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**Abstract**

Several deductive formalisms (e.g., sequent, nested sequent, labeled sequent, hypersequent calculi) have been used in the literature for the treatment of modal logics, and some connections between these formalisms are already known. Here we propose a general framework, which is based on a focused version of the labeled sequent calculus by Negri, augmented with some parametric devices allowing to restrict the set of proofs. By properly defining such restrictions and by choosing an appropriate polarization of formulas, one can obtain different, concrete proof systems for the modal logic K and for its extensions by means of geometric axioms. In particular, we show how to use the expressiveness of the labeled approach and the control mechanisms of focusing in order to emulate in our framework the behavior of a range of existing formalisms and proof systems for modal logic.

*Keywords:* Modal logic, sequent calculi, labeled proof systems, focusing.

## 1 Introduction

Modal proof theory is a notoriously difficult subject and several proposals for it have been given in the literature (a general account is in [6]). Such proposals range over a set of different proof formalisms (e.g., sequent, nested sequent, labeled sequent, hypersequent calculi), each of them presenting its own features and drawbacks. For instance, proof systems based on ordinary sequents present a good behavior in terms of proof search, but they are typically designed for a specific modal logic and lack modularity when one tries to capture modal logics with particular frame conditions. Moreover, cut-elimination for an important modal logic like S5 is problematic. For this reason, more sophisticated formalisms have been adapted or introduced, e.g., several hypersequent cut-free formulations have been given for S5, while nested and labeled sequents have been used for giving modular presentations of large classes of modal logics. Several results concerning correspondences and connections between the different formalisms are known [7,10,13].

We propose a general framework for emulating and comparing existing modal proof systems as well as for generating new proof systems. We shall do this in the familiar setting of *labeled deduction systems* [8] in which the axiomatization of a particular Kripke semantics is given. The resulting encoding of a modal logic

formula has its logical connectives and propositional atoms "cluttered" with additional (relational) atoms and assumptions that describe the reachability relationship of a class of Kripke models. While such clutter has been described as both "impure" and "semantic pollution" we provide here an additional defense of this approach to complement the defenses found in [19,21]. In particular, we introduce *focused* variants of sequent calculi: in such systems, we can build *synthetic* inference rules from Gentzen-style introduction and structural rules. Such synthetic rules are built from the clutter in controllable and rigid shapes. For example, when geometric formulas are used to axiomatize Kripke frames, the role of those formulas in proofs can be restricted to uses that correspond to synthetic inference rules [17,18]. By adding elements of polarization to the labeled sequent setting and by defining a few other parameters of the general framework, we are able to exploit the control mechanisms provided by focusing to reproduce proofs of the original calculi with precision.

The emulation of modal logic proof systems described in this paper can also be used to build proof checkers for modal logic formulas given checkers for first-order logic (such as those described in [4]) that do not have any special knowledge of modal operators and Kripke frames. In other words, the emulation results described here make it possible to build a modal logic proof checker using familiar proof search techniques such as backtracking search and (first-order) unification. A particularly challenging aspect of such emulation is, predictably, the promotion rule, such as the one for $K$ (in a one-sided sequent formulation):

$$\frac{\vdash \Gamma, B}{\vdash \Diamond\Gamma, \Box B} \ .$$

Since many introductions must be performed at once, this inference rule corresponds to more than one synthetic inference rule in our emulation. If $\Gamma$ contains $n$ occurrences of formulas, then we could emulate this one inference rules using $n+1$ synthetic rules as follows (reading proof rules from conclusion to premises): one of these rules performs the $\Box$-introduction (which corresponds to creating a new world that is assumed to be reachable) and $n$ of these rules perform the $\Diamond$-introduction rules (which correspond to moving all the assumptions of the form $\Diamond A$ to that new world). Notice that all of these $n$ inference rules can, in fact, be performed in parallel. We capture such parallel application of inference rules using synthetic inference rules built with *multifocusing*. As a result, we will capture this promotion rule via two synthetic inference rules: one for the $\Box$-introduction and one capturing all $\Diamond$-introductions.

We proceed as follows. After providing background notions concerning modal logic and focusing (Section 2), we present the general framework $LMF_*^X$ (Section 3) and prove some results about the emulation of existing modal proof systems (Section 4). In this paper, we restrict our attention to the emulation of ordinary and nested sequent systems. We remark, however, that the framework has been designed with the goal of capturing more modal calculi in a wider range of formalisms, as we discuss in the concluding remarks (Section 5), where we also sum up our contributions and propose some directions for future work.

| Axiom | Condition | First-Order Formula |
|---|---|---|
| T: $\Box A \supset A$ | Reflexivity | $\forall x.R(x,x)$ |
| 4: $\Box A \supset \Box\Box A$ | Transitivity | $\forall x,y,z.(R(x,y) \land R(y,z)) \supset R(x,z)$ |
| 5: $\Box A \supset \Box\Diamond A$ | Euclideaness | $\forall x,y,z.(R(x,y) \land R(x,z)) \supset R(y,z)$ |
| B: $A \supset \Box\Diamond A$ | Symmetry | $\forall x,y.R(x,y) \supset R(y,x)$ |
| D: $\Box A \supset \Diamond A$ | Seriality | $\forall x\exists y.R(x,y)$ |

Table 1
Axioms and corresponding first-order conditions on the accessibility relation $R$.

## 2 Background : Focusing and modal logic

### 2.1 Modal logic

We will consider *(propositional) modal formulas* in negation normal form based on a functionally complete set of classical connectives, a *modal operator* $\Box$, together with its dual $\Diamond$, and a denumerable set $\mathcal{P}$ of *propositional symbols*, according to the following grammar (where $P \in \mathcal{P}$):

$$A ::= P \mid \neg P \mid A \lor A \mid \top \mid A \land A \mid \bot \mid \Box A \mid \Diamond A,$$

The negation $\neg A$ of a formula $A$ is defined via the De Morgan laws (so the only formally negated formulas are the atoms), and $A \supset B$ is defined as usual as $\neg A \lor B$. $A$ is a $\Box$-*formula* ($\Diamond$-*formula*) if the main connective of $A$ is $\Box$ ($\Diamond$).

The semantics is defined by means of *Kripke frames*, *i.e.*, pairs $\mathcal{F} = (W, R)$ where $W$ is a non empty set of *worlds* and $R$ is a binary relation on $W$. A *Kripke model* is a triple $\mathfrak{M} = (W, R, V)$ where $(W, R)$ is a Kripke frame and $V : W \to 2^{\mathcal{P}}$ is a function that assigns to each world in $W$ a (possibly empty) set of propositional symbols.

*Truth* of a modal formula at a point $w$ in a Kripke structure $\mathfrak{M} = (W, R, V)$ is the smallest relation $\models$ satisfying:

$$\mathfrak{M}, w \models P \quad \text{iff} \quad P \in V(w)$$
$$\mathfrak{M}, w \models A \lor B \quad \text{iff} \quad \mathfrak{M}, w \models A \text{ or } \mathfrak{M}, w \models B$$
$$\mathfrak{M}, w \models A \land B \quad \text{iff} \quad \mathfrak{M}, w \models A \text{ and } \mathfrak{M}, w \models B$$
$$\mathfrak{M}, w \models \Box A \quad \text{iff} \quad \text{for all } w' \text{ s.t. } R(w, w') \; \mathfrak{M}, w' \models A$$
$$\mathfrak{M}, w \models \Diamond A \quad \text{iff} \quad \text{there exists } w' \text{ s.t. } R(w, w') \text{ and } \mathfrak{M}, w' \models A.$$

By extension, we write $\mathfrak{M} \models A$ when $\mathfrak{M}, w \models A$ for all $w \in W$ and we write $\models A$ when $\mathfrak{M} \models A$ for every Kripke structure $\mathfrak{M}$.

The former definition characterizes the basic modal logic $K$. Several further modal logics can be defined as extensions of $K$ by simply restricting the class of frames we consider. Many of the restrictions we are interested in are definable as formulas of first-order logic where the binary predicate $R(x,y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common frame logics, describing the corresponding frame property, together with the modal axiom capturing it [22]. We will refer to the logic satisfying a set of axioms $\{F_1, \ldots, F_n\}$ as $K\{F_1, \ldots, F_n\}$.

