Maximal cliques structure for cocomparability graphs and applications

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Jérémie Dusart[*], Michel Habib[*†] and Derek G. Corneil[‡]

November 7, 2016

Abstract

A cocomparability graph is a graph whose complement admits a transitive orientation. An interval graph is the intersection graph of a family of intervals on the real line. In this paper we investigate the relationships between interval and cocomparability graphs. This study is motivated by recent results [5, 13] that show that for some problems, the algorithm used on interval graphs can also be used with small modifications on cocomparability graphs. Many of these algorithms are based on graph searches that preserve cocomparability orderings.

First we propose a characterization of cocomparability graphs via a lattice structure on the set of their maximal cliques. Using this characterization we can prove that every maximal interval subgraph of a cocomparability graph G is also a maximal chordal subgraph of G. Although the size of this lattice of maximal cliques can be exponential in the size of the graph, it can be used as a framework to design and prove algorithms on cocomparability graphs. In particular we show that a new graph search, namely Local Maximal Neighborhood Search (LocalMNS) leads to an $O(n + mlogn)$ time algorithm to find a maximal interval subgraph of a cocomparability graph. Similarly we propose a linear time algorithm to compute all simplicial vertices in a cocomparability graph. In both cases we improve on the current state of knowledge.

Keywords: (co)-comparability graphs, interval graphs, posets, maximal antichain lattices, maximal clique lattices, graph searches.

1 Introduction

This paper is devoted to the study of *cocomparability graphs*, which are the complements of comparability graphs. *A comparability graph* is simply an undirected graph that admits a

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transitive acyclic orientation of its edges. Comparability graphs are well-studied and arise naturally in the process of modeling real-life problems, especially those involving partial orders. For a survey see [17, 33]. We also consider *interval graphs* which are the intersection graphs of a family of intervals on the real line. Comparability graphs and cocomparability graphs are well-known subclasses of perfect graphs [17]; and interval graphs are a well-known subclass of cocomparability graphs [3]. Clearly a given cocomparability graph G together with an acyclic transitive orientation of the edges of $\overline{G}$ (the corresponding comparability graph) can be equivalently represented by a poset P_G; thus new results in any of these three areas immediately translate to the other two areas. In this paper, we will often omit the translations but it is important to keep in mind that they exist.

A triple a, b, c of vertices forms an *asteroidal triple* if the vertices are pairwise independent, and every pair remains connected when the third vertex and its neighborhood are removed from the graph. An *asteroidal triple free graph* (AT-free for short) is a graph with no asteroidal triples. It is well-known that AT-free graphs strictly contain cocomparability graphs, see [17].

A classical way to characterize a cocomparability graph is by means of an *umbrella-free* total ordering of its vertices. In an ordering σ of G's vertices, an *umbrella* is a triple of vertices $x,\ y,\ z$ such that $x <_\sigma y <_\sigma z$, $xy,\ yz \notin E(G)$, and $xz \in E(G)$. It has been observed in [26] that a graph is a cocomparability graph if and only if it admits an umbrella-free ordering. We will also call an umbrella-free ordering a *cocomp ordering.* In a similar way, interval graphs are characterized by interval orderings, where an *interval ordering* σ is an ordering of the graph's vertices that does not admit a triple of vertices $x,\ y,\ z$ such that $x <_\sigma y <_\sigma z$, $xy \notin E(G)$, and $xz \in E(G)$. (Notice that an interval ordering is a cocomp ordering.) Other characterizations of interval graphs appear in theorem 2.14.

The paper studies the relationships shared by interval and cocomparability graphs and is motivated by some recents results:

- For the Minimum Path Cover (MPC) Problem (a minimum set of paths such that each vertex of G belongs to exactly one path in the set), Corneil, Dalton and Habib showed that the greedy MPC algorithm for interval graphs, when applied to a Lexicographic Depth First Search (LDFS) cocomp ordering provides a certifying solution for cocomparability graphs (see [5]).

- For the problem of producing a cocomp ordering (assuming the graph is cocomparability) Dusart and Habib showed that the multisweep Lexicographic Breadth First Search $(\text{LBFS})^+$ algorithm to find an interval ordering also finds a cocomp ordering ([13]). Note that $O(|V(G)|)$ LBFSs must be used in order to guarantee these results.

Other similar results will be surveyed in subsection 3.2. From these results, some natural questions arise: Do cocomparability graphs have some kind of hidden interval structure that allows the "lifting" of some interval graph algorithms to cocomparability graphs? What is the role played by graph searches LBFS and LDFS and are there other searches/problems where similar results hold?

As mentioned previously, interval graphs form a strict subclass of cocomparability graphs. It is also known that every minimal triangulation of a cocomparability graph is an interval graph [34, 31]. In section 2 of this article, we will show that we can equip the set of maximal cliques of a cocomparability graph with a lattice structure where every chain of the lattice forms an interval graph. Note that condition (iii) of theorem 2.14 states that a graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered so that for every vertex x, the cliques containing x appear consecutively. Thus, through the lattice, a cocomparability graph can be seen as a special composition of interval graphs. In particular, given a cocomparability graph G with P a transitive orientation of $\overline{G}$, the lattice $\mathcal{MA}(P)$ is formed on the set of maximal antichains of P (i.e., the maximal cliques of G). A graph $H = (V(H), E(H))$ with $E(H) \subseteq E(G)$ is a *maximal chordal* (respectively *interval*) *subgraph* if and only if H is a chordal graph and $\forall S \subseteq E(G) - E(H)$, $S \neq \emptyset$, $H' = (V(H), E(H) \cup S)$ is not a chordal (respectively interval) graph. Our final result of subsection 2.4 states that every maximal interval subgraph of a cocomparability graph is also a maximal chordal subgraph.

In sections 3 and 4 we turn our attention to algorithmic applications of the theory previously developed on the lattice $\mathcal{MA}(P)$. In section 3 we present algorithm Chainclique which on input a graph G and a total ordering σ of $V(G)$ returns an ordered set of cliques that collectively form an interval subgraph of G. We then introduce a new graph search (LocalMNS) that is very close to Maximal Neighborhood Search (MNS), which is a generalization of MCS, LDFS and LBFS. We show that Chainclique with σ being a LocalMNS cocomparability ordering of G returns a maximal interval subgraph of the cocomparability graph G; this algorithm also gives us a way to compute a minimal interval extension of a partial order (definitions given in subsection 1.1). Section 4 uses Chainclique to compute the set of simplicial vertices in a cocomparability graph.

Concluding remarks appear in section 5.

1.1 Notation

In this article, for graphs we follow standard notation; see, for instance, [17]. All the graphs considered here are finite, undirected, simple and with no loops. An edge between vertices u and v is denoted by uv, and in this case vertices u and v are said to be *adjacent*. $\overline{G}$ denotes the *complement* of $G = (V(G), E(G))$, i.e., $\overline{G} = (V(G), \overline{E(G)})$, where $uv \in \overline{E(G)}$ if and only if $u \neq v$ and $uv \notin E(G)$. Let $S \subseteq V(G)$ be a set of vertices of G. Then, the subgraph of G induced by S is denoted by $G[S]$, i.e., $G[S] = (S, F)$, where for any two vertices $u, v \in S$, $uv \in F$ if and only if $uv \in E(G)$. The set $N(v) = \{u \in V(G) | uv \in E(G)\}$ is called the *neighborhood* of the vertex $v \in V(G)$ in $G = (V(G), E(G))$. A vertex v is *simplicial* if $G[N(v) \cup \{v\}]$ is a clique. An *ordering* σ of $V(G)$ is a permutation of $V(G)$ where $\sigma(i)$ is the i'th vertex in σ; $\sigma^{-1}(x)$ denotes the position of x in σ. For two vertices u, v, we write that $u <_\sigma v$ if and only if $\sigma^{-1}(u) < \sigma^{-1}(v)$. For two vertices u, $v \in V(G)$, we say that u is *left* (respectively *right*) of v in τ if $u <_\tau v$ (respectively $v <_\tau u$).

For partial orders we use the following notation. A *partial order* (also known as a *poset*) $P = (X, \leq_P)$ is an ordered pair with a finite set X, the *ground set* of P, and with a binary relation $\leq_P$ satisfying reflexivity, anti-symmetry and transitivity. For x, $y \in X$, $x \neq y$, if

$x \leq_P y$ or $y \leq_P x$ then x, y are *comparable*, otherwise they are *incomparable* and denoted by $x \parallel_P y$. We will also use the *covering relation* in P denoted by $\prec_P$, satisfying $x \prec_P y$ if and only if $x \leq_P y$ and $\forall z$ such that $x \leq_P z \leq_P y$ then $x = z$ or $z = y$. In such a case we say that "*y covers x*" or that "*x is covered by y*".

A *chain* (respectively an *antichain*) is a partial order in which every pair (respectively no pair) is comparable. As mentioned previously, a given cocomparability graph G and a transitive orientation of the edges of $\overline{G}$ can be equivalently represented by a poset P_G. Note that a chain (respectively an antichain) in P_G corresponds to an independent set (respectively a clique) in G. An *extension* of a partial order $P = (X, \leq_P)$ is a partial order $P' = (X, \leq_{P'})$, where for $u, v \in X$, $u \leq_P v$ implies $u \leq_{P'} v$. In particular, if P' is a chain then P' is called a *linear extension* of P. An *interval extension* of a partial order is an extension that is also an interval order (*interval orders* are acyclic transitive orientations of the complement of interval graphs). P^- is the poset obtained from P by reversing all comparabilities.

A *lattice* is a particular partial order $L = (L, \leq_L)$[1] for which each two-element subset $\{a, b\} \subseteq L$ has a *join* (i.e., least upper bound) and a *meet* (i.e., greatest lower bound), denoted by $a \vee b$ and $a \wedge b$ respectively. This definition makes $\wedge$ and $\vee$ binary operations on L. All the lattices considered here are assumed to be finite. A *distributive* lattice is one in which the operations of join and meet distribute over each other; a *modular* lattice is a lattice that satisfies the following self-dual condition: $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ for all a. For other definitions on lattices, the reader is referred to [2, 18, 10, 37, 4].

2 Lattice characterization of cocomparability graphs

2.1 The maximal antichain lattice of a partial order

It is known from Birkhoff [2][2] that $\mathcal{A}(P)$, the set of all antichains of a partial order P, can be equipped with a lattice structure using the following relation between antichains: if A, B are two antichains in P then $A \leq_{\mathcal{A}(P)} B$ if and only if $\forall a \in A$, $\exists b \in B$ with $a \leq_P b$. Furthermore, it is also well-known that the lattice $\mathcal{A}(P) = (\mathcal{A}(P), \leq_{\mathcal{A}(P)})$ is a distributive lattice.

We now consider the relation $\leq_{\mathcal{MA}(P)}$, which is the restriction of the relation $\leq_{\mathcal{A}(P)}$ to the set of all maximal (with respect to set inclusion) antichains of P denoted by $\mathcal{MA}(P)$. Let us now review the main results known about $\mathcal{MA}(P) = (\mathcal{MA}(P), \leq_{\mathcal{MA}(P)})$.

The next lemma shows that the definition of $\leq_{\mathcal{MA}(P)}$ can be written symmetrically in the 2 antichains A and B, since they are maximal antichains.

Lemma 2.1. *[1] Let A, B be two maximal antichains of a partial order P.*

$A \leq_{\mathcal{MA}(P)} B$ if and only if $\forall a \in A$, $\exists b \in B$ with $a \leq_P b$ if and only if $\forall b \in B$, $\exists a \in A$ with $a \leq_P b$.

[1] Note that we use the same symbol, namely L for both the lattice and the ground set of the lattice; the exact meaning will be clear from the context.

[2] Indeed Birkhoff studied the ideal lattice of a partial order, but there exists a natural bijection between ideals and antichains.

Some helpful variations:

Lemma 2.2. *Let A, B be two maximal antichains of a partial order P.*

$A <_{\mathcal{MA}(P)} B$ if and only if $\forall a \in A$, $\exists b \in B$ with $a <_P b$ if and only if $\forall b \in B$, $\exists a \in A$ with $a <_P b$.

Lemma 2.3. *Let A, B be two maximal antichains of a partial order P such that $A \leq_{\mathcal{MA}(P)} B$.*

If $x \in A$, $y \in B$ then $x \leq_P y$ or $x \parallel_P y$.

Proof. We have two cases: either $A = B$ or $A \neq B$. In the first case, since A is an antichain and $A = B$, we get $\forall x \in A$, $\forall y \in A$, if $x \neq y$ then $x \parallel_P y$ and if $x = y$ then $x \leq_P y$.

In the second case, suppose for contradiction there exists $x \in A$, $y \in B$ such that $y <_P x$. Since we are in the case where A and B are different maximal antichains, and since x and y are comparable, necessarily we have $x \in A$-B, $y \in B$-A and also $A \leq_{\mathcal{MA}(P)} B$. Applying lemma 2.1 on A and B, there exists $z \in A$-B such that $z \leq_P y$. By transitivity of P, we establish that $z \leq_P x$, therefore contradicting A being an antichain. □

Now we focus on an interesting consecutiveness property in $\mathcal{MA}(P)$.

Proposition 2.4. *Let A, B, C be three maximal antichains of a partial order P such that $A \leq_{\mathcal{MA}(P)} B \leq_{\mathcal{MA}(P)} C$; then $A \cap C \subseteq B$.*

Proof. In the case where $A = B$ we have that $A \cap C \subseteq B$ and in the case $B = C$ we have that $A \cap C \subseteq B$. So we can assume that $A \neq B$, $B \neq C$ and as a consequence that $A <_{\mathcal{MA}(P)} B <_{\mathcal{MA}(P)} C$. Suppose for sake of contradiction that $A \cap C \not\subseteq B$. So there exists $x \in (A \cap C)$-B. Since x does not belong to B there must exist some $y \in B$ comparable to x. Using lemma 2.3 on A, B we establish that $x \leq_P y$. Again using lemma 2.3 on B, C we get that $y \leq_P x$. Since $x \leq_P y$ and $y \leq_P x$, necessarily $y = x$. Therefore x belongs to B which contradicts $x \in (A \cap C)$-B. □

The covering relation between maximal antichains has also been characterized.

Lemma 2.5. *[24] Let A, B be two different maximal antichains of a partial order P.*

$A \prec_{\mathcal{MA}(P)} B$ if and only if $\forall x \in A$-B and $\forall y \in B$-A, $x \prec_P y$.

