$. Then, $\Sigma(\overleftarrow{n}_{ii}) = 0$ for all $\overleftarrow{n}_{ii} \in \mathbb{N}$. Assume that $S_4 = \text{True}$ or $S_5 = \text{True}$. Then, when $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}^+_{ii}$, it holds that $\Sigma(\overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}^+_{ii}$, it holds that $\Sigma(\overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 4 is presented in Appendix C. ■

Theorem 4 highlights a necessary but not sufficient condition for improving the sum-rate by implementing feedback in transmitter-receiver pair i.

Remark 4: *A necessary but not sufficient condition for observing* $\Sigma(\overleftarrow{n}_{ii}) > 0$ *is to satisfy one of the following conditions: (a) both transmitter-receiver pairs are in LIR (Event* E_1*); or (b) both transmitter-receiver pairs are in HIR (Event* $\overline{E}_1$*).*

Finally, it follows from Corollary 3 that when S_4 or S_5 holds true, with $i \in \{1,2\}$ and $\overleftarrow{n}_{ii} > \overleftarrow{n}^+_{ii}$, aside from the fact that $\Sigma(\overleftarrow{n}_{ii}) > 0$, it also holds that $\Delta_1(\overleftarrow{n}_{ii}) > 0$ and $\Delta_2(\overleftarrow{n}_{ii}) > 0$.

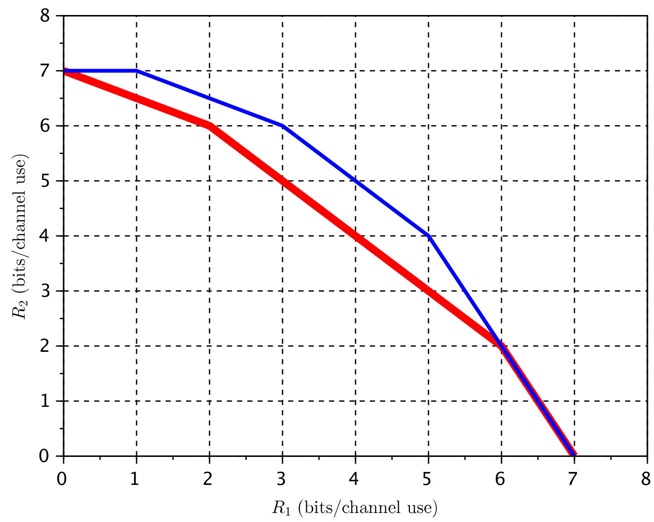

Figure 3: Capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(6,0)$ (thin blue line), with $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 7$, $n_{12} = 3$, $n_{21} = 5$.

4 Examples

Example 1 Consider an LD-IC-NOF with parameters $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 7$, $n_{12} = 3$, and $n_{21} = 5$.

In Example 1, both $S_{1,1}$ and $S_{1,2}$ hold true. Hence, from Theorem 1, when $\overleftarrow{n}_{11} > 5$ or $\overleftarrow{n}_{22} > 3$, there always exists an enlargement of the capacity region. More specifically, it follows from Theorem 2 and Theorem 3 that using feedback in transmitter-receiver pair 1, with $\overleftarrow{n}_{11} > 5$ or using feedback in transmitter-receiver pair 2, with $\overleftarrow{n}_{22} > 3$, both individual rates can be simultaneously improved, i.e., $\Delta_1(\overleftarrow{n}_{ii}) > 0$ and $\Delta_2(\overleftarrow{n}_{ii}) > 0$ with $i = 1$ or $i = 2$ respectively. Alternatively, note that S_4 holds true. Hence, it follows from Theorem 4 that using feedback in transmitter-receiver pair 1, with $\overleftarrow{n}_{11} > 5$ or using feedback in transmitter-receiver pair 2, with $\overleftarrow{n}_{22} > 3$, improves the sum-rate, i.e., $\Sigma(\overleftarrow{n}_{ii}) > 0$ with $i = 1$ or $i = 2$ respectively. These conclusions are observed in Figure 3, for the case $\overleftarrow{n}_{11} = 6$ and $\overleftarrow{n}_{22} = 0$, where the capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(6,0)$ (thin blue line) are plotted. Note that, when $\overleftarrow{n}_{11} = 6$, there always exist a rate pair $(R'_1, R'_2) \in \mathcal{C}(0,0)$ and a rate pair $(R_1, R_2) \in \mathcal{C}(6,0) \setminus \mathcal{C}(0,0)$ such that $R'_1 < R_1$ and $R'_2 = R_2$ (Theorem 2). Simultaneously, there always exist a rate pair $(R'_1, R'_2) \in \mathcal{C}(0,0)$ and a rate pair $(R_1, R_2) \in \mathcal{C}(6,0) \setminus \mathcal{C}(0,0)$ such that $R'_2 < R_2$ and $R'_1 = R_1$ (Theorem 3). Finally, note that for all rate pairs $(R'_1, R'_2) \in \mathcal{C}(0,0)$ there always exists a rate pair $(R_1, R_2) \in \mathcal{C}(6,0)$, for which $R_1 + R_2 > R'_1 + R'_2$ (Theorem 4).

Example 2 Consider an LD-IC-NOF with parameters $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 8$, $n_{12} = 6$, and $n_{21} = 5$.

In Example 2, the events $S_{1,1}$ and $S_{1,2}$ hold true; and the events S_4 and S_5 hold false. Hence, it follows from Theorem 4 that using feedback in either transmitter-receiver pair does not improve the sum-rate, i.e., for all $i \in \{1, 2\}$ and for all $\overleftarrow{n}_{ii} > 0$, $\Sigma(\overleftarrow{n}_{ii}) = 0$. These conclusions are observed in Figure 4, for the case $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 7$, where the capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(0,7)$ (thin blue line) are plotted. From Example 2, it becomes evident that

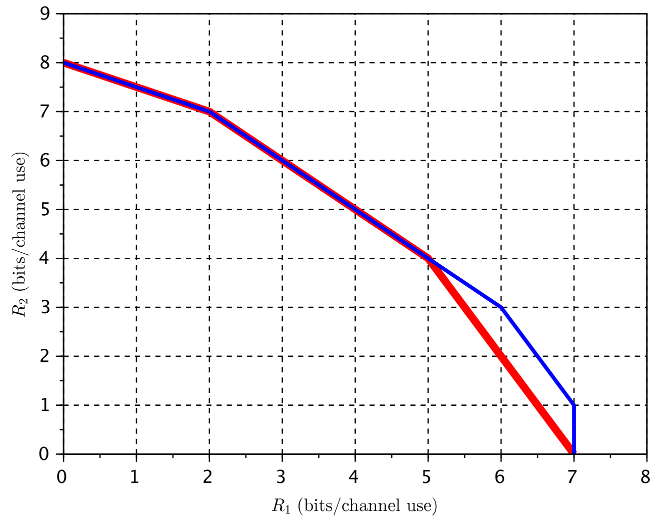

Figure 4: Capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(0,7)$ (thin blue line), with $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 8$, $n_{12} = 6$, $n_{21} = 5$.

when $S_{1,1}$ and $S_{1,2}$ hold true, S_4 and S_5 do not necessarily hold true. That is, the improvements on the individual rates, despite that they can be observed simultaneously, are not enough to improve the sum-rate beyond what is already achievable without feedback.

Example 3 Consider an LD-IC-NOF with parameters $\overrightarrow{n}_{11} = 5$, $\overrightarrow{n}_{22} = 1$, $n_{12} = 3$, and $n_{21} = 4$.