Asynchronous introduction rules

$$\overline{\mathcal{G} \vdash \Theta \Uparrow x : t^-, \Gamma} \ t^- \qquad \frac{\mathcal{G} \vdash \Theta \Uparrow x : A, \Gamma \quad \mathcal{G} \vdash \Theta \Uparrow x : B, \Gamma}{\mathcal{G} \vdash \Theta \Uparrow x : A \wedge^- B, \Gamma} \ \wedge^-$$

$$\frac{\mathcal{G} \vdash \Theta \Uparrow \Gamma}{\mathcal{G} \vdash \Theta \Uparrow x : f^-, \Gamma} \ f^- \qquad \frac{\mathcal{G} \vdash \Theta \Uparrow x : A, x : B, \Gamma}{\mathcal{G} \vdash \Theta \Uparrow x : A \vee^- B, \Gamma} \ \vee^- \qquad \frac{\mathcal{G} \cup \{xRy\} \vdash \Theta \Uparrow y : B, \Gamma}{\mathcal{G} \vdash \Theta \Uparrow x : \Box B, \Gamma} \ \Box$$

Synchronous introduction rules

$$\overline{\mathcal{G} \vdash \Theta \Downarrow x : t^+} \ t^+ \qquad \frac{\mathcal{G} \vdash \Theta \Downarrow x : A \quad \mathcal{G} \vdash \Theta \Downarrow x : B}{\mathcal{G} \vdash \Theta \Downarrow x : A \wedge^+ B} \ \wedge^+$$

$$\frac{\mathcal{G} \vdash \Theta \Downarrow x : A}{\mathcal{G} \vdash \Theta \Downarrow x : A \vee^+ B} \ \vee_1^+ \qquad \frac{\mathcal{G} \vdash \Theta \Downarrow x : B}{\mathcal{G} \vdash \Theta \Downarrow x : A \vee^+ B} \ \vee_2^+ \qquad \frac{\mathcal{G} \cup \{xRy\} \vdash \Theta \Downarrow y : B}{\mathcal{G} \cup \{xRy\} \vdash \Theta \Downarrow x : \Diamond B} \ \Diamond$$

Identity rules

$$\overline{\mathcal{G} \vdash x : \neg B, \Theta \Downarrow x : B} \ init \qquad \frac{\mathcal{G} \vdash \Theta \Uparrow x : B \quad \mathcal{G} \vdash \Theta \Uparrow x : \neg B}{\mathcal{G} \vdash \Theta \Uparrow \cdot} \ cut$$

Structural rules

$$\frac{\vdash \Theta, x : B \Uparrow \Gamma}{\vdash \Theta \Uparrow x : B, \Gamma} \ store \qquad \frac{\vdash \Theta \Uparrow x : B}{\vdash \Theta \Downarrow x : B} \ release \qquad \frac{\vdash x : B, \Theta \Downarrow x : B}{\vdash x : B, \Theta \Uparrow \cdot} \ decide$$

In *decide*, $B$ is a positive formula; in *release*, $B$ is a negative formula; in *store*, $B$ is a positive formula or a negative literal; in *init*, $B$ is a positive literal. In $\Box$, $y$ does not occur in $\Theta$ nor in $\Gamma$.

Fig. 1. *LMF*: a focused labeled proof system for the modal logic $K$

## 2.2 Focused labeled proof system for modal logic

This work takes up the labeled approach to the proof theory of modal logics which internalizes the Kripke semantics into the syntax to give sequent calculi for numerous modal logics [19,24]. Traditionally, labeled sequents are composed by both *labeled formulas* of the form $x : A$ and *relational atoms* of the form $xRy$, where $x, y$ range over a set of variables (called *labels*) and $A$ is a modal formula. A (one-sided) labeled sequent will therefore be of the form $\mathcal{G} \vdash \Gamma$ where $\mathcal{G}$ denotes a set of relational atoms, and $\Gamma$ a multiset of labeled formulas.

Here we present a variant of the focused labeled system that was introduced in [17]. In general, a *focused sequent calculus* is one where introduction rules are placed into one of two *phases*. The *asynchronous* phase contains all invertible introduction rules: during this phase, non-atomic formulas are decomposed without external information being supplied to them (that is, they decompose asynchronously). The *synchronous* phase contains introduction rules in which decomposition may require additional information to be supplied: for example, the $\Diamond$-introduction rule needs to check the context for the suitable relational atom. Thus, these inference rules need to synchronize with some source of

information such as an oracle or a proof certificate [4].

Figure 1 contains the (subset of rules for the logic $K$ of the) focused proof system *LMF* from [17] that was designed for a range of modal logics. The key features of this proof system, which follow the design of focused proof systems for classical and intuitionistic logic given in [15], are the following.

**Polarized formula**  *LMF* is a proof system of *polarized formulas* built using atomic formulas, the usual modalities $\square$ and $\lozenge$, and polarized versions of the logical connectives $\vee^-$, $\vee^+$, $\wedge^-$, $\wedge^+$, and constants $t^-$, $t^+$ for $\top$, and $f^-$, $f^+$ for $\bot$. The positive and negative versions of connectives and constants have identical truth conditions but different inference rules. All polarized formulas are either positive or negative: if a formula's top-level connective is $t^+$, $f^+$, $\vee^+$, $\wedge^+$, or $\lozenge$, then that formula is positive. Dually, if a formula's top-level connective is $t^-$, $f^-$, $\vee^-$, $\wedge^-$, or $\square$, then it is negative. In this way, every polarized formula is classified except for literals: to polarize them, we are allowed to fix the polarity of atomic formulas in any way we see fit. We may ask that all atomic formulas are positive, that they are all negative, or we can mix polarity assignments. In any case, if $P$ is a positive atomic formula, then it is a positive formula and $\neg P$ is a negative formula: conversely, if $P$ is a negative atomic formula, then it is a negative formula and $\neg P$ is a positive formula.

**Two sequent judgments**  Sequents in *LMF* are of the form $\mathcal{G} \vdash \Theta \Uparrow \Delta$ or $\mathcal{G} \vdash \Theta \Downarrow x : A$, where $\mathcal{G}$ is a set of relational atoms, $x$ is a label, $A$ is a polarized modal formula, $\Theta$ is a multiset of labeled polarized formulas (called the *storage*), and $\Delta$ is a list of labeled polarized formulas. The formula $x : A$ in $\Downarrow$ sequents is called the *focus* of that sequent. If $\Gamma$ is a multiset of formulas then $\lozenge\Gamma$ denotes the multiset $\{\lozenge B \mid B \in \Gamma\}$ and $x : \Gamma$ denotes the multiset of labeled formulas $\{x : B \mid B \in \Gamma\}$.

**Two phases of inference rules**  All the asynchronous inference rules of *LMF* have $\Uparrow$-sequents in their premises and conclusion while all the synchronous inference rules have $\Downarrow$-sequents in their premises and conclusion. The only rules that mix these sequents are the *release* and *decide* rules. A maximal sequence of asynchronous or synchronous inferences form *phases* with interfaces between phases given by instances of the *release* and *decide* rules. These phases form, in fact, macro-level (synthetic) inference rules constructed from collections of the smaller rules of the focused sequent calculus.

A polarized formula $B$ is a *bipolar formula* if $B$ is a positive formula and no positive subformula occurrence of $B$ is in the scope of a negative connective in $B$. A *bipole* is a pair of a synchronous phase below an asynchronous phase within *LMF*: thus, bipoles are macro inference rules in which the conclusion and the premises are $\Uparrow$-sequents with no formulas to the right of the up-arrow.