Proof. Suppose that $A \prec_{\mathcal{MA}(P)} B$ and let $y \in B$-A. Further suppose that y does not cover some $x \in A$-B. Note that either $x \leq_P y$ or $y \parallel_P x$. In the first case, $x \leq_P y$, and thus there exists z such that $x \leq_P z \leq_P y$. But then consider A' the set obtained from A by: exchanging x and z; deleting all vertices comparable with z and adding all successors of x incomparable with z. A' is a maximal antichain by construction and we have: $A \leq_{\mathcal{MA}(P)} A' \leq_{\mathcal{MA}(P)} B$, a contradiction.

In the second case, $y \parallel_P x$, where $x \in A$-B. Let us consider $A' = \{z \in A\text{-}y \mid z \prec_P y\}+y$. We complete A' as a maximal antichain $[A']$ by adding elements of $A \cup B$. Since $x, y \in [A']$, $[A'] \neq A$ and $[A'] \neq B$. Then we have: $A <_{\mathcal{MA}(P)} [A'] <_{\mathcal{MA}(P)} B$, a contradiction to $A \prec_{\mathcal{MA}(P)} B$.

Conversely, clearly we have $A \leq_{\mathcal{MA}(P)} B$. Let us now prove that B covers A; if not, there exists some maximal antichain A' such that $A <_{\mathcal{MA}(P)} A' <_{\mathcal{MA}(P)} B$. Let $y \in B\text{-}A'$; there exists $z \in A'$ with $z <_P y$. Using Proposition 2.4 we know that $A \cap B \subseteq A'$; then necessarily $z \notin A$. Thus there exists $x \in A\text{-}B$ with $x <_P z$, but then y is not covered by x a contradiction. □

We now introduce some new terminology in order to define the infimum and supremum on the lattice $\mathcal{MA}(P)$.

Definition 2.6. For a partial order $P = (X, \leq_P)$, $S \subseteq X$, $Max(S) = \{v \in S \mid \forall u \in S,\ u \leq_P v \text{ or } u \parallel_P v\}$. $Max(S)$ is the set of maximal elements of the partial order $P(S)$ induced by S. In the same way, $Min(S) = \{v \in S \mid \forall u \in S,\ v \leq_P u \text{ or } u \parallel_P v\}$. $Min(S)$ is the set of minimal elements of $P(S)$. And $Inc(S) = \{x \in X - S \mid \forall y \in S,\ x \parallel_P y\}$ is the set of incomparable elements to S.

Definition 2.7. For two antichains $A,\ B$ of a partial order $P = (X, \leq_P)$, let $S_{min}(A, B) = \{x \in A\text{-}B \mid \exists y \in B\text{-}A \text{ with } x <_P y\}$ and $S_{max}(A, B) = \{x \in A\text{-}B \mid \exists y \in B\text{-}A \text{ with } y <_P x\}$.

Since A, B are antichains, we necessarily have: $S_{min}(A, B) \cap S_{max}(A, B) = \emptyset$.

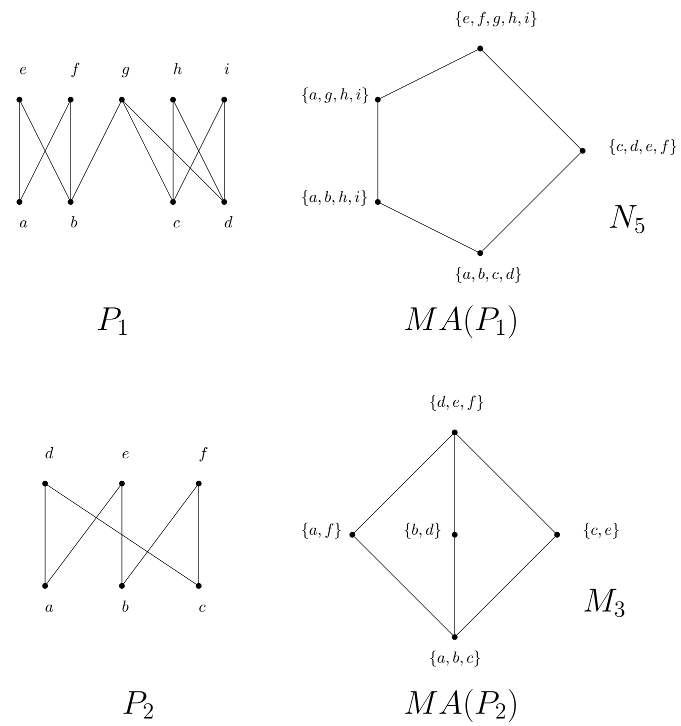

Figure 1: Two orders whose maximal antichain lattices are respectively N_5 and M_3, the smallest non distributive lattices.

As an example of these definitions consider the partial order P_1 of Figure 1. If we take the two maximal antichains $A = \{a, g, h, i\}$ and $B = \{c, d, e, f\}$, we see that $A \cap B = \emptyset$,

$S_{min}(A,B) = \{a\}$, $S_{max}(A,B) = \{g,h,i\}$, $S_{min}(B,A) = \{c,d\}$, $S_{max}(B,A) = \{e,f\}$. From these definitions we see:

Proposition 2.8. *Let A, B be two maximal antichains of a partial order P, then $(A \cap B), S_{min}(A,B)$ and $S_{max}(A,B)$ partition A.*

Proof. Let us consider a vertex x of A. We have two cases: either $x \in B$ or $x \notin B$. In the first case $x \in A \cap B$. In the second case, since B is a maximal antichain and $x \notin B$, there must exist $y \in B$ such that y is comparable to x. Since $y \in B$ and $x \notin B$, we deduce that $y \neq x$. Therefore we have two cases: either $x <_P y$ or $y <_P x$. In the first case, $x \in S_{min}(A,B)$ and in the second case $x \in S_{max}(A,B)$. Thus $A = (A \cap B) \cup S_{min}(A,B) \cup S_{max}(A,B)$ and by definition these 3 sets do not intersect. □

It should be noticed that in the well-known distributive lattice of antichains $\mathcal{A}(P)$, the infimum $A \wedge_{\mathcal{A}(P)} B = (A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A)$ and the supremum $A \vee_{\mathcal{A}(P)} B = (A \cap B) \cup S_{max}(A,B) \cup S_{max}(B,A)$.

In this definition $A \wedge_{\mathcal{A}(P)} B$ and $A \vee_{\mathcal{A}(P)} B$ are clearly antichains, but they are not maximal even if A, B are maximal. For example in P_1 in Figure 1 we have $\{a,g,h,i\} \vee_{\mathcal{A}(P)} \{c,d,e,f\} = \{e,f,h,i\}$ which is not maximal.

Therefore we can now define the infimum and supremum and $\mathcal{MA}(P)$ as follows:

Definition 2.9. For two maximal antichains A, B of a partial order P, we define the binary operators $\wedge_{\mathcal{MA}(P)}, \vee_{\mathcal{MA}(P)}$:

infimum: $A \wedge_{\mathcal{MA}(P)} B = (A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A) \cup Max(Inc((A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A))) = A \wedge_{\mathcal{A}(P)} B \cup Max(Inc(A \wedge_{\mathcal{A}(P)} B))$.

supremum: $A \vee_{\mathcal{MA}(P)} B = (A \cap B) \cup S_{max}(A,B) \cup S_{max}(B,A) \cup Min(Inc((A \cap B) \cup S_{max}(A,B) \cup S_{max}(B,A))) = A \vee_{\mathcal{A}(P)} B \cup Max(Inc(A \vee_{\mathcal{A}(P)} B))$.

Returning to the partial order P_1 of Figure 1 where $A = \{a,g,h,i\}$ and $B = \{c,d,e,f\}$ we see that $Max(Inc((A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A))) = \{b\}$. Therefore $\{a,g,h,i\} \wedge_{\mathcal{MA}(P)} \{c,d,e,f\} = \{a,b,c,d\}$. Similarly $\{a,g,h,i\} \vee_{\mathcal{MA}(P)} \{c,d,e,f\} = \{e,f,g,h,i\}$, whereas in $\mathcal{A}(P)$ we have: $\{a,g,h,i\} \vee_{\mathcal{A}(P)} \{c,d,e,f\} = \{e,f,h,i\} \subsetneq \{a,g,h,i\} \vee_{\mathcal{MA}(P)} \{c,d,e,f\}$.

Since the above supremum and infimum definitions are differently expressed compared to those of [1], for completeness we give a proof of the following theorem due to Berhendt.

Theorem 2.10. *[1] Let P be a partial order. $\mathcal{MA}(P) = (\mathcal{MA}(P), \wedge_{\mathcal{MA}(P)}, \vee_{\mathcal{MA}(P)})$ is a lattice.*

Proof. Let us first consider $A \wedge_{\mathcal{MA}(P)} B$. Clearly elements of $S_{min}(A,B)$ are incomparable with elements of $S_{min}(B,A)$. Therefore $(A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A)$ is an antichain of P. Adding to it $Max(Inc((A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A)))$ completes it as a maximal antichain.

Since A, B are maximal antichains, for every $x \in Inc((A \cap B) \cup S_{min}(A,B) \cup S_{min}(B,A))$ there exists $t \in S_{max}(A,B)$ and $z \in S_{max}(B,A)$ both comparable with x. If $t \leq_P x$, since there exists $y \in B$ such that $y \leq_P t$, it would imply by transitivity: $y \leq_p x$ which is impossible since $y \in S_{min}(B,A)$. Therefore $x \leq_P t$ and similarly one can obtain $x \leq_P z$.

Therefore we have:

$A \wedge_{\mathcal{MA}(P)} B \leq_{\mathcal{MA}(P)} A$ and $A \wedge_{\mathcal{MA}(P)} B \leq_{\mathcal{MA}(P)} B$.

Now let us consider a maximal antichain C, such that: $C \leq_{\mathcal{MA}(P)} A$ and $C \leq_{\mathcal{MA}(P)} B$.

But for every $c \in C$, there exists some $a \in A$ with $c \leq_P a$. If a does not belong to $(A \cap B) \cup S_{min}(A, B)$ then $a \in S_{max}(A, B)$ and so there exists $z \in Max(Inc((A \cap B) \cup S_{min}(A, B) \cup S_{min}(B, A)))$, with $c \leq_P a \leq_P z$. Thus $C \leq_{\mathcal{MA}(P)} A \wedge_{\mathcal{MA}(P)} A$. Symmetrically one can obtain $C \leq_{\mathcal{MA}(P)} A \wedge_{\mathcal{MA}(P)} B$.

Therefore this binary relation $\wedge_{\mathcal{MA}(P)}$ defined on maximal antichains behaves as an infimum relation on maximal antichains.

The proof is similar for $\vee_{\mathcal{MA}(P)}$.

□

Proposition 2.11. *Let A, B be two maximal antichains of a partial order P.*
Then $(A \cup B) \subseteq (A \vee_{\mathcal{MA}(P)} B) \cup (A \wedge_{\mathcal{MA}(P)} B)$.

Proof. Using the definition of $A \wedge_{\mathcal{MA}(P)} B$, we get that $(A \cap B) \cup S_{min}(A, B) \cup S_{min}(B, A) \subseteq (A \wedge_{\mathcal{MA}(P)} B)$ and symmetrically with $A \vee_{\mathcal{MA}(P)} B$ we get that $(A \cap B) \cup S_{max}(A, B) \cup S_{max}(B, A) \subseteq (A \vee_{\mathcal{MA}(P)} B)$.

By proposition 2.8, we know that $A = (A \cap B) \cup S_{min}(A, B) \cup S_{max}(A, B)$ and $B = (A \cap B) \cup S_{min}(B, A) \cup S_{max}(B, A)$, thereby showing $(A \cup B) \subseteq A \vee_{\mathcal{MA}(P)} B \cup (A \wedge_{\mathcal{MA}(P)} B)$. □

Corollary 2.12. *Let A, B be two maximal antichains of a partial order P where $x \in A - B$. Then we have two mutually exclusive cases: either $x \in (A \wedge_{\mathcal{MA}(P)} B)$ or $x \in (A \vee_{\mathcal{MA}(P)} B)$.*

Proof. Let A, B be two maximal antichains of a partial order P where $x \in A$-B. Then either $x \in (A \wedge_{\mathcal{MA}(P)} B)$ or otherwise (using proposition 2.11), necessarily $x \in (A \vee_{\mathcal{MA}(P)} B)$. From proposition 2.8, A-B is partitioned into $S_{min}(A, B) \subseteq A \wedge_{\mathcal{MA}(P)} B$ and $S_{max}(A, B) \subseteq A \vee_{\mathcal{MA}(P)} B$. Therefore the two cases are mutually exclusive.

□

There are two natural questions that arise concerning the lattice $\mathcal{MA}(P)$ for a given partial order P, namely:

- Does $\mathcal{MA}(P)$ have a particular lattice structure?
- What is the maximum size of $\mathcal{MA}(P)$ given n, the number of elements in P?

The answer to the first question is "no" since Markowsky in [28] and [29] showed that any finite lattice is isomorphic to the maximal antichain lattice of a height one partial order. This result has been rediscovered by Berhendt in [1]. This is summarized in the next theorem.

Theorem 2.13. *[1, 28, 29] Any finite lattice is isomorphic to the lattice $\mathcal{MA}(P)$ of some finite partial order.*

In particular, as shown in Figure 1 or using the previous theorem, $\mathcal{MA}(P)$ is not always distributive, thereby showing that $\mathcal{MA}(P)$ is not a sublattice of $\mathcal{A}(P)$ as already noticed in [1]. Jakubík in [24] studied for which partial orders P, $\mathcal{MA}(P)$ is modular.

For the second question, the size of $\mathcal{MA}(P)$ can be exponential in the number of elements of P. If we consider a poset P made up of k disjoint chains of length 2, $\mathcal{MA}(P)$ has exponential size since P has 2^k maximal antichains. The example of Figure 2 shows the $k = 2$ case. Furthermore Reuter showed in [36] that even the computation of the maximum length of a directed path (i.e., the height) in $\mathcal{MA}(P)$ is an NP-hard problem when only P is given as the input.