In Example 3, both $S_{2,1}$ in (29) and $S_{3,2}$ in (30) hold true. Hence, it follows from Theorem 1 that the capacity region can be enlarged by using feedback in transmitter-receiver pair 1 when $\overleftarrow{n}_{11} > 3$, whereas using feedback in transmitter-receiver pair 2 is useless. More specifically, it follows from Theorem 2 and Theorem 3 that using feedback in transmitter-receiver pair 1 does not improve the individual rate R_1 but R_2, i.e., $\Delta_1(\overleftarrow{n}_{11}) = 0$ and $\Delta_2(\overleftarrow{n}_{11}) > 0$. Note also that S_4 and S_5 hold false. Hence, it follows from Theorem 4 that using feedback in either transmitter-receiver pair does not improve the sum-rate, i.e., $\Sigma(\overleftarrow{n}_{11}) = 0$ and $\Sigma(\overleftarrow{n}_{22}) = 0$. These conclusions are observed in Figure 5, for the case $\overleftarrow{n}_{11} = 4$ and $\overleftarrow{n}_{22} = 0$, where the capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(4,0)$ (thin blue line) are plotted.

5 Implications on the Gaussian Interference Channel

Given a fixed tuple $\left(\overrightarrow{\mathrm{SNR}}_1, \overrightarrow{\mathrm{SNR}}_2, \mathrm{INR}_{12}, \mathrm{INR}_{21}\right)$, let $\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ be the achievable region of the G-IC-NOF described by Theorem 2 in [16] with parameters $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$; let $\overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ be the converse region of the G-IC-NOF described by Theorem 3 in [16] with parameters $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$; and let also $\mathcal{C}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ be the capacity region of the G-IC-NOF with parameters $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$.

These regions satisfy the following inclusions:

$$\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) \subseteq \mathcal{C}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) \subseteq \overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2). \tag{42}$$

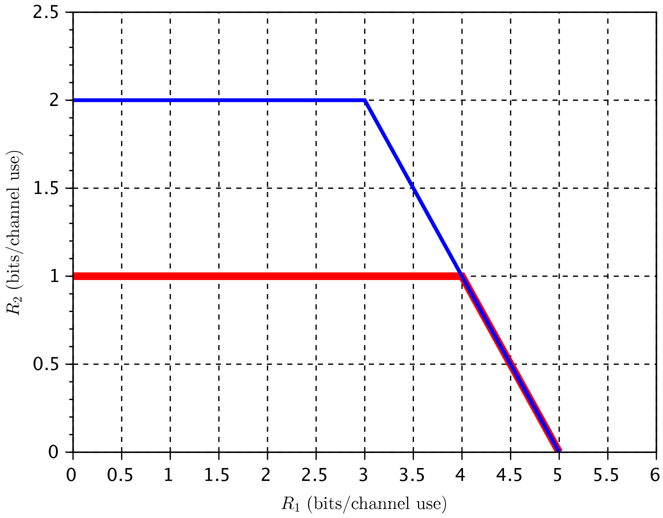

Figure 5: Capacity regions $\mathcal{C}(0,0)$ (thick red line) and $\mathcal{C}(4,0)$ (thin blue line), with $\overrightarrow{n}_{11} = 5$, $\overrightarrow{n}_{22} = 1$, $n_{12} = 3$, $n_{21} = 4$.

5.1 Improvement Metrics

In order to quantify the benefits of channel-output feedback in enlarging the achievable region $\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ or the converse region $\overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$, consider the following improvement metrics, which are similar to those defined in Sec. 3.2 for the LD-IC-NOF. The improvement metrics on the individual rates are defined as

$$\Delta_1^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \max_{0<R_2<R_2^*} \left\{ \sup_{(R_1,R_2)\in\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \{R_1\} - \sup_{(R_1^\dagger,R_2)\in\underline{\mathcal{R}}(0,0)} \{R_1^\dagger\} \right\}, \tag{43}$$

$$\Delta_2^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \max_{0<R_1<R_1^*} \left\{ \sup_{(R_1,R_2)\in\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \{R_2\} - \sup_{(R_1,R_2^\dagger)\in\underline{\mathcal{R}}(0,0)} \{R_2^\dagger\} \right\}, \tag{44}$$

$$\Delta_1^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \max_{0<R_2<R_2^\dagger} \left\{ \sup_{(R_1,R_2)\in\overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \{R_1\} - \sup_{(R_1^\dagger,R_2)\in\overline{\mathcal{R}}(0,0)} \{R_1^\dagger\} \right\}, \text{ and} \tag{45}$$

$$\Delta_2^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \max_{0<R_1<R_1^\dagger} \left\{ \sup_{(R_1,R_2)\in\overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \{R_2\} - \sup_{(R_1,R_2^\dagger)\in\overline{\mathcal{R}}(0,0)} \{R_2^\dagger\} \right\}, \tag{46}$$

with

$$R_1^* = \sup_{(r_1,r_2)\in\underline{\mathcal{R}}(0,0)} r_1, \tag{47}$$

$$R_2^* = \sup_{(r_1,r_2)\in\underline{\mathcal{R}}(0,0)} r_2, \tag{48}$$

$$R_1^\dagger = \sup_{(r_1,r_2)\in\overline{\mathcal{R}}(0,0)} r_1, \text{ and} \tag{49}$$

$$R_2^\dagger = \sup_{(r_1,r_2)\in\overline{\mathcal{R}}(0,0)} r_2. \tag{50}$$

Alternatively, the maximum improvements of the sum-rate $\Sigma^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ and $\Sigma^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ with respect to the case without feedback are

$$\Sigma^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \sup_{(R_1,R_2)\in\underline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \left\{ R_1 + R_2 \right\} - \sup_{(R_1^{\dagger},R_2^{\dagger})\in\underline{\mathcal{R}}(0,0)} \left\{ R_1^{\dagger} + R_2^{\dagger} \right\}, \text{ and} \quad (51)$$

$$\Sigma^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2) = \sup_{(R_1,R_2)\in\overline{\mathcal{R}}(\overleftarrow{\mathrm{SNR}}_1,\overleftarrow{\mathrm{SNR}}_2)} \left\{ R_1 + R_2 \right\} - \sup_{(R_1^{\dagger},R_2^{\dagger})\in\overline{\mathcal{R}}(0,0)} \left\{ R_1^{\dagger} + R_2^{\dagger} \right\}. \quad (52)$$

5.2 Approximate Thresholds on the Feedback SNRs

In Sec. 2.3, the connections between the LD-IC-NOF and the G-IC-NOF were discussed. Using these connections, a G-IC with fixed parameters $\left(\overrightarrow{\mathrm{SNR}}_1, \overrightarrow{\mathrm{SNR}}_2, \mathrm{INR}_{12}, \mathrm{INR}_{21}\right)$ is approximated by an LD-IC with parameters $\overrightarrow{n}_{11} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{SNR}}_1)\rfloor$, $\overrightarrow{n}_{22} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{SNR}}_2)\rfloor$, $\overrightarrow{n}_{12} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{INR}}_{21})\rfloor$ and $\overrightarrow{n}_{21} = \lfloor \frac{1}{2}\log_2(\overrightarrow{\mathrm{INR}}_{21})\rfloor$. From this observation, the results from Theorem 1 - Theorem 4 can used to determine the feedback SNR thresholds beyond which either an individual rate or the sum-rate is improved in the original G-IC-NOF. The procedure consists on using the equalities $\overleftarrow{n}_{ii} = \lfloor \frac{1}{2}\log_2\left(\overleftarrow{\mathrm{SNR}}_i\right)\rfloor$, with $i \in \{1,2\}$. Hence, the corresponding thresholds in the G-IC can be approximated by:

$$\overleftarrow{\mathrm{SNR}}_i^{*} = 2^{2\overleftarrow{n}_{ii}^{*}} \quad (53a)$$

$$\overleftarrow{\mathrm{SNR}}_i^{\dagger} = 2^{2\overleftarrow{n}_{ii}^{\dagger}} \text{ and} \quad (53b)$$

$$\overleftarrow{\mathrm{SNR}}_i^{+} = 2^{2\overleftarrow{n}_{ii}^{+}}. \quad (53c)$$