**Delays**  We shall find it important to break a sequence of negative or positive connectives by inserting *delays*: if $B$ is a polarized formula then we define $\partial^-(B)$ to be (always negative) $B \wedge^- t^-$ and $\partial^+(B)$ to be (always positive) $B \wedge^+ t^+$. From such a definition, the following rules can be derived:

$$\frac{\mathcal{G} \vdash \Theta \Uparrow x : B, \Delta}{\mathcal{G} \vdash \Theta \Uparrow x : \partial^-(B), \Delta} \ \partial^- \qquad\qquad \frac{\mathcal{G} \vdash \Theta \Downarrow x : B}{\mathcal{G} \vdash \Theta \Downarrow x : \partial^+(B)} \ \partial^+$$

To illustrate the use of delays, note that the sequent $xRy, yRz \vdash \cdot \Downarrow x : \Diamond\Diamond B$ must be the result of applying two $\Diamond$-introduction rules in a synchronous phase before further processing the formula $B$. In contrast, the sequent $xRy, yRz \vdash \cdot \Downarrow x : \Diamond\partial^-(\Diamond B)$ must be the conclusion of only one $\Diamond$-introduction rule and allows one to store an instance of $\Diamond B$ such that a separate occurrence of $\Diamond$ can take place elsewhere in the proof.

$$\frac{\dfrac{xRy, yRz \vdash \cdot \Downarrow z : B}{xRy, yRz \vdash \cdot \Downarrow y : \Diamond B} \ \Diamond}{xRy, yRz \vdash \cdot \Downarrow x : \Diamond\Diamond B} \ \Diamond \qquad \frac{\dfrac{\dfrac{\dfrac{\dfrac{xRy, yRz \vdash y : \Diamond B \Uparrow \cdot}{xRy, yRz \vdash \cdot \Uparrow y : \Diamond B} \ store}{xRy, yRz \vdash \cdot \Uparrow y : \partial^-(\Diamond B)} \ \partial^-}{xRy, yRz \vdash \cdot \Downarrow y : \partial^-(\Diamond B)} \ release}{xRy, yRz \vdash \cdot \Downarrow x : \Diamond\partial^-(\Diamond B)} \ \Diamond$$

The completeness of *LMF* is stated as follows [17]. We say that $\hat{B}$ is a *polarization* of the (unpolarized) $B$ if it results from placing superscripts $+$ and $-$ on the propositional connectives, assigning atomic formulas any mix of positive or negative polarization, and inserting any number of delays. Completeness is now the statement that if $B$ is an (unpolarized) modal logic theorem and $\hat{B}$ is *any* polarization of $B$, then $\vdash \cdot \Uparrow x : \hat{B}$ is provable in *LMF*. That is, the choice of polarization does not affect provability but it can have a big impact on the structure of proofs.

## 3    A focused labeled framework for modal logic

In this section, we present a multifocused version of *LMF* further augmented with some devices aimed at enabling the emulation of different modal proof systems. In order to motivate the need for such devices, consider the following typical sequent calculus rule for modal logic:

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A} \ .$$

Augmentation of the system *LMF* is driven by the following considerations of inference rules of this kind.

(i) As already noticed in Section 1, this rule works at the same time on one $\Box$-formula and on $n$ $\Diamond$-formulas. In order to process such $\Diamond$-formulas, in our labeled deduction setting, it is necessary to apply the $\Diamond$-introduction rule $n$ times. Since these applications do not interfere with each other, they can, in fact, be applied in parallel. For this reason, we move to a multifocused version of *LMF*, i.e., a variant where we can focus on several positive formulas at the same time. In this way, we can group all the $\Diamond$-introductions inside a single phase (in the following, we will sometimes call it a $\Diamond$-phase).

(ii) Intuitively, one can read this inference rule (reading from conclusion to premise) as moving from one world to another (reachable) world in a

suitable Kripke structure. Such a change of world becomes apparent when we consider the corresponding deduction steps in a labeled system, as, in this case, modal introduction rules will explicitly change the label of the formulas under consideration. In order to properly mimic the behavior of the original rule, in the labeled system we need to be able to force all the formulas involved in the rule to move to the same new world. We therefore modify the notion of a labeled formula to have the form $x\sigma : A$, for $\sigma$ a sequence of labels. Here $x$ indicates in which world such a formula holds, while the sequence $\sigma$ gets initialized when one multifocuses on the multiset of $\Diamond$-formulas and is used to drive future applications of $\Diamond$-rules. E.g., if $x : \Diamond\Gamma$ is on the left of $\Uparrow$, then we can multifocus on $xy : \Diamond\Gamma$ for a given $y$ reachable from $x$. This $y$ will be used as a witness in the application of a (properly modified) $\Diamond$-introduction rule, in such a way that at the end of the bipole, we will have the multiset $y : \Gamma$ on the left of $\Uparrow$.[1]

(iii) Finally, we observe that in *LMF*, when constructing a proof tree (going from the root towards the leaves), formulas we decide on are duplicated and stay in the storage (that is, on the left of $\Uparrow$ or $\Downarrow$). It follows that all along a proof, it is possible to switch freely from one label to another in the deduction process. On the contrary, in a sequent calculus rule like the one given above, only formulas having a modal operator as the main connective can be "promoted" to a different world. According to the Kripke-style interpretation presented, this amounts to considering a single world at a time, in such a way that when moving to a new one, formulas standing at previously encountered worlds are not accessible anymore. In order to emulate this aspect, labelled sequents are further decorated with a set $\mathcal{H}$ of labels, specifying which worlds are currently enabled, with the intended meaning that we can decide on a formula only if its label belongs to $\mathcal{H}$.[2]

In the following, we will formalize the intuitions given above, introduce some terminology and present the general framework $LMF_*^X$ (Figure 2).

In the rest of this paper, a *labeled formula* will have the form $\varphi \equiv x\sigma : A$, where $\sigma$ is a (possibly empty) sequence of labels. We say that $x$ is the *present* of $\varphi$ and $\sigma$ is the *future* of $\varphi$. An $LMF_*^X$ *sequent* has the form $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega$ or $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow \Omega$, where the *relational set (of the sequent)* $\mathcal{G}$ is a set of relational atoms, the *present (of the sequent)* $\mathcal{H}$ is a non-empty multiset of pairs $(x, \mathcal{F})$, where $x$ is a label and $\mathcal{F}$ is a set of labels, and $\Theta$ and $\Omega$ are multisets of labeled formulas. Intuitively, a pair $(x, \mathcal{F})$ specifies that $x$ is among the worlds we are currently working on and $\mathcal{F}$ indicates which worlds, among the reachable ones, are not accessible from $x$. E.g., if we are in the position of applying a *decide*$_*$

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[1] We note that for the emulation of the calculi presented in this paper, a future consisting of a single label is enough. We prefer, however, to present the framework in this more general version that allows for capturing also other behaviors.

[2] In fact, this representation of $\mathcal{H}$ as a set of labels is good for giving an insight of the technique, but it is slightly simpler than the one concretely used in the framework, which is formalized in the next paragraphs.

ASYNCHRONOUS INTRODUCTION RULES

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : t^-, \Omega} \; t_*^- \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : f^-, \Omega} \; f_*^-$$

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A, \Omega \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A \wedge^- B, \Omega} \; \wedge_*^- \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A, x : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A \vee^- B, \Omega} \; \vee_*^-$$

$$\frac{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Uparrow y : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : \Box B, \Omega} \; \Box_*$$

SYNCHRONOUS INTRODUCTION RULES

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : t^+} \; t_*^+ \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1, \Omega_1 \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_2, \Omega_2}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1 \wedge^+ B_2, \Omega_1, \Omega_2} \; \wedge_*^+$$

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_i, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1 \vee^+ B_2, \Omega} \; \vee_*^+, i \in \{1, 2\} \qquad \frac{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Downarrow y\sigma : B, \Omega}{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Downarrow xy\sigma : \Diamond B, \Omega} \; \Diamond_*$$

IDENTITY RULES

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} x : \neg B, \Theta \Downarrow x : B} \; init_* \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : B \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : \neg B}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} \; cut_*$$

STRUCTURAL RULES

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta, x : B \Uparrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : B, \Omega} \; store_* \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega'}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow \Omega} \; release_* \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} \; decide_*$$

RELATIONAL RULES

$$\frac{\mathcal{G} \cup \{yRy\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} T_* \quad \frac{\mathcal{G} \cup \{yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} D_* \quad \frac{\mathcal{G} \cup \{xRy, yRx\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} B_*$$

$$\frac{\mathcal{G} \cup \{xRy, yRz, xRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy, yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} \; 4_* \quad \frac{\mathcal{G} \cup \{xRy, xRz, yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy, xRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot} \; 5_*$$

In $store_*$, $B$ is a positive formula or a negative literal.
In $init_*$, $B$ is a positive literal.
In $\Box_*$, $y$ is different from $x$ and does not occur in $\mathcal{G}$ nor in $\Theta$.
In $decide_*$, if $x\sigma : A \in \Omega$ then $x : A \in \Theta$. Moreover, $\Omega$ contains only positive formulas of the form: $(i)$ $x\sigma : A$, where $A$ is not a $\Diamond$-formula and $(x, \mathcal{F}) \in \mathcal{H}$ for some $\mathcal{F}$; or $(ii)$ $zy\sigma : A$ where $A$ is a $\Diamond$-formula, $xRy, zRy \in \mathcal{G}$, $(x, \mathcal{F}) \in \mathcal{H}$ for some $\mathcal{F}$ and $y \notin \mathcal{F}$.
In $release_*$, $\Omega$ contains no positive formulas and $\Omega' = \{x : A \mid x\sigma : A \in \Omega\}$.
In $D_*$, $z$ is different from $y$ and does not occur in $\mathcal{G}$ and $\Theta$.