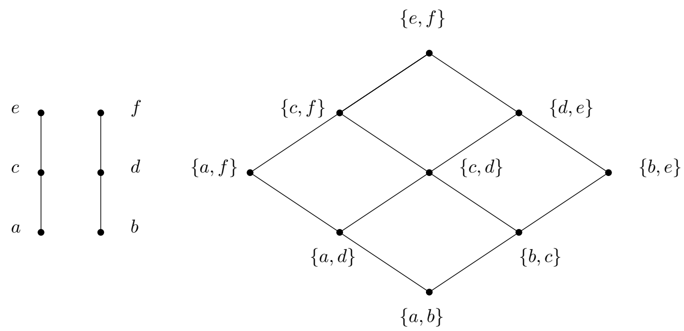

Figure 2: $\mathcal{MA}(P)$ for $k = 2$.

2.2 Maximal antichain lattice and interval orders

Following [17] interval graphs can be defined and characterized:

Theorem 2.14. *[16, 27]*

The following propositions are equivalent and characterize interval graphs.

(i) G can be represented as the intersection graph of a family of intervals of the real line.

(ii) There exists a total ordering τ of the vertices of V such that $\forall x, y, z \in G$ with $x \leq_\tau y \leq_\tau z$ and $xz \in E$ then $xy \in E$.

(iii) The maximal cliques of G can be linearly ordered such that for every vertex x of G, the maximal cliques containing x occur consecutively.

(iv) G contains no chordless 4-cycle and is a cocomparability graph.

(v) G is chordal and has no asteroidal triple.

As mentioned in the Introduction, an ordering of the vertices satisfying condition (ii) is called *an interval ordering* and *interval orders* are acyclic transitive orientations of the complement of interval graphs. Therefore we have the following characterization theorem:

Theorem 2.15. *The following propositions are equivalent and characterize interval orders:*

(i) P can be represented as a left-ordering of a family of intervals of the real line.

(ii) The successors sets are totally ordered by inclusion.

(iii) The predecessors sets are totally ordered by inclusion;

(iv) P has a maximal antichain path. (A maximal clique path is just a maximal clique tree T, reduced to a path).

(v) P does not contain a suborder isomorphic to ***2+ 2*** *(See Figure 3).*

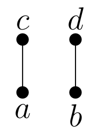

Figure 3: A $\mathbf{2}+\mathbf{2}$ partial order

In terms of the lattice $\mathcal{MA}(P)$, condition (iv) becomes:

Proposition 2.16. *[1] P is an interval order if and only if $\mathcal{MA}(P)$ is a chain.*

This result can be complemented by:

Theorem 2.17. *[21] The minimal interval order extensions of P are in a one-to-one correspondence with the maximal chains of $\mathcal{MA}(P)$.*

As a consequence developed in [21], the number of minimal interval orders extensions of P is a comparability invariant and is #P-complete to compute. For additional background information on interval orders, the reader is encouraged to consult Fishburn's monograph [15] or Trotter's survey article [38].

2.3 Maximal cliques structure of cocomparability graphs

In the following, we first characterize cocomparability graphs in terms of a particular lattice structure on its maximal cliques and show that it extends the well-known characterization of interval graphs by a linear ordering of its maximal cliques. Then, we state some corollaries on subclasses of cocomparability graphs. We let $\mathcal{C}(G)$ denote the set of maximal cliques of a graph G.

Theorem 2.18. $G = (V(G), E(G))$ *is a cocomparability graph if and only if* $\mathcal{C}(G)$ *can be equipped with a lattice structure* $\mathcal{L}$ *satisfying:*

(i) *For every* $A, B, C \in \mathcal{C}(G)$*, such that* $A \leq_{\mathcal{L}} B \leq_{\mathcal{L}} C$*, then* $A \cap C \subseteq B$*.*

(ii) *For every* $A, B \in \mathcal{C}(G)$*,* $(A \cup B) \subseteq (A \vee_{\mathcal{L}} B) \cup (A \wedge_{\mathcal{L}} B)$*.*

Proof. For the forward direction, let P be a partial order on $V(G)$ which corresponds to a transitive orientation of $\overline{G}$. Note that any maximal antichain of P forms a maximal clique of G. Let $\mathcal{L} = \mathcal{MA}(P)$. Proposition 2.4 shows that the first condition is satisfied. Proposition 2.11 shows that the second condition is satisfied.

For the reverse direction, we will prove the following claim, which shows that if there is a lattice structure on the maximal cliques of a graph G satisfying conditions (i) and (ii), then we can transitively orient $\overline{G}$ and thus G is a cocomparability graph.

Claim 2.19. *Let G be a graph and $\mathcal{C}(G)$ the set of maximal cliques of G such that $\mathcal{C}(G)$ can be equipped with a lattice structure $\mathcal{L}$ satisfying conditions (i) and (ii). Let $R_\mathcal{L}$ be a binary relation defined on $V(G)$ as follows:*

$xR_\mathcal{L}y$ if and only if $xy \notin E(G)$ and there exist 2 maximal cliques C', C'' of G with $C' \leq_\mathcal{L} C''$ and $x \in C'$, $y \in C''$.

Then $R_\mathcal{L}$ is a partial order on $V(G)$.

Proof. To show that $R_\mathcal{L}$ is a partial order, we start by showing that the relation is reflexive. Then we will show that $R_\mathcal{L}$ is antisymmetric and finally its transitivity.

Let x be a vertex of G. Because G is simple, we have that $xx \notin E(G)$. Let C_x be a maximal clique of G that contains x. We have that $C_x \leq_\mathcal{L} C_x$ and so we deduce that $xR_\mathcal{L}x$ which shows the reflexivity.

If we consider different vertices $x, y \in V(G)$ such that $xy \notin E(G)$, then x, y cannot be together in a maximal clique of G. Further, there exists at least two maximal cliques C_x, C_y such that $x \in C_x$ and $y \in C_y$. Assume $C_x \parallel_\mathcal{L} C_y$. Since x, y cannot belong together in the supremum or the infimum of C_x and C_y, using condition (ii) the supremum necessarily contains x (respectively y) and the infimum will contain y (respectively x). Hence, we can derive $yR_\mathcal{L}x$ (respectively $xR_\mathcal{L}y$) using the pair of maximal cliques C_y, $C_x \wedge_\mathcal{L} C_y$ (respectively the pair C_x, $C_x \wedge_\mathcal{L} C_y$).

To show the antisymmetry of $R_\mathcal{L}$, let us suppose for contradiction that $xR_\mathcal{L}y$, $yR_\mathcal{L}x$ and $x \neq y$. Then there exists C_x, C'_x, C_y, C'_y such that $x \in C_x$, $x \in C'_x$, $y \in C_y$, $y \in C'_y$, with $C_x \leq_\mathcal{L} C_y$ and $C'_y \leq_\mathcal{L} C'_x$. In the case where $C'_x \leq_\mathcal{L} C_y$ then the three maximal cliques $C'_y \leq_\mathcal{L} C'_x \leq_\mathcal{L} C_y$ contradict condition (i) and if $C_y \leq_\mathcal{L} C'_x$ then the three maximal cliques $C_x \leq_\mathcal{L} C_y \leq_\mathcal{L} C'_x$ contradict condition (i) and so we deduce that $C'_x \parallel_\mathcal{L} C_y$. But now using condition (ii) on C'_x, C_y, we deduce that in the supremum of C'_x and C_y we will find either x or y since they cannot belong together in a maximal clique. If it is y in $(C'_x \vee_\mathcal{L} C_y)$, we have $C'_y \leq_\mathcal{L} C'_x \leq_\mathcal{L} (C'_x \vee_\mathcal{L} C_y)$ that contradicts condition (i) and similarly if it is x in $(C'_x \vee_\mathcal{L} C_y)$, then $C_x \leq_\mathcal{L} C_y \leq_\mathcal{L} (C'_x \vee_\mathcal{L} C_y)$ contradicts condition (i). Thus if we have $xR_\mathcal{L}y$ and $yR_\mathcal{L}x$, we must have $x = y$.

Let us now examine the transitivity of $R_\mathcal{L}$. Let x, y, z be three different vertices such that $xR_\mathcal{L}y$ and $yR_\mathcal{L}z$. Let us assume for contradiction that $xz \in E(G)$. Therefore there exists a maximum clique C_{xz} of G such that $x, z \in C_{xz}$. Let C_y be a maximal clique that contains y. But now because y is not linked to x, y does not belong to C_{xz} and using corollary 2.12 on C_{xz} and C_y we have that either $y \in (C_{xz} \vee C_y)$ or $y \in (C_{xz} \wedge C_y)$. In the first case, by the definition of $R_\mathcal{L}$ we have that $zR_\mathcal{L}y$ and from the assumption, $yR_\mathcal{L}z$. So using the antisymmetry of $R_\mathcal{L}$ on z, y we have that $z = y$, which contradicts our choice of z, y being different vertices. In the second case, by the definition of $R_\mathcal{L}$, we have that $yR_\mathcal{L}x$ and from the assumption, $xR_\mathcal{L}y$. So using the antisymmetry on x, y, we have that $x = y$, which contradicts our choice of x, y being different vertices.

So assume that there exists three different vertices x, y, z such that $xR_\mathcal{L}y$ and $yR_\mathcal{L}z$. Now we show that $xR_\mathcal{L}z$. We just have proved that $xz \notin E(G)$. As $xR_\mathcal{L}y$, there is a maximal clique C_x and a maximal clique C_y such that $x \in C_x$, $y \in C_y$ and $C_x \leq_\mathcal{L} C_y$. Let C_z be a maximal clique such that $z \in C_z$. Using condition (ii) on C_z, C_y either $z \in (C_z \wedge_\mathcal{L} C_y)$ or

$z \in (C_z \vee_{\mathcal{L}} C_y)$. In the case where $z \in (C_z \wedge_{\mathcal{L}} C_y)$, using the definition of $R_{\mathcal{L}}$ on z, y and the cliques C_y, $(C_z \wedge_{\mathcal{L}} C_y)$ we get that $zR_{\mathcal{L}}y$. Since $yR_{\mathcal{L}}z$, using the antisymmetry of $R_{\mathcal{L}}$ we get $y = z$ thereby contradicting our assumption that y and z are different vertices. So z has to belong to $C_z \vee_{\mathcal{L}} C_y$. Now we have $C_x \leq_{\mathcal{L}} C_y \leq_{\mathcal{L}} C_z \vee C_y$ and using the definition of $R_{\mathcal{L}}$ on the vertices x, z and the cliques C_x, $C_z \vee_{\mathcal{L}} C_y$, we deduce that $xR_{\mathcal{L}}z$ which establishes its transitivity. $\square$

In fact with Claim 2.19, we have shown that $R_{\mathcal{L}}$ is a transitive orientation of $\overline{G}$, therefore G is a cocomparability graph. $\square$

Unfortunately as can be seen in Figure 4, not every lattice $\mathcal{L}$ satisfying the conditions (i) and (ii) of the previous theorem corresponds to a maximal antichain lattice $\mathcal{MA}(P)$ for some partial order P that gives a transitive orientation of $\overline{G}$. However by adding a simple condition, we can characterize when a lattice $\mathcal{L}$ is a lattice $\mathcal{MA}(P)$.

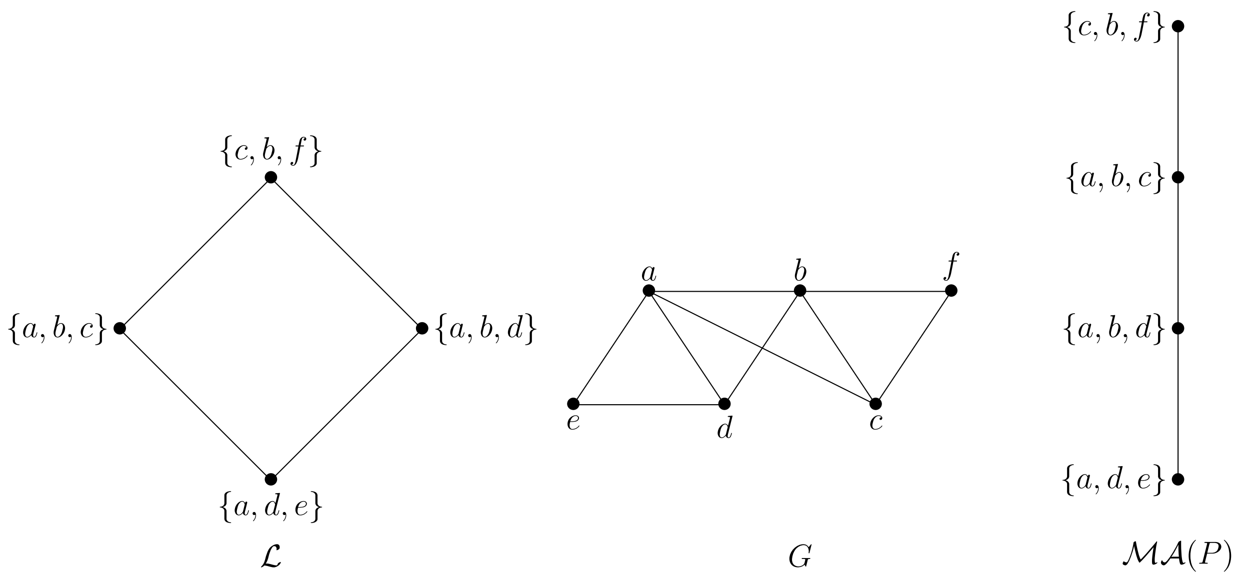

Figure 4: A graph G and a lattice $\mathcal{L}$ on $\mathcal{C}(G)$ that satisfies condition (i) and (ii) of Theorem 2.18. But $\mathcal{L}$ is not isomorphic to the lattice $\mathcal{MA}(P)$ for any partial order P that corresponds to a transitive orientation of $\overline{G}$. Since G is a prime interval graph, it has only one transitive orientation (up to reversal) which is an interval order and its maximal antichain lattice is a chain.

Theorem 2.20. *Let G be a cocomparability graph and let $\mathcal{L}$ be a lattice structure on $\mathcal{C}(G)$ satisfying conditions (i) and (ii) of theorem 2.18, then $\mathcal{L}$ is isomorphic to a lattice $\mathcal{MA}(P)$ with P a transitive orientation of $\overline{G}$ if and only if the following condition (iii) is also satisfied:*

(iii) *For every $A, B \in \mathcal{C}(G)$, $(A \cap B) \subseteq (A \vee_{\mathcal{L}} B)$ and $(A \cap B) \subseteq (A \wedge_{\mathcal{L}} B)$.*

Proof. Suppose that $\mathcal{L}$ is isomorphic to a lattice $\mathcal{MA}(P)$ with P a transitive orientation of $\overline{G}$. It is clear that (iii) is satisfied using the definition of $\wedge_{\mathcal{MA}(P)}$ and $\vee_{\mathcal{MA}(P)}$.