When the corresponding LD-IC-NOF is such that its capacity region can be improved when $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{*}$ (Theorem 1), for a given $i \in \{1,2\}$, it is expected that either the achievability or converse regions of the original G-IC-NOF become larger when $\overleftarrow{\mathrm{SNR}}_i > \overleftarrow{\mathrm{SNR}}_i^{*}$. Similarly, when the corresponding LD-IC-NOF is such that $\Delta_i(\overleftarrow{n}_{ii}) > 0$ or $\Delta_i(\overleftarrow{n}_{jj}) > 0$, it is expected to observe an improvement on the individual rate R_i by either using feedback in transmitter-receiver pair i, with $\overleftarrow{\mathrm{SNR}}_i > \overleftarrow{\mathrm{SNR}}_i^{\dagger}$ or by using feedback in transmitter-receiver pair j, with $\overleftarrow{\mathrm{SNR}}_j > \overleftarrow{\mathrm{SNR}}_j^{*}$. In the case of the sum-rate, when the corresponding LD-IC-NOF is such that $\Sigma(\overleftarrow{n}_{ii}) > 0$ using feedback in transmitter-receiver pair i, with $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{+}$, (Theorem 4), it is expected to observe an improvement on the sum-rate by using feedback in transmitter-receiver pair i, with $\overleftarrow{\mathrm{SNR}}_i > \overleftarrow{\mathrm{SNR}}_i^{+}$. Finally, when no improvement in a given metric is observed in the LD-IC-NOF, i.e., $\Delta_1(\overleftarrow{n}_{11}) = 0$, $\Delta_1(\overleftarrow{n}_{22}) = 0$, $\Delta_2(\overleftarrow{n}_{11}) = 0$, $\Delta_2(\overleftarrow{n}_{22}) = 0$, $\Sigma(\overleftarrow{n}_{11}) = 0$, or $\Sigma(\overleftarrow{n}_{22}) = 0$, only a negligible improvement (if any) is observed in the corresponding metric of the G-IC-NOF. For instance, when $\Delta_1(\overleftarrow{n}_{11}) = 0$, it is expected that $\Delta_1^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$ and $\Delta_1^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$, with $\epsilon > 0$ small. Similarly, when $\Delta_2(\overleftarrow{n}_{11}) = 0$, it is expected that $\Delta_2^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$ and $\Delta_2^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$. Finally, when $\Sigma(\overleftarrow{n}_{11}) = 0$, it is expected that $\Sigma^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$ and $\Sigma^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, 0) < \epsilon$.

5.3 Examples

The following examples highlight the relevance of the approximations in (53).

Example 4 Consider a G-IC with parameters $\overrightarrow{\mathrm{SNR}}_1 = 44\mathrm{dB}$, $\overrightarrow{\mathrm{SNR}}_2 = 44\mathrm{dB}$, $\mathrm{INR}_{12} = 20\mathrm{dB}$, and $\mathrm{INR}_{21} = 33\mathrm{dB}$.

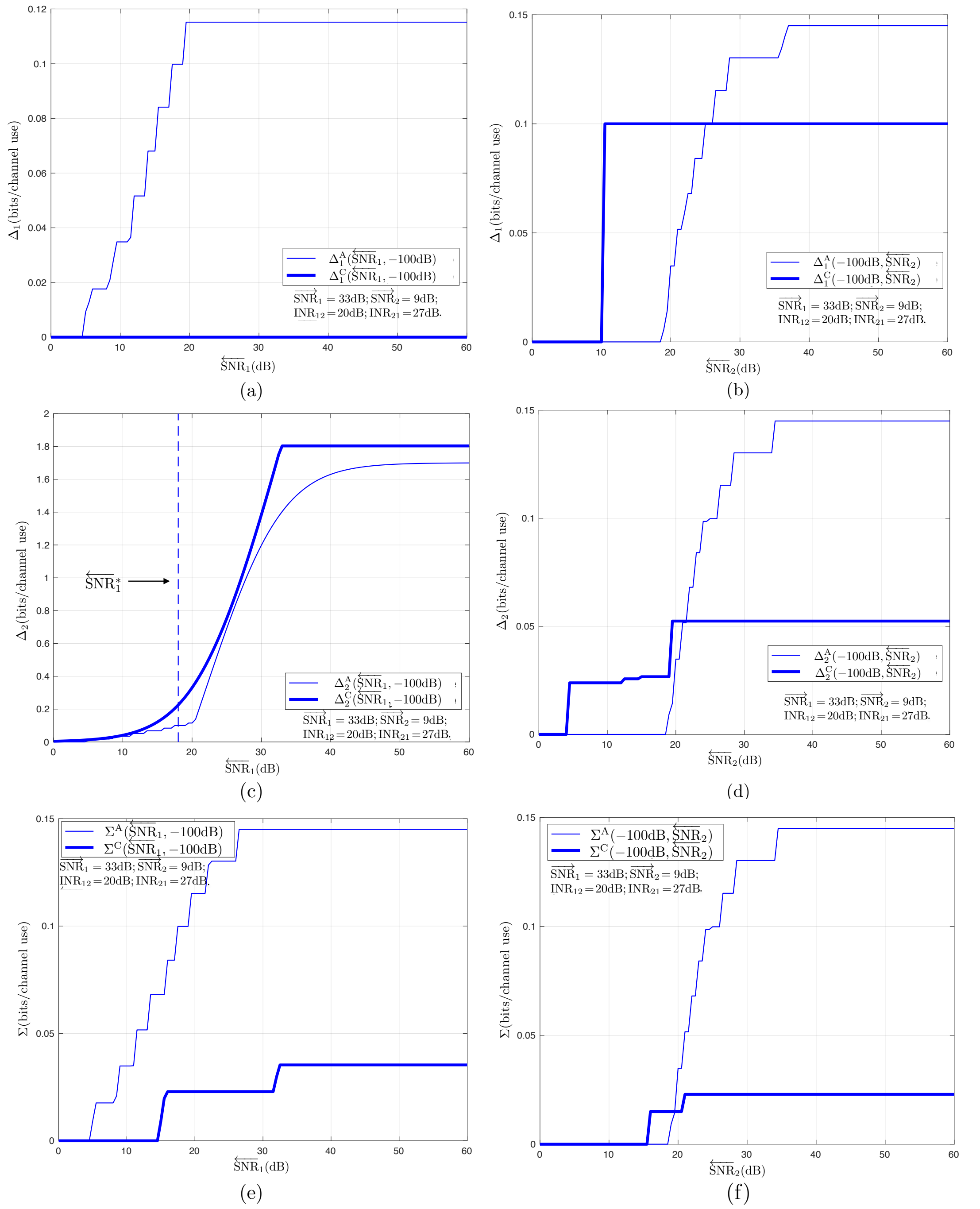

Figure 6: Improvement metrics Δ_i^{A}, Δ_i^{C}, Σ^{A}, and Σ^{C} as functions of $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$, with $i \in \{1, 2\}$, for Example 5.