Fig. 2. $LMF_*^X$: a focused labeled framework for modal logic.

(by proceeding bottom-up), then a pair $(x, \mathcal{F})$ contained in $\mathcal{H}$ says that: $(i)$ we can (multi)focus on non-$\Diamond$-formulas labeled with $x$; or $(ii)$ we can "move" to a $y$ reachable from $x$ (by (multi)focusing on $\Diamond$-formulas) if $y$ is not in the set $\mathcal{F}$ of *forbidden futures* for $x$. In this general formulation, the set $\mathcal{H}'$ that we get in the premise of the rule can be defined in an arbitrary way; specific ways of defining it will be proposed in next sections in order to obtain particular behaviors. In $decide_*$, we are also allowed to assign a future $\sigma$ to a formula $x : A$ in the storage, so that we actually focus on $x\sigma : A$. Such futures are relevant in the treatment of $\Diamond$-formulas. In fact, when we apply $\Diamond_*$ with respect to a formula $xy\tau : \Diamond A$, we are forced to "move" to the world $y$ thus getting $y\tau : A$ in the premise. Since futures of formulas are only relevant during the synchronous phase, applications of $release_*$ remove all such futures. The other rules of the system are simple adaptations of the ones in $LMF$.

We say that an $LMF_*^X$ sequent is a *synchronized sequent* if it has the form $\mathcal{G} \vdash_\mathcal{H} \Theta \Uparrow \cdot$. In fact, a sequent that contains no formulas on the right of the arrow is what we get at the end of a bipole. This is the moment when we can more easily compare the status of an $LMF_*^X$ proof with the status of the proof to be emulated, i.e., in a sense, synchronize the two proofs.

The parameter $X$ is a subset of $\{T, 4, 5, B, D\}$, specifying which modal logic we are considering. The system $LMF_*^\emptyset$ is a system for the logic $K$ and is obtained by including only the first four classes of rules (i.e., no relational rules). Any other system $LMF_*^X$ is obtained by adding to $LMF_*^\emptyset$ the set of relational rules $\{C_F \mid C \in X\}$.

**Theorem 3.1** *The system* $LMF_*^X$ *is sound and complete with respect to the logic* $KX$, *for any polarization of formulas.*

**Proof.** The system $LMF_*^X$ is a multifocused version of the system $LMF$ presented in [17] and recalled in Section 2.2, augmented with some devices for controlling the application of rules. Soundness follows from the fact that such devices can only introduce restrictions to the application of rules and multifocusing can be simulated in $LMF$ by several rule applications. Completeness is also a direct consequence of completeness of $LMF$, since in the liberal version presented in this section all new devices (including multifocusing) can just be ignored, or used in a trivial way, so that each proof in the previous system is also a valid proof in $LMF_*^X$. $\qquad\qquad\square$

In addition to being modular with respect to the relational properties considered, we can (and in the following will) obtain different concrete proof systems by properly specifying the behavior of the new devices introduced in $LMF_*^X$. These will be defined by specializing the rule $decide_*$, i.e., in particular, by playing with the following parameters:

- restrictions on the class of formulas on which multifocusing can be applied;
- restrictions on the definition of the future $\sigma$ of formulas in $\Omega$;
- restriction of the multiset $\mathcal{H}'$ (in the premise of $decide_*$).

IDENTITY AND STRUCTURAL RULES

$$\overline{\vdash \Gamma, P, \neg P} \; init \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta} \; cut \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \; contr$$

CLASSICAL CONNECTIVES RULES

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \; \wedge \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \; \vee$$

□-RULES

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta} \; \Box_K \qquad \frac{\vdash \Diamond\Gamma, \Gamma', A}{\vdash \Diamond\Gamma, \Box A, \Delta} \; \Box_{K4} \qquad \frac{\vdash \Diamond\Gamma, \Gamma', \Box\Sigma, A}{\vdash \Diamond\Gamma, \Box\Sigma, \Box A, \Delta} \; \Box_{K45}$$

◇-RULES

$$\frac{\vdash \Diamond A, A, \Sigma}{\vdash \Diamond A, \Sigma} \; \Diamond_T \qquad \frac{\vdash \Gamma}{\vdash \Diamond\Gamma, \Delta} \; \Diamond_D$$

In $\Box_{K4}$, $\Delta$ does not contain any formula whose main connective is $\Diamond$.
In $\Box_{K45}$, $\Delta$ does not contain any formula whose main connective is $\Diamond$ or $\Box$.
$\Gamma' \subseteq \Gamma$. $\neg A$ is the negation normal form of the negation of $A$.

Fig. 3. $OS^X$: a family of ordinary sequent proof systems for modal logic.

## 4 Emulation of other modal proof systems

In order to emulate proofs given in other proof calculi by means of the focused framework $LMF_*^X$, we need to: $(i)$ define a proper polarization of modal formulas; and $(ii)$ give a specialized version of the rule $decide_*$. As an illustration of this potentiality, we consider in this section some standard sequent and nested sequent calculi.

### 4.1 Ordinary sequent calculi

Several "ordinary" sequent systems have been proposed in the literature for different modal logics (a general account is, e.g., in [11,20]). In our treatment, we will use a formalization of a class of modal sequent systems, presented in Figure 3, which is adapted mainly from the presentations in [6,23]. It can be seen as a family of proof systems, where the system of a specific logic is obtained by adding to the base classical system (consisting of *identity*, *structural* and *classical connective* rules) one of the □-rules and any (possibly empty) set of ◇-rules. As the name of the rule suggests, the rule $\Box_K$ alone gives a system for the logic $K$. We replace it with $\Box_{K4}$ or $\Box_{K45}$ in case we want to capture logics characterized by transitive or both transitive and euclidean frames, respectively. The rules $\Diamond_T$ and $\Diamond_D$ can be further added, modularly, in order to get systems for those logics enjoying reflexivity and seriality, respectively. For instance, by adding $\Box_{K4}$ and $\Diamond_T$ to the base system, we get a system for the logic $S4$, while by adding $\Box_{K45}$ and $\Diamond_T$, we get a system for $S5$. Formulas are assumed to be in negation normal form.

First, we present a polarization that allows us to enforce the behavior of these rules in our framework. When translating a modal formula into a polarized one, we are often in a situation where we are interested in putting a delay in front of the formula only in the case when it is negative and not a literal. For that purpose, we define $A^{\partial^+}$, where $A$ is a modal formula in negation normal form, to be $A$ if $A$ is a literal or a positive formula and $\partial^+(A)$ otherwise. We extend such a notion to a multiset $\Gamma$ of formulas by defining $\Gamma^{\partial^+} = \{A^{\partial^+} | A \in \Gamma\}$. Then we define the translation $\lfloor \cdot \rfloor$ from modal formulas in negation normal form into polarized modal formulas as follows:

$$
\begin{array}{rclcrcl}
\lfloor P \rfloor & = & P & \qquad & \lfloor A \wedge B \rfloor & = & \lfloor A \rfloor^{\partial^+} \wedge^- \lfloor B \rfloor^{\partial^+} \\
\lfloor \neg P \rfloor & = & \neg P & & \lfloor A \vee B \rfloor & = & \lfloor A \rfloor^{\partial^+} \vee^- \lfloor B \rfloor^{\partial^+} \\
\lfloor \Box A \rfloor & = & \Box(\lfloor A \rfloor^{\partial^+}) & & \lfloor \Diamond A \rfloor & = & \Diamond(\partial^-(\lfloor A \rfloor^{\partial^+}))
\end{array}
$$

In the following, we will sometimes use the natural extension of this translation to multisets of modal formulas, i.e., $\lfloor \Gamma \rfloor = \{\lfloor A \rfloor \mid A \in \Gamma\}$.