Conversely, let us consider the partial order relation $R_{\mathcal{L}}$ defined in claim 2.19. We recall that $R_{\mathcal{L}}$ is defined on $V(G)$ as follows: $xR_{\mathcal{L}}y$ if and only if $xy \notin E(G)$ and there are maximal cliques C', C'' of G with $C' \leq_{\mathcal{L}} C''$ and $x \in C'$, $y \in C''$.

We now prove that $\mathcal{L}$ is isomorphic to the lattice $\mathcal{MA}(R_{\mathcal{L}})$. So for this purpose we will show that for two maximal cliques A, B, $A \leq_{\mathcal{L}} B$ if and only if $\forall x \in A$, $\exists y \in B$ with $xR_{\mathcal{L}}y$ which is the definition of $\mathcal{MA}(R_{\mathcal{L}})$. First we recall that since $R_{\mathcal{L}}$ is a transitive orientation of $\overline{G}$, any maximal clique of G corresponds to a maximal antichain in $R_{\mathcal{L}}$. Because both $\mathcal{L}$ and $R_{\mathcal{L}}$ are partial orders, the relations are reflexive and the case where $A = B$ is clear.

Let A, B be two different maximal cliques of G such that $A \leq_{\mathcal{L}} B$. Then $\forall x \in A\text{-}B$, x cannot be universal to $B\text{-}A$ because B is a maximal clique. Therefore there exists $y \in B\text{-}A$ such that $xy \notin E(G)$ and so y is comparable with x in $R_{\mathcal{L}}$. Furthermore we have that $x \neq y$ because $x \in A\text{-}B$ and $y \in B\text{-}A$. Using our definition of $R_{\mathcal{L}}$ on x, y and the cliques A, B we see that $xR_{\mathcal{L}}y$. For all $x \in A \cap B$ we also have that $xR_{\mathcal{L}}x$ and so if $A \leq_{\mathcal{L}} B$ then $\forall x \in A$, $\exists y \in B$ with $xR_{\mathcal{L}}y$.

Let A, B be two different maximal cliques of G, such that $\forall x \in A$, $\exists y \in B$ with $xR_{\mathcal{L}}y$. For the sake of contradiction assume that $A \parallel_{\mathcal{L}} B$. Let us consider $A \vee_{\mathcal{L}} B$. For a vertex $x \in A\text{-}B$, we know that there exists $y \in B\text{-}A$ such that $xR_{\mathcal{L}}y$ and so x must belong to $(A \wedge_{\mathcal{L}} B)$ otherwise if $x \in (A \vee_{\mathcal{L}} B)$, using the definition of $R_{\mathcal{L}}$ on A, $(A \vee_{\mathcal{L}} B)$ we have that $yR_{\mathcal{L}}x$ and so $x = y$ which contradicts our choice of x and y. But now we have that $A\text{-}B \subseteq (A \wedge_{\mathcal{L}} B)$ and using condition (iii) $(A \cap B) \subseteq (A \vee_{\mathcal{L}} B)$ so $A \subseteq (A \wedge_{\mathcal{L}} B)$. So either $A = (A \wedge_{\mathcal{L}} B)$ and so $A \leq_{\mathcal{L}} B$ which contradicts our choice of A and B, or $A \subsetneq (A \wedge_{\mathcal{L}} B)$ which contradicts that A is a maximal antichain. □

The following corollary enlightens the relationship between a lattice satisfying (i) and (ii) and a lattice satisfying (i),(ii) and (iii).

Corollary 2.21. *For every lattice $\mathcal{L}$ associated to the maximal cliques of a cocomparability graph G and satisfying (i) and (ii) there exists a transitive orientation $R_{\mathcal{L}}$ of $\overline{G}$ such that $\mathcal{MA}(R_{\mathcal{L}})$ is an extension of $\mathcal{L}$.*

Proof. Using claim 2.19, for a given lattice structure $\mathcal{L}$ associated with a cocomparability graph and satisfying the conditions of theorem 2.18, we can define a partial order $R_{\mathcal{L}}$. From the previous proof, we know that if $A \leq_{\mathcal{L}} B$ then $\forall x \in A$, $\exists y \in B$ with $xR_{\mathcal{L}}y$ and so $A \leq_{\mathcal{MA}(R_{\mathcal{L}})} B$. Therefore $\mathcal{MA}(R_{\mathcal{L}})$ is an extension of $\mathcal{L}$. □

It should be noticed that the last two theorems 2.18 and 2.20 can be easily rewritten into a characterization of comparability graphs just by exchanging maximal cliques into maximal independent sets. Let us now study the particular case of interval graphs.

Corollary 2.22. *[16] G is an interval graph if and only if $\mathcal{C}(G)$ can be equipped with a total order T satisfying for every C_i, C_j, C_k maximal cliques such that $C_i \leq_T C_j \leq_T C_k$, then $C_i \cap C_k \subseteq C_j$.*

Proof. Using the last two theorems, we know that if G is a cocomparability graph then the set of maximal cliques of G can be equipped with a lattice structure $\mathcal{L}$ satisfying conditions (i), (ii), (iii) and isomorphic to a lattice $\mathcal{MA}(P)$ with P a transitive orientation of $\overline{G}$. From property 2.16 $\mathcal{MA}(P)$ is a chain if and only if P is an interval order. Since $\mathcal{MA}(P)$ is a chain, it should be noticed that conditions (ii) and (iii) are always satisfied and can be omitted. Therefore only condition (i) remains. □

Applied to permutation graphs[3] the characterization theorems yield:

Corollary 2.23. *G is a permutation graph if and only if there exists a lattice structure satisfying (i), (ii) and (iii) on the set of its maximal cliques and a lattice structure satisfying (i), (ii) and (iii) on the set of its maximal independent sets.*

Proof. We know from [14] that G is a permutation graph if and only if G is a cocomparability and a comparability graph and the result follows. □

2.4 Maximal chordal and interval subgraphs

As mentioned in theorem 2.17, for any partial order P there is a bijection between maximal chains in $\mathcal{MA}(P)$ and minimal interval extensions of P. Therefore theorem 2.20 also yields a bijection between maximal interval subgraphs of a cocomparability graph and the minimal interval extensions of P (acyclic transitive orientations of $\overline{G}$). This bijection will be heavily used in the algorithms of the following sections. It should also be noticed that theorem 2.20 gives another proof of the fact that the number of minimal interval extensions of a partial order is a comparability invariant (i.e., it does not depend on the chosen acyclic transitive orientation).

Let G be a cocomparability graph and σ a cocomp ordering of G. We define P_σ as the transitive orientation of $\overline{G}$ obtained using σ. For a chain $\mathcal{C} = C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$, $G_\mathcal{C} = (V(G), E(\mathcal{C}))$ denotes the graph formed by the cliques $C_1, \ldots, C_k$. For a vertex x, $N_\mathcal{C}(x)$ is the neighborhood of x in the graph $G_\mathcal{C}$.

Proposition 2.24. *Consider a maximal chain of $\mathcal{MA}(P_\sigma)$, $\mathcal{C} = C_1 \prec_{\mathcal{MA}(P_\sigma)} C_2 \cdots \prec_{\mathcal{MA}(P_\sigma)} C_k$. Such a chain forms a maximal interval subgraph of G.*

Proof. The sequence $C_1, C_2 \ldots C_k$ forms a chain of maximal cliques that respects proposition 2.4 (consecutiveness condition). So using corollary 2.22, we deduce that this chain forms a maximal interval subgraph of G. □

Therefore, we can see a cocomparability graph as a union of interval subgraphs.

In this subsection, we now show that for cocomparability graphs a maximal chain of $\mathcal{MA}(P)$ not only forms a maximal interval subgraph but also a maximal chordal subgraph.

[3] A graph is a permutation graph if and only if it is the intersection of line segments whose endpoints lie on two parallel lines.

Proposition 2.25. *Let G be a cocomparability graph and let σ be a cocomp ordering. Then $\mathcal{C} = C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$ is a maximal chain of $\mathcal{MA}(P_\sigma)$ if and only if the following conditions are satisfied*

- *C_1 is the set of sources of P_σ*
- *$1 < i \leq k$, $C_{i-1} \prec_{\mathcal{MA}(P)} C_i$ (i.e., C_i covers C_{i-1})*
- *C_k is the set of sinks of P_σ*

Proof. For the forward direction, let $\mathcal{C} = C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$ be a maximal chain of cliques of $\mathcal{MA}(P_\sigma)$. Let C_S be the set of sources of P_σ. Since every source is incomparable with all the other sources, C_S is an antichain of P_σ. Every element that is not a source is comparable to at least one source and so C_S is a maximal antichain. We now show that for every maximal antichain A of P_σ, $C_S \leq_{\mathcal{MA}(P_\sigma)} A$. Let A be a maximal antichain of P_σ; in the case $A = C_S$ then $C_S \leq_{\mathcal{MA}(P_\sigma)} A$ and so we take $A \neq C_S$. For the sake of contradiction assume that $C_S \not\leq_{\mathcal{MA}(P_\sigma)} A$. So we have two cases: either $A <_{\mathcal{MA}(P_\sigma)} C_S$ or $A \parallel_{\mathcal{MA}(P_\sigma)} C_S$. In the first case, let $y \in C_S - A$ and using lemma 2.1 on A and C_S we know there exists $x \in A - C_S$ such that $x <_{P_\sigma} y$. But now because $x <_{P_\sigma} y$ we contradict the fact that y is a source. In the second case, we know that there exists $(A \wedge_{\mathcal{MA}(P)} C_S)$ such that $(A \wedge_{\mathcal{MA}(P)} C_S) \leq_{\mathcal{MA}(P_\sigma)} C_S$. Since $A \parallel_{\mathcal{MA}(P_\sigma)} C_S$ we have $(A \wedge_{\mathcal{MA}(P)} C_S) <_{\mathcal{MA}(P_\sigma)} C_S$. But now we are back in the first case, which again gives us a contradiction. So for every maximal antichain A of P_σ, $C_S \leq_{\mathcal{MA}(P_\sigma)} A$ and so we have that $C_S \leq_{\mathcal{MA}(P_\sigma)} C_1$. Now if $C_S \neq C_1$ then we can add C_S at the beginning of the chain $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$, thereby contradicting the maximality of $\mathcal{C}$. Thus $C_S = C_1$.

Now assume for contradiction that for some $1 < i \leq k$, C_i does not cover C_{i-1}. Then there exists a maximal antichain B of P_σ such that $C_{i-1} <_{\mathcal{MA}(P_\sigma)} B <_{\mathcal{MA}(P_\sigma)} C_i$. But now the chain $C_1 \leq_{\mathcal{MA}(P_\sigma)} \ldots C_{i-1} <_{\mathcal{MA}(P_\sigma)} B <_{\mathcal{MA}(P_\sigma)} C_i \cdots \leq_{\mathcal{MA}(P_\sigma)} C_k$ contains $\mathcal{C}$ as a subchain which contradicts the maximality of $\mathcal{C}$.

Let C_P be the set of sinks of P_σ. Using the same argument as in the case of the set of sources, we can deduce that for every maximal antichain A of P_σ, $A \leq_{\mathcal{MA}(P_\sigma)} C_P$. Now if $C_P \neq C_k$ then we can add C_P at the end of the chain $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$, thereby contradicting the maximality of $\mathcal{C}$. Thus $C_P = C_k$.

Conversely, assume for contradiction that $\mathcal{C}$ is not a maximal chain of cliques. Then we can add a maximal clique B to $\mathcal{C}$. There are three cases: B can be added at the beginning of $\mathcal{C}$; B can be added at the end of $\mathcal{C}$ or B can be added in the middle of $\mathcal{C}$. In the first case we have $B <_{\mathcal{MA}(P_\sigma)} C_1$, which as shown previously contradicts C_1 being the set of sources. Similarly the case where $C_k <_{\mathcal{MA}(P_\sigma)} B$ contradicts C_k being the set of sinks. In the last case, we have $C_{i-1} <_{\mathcal{MA}(P_\sigma)} B <_{\mathcal{MA}(P_\sigma)} C_i$ for some index i such that $1 < i \leq k$. But this contradicts $C_{i-1} \prec_{\mathcal{MA}(P)} C_i$, which concludes the proof. $\square$

Theorem 2.26. *Every maximal interval subgraph of a cocomparability graph is also a maximal chordal subgraph.*

Proof. Let G be a cocomparability graph and let σ be a cocomp ordering. We just need to prove that a maximal chain of $\mathcal{MA}(P_\sigma)$ is a maximal interval subgraph and a maximal chordal subgraph.

By proposition 2.24, given a maximal chain of $\mathcal{MA}(P_\sigma)$: $\mathcal{C} = C_1 \prec_{\mathcal{MA}(P_\sigma)} C_2 \cdots \prec_{\mathcal{MA}(P_\sigma)} C_k$; then $G_\mathcal{C}$ is a maximal interval subgraph.

Assume for contradiction that $G_\mathcal{C}$ is not a maximal chordal subgraph. Let S be a set of edges such that the graph $H = (V(G), E(\mathcal{C}) \cup S)$ is a maximal chordal subgraph. In the proof, we will carefully choose an edge $uv \in S$ and show that we can find an induced path from u to v of length at least 3 in H. Therefore it will prove that $G_\mathcal{C}$ is a maximal chordal subgraph. Since interval graphs are a subclass of chordal graphs and $G_\mathcal{C}$ is an interval graph, we will deduce that $G_\mathcal{C}$ is also a maximal interval subgraph.

We start by proving two claims.

Claim 2.27. *Let G be a cocomparability graph, let σ be a cocomp ordering and let u, v be two vertices of G such that $uv \in E$.*

If C_u, C_v are maximal cliques of G such that $u \in C_u$, $v \notin C_u$, $v \in C_v$, $u \notin C_v$, $C_u <_{\mathcal{MA}(P_\sigma)} C_v$ then there exists a maximal clique C_{uv} such that $u, v \in C_{uv}$ and $C_u <_{\mathcal{MA}(P_\sigma)} C_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$.