The linear deterministic approximation to the G-IC in Example 4 is the one presented in Example 1. Hence, $\overleftarrow{n}^{*}_{11} = \overleftarrow{n}^{\dagger}_{11} = \overleftarrow{n}^{+}_{11} = 5$ and $\overleftarrow{n}^{*}_{22} = \overleftarrow{n}^{\dagger}_{22} = \overleftarrow{n}^{+}_{22} = 3$. This implies that

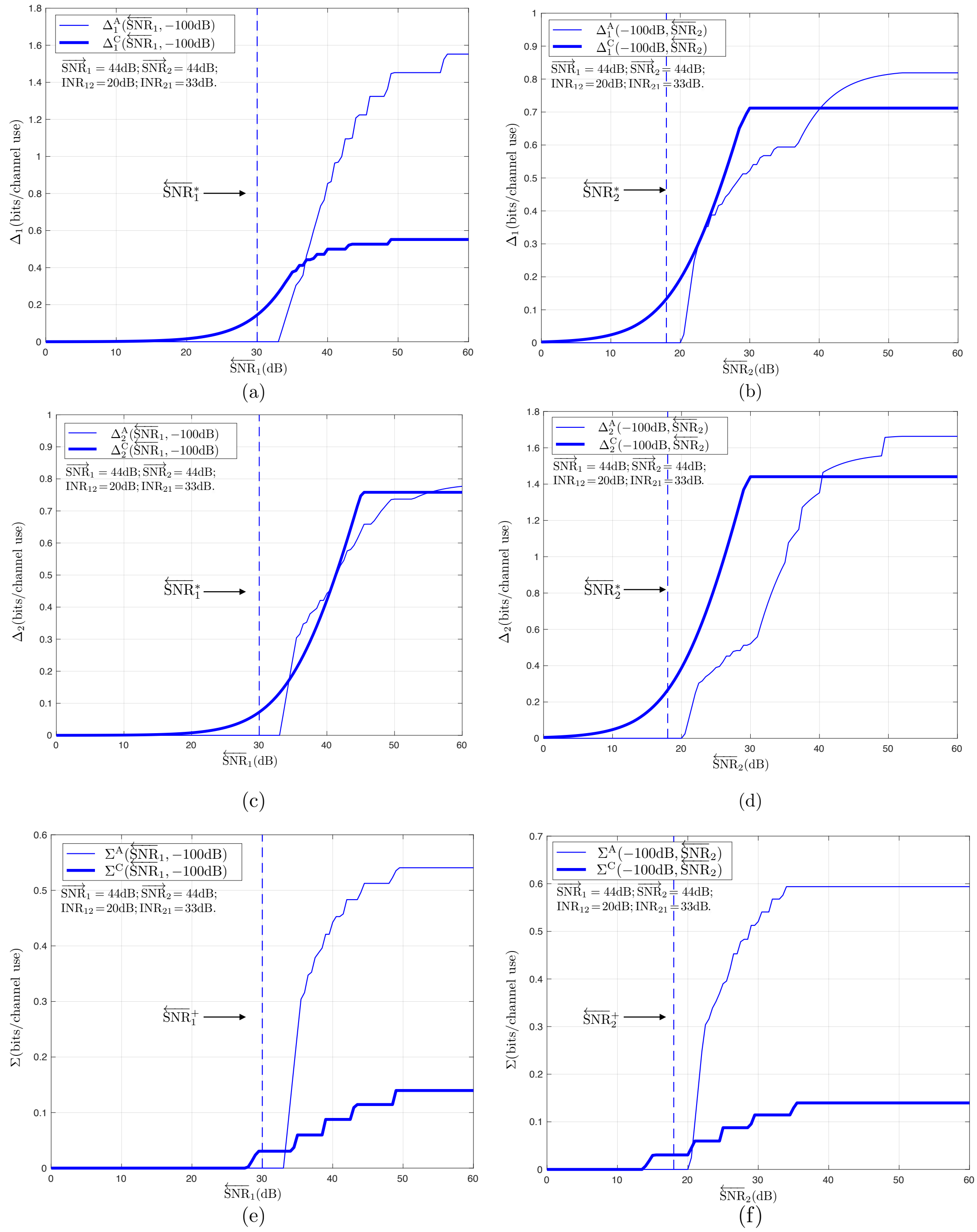

Figure 7: Improvement metrics Δ_i^{A}, Δ_i^{C}, Σ^{A}, and Σ^{C} as functions of $\overleftarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_2$, with $i \in \{1, 2\}$, for Example 4.

$\overleftarrow{\mathrm{SNR}}_1^* = \overleftarrow{\mathrm{SNR}}_1^\dagger = \overleftarrow{\mathrm{SNR}}_1^+ = 30$dB and $\overleftarrow{\mathrm{SNR}}_2^* = \overleftarrow{\mathrm{SNR}}_2^\dagger = \overleftarrow{\mathrm{SNR}}_2^+ = 18$dB.

Figure 7 shows that significant improvements on the metrics $\Delta_i^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$, $\Delta_i^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$, $\Sigma^{\mathrm{A}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ and $\Sigma^{\mathrm{C}}(\overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2)$ are obtained when the feedback SNRs are beyond the corresponding thresholds. More importantly, negligible effects are observed when $\overleftarrow{\mathrm{SNR}}_1 < \overleftarrow{\mathrm{SNR}}_1^*$ and $\overleftarrow{\mathrm{SNR}}_2 < \overleftarrow{\mathrm{SNR}}_2^*$.

Example 5 Consider a G-IC with parameters $\overrightarrow{\mathrm{SNR}}_1 = 33$dB, $\overrightarrow{\mathrm{SNR}}_2 = 9$dB, $\mathrm{INR}_{12} = 20$dB, and $\mathrm{INR}_{21} = 27$dB.

The linear deterministic approximation to the G-IC in Example 5 is the one presented in Example 3. Hence, $\overleftarrow{n}_{11}^* = 3$, which implies that $\overleftarrow{\mathrm{SNR}}_1^* = 18$dB. It follows from the LD-IC that using feedback in transmitter-receiver pair 1 exclusively increases the individual rate R_2. This is observed in Figure 6c. Note that the improvement in the individual rate R_2 for all $\overleftarrow{\mathrm{SNR}}_1 < \overleftarrow{\mathrm{SNR}}_1^*$ is negligible. Significant improvement is observed only beyond the threshold $\overleftarrow{\mathrm{SNR}}_1^*$.

Note also that using feedback in either transmitter-receiver pair does not improve the rate R_1 in the LD-IC-NOF, i.e., $\Delta_1(\overleftarrow{n}_{11}) = \Delta_1(\overleftarrow{n}_{22}) = 0$. This is also verified in the G-IC-NOF by Figure 6a, Figure 6b, and Figure 6d, where $\Delta_1^{\mathrm{A}}\left(-100\mathrm{dB}, \overleftarrow{\mathrm{SNR}}_2\right) < 0.15$ and $\Delta_1^{\mathrm{C}}\left(-100\mathrm{dB}, \overleftarrow{\mathrm{SNR}}_2\right) < 0.1$.

Finally, note that using feedback in either transmitter-receiver pair does not increase the sum-rate in the LD-IC-NOF, i.e., $\Sigma(\overleftarrow{n}_{11}) = \Sigma(\overleftarrow{n}_{22}) = 0$. This is also verified in the G-IC-NOF by Figure 6e and Figure 6f, where $\Sigma^{\mathrm{A}}\left(\overleftarrow{\mathrm{SNR}}_1, -100\mathrm{dB}\right) < 0.15$, $\Sigma^{\mathrm{C}}\left(\overleftarrow{\mathrm{SNR}}_1, -100\mathrm{dB}\right) < 0.05$, $\Sigma^{\mathrm{A}}\left(-100\mathrm{dB}, \overleftarrow{\mathrm{SNR}}_2\right) < 0.15$, and $\Sigma^{\mathrm{C}}\left(-100\mathrm{dB}, \overleftarrow{\mathrm{SNR}}_2\right) < 0.05$.