We note that the use of delays in this translation is motivated by the desire of keeping the correspondence between rule inferences in the emulated calculus and bipoles in our framework as strict as possible. In a sense, delays ensure that in a single phase we do not emulate more than one rule application of the original proof system.

Furthermore, we specialize the rule $decide_*$ as follows:

$$
\frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\{(x,\mathcal{F})\}} \Theta \Uparrow \cdot} \ decide_{OS}
$$

where (in addition to the general conditions of Figure 2) we have that either:

(i) there exists $y$ s.t.:
  - $xRy \in \mathcal{G}$;
  - if $x \neq y$, then $\mathcal{H}' = \{(y, \mathcal{F} \cup \{x\})\}$ and $\Omega$ is a multiset of formulas of the form $zy : \Diamond A$, s.t. $zRy \in \mathcal{G}$, $z \in \mathcal{F}$;
  - if $x = y$ then $\mathcal{H}' = \{(x, \mathcal{F})\}$ and $\Omega = \{xx : \Diamond A\}$ for some $A$; or

(ii) $\Omega = \{x : A\}$ for some $A$ and $\mathcal{H}' = \{(x, \mathcal{F})\}$.

Intuitively, the specialization with respect to the general framework consists in: $(i)$ restricting the use of multifocusing to $\Diamond$-formulas; $(ii)$ forcing such $\Diamond$-formulas to be labeled with the same future; $(iii)$ when moving to a new label, adding the current label to the set of forbidden futures.

The structure of the proofs obtained by using these restrictions can be described as a sequence of blocks, each of which is related to a specific world (label). For each such a block, we first apply a number of classical and $\Box$-introductions on the given world (and some relational rules, if we are beyond $K$) and then move to a new one by means of a $\Diamond$-phase. The mechanism that we use, in $decide_{OS}$, for updating the present $\mathcal{H}$ of the sequent ensures that we never go back to an already encountered world.

We call $LMF_{OS}^X$ the system obtained from $LMF_*^X$ by replacing the rule $decide_*$ with the rule $decide_{OS}$. It is easy to notice that, given the polarization above and this new rule, we can in fact restrict $LMF_{OS}^X$ to deal with sequents

whose present is always a singleton and such that the future of each labeled formula has length at most 1. In the rest of this section, for simplicity, we will then write sequents using the following notation: $\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Updownarrow \Omega$.

As remarked in the discussion at the beginning of Section 3, even for a simple modal rule like $\Box_K$, at least two corresponding bipoles (one involving the $\Box$-formula and one involving $\Diamond$-formulas) are necessary in our framework. This means that in an $LMF_{OS}^X$ proof, we can encounter synchronized sequents that do not correspond "precisely" to any sequent in the original proof, e.g., because (by reading the $LMF_{OS}^X$ proof bottom-up) we are at a stage where a $\Box$-rule has been applied but the corresponding $\Diamond$-phase has not started yet. We will thus base our correspondence results on an interpretation that takes this fact into account. In the case of logics whose frames enjoy transitivity, such an interpretation will also have to consider that in the rule $\Box_{K4}$, $\Diamond$-formulas stay in the sequent when going from conclusion to premise, and such a behavior can only be captured in $LMF_{OS}^X$ by applying more than one step. Formally, we define the interpretation $\mathcal{I}_{OS}^X(\cdot)$ of synchronized sequents as multisets of modal formulas (where the $X$ denotes the fact that the interpretation is also parametric in the logic considered) as follows:

$$\mathcal{I}_{OS}^X(\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot) = \begin{cases} \bullet \; \{A \mid x : \lfloor A \rfloor^{\partial^+} \in \Theta\} \cup \{\Box B \mid y : \partial^+(\lfloor B \rfloor) \in \Theta, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\}, \\ \quad \text{if } 4 \notin X \\ \bullet \; \{A \mid x : \lfloor A \rfloor^{\partial^+} \in \Theta\} \cup \{\Box B \mid y : \partial^+(\lfloor B \rfloor) \in \Theta, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\} \cup \\ \quad \{\Diamond C \mid z : \lfloor \Diamond C \rfloor \in \Theta, zRx \in \mathcal{G}^*, z \in \mathcal{F}\}, \quad \text{otherwise} \end{cases}$$

where $\mathcal{G}^*$ denotes the closure of $\mathcal{G}$ with respect to those properties among reflexivity, transitivity and euclideaness contained in $X$.

We notice that in an $LMF_{OS}^X$ derivation (reading from the end-sequent upwards), when we decide on a formula, we keep a copy of it in the storage, i.e., we implicitly apply a contraction. For this reason, we have that an $LMF_{OS}^X$ derivation tends to keep some information that is lost in the corresponding $OS^X$ derivation (again, reading bottom-up). We define a notion of extension of a sequent that will help compare the two systems. Given a synchronized sequent $S \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ and an $OS$ sequent $\vdash \Gamma$, we say that $S$ *extends* $\vdash \Gamma$ if there exists $S' \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta' \Uparrow \cdot$ such that $\Theta \supseteq \Theta'$ and $\mathcal{I}_{OS}^X(S') = \Gamma$. Furthermore, we say that a synchronized sequent $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ is in *OS form* if for all $x : A \in \Theta$, $A = \lfloor B \rfloor$ for some modal formula $B$.

**Lemma 4.1** *Let $\dfrac{S_1}{S} \; r \; \left( \dfrac{S_1 \quad S_2}{S} \; r \right)$ be an application of a non-structural rule in $OS^X$. Then for any synchronized sequent $S'$ that is in OS form and extends $S$, there exists a derivation $\begin{matrix} S_1' \\ \vdots \\ S' \end{matrix} \; \left( \begin{matrix} S_1' \quad S_2' \\ \vdots \\ S' \end{matrix} \right)$ in $LMF_{OS}^X$, such that $S_1'$ is in OS form and extends $S_1$ ($S_1'$, $S_2'$ are in OS form and extend $S_1$, $S_2$, respectively). Furthermore, if $\dfrac{}{S} \; r$ is a rule application in $OS^X$, then for any synchronized sequent $S'$ in OS form extending $S$, there exists a proof of $S'$ in $LMF_{OS}^X$.*

**Proof.** The proof proceeds by considering all the non-structural rules of $OS^X$.

The cases for initial and the introduction of classical connectives are trivial and we omit them. We consider some key cases. We will also use the rules $\partial_*^-$ and $\partial_*^+$, which are trivial adaptations of $\partial^-$ and $\partial^+$ to the case of $LMF_*^X$ sequents.

(1) Let us take an application of the rule $\Box_K$:

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta} \ \Box_K$$

where $\Delta$ does not contain any formula whose main connective is $\Diamond$. Now assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in $OS$ form and extends $\vdash \Diamond\Gamma, \Box A, \Delta$. Notice that we are in the case when 4 does not occur in $X$. It follows that $x : \lfloor \Diamond\Gamma \rfloor \subseteq \Theta$. We have two cases: either $(a)$ $x : \partial^+(\lfloor \Box A \rfloor) \in \Theta$ or $(b)$ $y : \lfloor A \rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}$. Then the $LMF_*^X$ derivation corresponding to this rule application consists in the following steps (reading the derivation bottom-up):

(i) decide on $x : \partial^+(\lfloor \Box A \rfloor)$, ending up by adding $xRy$ to $\mathcal{G}$ (note that this step is only required if we are in case $(a)$);

$$\frac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow y : \lfloor A \rfloor^{\partial^+}} \ store_*}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow x : \Box\lfloor A \rfloor^{\partial^+}} \ \Box_*}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^+(\Box\lfloor A \rfloor^{\partial^+})} \ \partial_*^+, release_*}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot} \ decide_{OS}$$

(ii) multifocus on $x : \lfloor \Diamond\Gamma \rfloor$ choosing $y$ as the future.