Proof. Since $uv \in E$ there must exist at least one maximal clique C^0_{uv} of G that contains u and v. We now define $C^1_{uv} = C^0_{uv} \vee_{\mathcal{MA}(P_\sigma)} C_u$ and show that u, v belong to C^1_{uv}. We have three cases: $C_u <_{\mathcal{MA}(P_\sigma)} C^0_{uv}$; $C^0_{uv} <_{\mathcal{MA}(P_\sigma)} C_u$; $C_u \parallel_{\mathcal{MA}(P_\sigma)} C^0_{uv}$. In the first case, we see that $C^1_{uv} = C^0_{uv}$ and so u, v belong to C^1_{uv}. In the second case, v belongs to C^0_{uv} and C_v but not to C_u and so $C^0_{uv} <_{\mathcal{MA}(P_\sigma)} C_u <_{\mathcal{MA}(P_\sigma)} C_v$ contradicts proposition 2.4 (consecutiveness condition). Therefore this case cannot happen. In the last case, using the definition of $\vee_{\mathcal{MA}(P_\sigma)}$ on C^0_{uv} and C_u, we see that u must belong to C^1_{uv} because it belongs to $C^0_{uv} \cap C_u$. Using again the definition of $\vee_{\mathcal{MA}(P_\sigma)}$ on C^0_{uv} and C_u, we see that v must belong to C^1_{uv} otherwise v would have to belong to $C^0_{uv} \wedge_{\mathcal{MA}(P_\sigma)} C_u$ and $(C^0_{uv} \wedge_{\mathcal{MA}(P_\sigma)} C_u) <_{\mathcal{MA}(P_\sigma)} C_u <_{\mathcal{MA}(P_\sigma)} C_v$ would contradict proposition 2.4 (consecutiveness condition). Thus u, v belong to C^1_{uv}.

We finish the proof of the claim by showing that $C_{uv} = C^1_{uv} \wedge_{\mathcal{MA}(P_\sigma)} C_v$ satisfies u, $v \in C_{uv}$ and $C_u <_{MA(P_\sigma)} C_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$. From the choice of C_{uv} we know $C_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$. We have three cases: $C^1_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$; $C_v <_{\mathcal{MA}(P_\sigma)} C^1_{uv}$; $C^1_{uv} \parallel_{\mathcal{MA}(P_\sigma)} C_v$. In the first case, we see that $C_{uv} = C^1_{uv}$ and so $u, v \in C_{uv}$ and $C_u <_{\mathcal{MA}(P_\sigma)} C_{uv}$. In the second case, u belongs to C^1_{uv} and C_u but not to C_v and so $C_u <_{\mathcal{MA}(P_\sigma)} C_v <_{\mathcal{MA}(P_\sigma)} C^1_{uv}$ contradicts proposition 2.4 (consecutiveness condition). Therefore this case cannot happen. In the last case, using the definition of $\wedge_{\mathcal{MA}(P_\sigma)}$ on C^1_{uv} and C_v we see that v must belong to C_{uv} since it belongs to $C^1_{uv} \cap C_v$. Using theorem 2.18 on C^1_{uv} and C_u, we get that u must belong to C_{uv} otherwise u would have to belong to $(C^1_{uv} \vee_{\mathcal{MA}(P_\sigma)} C_v)$ and $C_u <_{\mathcal{MA}(P_\sigma)} C_v <_{\mathcal{MA}(P_\sigma)} (C^1_{uv} \vee_{\mathcal{MA}(P_\sigma)} C_v)$ would contradict proposition 2.4 (consecutiveness condition). Since C_{uv} is defined as $C^1_{uv} \wedge_{\mathcal{MA}(P_\sigma)} C_v$ by the definition of the lattice $C_u <_{\mathcal{MA}(P_\sigma)} C_{uv}$. $\square$

We now introduce some terminology. Let $\mathcal{C} = C_1 \prec_{\mathcal{MA}(P_\sigma)} C_2 \cdots \prec_{\mathcal{MA}(P_\sigma)} C_k$ be a maximal chain of $\mathcal{MA}(P_\sigma)$; for every vertex $x \in V(G)$ we define $first_\mathcal{C}[x]$ (respectively $last_\mathcal{C}[x]$) as the first (respectively last) index of a clique of $\mathcal{C}$ that contains x.

Since a maximal interval subgraph is obviously a spanning subgraph, these functions are well-defined. Furthermore when there is no ambiguity on $\mathcal{C}$, we simply denote these values by $first[x]$ and $last[x]$.

Claim 2.28. *Let G be a cocomparability graph, σ be a cocomp ordering, $\mathcal{C} = C_1 \prec_{\mathcal{MA}(P_\sigma)} C_2 \cdots \prec_{\mathcal{MA}(P_\sigma)} C_k$ be a maximal chain of $\mathcal{MA}(P_\sigma)$ and u, v be two vertices of G such that $uv \in E$.*

If $last[u] < first[v]$ then $\exists x,\ y$ such that $first[x] \leq last[u] < first[y] \leq last[x] < first[v]$, $last[x] \leq last[y]$ and $yv \in E$.

Proof. Let $C_u = C_{last[u]}$ and $C_v = C_{first[v]}$. Since $u \in N(v) - N_{\mathcal{C}}(v)$, we deduce that $uv \notin E(\mathcal{C})$. Furthermore, since $u <_\sigma v$, C_u, $C_v \in \mathcal{C}$, we see that $C_u <_{\mathcal{MA}(P_\sigma)} C_v$. Using the previous claim on C_u, C_v, we deduce there exists C_{uv} a maximal clique of G such that $C_u <_{\mathcal{MA}(P_\sigma)} C_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$. Since $C_1, \ldots, C_k$ is a maximal chain of cliques and $uv \notin E(\mathcal{C})$ we further know that $C_{uv} \notin \mathcal{C}$. Since $\mathcal{C}$ is a maximal chain, $C_u <_{\mathcal{MA}(P_\sigma)} C_{uv} <_{\mathcal{MA}(P_\sigma)} C_v$ and $C_{uv} \notin \mathcal{C}$, we know that there exists a maximal clique D_1 in $\mathcal{C}$ such that $C_u <_{\mathcal{MA}(P_\sigma)} D_1 <_{\mathcal{MA}(P_\sigma)} C_v$ and D_1 covers C_u otherwise we contradict the maximality of the chain. We have three cases: $C_{uv} <_{\mathcal{MA}(P_\sigma)} D_1$; $D_1 <_{\mathcal{MA}(P_\sigma)} C_{uv}$; $D_1 \parallel_{\mathcal{MA}(P_\sigma)} C_{uv}$. In the first case, this contradicts the assumption that D_1 covers C_u. So this case cannot happen. In the second case, we have $u \in C_u, C_{uv}$ and $u \notin D_1$ which contradicts proposition 2.4 (consecutiveness condition) and so this case cannot happen. So we are left with the last case. Since D_1 covers C_u and $D_1 \parallel_{\mathcal{MA}(P_\sigma)} C_{uv}$ we now show that $D_1 \wedge_{\mathcal{MA}(P_\sigma)} C_{uv} = C_u$. Assume it is not the case; then we would have $C_u <_{\mathcal{MA}(P_\sigma)} (D_1 \wedge_{\mathcal{MA}(P_\sigma)} C_{uv}) <_{\mathcal{MA}(P_\sigma)} D_1$ which contradicts that D_1 covers C_u. So we are in the situation described in Figure 5.

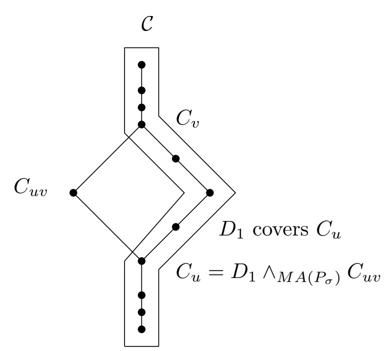

Figure 5:

Since we chose C_v as the first maximal clique in $\mathcal{C}$ that contains v, v is not universal to D_1 and let x be a vertex of maximum last value among $D_1 - N(v)$. So $last[x] < first[v]$ Let us show that x belongs to C_u. Assume that x belongs to $C_{uv} \vee_{\mathcal{MA}(P_\sigma)} D_1$, then $v \notin C_{uv} \vee_{\mathcal{MA}(P_\sigma)} D_1$ and using proposition 2.11 we deduce that $v \in C_{uv} \wedge_{\mathcal{MA}(P_\sigma)} D_1$. But now v belongs to C_u which contradicts that uv does not belong to $G_{\mathcal{C}}$. Thus x is a vertex such that $first[x] \leq last[u] < last[x] < first[v]$.

Let $C_x = C_{last[x]}$. Since $\mathcal{C}$ is an interval graph and $v \notin N(x)$, we see that $C_x <_{\mathcal{MA}(P_\sigma)} C_v$. Using the same argument as in the case of D_1, we also see that $C_x \parallel_{\mathcal{MA}(P_\sigma)} C_{uv}$. From the lattice definition we have that $(C_x \vee_{\mathcal{MA}(P_\sigma)} C_{uv}) \leq_{MA(P_\sigma)} C_v$ and $C_u \leq_{\mathcal{MA}(P_\sigma)} (C_x \wedge_{\mathcal{MA}(P_\sigma)} C_{uv})$. Using proposition 2.4 (consecutiveness condition) on $C_{uv} \leq_{\mathcal{MA}(P_\sigma)} (C_x \vee_{\mathcal{MA}(P_\sigma)} C_{uv}) \leq_{\mathcal{MA}(P_\sigma)} C_v$ we see that $v \in (C_x \vee_{\mathcal{MA}(P_\sigma)} C_{uv})$. Using proposition 2.4 (consecutiveness condition) on $C_u \leq_{\mathcal{MA}(P_\sigma)} (C_x \wedge_{\mathcal{MA}(P_\sigma)} C_{uv}) \leq_{\mathcal{MA}(P_\sigma)} C_{uv}$ we see that $u \in (C_x \wedge_{\mathcal{MA}(P_\sigma)} C_{uv})$. Since $u \notin C_x$, u is not universal to C_x and let y be a vertex of maximum last value among $C_x - N(u)$. So $last[u] < first[y] \leq last[x]$ and $last[x] \leq last[y]$. Since $u \in (C_x \wedge_{\mathcal{MA}(P_\sigma)} C_{uv})$ and $y \notin N(u)$, using proposition 2.11 we deduce that $y \in C_x \wedge_{\mathcal{MA}(P_\sigma)} C_{uv}$ and so $y \in N(v)$. So we have $yv \in E$. $\square$

We now carefully choose an edge $uv \in S$ and show that we can find an induced path of length at least 3 in H from u to v. Let uv be an edge of S such that $last[u] < first[v]$ and $\nexists x, y \in S$, $last[x] < first[y]$ and $((last[u] < last[x]$ and $first[y] \leq first[v])$ or $(last[u] \leq last[x]$ and $first[y] < first[v]))$.

Using the previous claim on u and v, we know that there exists x_1 and y_1 such that $first[x_1] \leq last[u] < first[y_1] \leq last[x_1] < first[v]$, $last[x_1] \leq last[y_1]$ and $y_1v \in E$. We choose x_1 and y_1 to be the vertices of maximum *last* values among the ones satisfying the conditions. By our choice of uv we know that $x_1v \notin E(H)$ and $uy_1 \notin E(H)$. Now we have two cases: either $first[v] \leq last[y_1]$ or $last[y_1] < first[v]$. In the first case, we have that u, x_1, y_1, v is an induced path of length 3 in H, and so an induced C_4, which contradicts H being a chordal graph. In the second case, we apply the previous claim on y_1, v and deduce that there exists x_2 and y_2 such that $first[x_2] \leq last[y_1] < first[y_2] \leq last[x_2] < first[v]$, $last[x_2] \leq last[y_2]$ and $y_2v \in E$. We choose x_2 and y_2 to be the vertices of maximum *last* values among the ones satisfying the conditions. By our choice of uv we know that $x_2v \notin E(H)$, $uy_2 \notin E(H)$ and $y_1, y_2 \notin E(H)$. Since we chose x_1 to be the vertex of maximum *last* value we know that $x_2u \notin E(H)$. Since we chose y_1 to be a vertex of maximum *last* value we know that $x_1x_2 \notin E$. Now we again have two cases: either $first[v] \leq last[y_1]$ or $last[y_1] < first[v]$. In the first case, u, x_1, y_1, x_2, y_2, v is an induced path of length 5 in H, and so there is an induced C_6, which contradicts H being chordal. In the second case, we do the same argument again. By continuing in this fashion, we always find an induced path from u to v of length at least 3 in H. Therefore H cannot be chordal.

$\square$

The statement of theorem 2.26 begs the question of whether all maximal chordal subgraphs of a cocomparability graph are interval subgraphs. As shown in Figure 6 this is not the case. This naturally leads to the question: what is the complexity to compute a maximum interval subgraph (i.e., having a maximum number of edges) of a cocomparability graph? Unfortunately it has been shown in [12] that it is NP-hard.

It is interesting to compare theorem 2.26 to a result implicit in [20] but stated in [35] that says the following: every minimal triangulation (or chordalization) of a cocomparability graph is an interval graph. As a corollary, treewidth and pathwidth are equal for cocomparability graphs.

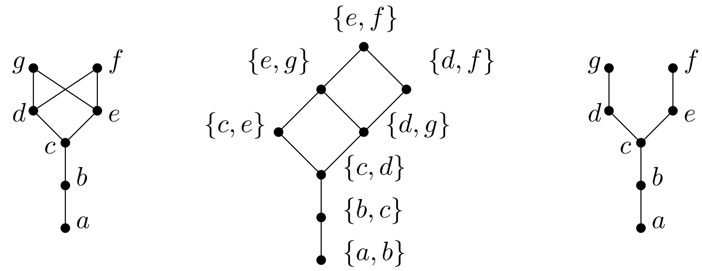

Figure 6: From left to right: a cocomparability graph, along with one of its lattices and a maximal chordal subgraph that is not an interval graph since it contains an asteroidal triple (a, f, g).

3 Algorithmic aspects

The problem of finding a maximal chordal subgraph of an arbitrary graph has been studied in [11] and an algorithm with complexity $O(nm)$ has been proved. In this section, using a new graph search we will improve this to $O(n + mlogn)$ for cocomparability graphs.