6 Generalized Degrees of Freedom

This section focuses on the analysis of the number of GDoF of the LD-IC-NOF for studying the case in which feedback is simultaneously implemented in both transmitter-receiver pairs. Moreover, the analysis is only performed for the symmetric case, i.e., $\overrightarrow{n} = \overrightarrow{n}_{11} = \overrightarrow{n}_{22}$, $m = n_{12} = n_{21}$, and $\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}$, with $(\overrightarrow{n}, m, \overleftarrow{n}) \in \mathbb{N}^3$. The results in Lemma 1 allow a more general analysis of the GDoF, e.g., non-symmetric case. However, the symmetric case captures some of the most important insights about how the capacity region is enlarged when feedback is used in both transmitter-receiver pairs.

Essentially, given the parameters $\overrightarrow{n}$, m and $\overleftarrow{n}$, with $\alpha = \frac{m}{\overrightarrow{n}}$ and $\beta = \frac{\overleftarrow{n}}{\overrightarrow{n}}$, the number of GDoF, denoted by $D(\alpha, \beta)$, is the ratio between the symmetric capacity, i.e., $C_{\mathrm{sym}}(\overrightarrow{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\overrightarrow{n}, \overrightarrow{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}$, and the individual interference-free point-to-point capacity, i.e., $\overrightarrow{n}$, when $(\overrightarrow{n}, m, \overleftarrow{n}) \to (\infty, \infty, \infty)$ at constant ratios $\alpha = \frac{m}{\overrightarrow{n}}$ and $\beta = \frac{\overleftarrow{n}}{\overrightarrow{n}}$. More specifically, the number of GDoF is

$$D(\alpha, \beta) = \lim_{\overrightarrow{n}, m, \overleftarrow{n} \to \infty} \frac{C_{\mathrm{sym}}(\overrightarrow{n}, m, \overleftarrow{n})}{\overrightarrow{n}}. \tag{54}$$

Theorem 5 determines the number of GDoF for the two-user LD-IC-NOF.

Theorem 5 The number of GDoF for the two user symmetric LD-IC-NOF with parameters α

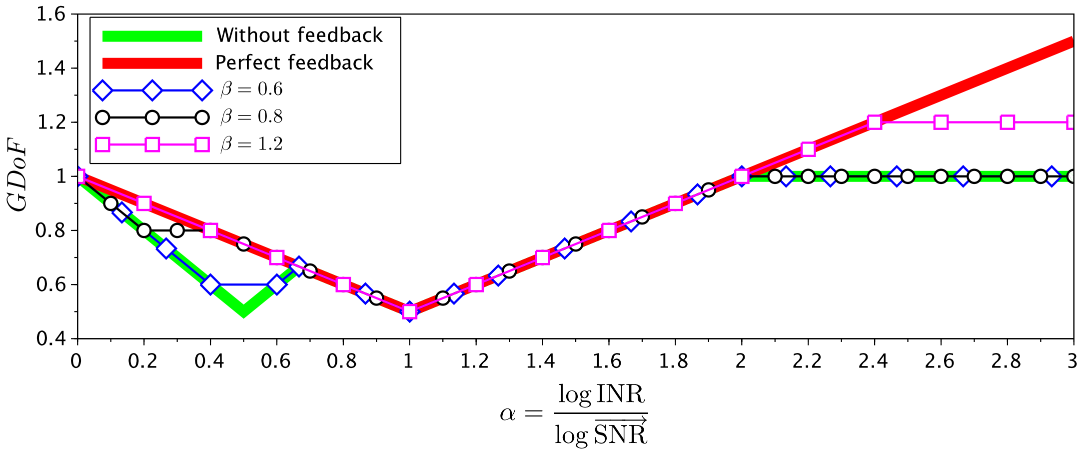

Figure 8: Generalized Degrees of Freedom (GDoF) as a function of parameters α and β, with $0 \leqslant \alpha \leqslant 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$, of the symmetric LD-IC-NOF. The plot without feedback is obtained from [8] and the plot with perfect-output feedback is obtained from [14].

and β is given by

$$D(\alpha,\beta)=\min\Bigg(\max(1,\alpha), \max\left(1, \beta-(1-\alpha)^+\right), \frac{1}{2}\left(\max(1,\alpha)+(1-\alpha)^+\right), \max\left((1-\alpha)^+, \alpha, 1-(\max(1,\alpha)-\beta)^+\right), \frac{1}{3}\left(\max(1,\alpha)+(1-\alpha)^+ + \max\left(\left(1-\alpha\right)^+, \alpha, 1-(\max(1,\alpha)-\beta)^+\right)\right)\Bigg). \quad (55)$$

Proof: The proof of Theorem 5 is presented in Appendix D. ■

The result in Theorem 5 can also be obtained from Theorem 1 in [15]. The following properties are a direct consequence of Theorem 5.

Corollary 4 *The number of GDoF for the two user symmetric LD-IC-NOF with parameters α and β satisfies the following properties:*

$$\forall \alpha \in \left[0, \frac{2}{3}\right] \text{ and } \beta \leqslant 1, \quad \max\left(\frac{1}{2}, \beta\right) \leqslant D(\alpha,\beta) \leqslant 1, \quad (56a)$$

$$\forall \alpha \in \left[0, \frac{2}{3}\right] \text{ and } \beta > 1, \quad D(\alpha,\beta) = 1 - \frac{\alpha}{2}, \quad (56b)$$

$$\forall \alpha \in \left(\frac{2}{3}, 2\right] \text{ and } \beta \in [0,\infty), \quad D(\alpha,0) = D(\alpha,\beta) = D(\alpha,\max(1,\alpha)), \quad (56c)$$

$$\forall \alpha \in (2,\infty) \text{ and } \beta \geqslant 1, \quad 1 \leqslant D(\alpha,\beta) \leqslant \min\left(\frac{\alpha}{2}, \beta\right), \quad (56d)$$

$$\forall \alpha \in (2,\infty) \text{ and } \beta < 1, \quad D(\alpha,\beta) = 1. \quad (56e)$$

Properties (56a) and (56b) highlight the fact that the existence of feedback links in the symmetric LD-IC in the VWIR and WIR does not have any impact in the GDoF when $\beta \leqslant \frac{1}{2}$, and the GDoF is equal to the case with perfect-output feedback when $\beta > 1$. Property (56c) underlines that in the symmetric LD-IC in MIR and SIR, the number of GDoF is identical in both extreme cases: without feedback ($\beta = 0$) and with perfect-output feedback $\big(\beta = \max(1, \alpha)\big)$. Finally, from (56d) and (56e), it follows that for observing an improvement in the GDoF of the LD-IC-NOF in VSIR, the following condition must be met: $\beta > 1$. That is, the number of bit-pipes in the feedback links must be strictly bigger than the number of bit-pipes in the direct links.

Figure 8 shows the number of GDoF for the two user symmetric LD-IC-NOF for the case in which $0 \leqslant \alpha \leqslant 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$.