$$\frac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta, y : \lfloor A \rfloor^{\partial^+}, y : \lfloor \Gamma \rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Uparrow y : \partial^-(\lfloor \Gamma \rfloor^{\partial^+})} \ \partial_*^-, store_*}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Downarrow y : \partial^-(\lfloor \Gamma \rfloor^{\partial^+})} \ release_*}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Downarrow xy : \Diamond\partial^-(\lfloor \Gamma \rfloor^{\partial^+})} \ \Diamond_*}{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Uparrow \cdot} \ decide_{OS}$$

(2) Let us consider an application of the rule $\Box_{K4}$:

$$\frac{\vdash \Diamond\Gamma, \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta} \ \Box_{K4}$$

where $\Delta$ does not contain any formula whose main connective is $\Diamond$. Assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in $OS$ form and extends $\vdash \Diamond\Gamma, \Box A, \Delta$. As in $(i)$, we can have two cases: either $(a)$ $x : \partial^+(\lfloor \Box A \rfloor) \in \Theta$ or $(b)$ $y : \lfloor A \rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}^*$. Moreover, for each $B \in \Gamma$, one of the following two cases holds: either $(c)$ $x : \lfloor \Diamond B \rfloor \in \Theta$ or $(d)$ $z : \lfloor \Diamond B \rfloor \in \Theta$ and $zRx \in \mathcal{G}^*$ for some $z$. After possible applications of relational rules that lead to a sequent containing $xRy$ (if we are in case $(b)$) and $zRx$ (if we are in case $(d)$), the $LMF_*^X$ derivation corresponding to this rule application consists in the following bipoles:

(i) decide on $x : \partial^+(\lfloor \Box A \rfloor)$, ending up by adding $xRy$ to $\mathcal{G}$ (note that this

step is only required if we are in case $(a)$);

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A \rfloor^{\partial^+} \Uparrow \cdot
      }{
        \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow x : \Box \lfloor A \rfloor^{\partial^+}
      } \; \Box_*, store_*
    }{
      \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^+ (\Box \lfloor A \rfloor^{\partial^+})
    } \; \partial_*^+, release_*
  }{
    \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot
  } \; decide_{OS}
}{}
$$

(ii) for those $B \in \Gamma$ such that case $(d)$ holds, we apply the rule $4_*$ to $zRx$ and $xRy$ (ending up by adding $zRy$ to the relation set);

$$
\cfrac{
  \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot
}{
  \mathcal{G} \cup \{zRx, xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot
} \; 4_*
$$

(iii) multifocus on all the $w : \lfloor \Diamond B \rfloor$ such that $wRy$ is in the relation set and $B \in \Gamma$, choosing $y$ as the future.

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta, y : \lfloor B \rfloor^{\partial^+}, \Omega'' \Uparrow \cdot
      }{
        \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta \Uparrow y : \partial^- (\lfloor B \rfloor^{\partial^+}), \Omega'
      } \; \partial_*^-, store_*
    }{
      \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta \Downarrow y : \partial^- (\lfloor B \rfloor^{\partial^+}), \Omega'
    } \; release_*
  }{
    \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F} \cup \{x\}} \Theta \Downarrow wy : \Diamond \partial^- (\lfloor B \rfloor^{\partial^+}), \Omega
  } \; \Diamond_*
}{
  \mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot
} \; decide_{OS}
$$

(3) Let us consider an application of the rule $\Box_{K45}$:

$$
\cfrac{
  \vdash \Diamond \Gamma, \Gamma, \Box \Sigma, A
}{
  \vdash \Diamond \Gamma, \Box \Sigma, \Box A, \Delta
} \; \Box_{K45}
$$

where $\Delta$ does not contain any formula whose main connective is $\Diamond$ or $\Box$. Assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in $OS$ form and extends $\vdash \Diamond \Gamma, \Box \Sigma, \Box A, \Delta$. We focus on the treatment of the formulas in $\Sigma$, which is the difference with respect to case $(ii)$. Let $B \in \Sigma$. By hypothesis, either $(a)$ $x : \partial^+ (\lfloor \Box B \rfloor) \in \Theta$ or $(b) y : \lfloor B \rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}^*$. If we are in case $(a)$, then an application of $\Box_*$ followed by an application of $5_*$ will eventually lead to a synchronized sequent $S_1'$ such that $\Box B \in \mathcal{I}_{OS}^X (S_1')$. If we are in case $(b)$, then an application of $5_*$, plus possible relational rules to get $xRy$ in the relational set, will suffice.

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \mathcal{G} \cup \{xRy, xRu, yRu\} \vdash_{x,\mathcal{F}} \Theta, u : \lfloor B \rfloor^{\partial^+} \Uparrow \cdot
        }{
          \mathcal{G} \cup \{xRy, xRu\} \vdash_{x,\mathcal{F}} \Theta, u : \lfloor B \rfloor^{\partial^+} \Uparrow \cdot
        } \; 5_*
      }{
        \mathcal{G} \cup \{xRy, xRu\} \vdash_{x,\mathcal{F}} \Theta \Uparrow x : \Box \lfloor B \rfloor^{\partial^+}
      } \; \Box_*, store_*
    }{
      \mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^+ (\Box \lfloor B \rfloor^{\partial^+})
    } \; \partial_*^+, release_*
  }{
    \mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot
  } \; decide_{OS}
}{}
$$

(4) Let us consider an application of the rule $\Diamond_T$:

$$
\cfrac{
  \vdash \Diamond A, A, \Sigma
}{
  \vdash \Diamond A, \Sigma
} \; \Diamond_T
$$

and assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in $OS$ form and extends $\vdash \Diamond A, \Sigma$. We have that either $(a)$ $x : \lfloor \Diamond A \rfloor \in \Theta$ or $(b)$ we are in a case where $X$ contains

4 and $z : \lfloor \Diamond A \rfloor \in \Theta$ and $zRx \in \mathcal{G}^*$. After possible applications of relational rules that lead to a sequent whose relational set contains $zRx$ (if we are in case $(b)$), the $LMF_*^X$ derivation corresponding to this rule application consists in the following bipoles (reading the derivation bottom-up):

(i) if we are in case $(a)$, apply the rule $T_*$ in order to add $xRx$ to $\mathcal{G}$; then decide on $x : \lfloor \Diamond A \rfloor$;

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRx\} \vdash_{x,\mathcal{F}} \Theta, x : \lfloor A \rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{xRx\} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^-(\lfloor A \rfloor^{\partial^+})} \ release_*, \partial_*^-, store_*}{\mathcal{G} \cup \{xRx\} \vdash_{x,\mathcal{F}} \Theta \Downarrow xx : \Diamond \partial^-(\lfloor A \rfloor^{\partial^+})} \ \Diamond_*}{\mathcal{G} \cup \{xRx\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot} \ decide_{OS}}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot} \ T_*$$

(ii) if we are in case $(b)$, then decide on $z : \lfloor \Diamond A \rfloor$ and choose $x$ as the future.

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{zRx\} \vdash_{x,\mathcal{F}} \Theta, x : \lfloor A \rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{zRx\} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^-(\lfloor A \rfloor^{\partial^+})} \ release_*, \partial_*^-, store_*}{\mathcal{G} \cup \{zRx\} \vdash_{x,\mathcal{F}} \Theta \Downarrow zx : \Diamond \partial^-(\lfloor A \rfloor^{\partial^+})} \ \Diamond_*}{\mathcal{G} \cup \{zRx\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot} \ decide_{OS}$$

(5) Let us consider an application of the rule $\Diamond_D$: $\dfrac{\vdash \Gamma}{\vdash \Diamond \Gamma, \Delta} \ \Diamond_D$ where $\Delta$ does not contain any formula whose main connective is $\Diamond$. Assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in $OS$ form and extends $\vdash \Diamond \Gamma, \Delta$. For each $B \in \Gamma$, one of the following two cases holds: either $(a)$ $x : \lfloor \Diamond B \rfloor \in \Theta$ or $(b)$ $z : \lfloor \Diamond B \rfloor \in \Theta$ and $zRx \in \mathcal{G}^*$ (note that this is only possible if $X$ contains 4). After possible applications of relational rules that lead to a sequent whose relational set contains $zRx$ (if we are in case $(b)$), the $LMF_*^X$ derivation corresponding to this application consists in the following bipoles (reading the derivation bottom-up): (i) apply the rule $D_*$ in order to add $xRy$ to $\mathcal{G}$ for some "fresh" $y$; (ii) for those $B \in \Gamma$ such that case $c$ holds, apply the rule $4_*$ to $zRx$ and $xRy$ (ending up by adding $zRy$ to the relation set); (iii) multifocus on all the $w : \lfloor \Diamond B \rfloor$ such that $wRy$ is in the relation set and $B \in \Gamma$, choosing $y$ as the future. $\square$