3.1 Graph searches and cocomparability graphs

In the introduction we presented two problems on cocomparability graphs solvable by graph searching where these algorithms are very similar to a corresponding algorithm on interval graphs. In subsection 3.2 we present other problems where this "lifting" technique provides new easily implementable cocomparability graph algorithms. All of the algorithms that we mention use a technique called the "+ tie-break rule" in which a total ordering τ of $V(G)$ is used to break ties in a particular graph search $\mathcal{S}$. In particular, the next chosen vertex in $\mathcal{S}$ will be the *rightmost* tied vertex in τ. Such a tie-breaking search will be denoted $\mathcal{S}^+(\tau)$. Many of these examples use that fact that some searches (most notably LDFS) when applied as a "+-sweep" to a cocomp ordering produce a vertex ordering that is also a cocomp ordering. In fact, in [6] there is a characterization of the graph searches that have this property of preserving a cocomp ordering. Given a cocomparability graph G, computing a cocomp ordering can be done in linear time, [30]. This algorithm, however, is quite involved and other algorithms with a running time in $O(n+m\log(n))$ are easier to implement [30, 19]. It should be noticed that up to now, it is not known if one can check if an ordering is a cocomp ordering in less than boolean matrix multiplication time.

3.2 Other examples of graph searches on cocomparability graphs

Following the two examples presented in the introduction we now present three other examples of search based algorithms for other problems on cocomparability graphs:

- Let x be the last vertex of an arbitrary LBFS of cocomp graph G and let y be the last vertex of an arbitrary LBFS starting at x. Then $\{x, y\}$ forms a *dominating pair* in the sense that for all $[x, y]$ paths in G, every vertex of G is either on the path or has a neighbor on the path [7].[4]

- Let σ be an LDFS cocomp ordering of graph G. Then a simple dynamic programming algorithm for finding a longest path in an interval graph also solves the longest path problem on cocomparability graphs where σ is part of the input to the algorithm [32].

- Let σ be an LDFS cocomp ordering of graph G. Then a simple greedy algorithm for finding the maximum independent set (MIS) in an interval graph also solves the problem on cocomparability graphs where σ is part of the input to the algorithm [6]. It is well known that for any graph $G = (V(G), E(G))$ with MIS $X \subseteq V(G)$, the set $Y = V(G) \setminus X$ forms a minimum cardinality vertex cover (i.e., every edge in $E(G)$ has at least one endpoint in Y). The MIS algorithm in [6] certifies the constructed MIS by constructing a clique cover (i.e., a set of cliques such that each vertex belongs to exactly one clique in the set) of the same cardinality as the MIS. Recall that cocomparability graphs are perfect.

The last two algorithms in the list as well as the two in the introduction suggest the existence of an interesting relationship between interval and cocomparability graphs. We believe that the basis of this relationship is the lattice $\mathcal{MA}(P)$, which characterizes cocomparability graphs and shows that a cocomparability graph can be seen as a composition of interval graphs (i.e., the maximal chains of cliques of $\mathcal{MA}(P)$).

3.3 Computing interval subgraphs of a cocomparability graph

In this section, we develop an algorithm that computes a maximal chain of the lattice $\mathcal{MA}(P_\sigma)$ and show that it forms a maximal interval and chordal subgraph. The problem of finding a maximal interval subgraph is the dual of the problem of finding a minimal interval completion (see [9, 23]). The algorithm that we are going to present uses a new graph search that we call LocalMNS, since it shares a lot of similarities with MNS.

This algorithm also gives us a way to compute a minimal interval extension of a partial order. An interval extension of a partial order is an extension that is also an interval order. In [21, 22], it has been proved that the maximal chains of $\mathcal{MA}(P)$ are in a one-to-one correspondence with the minimal interval extensions. Therefore, our algorithm also allows us to compute a minimal interval extension of a partial order in $O(n + mlogn)$ time.

This section is organized as follows. First we present a greedy algorithm, called Chainclique with input a total ordering of an arbitrary graph's vertex set, that computes an interval subgraph. This idea has already been described in [8] for extracting the maximal cliques of an interval graph from an interval ordering. Here we generalize it in order to accept as input any graph and any ordering. In subsection 3.4, we present a new graph search named

[4]In fact this result was proved for the larger family of *asteroidal triple-free (AT-free) graphs* and was the first use of LBFS outside the chordal graph family.

LocalMNS. We will also prove that applying algorithm Chainclique on a LocalMNS cocomp ordering produces a maximal chain of $\mathcal{MA}(P)$. In subsection 2.4, we have shown that such a maximal chain of $\mathcal{MA}(P)$ forms a maximal interval and chordal subgraph.

Definition 3.1. Let $G = (V(G), E(G))$ be a graph and σ an ordering of $V(G)$.

A graph $H = (V(G), E(H))$ with $E(H) \subseteq E(G)$ is a *σ-maximal interval subgraph* for the ordering σ if and only if σ is an interval ordering for the graph H and $\forall S \subseteq E(G) - E(H)$, $S \neq \emptyset$, σ is not an interval ordering for the graph $H' = (V(G), E(H) \cup S)$.

Algorithm 1: Chainclique(G, σ)

Data: $G = (V(G), E(G))$ and a vertex ordering σ
Result: a chain of cliques C_1,...,C_j
$j \leftarrow 0$;
$i \leftarrow 1$;
$C_0 \leftarrow \emptyset$;
while $i \leq |V|$ **do**
  $j \leftarrow j + 1$ %{Starting a new clique}%;
  $C_j \leftarrow \{\sigma(i)\} \cup (N(\sigma(i)) \cap C_{j-1})$;
  $i \leftarrow i + 1$;
  while $i \leq |V|$ *and* $\sigma(i)$ *is universal to* C_j **do**
    $C_j \leftarrow C_j \cup \{\sigma(i)\}$ %{Augmenting the clique}%;
    $i \leftarrow i + 1$;

Output $C_1, \ldots, C_j$;

As we will prove, Chainclique(G, σ) computes a σ-maximal interval subgraph for an arbitrary given graph G. To this end, Chainclique(G, σ) computes a sequence of cliques that respects the consecutiveness condition. Chainclique(G, σ) tries to increase the current clique and when it cannot, it creates a new clique and sets it to be the new current clique. Another way to see it is that Chainclique(G, σ) discards all the edges $xz \in E(G)$ such that $\exists y$, $x <_\sigma y <_\sigma z$ and $xy \notin E(G)$.

It should be noticed that the cliques produced by Chainclique(G, σ) are not necessarily maximal ones, for example take a P_3 on the 3 vertices u, v, w with the edges uv and vw. Chainclique(P_3, σ) with $\sigma = u,\ w,\ v$, produces the cliques: $\{u\}, \{w,\ v\}$. It should also be noted that the algorithm works on an arbitrary graph and with an arbitrary ordering. For an example, let us consider the graph H of Figure 7 and the ordering $\tau = v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6,\ v_7,\ v_8,\ v_9,\ v_{10},\ v_{11},\ v_{12},\ v_{13},\ v_{14},\ v_{15}$. Chainclique$(H, \tau)$ outputs $\{v_1,\ v_2,\ v_3\}$, $\{v_2,\ v_3,\ v_4\}$, $\{v_3,\ v_4,\ v_5\}$, $\{v_5,\ v_6\}$, $\{v_6,\ v_7,\ v_8\}$, $\{v_7,\ v_8,\ v_9\}$, $\{v_8,\ v_9,\ v_{10}\}$, $\{v_{11},\ v_{12},\ v_{13}\}$, $\{v_{11},\ v_{13},\ v_{14}\}$, $\{v_{13},\ v_{14},\ v_{15}\}$. For the graph H presented in Figure 7, $E(H) - E(\mathcal{C}_H) = \{\{v_7,\ v_{11}\},\ \{v_4,\ v_{12}\}\}$.

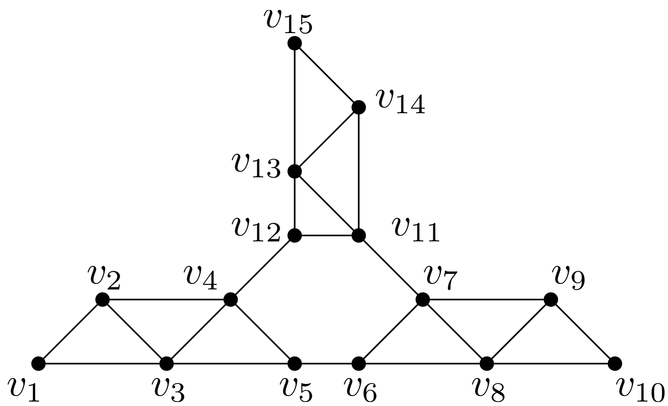

Figure 7: The graph H

Now let us prove that Chainclique(G, σ) allows us to obtain a maximal interval subgraph for an ordering σ. The proof is organized as follows. In the first proposition we prove that Chainclique(G, σ) outputs a sequence of cliques that respects the consecutiveness property. In the second proposition, we prove that the ordering given to Chainclique is an interval ordering for the sequence of cliques. In the last proposition we prove that the graph formed by the sequence is a maximal interval subgraph for the ordering.

Proposition 3.2. *For a graph G and an ordering σ, Chainclique(G, σ) outputs a sequence of cliques $\mathcal{C} = C_1, \ldots, C_k$ such that for every C_e, C_f, C_g, $1 \leq e \leq f \leq g \leq k$, $C_e \cap C_g \subseteq C_f$.*

Proof. We do the proof by induction on the cliques of $\mathcal{C}$ and the induction hypothesis is that at each step j if $x \in C_{j-1}$-C_j then $x \notin C_{j'}$, $j' \geq j$. Since $C_0 = \emptyset$, the hypothesis is true for the initial case, $j = 1$.

Assume that the hypothesis is true for the first $j \geq 1$ cliques. When we start to build the clique C_{j+1}, we add a vertex that has not been considered before and its neighborhood in C_j. By doing so, we cannot add a vertex x to C_{j+1} such that $x \in C_i$-C_j and $i < j$. When we increase the clique, we only add vertices that have not been considered before and so we cannot add a vertex x such that $x \in C_i$-C_j and $i < j$, in C_{j+1}. Therefore the induction hypothesis is also verified at step $j + 1$. $\square$

Therefore using the characterization of interval graphs of corollary 2.22, Chainclique(G, σ) outputs a sequence of cliques that defines an interval subgraph.

Proposition 3.3. *For a graph G and an ordering σ, Chainclique(G, σ) outputs a sequence of cliques $\mathcal{C} = C_1, \ldots, C_k$ such that σ is an interval ordering for $G_{\mathcal{C}}$ and $\forall x \in C_i$-C_j, $\forall y \in C_j$-C_i, $i < j$, $x <_\sigma y$.*

Proof. Assume for contradiction that σ is not an interval ordering for $G_{\mathcal{C}}$. So there exists $u <_\sigma v <_\sigma w$ such that $uv \notin E(\mathcal{C})$ and $uw \in E(\mathcal{C})$. Let C_u be the first clique in which u

appears, C_v be the first clique in which v appears and C_{uw} the first clique that contains both u and w. Because Chainclique(G, σ) considers the vertices in the order they appear in σ, the clique C_u must appear in $\mathcal{C}$ before the clique C_v. Using the same argument the clique C_v must appear before the clique C_{uw}. But now C_u, C_v, C_{uw} contradict proposition 3.2, since $u \notin C_v$.

Now assume for contradiction that $\exists x \in C_i\text{-}C_j$, $\exists y \in C_j\text{-}C_i$, $i < j$, $y <_\sigma x$. Now the vertices are considered by Chainclique(G, σ) in the order they appear in σ. Since $y <_\sigma x$, let C_g be the first clique in which y appears. We see that $g \leq i$. Since y belongs to C_g and C_j, using proposition 3.2 we know that $y \in C_i$. Therefore $y \notin C_j\text{-}C_i$, which contradicts our choice of y. Thus $\forall x \in C_i\text{-}C_j$, $\forall y \in C_j\text{-}C_i$, $i < j$, $x <_\sigma y$. □

We are ready to prove that the graph formed by the sequence is a σ-maximal interval subgraph.

Proposition 3.4. *For a graph G and an ordering σ, Chainclique(G, σ) outputs a sequence of cliques $\mathcal{C} = C_1, \ldots, C_k$ that induces a maximal interval subgraph for the ordering σ.*

Proof. Assume for contradiction that $\mathcal{C} = C_1, \ldots, C_k$ does not form a σ-maximal interval subgraph. Therefore there exists a non empty set of edges S such that σ is an interval ordering for the graph $H = (V(G), E(\mathcal{C}) \cup S)$. Let uv be an edge of S and assume without loss of generality that $u <_\sigma v$. Let C_i be the last clique of $\mathcal{C}$ containing u and consider w the first vertex of C_{i+1}, as chosen by Chainclique; clearly $uw \notin E$ and thus $w \neq v$. Now $u <_\sigma w <_\sigma v$ contradicts σ being an interval ordering for the graph H. □

Proposition 3.5. *Chainclique*(G, σ) *has complexity* $O(n + m)$.

Proof. All the tests can be performed by visiting once the neighborhood of a vertex. Since the sequence of cliques forms a subgraph of G, its size is bounded by m. Therefore, Chainclique(G, σ) has complexity $O(n + m)$. □

3.4 Computing a maximal chain in the lattice

In this subsection, we introduce a new graph search that will be used as a preprocessing step in the computation of a maximal chain of $\mathcal{MA}(P)$. This graph search will be called LocalMNS and when we use Chainclique(G, σ) on a LocalMNS cocomp ordering σ we will obtain a maximal chain of $\mathcal{MA}(P)$.

First, we start by looking at the behavior of Chainclique(G, σ) in which σ is a LBFS or a LDFS ordering. Let us consider the graph in Figure 8. Applying the algorithm Chainclique(G, σ) on the LDFS ordering $\sigma = 1, 3, 2, 4, 6, 5$, we get the chain of cliques $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{4, 5, 6\}$ which is not a maximal chain of $\mathcal{MA}(P)$. A similar result holds using the LBFS ordering $\tau = 2, 3, 1, 4, 6, 5$. Thus, LBFS and LDFS do not help us find a maximal chain of cliques of $\mathcal{MA}(P)$ using Chainclique(G, σ). This motivates the introduction of LocalMNS (algorithm 2).