7 Conclusions

In this research report, for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, the exact values on the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ of the two-user LD-IC-NOF beyond which the capacity region can be enlarged are characterized. That is, the exact values of $\overleftarrow{n}_{11}$ (resp. $\overleftarrow{n}_{22}$) for which $\mathcal{C}(0,0) \subset \mathcal{C}(\overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(0,0) \subset \mathcal{C}(0, \overleftarrow{n}_{22})\big)$ holds with strict inclusion. Using these results from the LD approximation, the SNRs in the feedback links beyond which feedback plays a significant role in terms of increasing the individual rates or the sum-rate in the G-IC are identified. The relevance of this work lies on the fact that it allows identifying a number of scenarios in any G-IC for which one of the following statements is true: (a) Feedback does not enlarge the capacity region; (b) Feedback enlarges the capacity region and the sum-rate is higher than the largest sum-rate without feedback; and (c) Feedback enlarges the capacity region but no significant improvement is observed in the sum-rate.

Appendices

A Proof of Theorem 1: Enlargement of the Capacity Region by Using Feedback in one Transmitter-Receiver Pair

The proof of Theorem 1 is obtained by comparing $\mathcal{C}(\overleftarrow{n}_{11},0)$ $\big($resp. $\mathcal{C}(0,\overleftarrow{n}_{22})\big)$ and $\mathcal{C}(0,0)$, with fixed parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, and n_{21}. More specifically, for each tuple $\big(\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, $n_{21}\big)$, the exact value $\overleftarrow{n}_{11}^{*}$ (resp $\overleftarrow{n}_{22}^{*}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^{*}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}_{22}^{*}$) ensures $\mathcal{C}(0,0) \subset \mathcal{C}(\overleftarrow{n}_{11},0)$ (resp. $\mathcal{C}(0,0) \subset \mathcal{C}(0,\overleftarrow{n}_{22})$) is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, that is,

$$\alpha_1 = \frac{n_{12}}{\overrightarrow{n}_{11}} \leqslant \frac{1}{2} \text{ and } \alpha_2 = \frac{n_{21}}{\overrightarrow{n}_{22}} \leqslant \frac{1}{2}. \tag{57}$$

Under conditions (57), it follows from Theorem 1 in [16] that $\mathcal{C}(0,0)$ is the set of non-negative rate pairs (R_1,R_2) that satisfy

$$R_1 \leqslant \overrightarrow{n}_{11} \triangleq \theta_1, \tag{58a}$$
$$R_2 \leqslant \overrightarrow{n}_{22} \triangleq \theta_2, \tag{58b}$$
$$R_1+R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22},n_{12}\right)+\overrightarrow{n}_{11}-n_{12}, \max\left(\overrightarrow{n}_{11},n_{21}\right)+\overrightarrow{n}_{22}-n_{21}\right) \triangleq \theta_3, \tag{58c}$$
$$R_1+R_2 \leqslant \max\left(\overrightarrow{n}_{11}-n_{12},n_{21}\right)+\max\left(\overrightarrow{n}_{22}-n_{21},n_{12}\right) \triangleq \theta_4, \tag{58d}$$
$$2R_1+R_2 \leqslant \max\left(\overrightarrow{n}_{11},n_{21}\right)+\overrightarrow{n}_{11}-n_{12}+\max\left(\overrightarrow{n}_{22}-n_{21},n_{12}\right) \triangleq \theta_5, \tag{58e}$$
$$R_1+2R_2 \leqslant \max\left(\overrightarrow{n}_{22},n_{12}\right)+\overrightarrow{n}_{22}-n_{21}+\max\left(n_{21},\overrightarrow{n}_{11}-n_{12}\right) \triangleq \theta_6. \tag{58f}$$

Note that for all $(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{22}) \in \mathbb{N}^5$ and $\overleftarrow{n}_{11} > \max(\overrightarrow{n}_{11},n_{12})$, it follows that $\mathcal{C}(\overleftarrow{n}_{11},\overleftarrow{n}_{22}) = \mathcal{C}(\max(\overrightarrow{n}_{11},n_{12}),\overleftarrow{n}_{22})$. Hence, in the following, the analysis is restricted to the following condition:

$$\overleftarrow{n}_{11} \leqslant \max\left(\overrightarrow{n}_{11},n_{12}\right). \tag{59}$$

Under conditions (57) and (59), it follows from Theorem 1 in [16] that $\mathcal{C}(\overleftarrow{n}_{11},0)$ is the set of non-negative rate pairs (R_1,R_2) that satisfy

$$R_1 \leqslant \overrightarrow{n}_{11}, \tag{60a}$$
$$R_2 \leqslant \overrightarrow{n}_{22}, \tag{60b}$$
$$R_1+R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22},n_{12}\right)+\overrightarrow{n}_{11}-n_{12}, \max\left(\overrightarrow{n}_{11},n_{21}\right)+\overrightarrow{n}_{22}-n_{21}\right), \tag{60c}$$
$$R_1+R_2 \leqslant \max\left(\overrightarrow{n}_{11}-n_{12},n_{21},\overleftarrow{n}_{11}\right)+\max\left(\overrightarrow{n}_{22}-n_{21},n_{12}\right) \triangleq \theta_7, \tag{60d}$$
$$2R_1+R_2 \leqslant \max\left(\overrightarrow{n}_{11},n_{21}\right)+\overrightarrow{n}_{11}-n_{12}+\max\left(\overrightarrow{n}_{22}-n_{21},n_{12}\right), \tag{60e}$$
$$R_1+2R_2 \leqslant \max\left(\overrightarrow{n}_{22},n_{12}\right)+\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},n_{21},\overleftarrow{n}_{11}\right) \triangleq \theta_8. \tag{60f}$$

When comparing $\mathcal{C}(0,0)$ and $\mathcal{C}(\overleftarrow{n}_{11},0)$, note that (58a), (58b), (58c), and (58e) are equivalent to (60a), (60b), (60c), and (60e), respectively. Under these observations, the region $\mathcal{C}(\overleftarrow{n}_{11},0)$ is greater than the region $\mathcal{C}(0,0)$ if at least one of the following conditions is true:

$$\min(\theta_3,\theta_4,\theta_1+\theta_2,\theta_5,\theta_6)<\theta_7<\min(\theta_3,\theta_1+\theta_2,\theta_5,\theta_8), \tag{61a}$$
$$\min(\theta_6,\theta_1+2\theta_2,\theta_2+\theta_3,\theta_4+\theta_2)<\theta_8<\min\left(\theta_1+2\theta_2,\theta_2+\theta_3,\theta_2+\theta_7\right). \tag{61b}$$

Condition (61a) implies that the active sum-rate bound in $\mathcal{C}(\overleftarrow{n}_{11}, 0)$ is greater than the active sum-rate bound in $\mathcal{C}(0,0)$. Condition (61b) implies that the active weighted sum-rate bound on R_1+2R_2 in $\mathcal{C}(\overleftarrow{n}_{11}, 0)$ is greater than the active weighted sum-rate bound on R_1+2R_2 in $\mathcal{C}(0,0)$.