**Theorem 4.2** *Let $\Pi$ be an $OS^X$ derivation of a sequent $S \equiv \vdash A$ from the sequents $S_1, \ldots, S_n$ and let $S' \equiv \emptyset \vdash_{\{x,\emptyset\}} x : (\lfloor A \rfloor)^{\partial^+} \Uparrow \cdot$ for some $x$. Then there exists an $LMF_{OS}^X$ derivation $\Pi'$ of $S'$ from $S_1', \ldots, S_n'$, where $S_1', \ldots, S_n'$ extend $S_1, \ldots, S_n$, respectively. Moreover, $\Pi'$ is such that each rule application in $\Pi$, deriving a sequent $\hat{S}$, corresponds to a sequence $s$ of bipoles in $\Pi'$ such that $s$ ends with a synchronized sequent $\hat{S}'$ extending $\hat{S}$.*

**Proof.** For simplicity, we assume that in $\Pi$ the rule *contr* is only applied to a given formula immediately below a rule that introduces an occurrence of such a formula. We proceed bottom-up by starting from the root of $\Pi$ and build $\Pi'$ by repeatedly applying Lemma 4.1. At each step, we get as leaves sequents that are extensions of the ones in $\Pi$, so that Lemma 4.1 can be applied again. $\square$

We say that a synchronized sequent $S \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ is a *contraction of an OS sequent* $\vdash \Gamma$ if $S$ is in $OS$ form, $\Gamma$ contains $\mathcal{I}_{OS}^X(S)$ and for each formula $A$ in $\Gamma$ there is at least one occurrence of $A$ in $\mathcal{I}_{OS}^X(S)$.

**Lemma 4.3** *Let $S' \equiv \mathcal{G} \vdash_{\{x,\mathcal{F}\}} \Theta \Uparrow \cdot$ be a synchronized sequent in OS form. For each derivation of the form* $\begin{array}{c} S_1' \\ \vdots \\ S' \end{array} \left( \begin{array}{cc} S_1' & S_2' \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{S'} \end{array} \right)$ *in $LMF_{OS}^X$ that is a bipole, there exists an OS sequent $S$ such that: (i) $S'$ is a contraction of $S$; and (ii) if $\mathcal{I}_{OS}^X(S_1') \neq \mathcal{I}_{OS}^X(S')$ ($\mathcal{I}_{OS}^X(S_1') \neq \mathcal{I}_{OS}^X(S')$ and $\mathcal{I}_{OS}^X(S_2') \neq \mathcal{I}_{OS}^X(S')$), then there exists a rule application $\dfrac{S_1}{S}$ $\left( \dfrac{S_1 \quad S_2}{S} \right)$ in $OS^X$ such that $\mathcal{I}_{OS}^X(S_1') = S_1$ ($\mathcal{I}_{OS}^X(S_1') = S_1$ and $\mathcal{I}_{OS}^X(S_2') = S_2$). Furthermore, for each proof of $S'$ that is a bipole, there exist: (i) an OS sequent $S$ such that $S'$ is a contraction of $S$; and (ii) a rule application $\overline{S}$ in $OS^X$.*

**Proof.** We can distinguish cases according to the main connective of the formula(s) on which we decide. The case of classical connectives is trivial, since we have that there is an exact correspondence between a bipole in $LMF_{OS}^X$ and a rule application in $OS^X$. The case of a formula with $\Box$ as the main connective is also simple, because we have that $\mathcal{I}_{OS}^X(S') = \mathcal{I}_{OS}^X(S_1')$. Relational rules do not change interpretation of the sequent either. If we consider a decide on a multiset of formulas, whose main operator is $\Diamond$, we have that, by inspecting the cases arising from condition $(i)$ in the definition of the rule $decide_{OS}$, one can see that this corresponds to an application of $\Box_K$, $\Box_{K4}$, $\Box_{K45}$, $\Diamond_T$ or $\Diamond_D$ according to the logic considered and the label chosen as the next one.      $\Box$

**Theorem 4.4** *Let $\Pi'$ be a proof of a sequent $S' \equiv \emptyset \vdash_{\{x,\emptyset\}} x : (\lfloor A \rfloor)^{\partial^+} \Uparrow \cdot$ for some $x$. Then there exists a proof $\Pi$ of a sequent $S$ in $OS^X$, where $S'$ is a contraction of $S$, such that each bipole in $\Pi'$ corresponds to one rule application in $\Pi$, plus possible applications of $contr$.*

**Proof.** We proceed top-down starting from the leaves of $\Pi'$ and build $\Pi$ by repeatedly applying Lemma 4.3. At each step, we get as the conclusion of an $OS^X$ rule a sequent $S^*$ such that the one obtained in the corresponding step of $\Pi'$ is a contraction of $S^*$. By applying $contr$, we transform the $OS^X$ derivation built so far and remove possible undesired multiple occurrences of a formula.$\Box$

**Theorem 4.5** *The system $LMF_{OS}^X$ is sound and complete for the logic $KX$.*

**Proof.** Soundness is obvious, since $LMF_{OS}^X$ is just a restriction of $LMF_*^X$. Completeness follows from Theorem 4.2 and completeness of the system $OS^X$.$\Box$

## 4.2 A different formulation for ordinary sequent systems

The system $LMF_{OS}^X$ is designed with the aim of emulating the behavior of $OS^X$ as much as possible in a rule-by-rule way. However, we can also give a different polarization (obtained by introducing delays less intensively) that makes the role of focusing even more significant, i.e., such that a bipole in the focused system corresponds now to a larger, but well identified, block of an

$OS^X$ derivation. In fact, as observed in the previous section, we can read an $LMF_{OS}^X$ derivation (from the root upwards) as composed of blocks, where at each block we first apply all the classical reasoning and the $\square$-rules relative to a given world and then execute a $\lozenge$-phase (thus moving to a different world). With the polarization given below, we can group all such classical+$\square$ reasoning in a single asynchronous phase, so that (at least for the logic $K$) each block will correspond to exactly two phases. We define $\lfloor \cdot \rfloor_{OS'}$ as follows:

$$\begin{array}{llllll}
\lfloor P \rfloor_{OS'} & = & P & \lfloor A \wedge B \rfloor_{OS'} & = & \lfloor A \rfloor_{OS'} \wedge^- \lfloor B \rfloor_{OS'} \\
\lfloor \neg P \rfloor_{OS'} & = & \neg P & \lfloor A \vee B \rfloor_{OS'} & = & \lfloor A \rfloor_{OS'} \vee^- \lfloor B \rfloor_{OS'} \\
\lfloor \square A \rfloor_{OS'} & = & \square(\lfloor A \rfloor_{OS'}{}^{\partial^+}) & \lfloor \lozenge A \rfloor_{OS'} & = & \lozenge(\partial^-(\lfloor A \rfloor_{OS'}))
\end{array}$$

In this setting, each new bipole is started by choosing a successor $y$ of the current world and by multifocusing on $\lozenge$-formulas labeled with a world from which $y$ is reachable and non-$\lozenge$-formulas labeled with $y$, i.e., we define:

$$\frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\{(x,\mathcal{F})\}} \Theta \Uparrow \cdot} \; decide_{OS'}$$

where (in addition to the general conditions of Figure 2) we have that either (i) $\Omega = \{x : B\}$ for $B$ a positive literal and $\mathcal{H}' = \{(x, \mathcal{F})\}$ (special case for closing branches); or (ii) there exists $xRy \in \mathcal{G}$ such that:

- if $x \neq y$, then $\mathcal{H}' = \{(y, \mathcal{F} \cup \{x\})\}$ and $\Omega$ is a multiset of formulas of the form
  - $zy : \lozenge A$, s.t. $zRy \in \mathcal{G}$, $z \in \mathcal{F}$; or
  - $y : B$ for $B$ positive but not a $\lozenge$-formula;

- if $x = y$ then $\mathcal{H}' = \{(x, \mathcal{F})\}$ and $\Omega$ is a multiset of formulas of the form $xx : \lozenge A$ or $x : B$ for $B$ positive but not a $\lozenge$-formula.