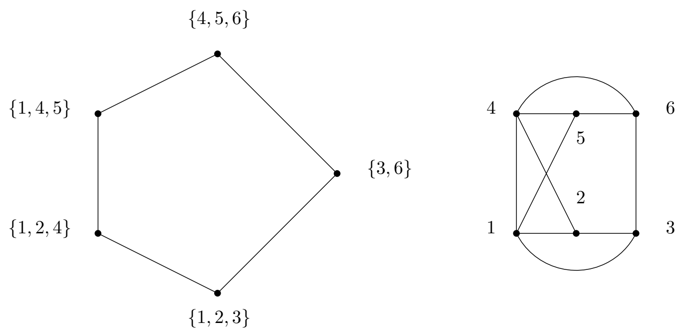

Figure 8: $\mathcal{MA}(P)$ and the corresponding cocomparability graph G

Algorithm 2: LocalMNS

Data: $G = (V, E)$
Result: a total ordering σ such that $\sigma(i)$ is the i'th visited vertex
$D_1 \leftarrow \emptyset$;
$V' \leftarrow V$ %{V' is the set of unchosen vertices}%;
$X \leftarrow \emptyset$ %{X is the set of chosen vertices}%;
;
for $i = 1$ *to* $|V|$ **do**
  v is chosen as a vertex from V' with maximal neighborhood in D_i;
  $\sigma(i) \leftarrow v$;
  $V' \leftarrow V' - \{v\}$;
  $X \leftarrow X \cup \{v\}$;
  $D_{i+1} \leftarrow \{v\} \cup (N(v) \cap D_i)$; % Note: $x \in D_i$-$D_{i+1} \rightarrow x \notin D_j, j > i$%;

This algorithm is very similar to the standard Maximal Neighborhood Search (MNS) algorithm. The only difference is in LocalMNS we are considering the neighborhood of the unvisited vertices only in D_i, which can be a strict subset of X (the visited vertices) and in the case of MNS we are considering the neighborhood in X. This is the reason for the name LocalMNS. Let us look at the behavior of $LocalMNS^+$ on the example of Figure 8. Let $\tau = 5, 6, 4, 2, 3, 1$ be a cocomp ordering. $\sigma = LocalMNS^+(G, \tau) = 1, 3, 2, 4, 5, 6$ is a cocomp ordering and $Chainclique(G, \sigma)$ computes the maximal chain $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 4, 5\}$ and $\{4, 5, 6\}$.

Proposition 3.6. *LocalMNS can be implemented in linear time.*

Proof. It is well known that MNS can be implemented in linear time via MCS. So we will use a LocalMCS to compute LocalMNS. LocalMCS works the same ways as LocalMNS except that at each step i, instead of choosing a vertex of maximal neighborhood in D_i, we choose a vertex with maximum degree in D_i.

To implement LocalMCS we use a partition refinement technique. We use an ordered partition in which each part contains the vertices of V' having a given degree in D_i. This structure can be easily maintained when D_{i+1} is formed from D_i. If a vertex $x \in D_i$ does not appear in D_{i+1}, then $x \notin D_j, j > i$ and thus, during the execution of LocalMCS, we visit the neighborhood of each vertex at most 2 times. Therefore LocalMCS is in $O(n + m)$. □

Proposition 3.7. *LocalMNS*$^+(G, \sigma)$ *can be implemented in* $O(n + mlogn)$.

Proof. As in the previous proposition $LocalMNS^+$ can be implemented via $LocalMCS^+$. We will also use an ordered partition in which each part contains the vertices of V' having a given degree in D_i. But each part has to be ordered with respect to σ. This is the bottleneck of this algorithm. To handle this difficulty each part will be represented by a tree data structure. This leads to an algorithm in O(n+mlogn). □

We now prove that Chainclique(G, σ) on a LocalMNS cocomp ordering outputs a maximal chain of cliques. Let G be a cocomparability graph and τ a cocomp ordering of G. The proof is organized as follows. We first show that $\sigma = LocalMNS^+(G, \tau)$ is a cocomp ordering. Then we describe the structure of a maximal chain of $\mathcal{MA}(P_\sigma)$ and its relation to P_σ. Finally we prove that $Chainclique(G, \sigma)$ outputs a maximal chain of $\mathcal{MA}(P_\sigma)$.

Lemma 3.8. *If G is a cocomparability graph, then τ is a cocomp ordering if and only if $\sigma = LocalMNS^+(G, \tau)$ is a cocomp ordering.*

Proof. Note that this lemma can be stated as a corollary of the characterization in [6] of the searches that preserve being a cocomp ordering. Instead we give a direct proof.

First we show that σ and τ satisfy the "flipping property" insofar as two nonadjacent vertices u, v must be in different relative orders in the two searches. To prove this, we assume that $u <_\sigma v$ and $u <_\tau v$ where, without loss of generality, u is the leftmost vertex in σ that has such a non flipping non neighbor v. Now, because of the "+" rule in order for $u <_\sigma v$ at the time u was selected by σ, there must exist a previously visited vertex w in σ such that $uw \in E(G)$, $vw \notin E(G)$. Note that $w <_\sigma u <_\sigma v$. By the choice of u in σ, we see that $v <_\tau w$ and thus there is an umbrella $u <_\tau v <_\tau w$ in τ, contradicting τ being a cocomp ordering.

Now assume that τ is a cocomp ordering but σ is not. Let $a <_\sigma b <_\sigma c$ be an umbrella in σ where $ac \in E(G), ab, bc \notin E(G)$. By the "flipping property", $b <_\tau a$ and $c <_\tau b$ thereby showing that $c <_\tau b <_\tau a$ forms an umbrella in τ contradicting τ being a cocomp ordering.

The rest of the proof follows immediately. □

Let us introduce some terminology to help us describe the behavior of Chainclique on an ordering σ. Let j_i be the first value of j such that $\sigma(i)$ belongs to C_j (i.e., C_{j_i} is the leftmost clique containing $\sigma(i)$. Let $C_j^1, \ldots, C_j^{l_j}$ be the sequence to build the clique C_j. Let p_i be the first value of p such that $\sigma(i)$ belongs to the clique $C_{j_i}^p$.

We are ready to prove that Chainclique(G, σ) where σ is a LocalMNS cocomp ordering outputs a maximal chain of $\mathcal{MA}(P_\sigma)$. The proof is organized as follows. The next claim links Chainclique(G, σ) and LocalMNS. In proposition 3.10, we prove that the cliques output by Chainclique(G, σ) where σ is a LocalMNS cocomp ordering are maximal cliques. In theorem 3.11, we prove that the chain is a maximal chain of $\mathcal{MA}(P_\sigma)$.

Claim 3.9. *Let G be a cocomparability graph and let τ be a cocomp ordering.*

If $\sigma = LocalMNS^+(G, \tau)$ and $Chainclique(G, \sigma) = C_1, \ldots, C_k$ then for all values of i the set D_{i+1} computed by LocalMNS equals the set $C_{j_i}^{p_i}$ computed by $Chainclique(G, \sigma)$.

Proof. The proof is by induction. The inductive hypothesis is H_i: for all $i \geq 1$, $D_{i+1} = C_{j_i}^{p_i}$.

Since $D_2 = \{\sigma(1)\}$ and $C_1^1 = \{\sigma(1)\}$, H_1 is trivially true. Assume the hypothesis is true for the first i-1 vertices with $i > 1$. We have two cases: $\sigma(i)$ is complete to D_i or not. In the first case, LocalMNS increases the set D_i by adding $\sigma(i)$ and Chainclique(G, σ) will increase the clique $C_{j_{i-1}}^{p_{i-1}}$ by adding $\sigma(i)$. Therefore using H_i, we deduce that $D_{i+1} = C_{j_i}^{p_i}$. In the second case, LocalMNS sets $D_{i+1} = D_i \cap N(\sigma(i))$. In the same way, $Chainclique(G, \sigma)$ creates a new clique $C_{j_i}^1 = C_{j_{i-1}}^{p_{i-1}} \cap N(\sigma(i))$. Therefore using H_i, we deduce that $D_{i+1} = C_{j_i}^{p_i}$. □

Proposition 3.10. *Let G be a cocomparability graph and let τ be a cocomp ordering.*

If $\sigma = LocalMNS^+(G, \tau)$ then $Chainclique(G, \sigma) = C_1, \ldots, C_k$ is a chain of maximal cliques of $\mathcal{MA}(P_\sigma)$ such that $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 <_{\mathcal{MA}(P_\sigma)} \cdots <_{\mathcal{MA}(P_\sigma)} C_k$.

Proof. We start by proving that $\mathcal{C} = C_1, \ldots, C_k$ are all maximal cliques of G and then $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 <_{\mathcal{MA}(P_\sigma)} \cdots <_{\mathcal{MA}(P_\sigma)} C_k$.

Assume for contradiction that some cliques are not maximal and let C_g be the first clique of the chain which is not a maximal clique of G. Let w be a vertex complete to C_g but $w \notin C_g$ and let w be the rightmost such vertex in σ. Let v be the first vertex in σ of $C_g - C_{g-1}$. We have two cases: either $v <_\sigma w$ or $w <_\sigma v$.

In the first case, let $u = \sigma(i)$ be first vertex of $C_{g+1} - C_g$. Since $w \notin C_g$, we must have $u <_\sigma w$. Using claim 3.9, we see that in LocalMNS$^+$ at step $i - 1$, $D_i = C_g$. But now at the time u was chosen, w is complete to D_i but u is not, thereby, contradicting LocalMNS$^+$ choosing u.

In the second case, let C_h be the last clique in the chain with w. We must have that $h < g$ since $w <_\sigma v$. Since w is a neighbor of v, w is not in C_{g-1} (otherwise $w \in C_g$) and we have that $h + 1 < g$. So now let us consider the first vertex x of $C_{h+1} - C_h$ in the ordering σ. Since w does not belong to C_{h+1}, we know that $wx \notin E$ and since w is universal to C_g we have that $x \notin C_g$. Because w is the rightmost complete vertex to C_g, x must not be adjacent to some vertex y of C_g. Now either $x <_\sigma y$ or $y <_\sigma x$. In the first case we know that $w <_\sigma x <_\sigma y$ and w, x, y is an umbrella. Therefore σ is not a cocomp ordering, which is a contradiction to lemma 3.8. In the second case since y appears before x, we have that $first[y] \leq h + 1$. But now y belongs to $C_{first[y]}$ and C_g and so using property 3.2, we know that $y \in C_{h+1}$. This is a contradiction to $xy \notin E$. Thus all the cliques in $\mathcal{C}$ are maximal cliques of G.

Let us now prove that $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 \cdots <_{\mathcal{MA}(P_\sigma)} C_k$. For this purpose, we show that if $i < j$ then $\forall x \in C_i$, $\exists y \in C_j$ such that $xy \notin E$, $x \leq_\sigma y$. Assume first that $x \in C_i \cap C_j$ then we have that $xx \notin E$ and $x \leq_\sigma x$. Now assume that $x \in C_i - C_j$. Since C_j is a maximal clique of G and x does not belong to C_j, there exists $y \in C_j - C_i$ such that $xy \notin E$. We know by proposition 3.3 that $\forall x \in C_i - C_j$, $\forall y \in C_j - C_i$, $i < j$, $x <_\sigma y$. Therefore $x <_\sigma y$ and so $C_i <_{\mathcal{MA}(P_\sigma)} C_j$. □

Let us now prove that Chainclique forms a maximal chain.

Theorem 3.11. *For a cocomparability graph G and a cocomp ordering τ of G, if $\sigma = LocalMNS^+(G,\tau)$ then $Chainclique(G,\sigma)$ is a maximal chain of maximal cliques of $\mathcal{MA}(\sigma)$.*

Proof. Let $\mathcal{C} = C_1, \ldots, C_k$ be the chain of cliques output by $Chainclique(G,\sigma)$. In proposition 3.10, we proved that $\mathcal{C}$ forms a chain of maximal cliques of $\mathcal{MA}(P_\sigma)$. Thereby we only have to prove now that the chain is maximal.

We will show in the next three claims that C_1 is the set of sources of P_σ, that C_j covers C_{j-1} and finally that C_k is the set of sinks of P_σ. Using proposition 2.25, we will be able to deduce that $C_1 <_{\mathcal{MA}(P_\sigma)} C_2 <_{\mathcal{MA}(P_\sigma)} \cdots <_{\mathcal{MA}(P_\sigma)} C_k$ is a maximal chain of $\mathcal{MA}(P_\sigma)$.

Claim 3.12. *C_1 is the set of sources of P_σ.*

Proof. For the initial case, let C_S be the set of sources of P_σ. We will show that $C_1 = C_S$ by proving that σ starts with all the vertices of C_S and only them. Since σ is a linear extension of P_σ, σ starts with at least one source. So we suppose without loss of generality that σ starts with a set of sources $S \subsetneq C_S$ and $S \neq \emptyset$. Now assume for contradiction that after S, we have a vertex x such that $x \notin C_S$. Let $i = \sigma^{-1}(x)$. All the sources after x are complete to S and so for LocalMNS to choose x, x must also be universal to S since at this step D_i is equal to S. Now since x does not belong to C_S, there is a vertex $v \in C_S$ that is comparable to x and because C_S is the set of sources of P_σ and since σ is a linear extension of P_σ, we must have that $v <_\sigma x$. And so v must belong to S. But now x cannot be complete to S, which is a contradiction. So σ starts with all the sources and only them. □

Claim 3.13. *$C_{j-1} \prec_{\mathcal{MA}(P_\sigma)} C_j$ for $1 < j \leq k$.*

Proof. Assume for contradiction that C_j does not cover C_{j-1} and that C_j is leftmost with this property. So C_g covers C_{g-1} for $1 < g \leq j-1$. Let A be a maximal clique of the lattice such that A covers C_{j-1} and $A <_{\mathcal{MA}(P_\sigma)} C_j$. Using proposition 2.4 on C_{j-1}, C_j and A, we deduce that $C_{j-1} \cap C_j \subsetneq A$. We have two cases: either $A \subseteq C_j \cup C_{j-1}$ or $A \not\subset C_j \cup C_{j-1}$.