To simplify the inequalities containing the operator $\max(\cdot,\cdot)$ in (60) and (58), the following 4 cases are identified:

$$\text{Case 1}: \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} < n_{12}; \quad (62)$$
$$\text{Case 2}: \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} \geqslant n_{12}; \quad (63)$$
$$\text{Case 3: } \overrightarrow{n}_{11} - n_{12} \geqslant_{21} \quad \text{and } \overrightarrow{n}_{22} - n_{21} < n_{12}; \text{ and} \quad (64)$$
$$\text{Case 4: } \overrightarrow{n}_{11} - n_{12} \geqslant n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} \geqslant n_{12}. \quad (65)$$

<u>Case 1:</u> Under assumptions (57) and (62), this case is not possible.
<u>Case 2</u>: Under assumptions (57) and (63), this case is possible.
Plugging (63) into (60) yields:

$$R_1 + R_2 \leqslant \min\left(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{22} - n_{21}\right), \quad (66a)$$
$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \quad (66b)$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \quad (66c)$$

Plugging (63) into (58) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22}, \quad (67a)$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22}. \quad (67b)$$

To simplify the inequalities containing the operator $\max(\cdot,\cdot)$ in (66), the following 2 cases are identified:

$$\text{Case 2a}: \overrightarrow{n}_{11} > n_{21}; \text{ and} \quad (68)$$
$$\text{Case 2b}: \overrightarrow{n}_{11} \leqslant n_{21}. \quad (69)$$

<u>Case 2*a*</u>: Plugging (68) into (66) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}, \quad (70a)$$
$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \quad (70b)$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \quad (70c)$$

Comparing inequalities (70a) and (70b) with inequality (67a), it can be verified that $\min\Big(\overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}, \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}\Big) > \overrightarrow{n}_{22}$, i.e., condition (61a) holds, when $\overleftarrow{n}_{11} > n_{21}$. Comparing inequalities (70c) and (67b), it can be verified that $2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22}$, i.e., condition (61b) holds, when $\overleftarrow{n}_{11} > n_{21}$. Therefore, $\overleftarrow{n}^{*}_{11} = n_{21}$ under assumptions (57), (59), (63), and (68).

<u>Case 2*b*:</u> Plugging (69) into (66) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22}, \quad (71a)$$
$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \quad (71b)$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \quad (71c)$$

Comparing inequalities (71a) and (71b) with inequality (67a), it can be verified that $\min\Big(\overrightarrow{n}_{22}, \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}\Big) = \overrightarrow{n}_{22}$, i.e., condition (61a) does not hold, for all $\overleftarrow{n}_{11} \in \mathbb{N}$. Comparing inequalities (71c) and (67b) it can be verified that $2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22}$, when $\overleftarrow{n}_{11} > n_{21}$, which implies that $\overleftarrow{n}_{11} > \max\left(\overrightarrow{n}_{11}, n_{12}\right)$. However, under the assumptions (57), (59), (63), and (69), the bounds (67b) and (71c) are not active. Hence, condition (61b) does not hold. Therefore, for all $\overleftarrow{n}_{11} \in \mathbb{N}$, the capacity region cannot be enlarged under assumptions (57), (59), (63), and (69).

Case 3: Under assumptions (57) and (64), this case is possible.

Plugging (64) into (60) yields:

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{11} - n_{12}, \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}\right), \tag{72a}$$
$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}, \tag{72b}$$
$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{72c}$$

Plugging (64) into (58) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{73a}$$
$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{73b}$$

To simplify the inequalities containing the operator $\max(\cdot, \cdot)$ in (72) and (73), the following 2 cases are identified:

$$\text{Case 3a}: \overrightarrow{n}_{22} > n_{12}; \text{ and} \tag{74}$$
$$\text{Case 3b}: \overrightarrow{n}_{22} \leqslant n_{12}. \tag{75}$$

Case 3a: Plugging (74) into (72) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \tag{76a}$$
$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}, \tag{76b}$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{76c}$$

Plugging (74) into (73) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{77a}$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{77b}$$

Comparing inequalities (76a) and (76b) with inequality (77a), it can be verified that $\min\Big(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}\Big) > \overrightarrow{n}_{11}$, i.e., condition (61a) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Comparing inequalities (76c) and (77b), it can be verified that $2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}$, i.e., condition (61b) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Therefore, $\overleftarrow{n}^{*}_{11} = \overrightarrow{n}_{11} - n_{12}$ under assumptions (57), (59), (64), and (74).

Case 3b: Plugging (75) into (72) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{78a}$$
$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}, \tag{78b}$$
$$R_1 + 2R_2 \leqslant n_{12} + \overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{78c}$$

Plugging (74) into (73) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{79a}$$
$$R_1 + 2R_2 \leqslant \overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11}. \tag{79b}$$

Comparing inequalities (78a) and (78b) with inequality (79a), it can be verified that $\min\Big(\overrightarrow{n}_{11}, \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}\Big) = \overrightarrow{n}_{11}$, i.e., condition (61a) does not hold, for all $\overleftarrow{n}_{11} \in \mathbb{N}$. Comparing inequalities (78c) and (79b), it can be verified that $n_{12} + \overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) > \overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11}$, i.e., condition (61b) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Therefore, $\overleftarrow{n}^*_{11} = \overrightarrow{n}_{11} - n_{12}$ under conditions (57), (59), (64), and (75).

Case 4: Under conditions (57) and (65), this case is possible.

Plugging (65) into (60) yields:

$$R_1 + R_2 \leqslant \min\left(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}\right), \tag{80a}$$
$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \tag{80b}$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{80c}$$

Plugging (65) into (58) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11} - n_{12} + \overrightarrow{n}_{22} - n_{21}, \tag{81a}$$
$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{81b}$$

Comparing inequalities (80a) and (80b) with inequality (81a), it can be verified that $\min\Big(\min\left(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}\right), \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}\Big) > \overrightarrow{n}_{11} - n_{12} + \overrightarrow{n}_{22} - n_{21}$, i.e., condition (61a) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Comparing inequalities (80c) and (81b), it can be verified that: $2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}$, i.e., condition (61b) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$.

Therefore, $\overleftarrow{n}^*_{11} = \overrightarrow{n}_{11} - n_{12}$ under conditions (57), (59), and (65).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that when $\overleftarrow{n}_{11} > \overleftarrow{n}^*_{11}$ and $\overrightarrow{n}_{11} > n_{21}$ (event $E_{8,1}$ is True) with with $\overleftarrow{n}^*_{11} = \max\left(\overrightarrow{n}_{11} - n_{12}, n_{21}\right)$, then $\mathcal{C}(0,0) \subset \mathcal{C}(\overleftarrow{n}_{11}, 0)$. Otherwise $\mathcal{C}(0,0) = \mathcal{C}(\overleftarrow{n}_{11}, 0)$. Note that when events E_1 and $E_{8,1}$ hold simultaneously true, then the event $S_{1,1}$ is true, which verifies the statement of Theorem 1. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. ∎

B Proof of Theorem 2: Improvement of the Individual Rate R_i by Using Feedback in Link i

The proof of Theorem 2 is obtained by comparing $\mathcal{C}(\overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(0, \overleftarrow{n}_{22})\big)$ and $\mathcal{C}(0,0)$, for all possible parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{11}$ (resp. $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{22}$). More specifically, for each tuple $\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\right)$, the exact value $\overleftarrow{n}^{\dagger}_{11}$ (resp $\overleftarrow{n}^{\dagger}_{22}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}^{\dagger}_{11}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}^{\dagger}_{22}$) ensures an improvement on R_1 (resp. R_2) , i.e., $\Delta_1(\overleftarrow{n}_{11}, 0) > 0$ (resp. $\Delta_2(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22}) > 0$), is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (57) hold. Under these conditions, the capacity regions $\mathcal{C}(0,0)$ and $\mathcal{C}(\overleftarrow{n}_{11},0)$ are given by (58) and (60), respectively. When comparing $\mathcal{C}(0,0)$ and $\mathcal{C}(\overleftarrow{n}_{11},0)$, note that (58a), (58b), (58c), and (58e) are equivalent to (60a), (60b), (60c), and (60e), respectively. In this case any improvement on R_1 is produced by an improvement on R_1+R_2 (condition (61a)) or $2R_1+R_2$ (condition (61a)), and thus, the proof of Theorem 2 in these particular interference regimes follows exactly the same steps in Theorem 1. This completes the proof. ■

C Proof of Theorem 4: Improvement of the Sum-Rate Capacity by Using Feedback in one Transmitter-Receiver Pair