We remark that with such a polarization and such a version of the decide rule, we obtain an instantiation of the framework which behaves very similarly to the focused system of [14].

### 4.3  Nested sequent calculi

Nested sequents (first introduced by Kashima [12], and then independently rediscovered by Poggiolesi [20], as *tree-hypersequents*, and by Brünnler [2]) are an extension of ordinary sequents to a structure of tree, where each [ ]-node represents the scope of a modal $\square$. We write a nested sequent as a multiset of formulas and *boxed sequents*, according to the following grammar, where $A$ can be any modal formula in negative normal form: $\mathcal{N} ::= \emptyset \mid A, \mathcal{N} \mid [\mathcal{N}], \mathcal{N}$

In a nested sequent calculus, a rule can be applied at any depth in this tree structure, that is, inside a certain nested sequent context. A *context* written as $\mathcal{N}\{ \} \cdots \{ \}$ is a nested sequent with a number of holes occurring in place of formulas (and never inside a formula). Given a context $\mathcal{N}\{ \} \cdots \{ \}$ with $n$ holes, and $n$ nested sequents $\mathcal{M}_1, \ldots, \mathcal{M}_n$, we write $\mathcal{N}\{\mathcal{M}_1\} \cdots \{\mathcal{M}_n\}$ to denote the nested sequent where the $i$-th hole in the context has been replaced by $\mathcal{M}_i$, with the understanding that if $\mathcal{M}_i = \emptyset$ then the hole is simply removed.

We are going to consider the nested sequent system (on Figure 4) introduced by Brünnler in [2], that we call here $NS^X$. The first two categories of rules

IDENTITY AND STRUCTURAL RULES

$$\frac{}{\mathcal{N}\{P, \neg P\}} \; init \qquad\qquad \frac{\mathcal{N}\{A\} \quad \mathcal{N}\{\neg A\}}{\mathcal{N}\{\emptyset\}} \; cut$$

CONNECTIVES RULES

$$\frac{\mathcal{N}\{A\} \quad \mathcal{N}\{B\}}{\mathcal{N}\{A \wedge B\}} \; \wedge \qquad \frac{\mathcal{N}\{A, B\}}{\mathcal{N}\{A \vee B\}} \; \vee \qquad \frac{\mathcal{N}\{A\}}{\mathcal{N}\{\Box A\}} \; \Box \qquad \frac{\mathcal{N}\{\Diamond A, [A, \mathcal{M}]\}}{\mathcal{N}\{\Diamond A, [\mathcal{M}]\}} \; \Diamond$$

$\Diamond$-RULES

$$\frac{\mathcal{N}\{\Diamond A, A\}}{\mathcal{N}\{\Diamond A\}} \; T^\Diamond \qquad \frac{\mathcal{N}\{\Diamond A, [A]\}}{\mathcal{N}\{\Diamond A\}} \; D^\Diamond \qquad \frac{\mathcal{N}\{A, [\Diamond A, \mathcal{M}]\}}{\mathcal{N}\{[\Diamond A, \mathcal{M}]\}} \; B^\Diamond$$

$$\frac{\mathcal{N}\{\Diamond A, [\Diamond A, \mathcal{M}]\}}{\mathcal{N}\{\Diamond A, [\mathcal{M}]\}} \; 4^\Diamond \qquad \frac{\mathcal{N}\{\Diamond A\}\{\Diamond A\}}{\mathcal{N}\{\Diamond A\}\{\emptyset\}} \; 5^\Diamond$$

In $5^\Diamond$, the first hole in $\mathcal{N}\{\ \}\{\ \}$ can not occur at the root of the sequent tree.

Fig. 4. $NS^X$: a family of nested sequent proof systems for modal logic.

constitute a complete system for the modal logic $K$. It can then be extended modularly by a subset $X^\Diamond$ of the $\Diamond$-rules to give a complete system for any logic built from *45-closed*[3] set of axioms $X$ among $D$, $T$, $B$, 4 and 5.

We want to specify the general framework $LMF_*^X$ in order to emulate the proofs produced by $NS^X$. We can use here the same polarization $\lfloor \cdot \rfloor$ presented for ordinary sequents in Section 4.1 and specialize the rule $decide_*$ as follows:

$$\frac{\mathcal{G} \vdash_{\{(x, \emptyset) | x \in \mathcal{L}\}} \Theta \Downarrow x : A}{\mathcal{G} \vdash_{\{(x, \emptyset) | x \in \mathcal{L}\}} \Theta \Uparrow \cdot} \; decide_{NS}$$

where, as defined in Section 3, $\mathcal{L}$ denotes the set of all labels.

We can also use $LMF_*^X$ in order to emulate the behavior of focused nested sequent calculi, like the one in [3]. Such a system can be captured by defining a polarization that does not apply delays intensively, such as the one given in Section 4.2 for an ordinary sequent focused system.

## 5  Concluding remarks

We have presented $LMF_*^X$ as a general framework for emulating the behavior of several known modal proof systems based on ordinary sequents and on nested sequents. Our framework relies on using both labeled sequents (which injects semantics-related items into sequents) as well as focused proof rules (which

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[3] $X$ is said to be 45-closed: if whenever 4 is derivable in $K + X$, $4 \in X$, and whenever 5 is derivable in $K + X$, $5 \in X$. This condition is not restrictive as any logic obtained from a combination of axioms among $D$, $T$, $B$, 4 and 5, i.e. any logic in the so called $S_5$-cube, always has an equivalent 45-closed axiomatization.

can organize those injected items to support more high-level proof rules). The emulation of ordinary sequents is interesting because such calculi are proved to be optimal from the point of view of the efficiency of proof search. By decorating the sequents used in our framework with information (the *present* of the sequent) that specifies which world we are currently working on and which worlds are not reachable anymore, we are able to reproduce the mechanism that constrains (and improves) proof search in such calculi. Lellmann and Pimentel in [14] also use focusing but employ a notion of decorated sequent in order to constrain the construction of proofs in a fashion that only captures ordinary sequents (also, without invoking multifocusing). Section 4.2 shows how to instantiate our parametric framework so as to emulate the proofs of [14].

By analyzing the case of ordinary sequents, we conclude that modal rules in such a setting correspond to the application of two bipoles in our (1-sided) focused framework: the first bipole concerns a formula whose main connective is a $\Box$, while the second phase multifocuses on formulas with $\Diamond$ as the main connective. In the case of logics extending $K$, additional bipoles capturing the application of relational rules may also be required. The case of nested sequents illustrates the use of sequents decorated with a present that can contain more than just one world.

We believe that our framework is general enough to capture modal proof systems defined in other formalisms, such as prefixed tableaux systems [5,9] and 2-sequents [16] and their generalization to linear nested sequents [13]. In particular, we are currently working on formalizing a parametrization of $LMF_*^X$ that can capture the modal hypersequent systems of, e.g., [1]. The basic idea consists in (1) using a present which is a multiset, (2) representing external structural rules as operations on such a present, and (3) viewing modal communication rules as a combination of relational and modal rules.

We have shown that $LMF_*^X$, when properly instantiated, can emulate several modal proof systems with high precision: individual modal inference rules correspond to certain chains of bipoles in the encoded $LMF_*^X$ system *and vice versa*. Thus implementations of the $LMF_*^X$ proof system can be seen as providing a theorem prover and a proof checker for the emulated proof systems. Although the $LMF_*^X$ proof system imposes a lot of structure on the search for proofs, several important details are free to be implemented in differing ways. For example, one is free to implement the closure of the underlying world structure $\mathcal{G}^*$ via saturated bottom-up or top-down proof search.

While we have concentrated here on emulating existing calculi, we believe that $LMF_*^X$ can be used to develop new and original (focused) proof systems for modal logics, all achieved by properly tuning the parametrical aspects of the framework: this is the object of ongoing research.

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