The proof of Theorem 4 is obtained by comparing $\mathcal{C}(\overleftarrow{n}_{11},0)$ (resp. $\mathcal{C}(0,\overleftarrow{n}_{22})$) and $\mathcal{C}(0,0)$, for all possible parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{11}$ (resp. $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{22}$). More specifically, for each tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, the exact value $\overleftarrow{n}_{11}^{+}$ (resp $\overleftarrow{n}_{22}^{+}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^{+}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}_{22}^{+}$) ensures an improvement on R_1+R_2, i.e., $\Sigma(\overleftarrow{n}_{11},0)>0$ (resp. $\Sigma(0,\overleftarrow{n}_{22})>0$), is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (57) hold. Under these conditions, the capacity regions $\mathcal{C}(0,0)$ and $\mathcal{C}(\overleftarrow{n}_{11},0)$ are given by (58) and (60), respectively. When comparing $\mathcal{C}(0,0)$ and $\mathcal{C}(\overleftarrow{n}_{11},0)$, note that (58a), (58b), (58c), and (58e) are equivalent to (60a), (60b), (60c), and (60e), respectively.

In this case, the proof is focused on any improvement on R_1+R_2 (condition (61a)), and thus, the proof of Theorem 4 in these particular interference regimes follows exactly the same steps in Theorem 1.

From the analysis presented in Appendix A, it follows that:

Case 2*a*: condition (61a) holds true, when $\overleftarrow{n}_{11} > n_{21}$ under assumptions (57), (59), (63), and (68).

Case 2*b*: condition (61a) does not hold true, under assumptions (57), (63), and (69).

Case 3*a*: condition (61a) holds true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (57), (59), (64), and (74).

Case 3*b*: condition (61a) does not hold true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (57), (59), (64), and (75).

Case 4: condition (61a) holds true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (57), (59), and (65).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that when $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^{+}$, $\overrightarrow{n}_{11} > n_{21}$ (event $E_{8,1}$ is True), $\overrightarrow{n}_{22} > n_{12}$ (event $E_{8,2}$ is True), $\overrightarrow{n}_{11} + \overrightarrow{n}_{22} > n_{12} + 2n_{21}$ (event $E_{10,1}$ is True), and $\overrightarrow{n}_{11} + \overrightarrow{n}_{22} > n_{21} + 2n_{12}$ (event $E_{10,2}$ is True) with $\overleftarrow{n}_{11}^{+} = \max(\overrightarrow{n}_{11} - n_{12}, n_{21})$, then $\Sigma(\overleftarrow{n}_{11},0) > 0$. Otherwise $\Sigma(\overleftarrow{n}_{11},0) = 0$. Note that when events E_1, $E_{8,1}$, $E_{8,2}$, $E_{10,1}$, and $E_{10,2}$ hold simultaneously true, then the event S_4 is true, which verifies the statement of Theorem 4. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. ■

D Proof of Theorem 5: Generalized Degrees of Freedom

This appendix provides a proof to Theorem 5 for the two user LD-IC-NOF.

Proof:
Lemma 1 fully characterizes the set $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

Lemma 1 (Theorem 1 in [16]) The capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user LD-IC-NOF is the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), \tag{82a}$$

$$R_i \leqslant \min\Big(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\Big(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\Big)\Big), \tag{82b}$$

$$R_1 + R_2 \leqslant \min\Big(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \left(\overrightarrow{n}_{11} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{21}\right)^+\Big), \tag{82c}$$

$$\begin{aligned} R_1 + R_2 \leqslant &\max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}, \overrightarrow{n}_{11} - \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\Big) \\ &+ \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}, \overrightarrow{n}_{22} - \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\Big), \end{aligned} \tag{82d}$$

$$\begin{aligned} 2R_i + R_j \leqslant &\max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ \\ &+ \max\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}, \overrightarrow{n}_{jj} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big). \end{aligned} \tag{82e}$$

Under symmetric conditions i.e., $\overrightarrow{n} = \overrightarrow{n}_{11} = \overrightarrow{n}_{22}$, $m = n_{12} = n_{21}$ and $\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}$, from (82a) and (82b) with $i = 1$ and $j = 2$, it follows that:

$$R_1 \leqslant \min\Big(\max\left(\overrightarrow{n}, m\right), \max\Big(\overrightarrow{n}, \overleftarrow{n} - \left(\overrightarrow{n} - m\right)^+\Big)\Big) \triangleq a_1; \tag{83}$$

from (82c) and (82d), it follows that:

$$\begin{aligned} R_1 + R_2 \leqslant &\min\Big(\max\left(\overrightarrow{n}, m\right) + \left(\overrightarrow{n} - m\right)^+, 2\max\Big(\left(\overrightarrow{n} - m\right)^+, m, \overrightarrow{n} - \left(\max\left(\overrightarrow{n}, m\right) - \overleftarrow{n}\right)^+\Big)\Big) \\ \triangleq &a_2; \end{aligned} \tag{84}$$

and from (82e) with $i = 1$ and $j = 2$, it follows that:

$$2R_1 + R_2 \leqslant \max\left(\overrightarrow{n}, m\right) + \left(\overrightarrow{n} - m\right)^+ + \max\Big(\left(\overrightarrow{n} - m\right)^+, m, \overrightarrow{n} - \left(\max\left(\overrightarrow{n}, m\right) - \overleftarrow{n}\right)^+\Big) \triangleq a_3 \tag{85}$$

The symmetric sum-capacity, $C_{\text{sym}}(\overrightarrow{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\overrightarrow{n}, \overrightarrow{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}$, can be obtained from (83), (84) and (85) as follows

$$\begin{aligned} C_{\text{sym}} =& \min\left(a_1, \frac{a_2}{2}, \frac{a_3}{3}\right) \\ =& \min\Bigg(\max\left(\overrightarrow{n}, m\right), \max\Big(\overrightarrow{n}, \overleftarrow{n} - \left(\overrightarrow{n} - m\right)^+\Big), \frac{1}{2}\Big(\max\left(\overrightarrow{n}, m\right) + \left(\overrightarrow{n} - m\right)^+\Big), \\ &\max\Big(\left(\overrightarrow{n} - m\right)^+, m, \overrightarrow{n} - \left(\max\left(\overrightarrow{n}, m\right) - \overleftarrow{n}\right)^+\Big), \frac{1}{3}\Big(\max\left(\overrightarrow{n}, m\right) + \left(\overrightarrow{n} - m\right)^+ \\ &\max\Big(\left(\overrightarrow{n} - m\right)^+, m, \overrightarrow{n} - \left(\max\left(\overrightarrow{n}, m\right) - \overleftarrow{n}\right)^+\Big)\Big)\Bigg). \end{aligned} \tag{86}$$

Plugging (86) into (54) yields

$$D_{\mathrm{sym}}(\alpha,\beta) = \min\Bigg(\max(1,\alpha), \max\Big(1, \beta - (1-\alpha)^+\Big), \frac{1}{2}\Big(\max(1,\alpha) + (1-\alpha)^+\Big), \max\Big((1-\alpha)^+, \alpha, 1 - (\max(1,\alpha) - \beta)^+\Big), \frac{1}{3}\Big(\max(1,\alpha) + (1-\alpha)^+ + \max\Big((1-\alpha)^+, \alpha, 1 - (\max(1,\alpha) - \beta)^+\Big)\Big)\Bigg). \quad (87)$$

where $\alpha = \frac{m}{\overrightarrow{n}}$ and $\beta = \frac{\overleftarrow{n}}{\overrightarrow{n}}$ and this completes the proof. ■

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