In the first case, since $C_{j-1} \cap C_j \subseteq A$, we have $C_{j-1} \cap C_j \subseteq A \cap C_{j-1}$. Assume for contradiction that $C_{j-1} \cap C_j = C_{j-1} \cap A$. Using $A \subset C_j \cup C_{j-1}$ and $C_{j-1} \cap C_j = C_{j-1} \cap A$ we can deduce that $A \subset C_j$, which contradicts the maximality of A. Therefore $C_{j-1} \cap C_j \subsetneq A \cap C_{j-1}$. Let $v = \sigma(i)$ be the leftmost vertex of $C_j - C_{j-1}$ in σ. We again have two cases: either $v \in A$ or $v \notin A$. In the first case, Chainclique set $C_j^1 = (N(v) \cap C_{j-1}) \cup \{v\}$. But since $C_{j-1} \cap C_j \subsetneq A \cap C_{j-1}$ and $v \in A$, we have $C_{j-1} \cap C_j \subsetneq N(v) \cap C_{j-1}$ and so $C_j^1 \not\subset C_j$. This is a contradiction to the behavior of Chainclique. In the second case, using claim 3.9 on v we deduce that the set D_i of LocalMNS equals C_{j-1}. But now let $x \in A - C_{j-1}$. Since

$C_{j-1} \cap C_j \subsetneq A \cap C_{j-1}$, we have that $N(v) \cap D_i \subsetneq N(x) \cap D_i$. This is a contradiction to the choice of LocalMNS.

In the second case, let $x \in A - (C_j \cup C_{j-1})$. Assume for contradiction that there exists a maximal clique B such that $x \in B$ and $B <_{\mathcal{MA}(P_\sigma)} C_{j-1}$. Now $x \in B, A$ and $x \notin C_{j-1}$ contradicting proposition 2.4. Therefore x cannot appear in σ before the last vertex of C_{j-1}. Since $x \notin C_j$, $\exists y \in C_j$ such that $xy \notin E$. Using lemma 2.5 on x, y we deduce that $x <_{P_\sigma} y$ and since σ is a linear extension of P_σ we know that $x <_\sigma y$. But now since x appears after the last vertex of C_{j-1}, before the last of C_j and x is not complete to C_j, Chainclique must build a clique in the sequence between C_{j-1} and C_j, which is a contradiction.

□

Claim 3.14. *C_k is the set of sinks of P_σ.*

Proof. For the final case, we show that C_k is the set of sinks of P_σ. Assume for contradiction that x is a sink of P_σ and x does not belong to C_k. All the vertices belong to at least one clique of $\mathcal{C}$ and let $C_g = C_{last[x]}$. Since $x \notin C_k$, we have $g < k$. So x does not belong to C_{g+1}. But now let y be the first vertex in σ of $C_{g+1} - C_g$. Since $x \notin C_{g+1}$ we have $xy \notin E$ and $x <_\sigma y$. But this contradicts that x is a sink. □

□

Corollary 3.15. *Let G be a cocomparability graph, then a maximal interval subgraph of G can be computed in $O(n + mlogn)$.*

To finish let us now show that any maximal chain of $\mathcal{MA}(P_\sigma)$ can be computed by LocalMNS.

Theorem 3.16. *For a cocomparability graph G and a transitive orientation P of $\overline{G}$, every maximal chain of $\mathcal{MA}(P)$ can be computed using $Chainclique(G, \sigma)$ on some cocomp LocalMNS ordering σ.*

Proof. Let $C_1 \prec_{\mathcal{MA}(P)} \dots \prec_{\mathcal{MA}(P)} C_k$ be a maximal chain of $\mathcal{MA}(P)$. Let us take the ordering τ as the interval ordering for this maximal chain of cliques. Now using $LocalMNS^+$ on τ^{-1}, we get τ. So τ is a LocalMNS cocomp ordering and using Chainclique(G, τ) we get $C_1 <_{\mathcal{MA}(P)} \dots <_{\mathcal{MA}(P)} C_k$. □

Using Theorem 2.26 we immediately have:

Corollary 3.17. *A maximal chordal subgraph of a cocomparability graph G can be computed with complexity $O(n + mlogn)$.*

Proof. Preuve dplacer au dessus ?

The algorithm consists of finding a cocomp ordering, then performing a $LocalMCS^+$ and then using Chainclique. A cocomp ordering can be found in $(O(n+m)$ [30] and Chainclique has complexity $O(n + m)$. So the bottleneck of this algorithm lies in $LocalMCS^+$. Using proposition 3.7 the full algorithm can be computed in $O(n + mlogn)$ time.

□

4 Computing all simplicial vertices

In order to compute simplicial vertices we need to consider some particular maximal cliques, called **fully comparable cliques**.

Definition 4.1. In a lattice $\mathcal{L} = (\mathcal{X}, \leq_{\mathcal{L}})$, an element $e \in X$ is said to be *fully comparable* if and only if for every $u \in X$, either $e \leq_{\mathcal{L}} u$ or $u \leq_{\mathcal{L}} e$.

We now prove that if σ is an MNS cocomp ordering of G then all the fully comparable cliques of $MA(P_\sigma)$ belong to the sequence of cliques obtained using ChainClique(G,σ). As we will show, these cliques play a decisive role in the problem of finding the simplicial vertices of G.

Theorem 4.2. *Let G be a cocomparability graph and σ a MNS cocomp ordering of the vertices of G. If C_b is a maximal clique such that C_b is fully comparable in $\mathcal{MA}(P_\sigma)$ then C_b is a maximal clique of the chain output by $ChainClique(G, \sigma)$.*

Proof. Let C_b be a fully comparable maximal clique in $\mathcal{MA}(P_\sigma)$. We define V_{C_b} to be $\{x \in V | \exists C_x$ such that $x \in C_x$ and $C_x \leq_{\mathcal{MA}(P_\sigma)} C_b\}$.

Let us first prove that the ordering σ starts with all the vertices of V_{C_b}. For the sake of a contradiction, let's assume that we have in σ a vertex v such that $\exists C_v$, $v \in C_v$ and $C_v >_{\mathcal{MA}(P_\sigma)} C_b$ before a vertex $x \in V_{C_b}$ and v is the leftmost such vertex in σ. Let C_x be a maximal clique such that $x \in C_x$ and $C_x \leq_{\mathcal{MA}(P_\sigma)} C_b$. We have two cases, either $xv \notin E$ or $xv \in E$.

Case 1: $xv \notin E$. Using lemma 2.5 on x, v, we deduce that $x <_{P_\sigma} v$ and since σ is a linear extension of P_σ, we know that $x <_\sigma v$ contradicting our choice of v.

Case 2: $xv \in E$. We prove that $N(v) \cap V_{C_b} \subset N(x) \cap V_{C_b}$. Since $xv \in E$ there exists a maximal clique D such that $\{x, v\} \subset D$ and since $v \notin V_{C_b}$ we know $C_b <_{\mathcal{MA}(P_\sigma)} D$. Using proposition 2.4 on C_x, C_b, D we deduce that $x \in C_b$. Let u be a vertex of $N(v) \cap V_{C_b}$. Since $uv \in E$ there exists a maximal clique A such that $\{u, v\} \subset A$ and since $v \notin V_{C_b}$ we know $C_b <_{\mathcal{MA}(P_\sigma)} A$. Let C_u be a maximal clique such that $u \in C_u$ and $C_u \leq_{\mathcal{MA}(P_\sigma)} C_b$. Using proposition 2.4 on C_u, C_b, A we deduce that $u \in C_b$. Therefore $ux \in E$ and so $N(v) \cap V_{C_b} \subset N(x) \cap V_{C_b}$. But now at the time when v was chosen, the label of v can only be equal to the label of x. Now C_b is a maximal clique and since $v \notin C_b$, there must exist a vertex $w \in C_b$ such that $wv \notin E$. Since $C_v >_{\mathcal{MA}(P_\sigma)} C_b$, necessarily $w <_\sigma v$. Since $x \in C_b$, $wx \in E$ and so the label of x is strictly greater than the label of v when v was chosen which is a contradiction to the choice of MNS. Thus σ starts with all the vertices of V_{C_b}.

Since σ starts with all the vertices of V_{C_b}, the ordering of the vertices of V_{C_b} induced by σ is a MNS cocomp ordering for the graph induced by V_{C_b}. Let P_{C_b} be the transitive orientation of the complement of the graph induced by V_{C_b} obtained using σ. To prove that C_b belongs to the interval graph computed by $ChainClique$ we will prove that the last clique that ChainClique computes using the ordering induced by the vertices of V_{C_b} is the set of sinks of P_{C_b}, which is equal to C_b. Let $C_1, \ldots, C_k$ be the chain of cliques that ChainClique computes using the ordering induced by the vertices of V_{C_b}. Assume for contradiction that

x is a sink and $x \notin C_k$. Let C_g be the last clique that contains x. Since $x \notin C_k$ we have that $g < k$. Let y be the first vertex in σ that belongs to $C_{g+1} - C_g$. Since $x \notin C_{g+1}$, we must have that $xy \notin E$ and so $x <_{P_{C_b}} y$, therefore contradicting the assumption that x is a sink. □

This result does not hold if σ is not a MNS cocomp ordering. For example just take a P_3, u, v, w and the ordering $u < w < v$ (see Figure 9). The algorithm cannot output $\{u, v\}$ which satisfies the property.

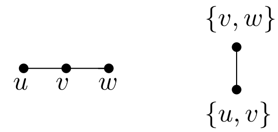

Figure 9: A P_3 and its lattice.

Theorem 4.3. *Let G be a cocomparability graph and σ a cocomp ordering. If v is a simplicial vertex then there exists a maximal clique C_v such that $v \in C_v$ and C_v is fully comparable in $\mathcal{MA}(P_\sigma)$.*

Proof. Clearly a simplicial vertex belongs to a unique maximal clique. For a simplicial vertex v let us denote by C_v the maximal clique such that $v \in C_v$. Assume that there exists a maximal clique D such that $D \parallel_{\mathcal{MA}(P_\sigma)} C_v$. Since $\mathcal{MA}(P_\sigma)$ is a lattice, there exists $D \vee_{\mathcal{MA}(P_\sigma)} C_v$ and $D \wedge_{\mathcal{MA}(P_\sigma)} C_v$ two other maximal cliques. Using the definition of $D \vee_{\mathcal{MA}(P_\sigma)} C_v$ and $D \wedge_{\mathcal{MA}(P_\sigma)} C_v$, v belongs to either $D \vee_{\mathcal{MA}(P_\sigma)} C_v$ or $D \wedge_{\mathcal{MA}(P_\sigma)} C_v$. Therefore C_v is not the only maximal clique that contains v, contradicting v is a simplicial vertex. □

Let us now study an algorithm to find all the simplicial vertices in a cocomparability graph. The correctness of the algorithm mainly relies on theorems 4.2 and 4.3. To compute the simplicial vertices we need for each vertex a couple of values. We compute the value $first$ and $last$ defined by $first[v] = min\{i | v \in C_i\}$ and $last[v] = max\{i | v \in C_i\}$. The value $forward[v]$ will either be $last[v]$ or the index of the last clique in which v has a neighbor not in the interval graph and so we define $forward[v]$ to be $max\{\{first[u] | u \in N(v)$ and $first[u] > last[v]\} \cup \{last[v]\}\}$. Note that $forward[v] \geq last[v]$.

The value $backward[v]$ will either be $first[v]$ or the index of the first clique in which v has a neighbor not in the interval graph and so we define $backward[v]$ to be $min\{\{last[u] | u \in N(v)$ and $last[u] < first[v]\} \cup \{first[v]\}\}$. Note that $backward[v] \leq first[v]$.

Theorem 4.4. *Let G be a cocomparability graph and σ a MNS cocomp ordering of the vertices of G. Then algorithm 3 outputs all the simplicial vertices of G.*

Algorithm 3: Simplicial vertices in a cocomparability graph

Data: $G = (V(G), E(G))$ and an MNS cocomp ordering σ of $V(G)$
Result: The simplicial vertices of G
Compute the sequence $C_1, ..., C_k$ using $ChainClique(G, \sigma)$;
Compute $first$, $last$, $backward$, $forward$ for every vertex of $V(G)$;
$S \leftarrow \emptyset$ %{The simplicial vertices}%;
for ($i \leftarrow 1$ **to** n) **do**
  if $forward[\sigma(i)] = backward[\sigma(i)]$ **then** $S \leftarrow S \cup \{\sigma(i)\}$;
Output S;

Proof. Using theorem 4.2 we deduce that all the simplicial vertices and their neighborhood belong to one clique of the sequence output by ChainClique. So to test if a vertex is a simplicial vertex, we have to check that a vertex belongs to only one clique in the sequence and has no neighbors in the rest of the interval graph. By definition, for any vertex v we have that $last[v] \leq forward[v]$ and $backward[v] \leq first[v]$. So for any simplicial vertex $\sigma(i)$ we have that $forward[\sigma(i)] = backward[\sigma(i)] = last[\sigma(i)] = first[\sigma(i)]$. If a vertex $\sigma(i)$ is not simplicial then either it belongs to more than one clique in the sequence and so $last[\sigma(i)] \neq first[\sigma(i)]$ implying $forward[\sigma(i)] \neq backward[\sigma(i)]$ or it has a neighbor outside the sequence and so $last[\sigma(i)] < forward[\sigma(i)]$ or $backward[\sigma(i)] < first[\sigma(i)]$ implying $forward[\sigma(i)] \neq backward[\sigma(i)]$. Therefore our algorithm successfully finds all the simplicial vertices of a cocomparability graph. □

Theorem 4.5. *Simplicial vertices can be computed in linear time on a cocomparability graph, when a cocomp ordering is provided.*

Proof. To apply algorithm 3, we need a MNS cocomp ordering. To obtain such an ordering, if a cocomp ordering σ is provided, we can simply apply $\text{LBFS}^+(G, \sigma)$. This can be done in linear time. $ChainClique$ has complexity $O(n + m)$. Enumerating all the cliques takes $O(n + m)$. Computing $first$ and $last$ can be done by enumerating all the cliques of the spanning interval graph. Computing the $forward$ and $backward$ functions can be done by enumerating for each vertex its neighborhood after computing $first$ and $last$ and so can be done in $O(n + m)$. Enumerating the vertices can be done in $O(n)$.

□

5 Conclusions and perspectives

In sections 1 and 3.2 we presented a number of examples of problems where simple interval graph algorithms can be "lifted" to similar algorithms for cocomparability graphs. These algorithms are typically based on cocomp orderings produced by graph searches, most notably LDFS and LBFS. The underlying question is whether there is a structural feature that indicates which problems can be "lifted" in this way. In an attempt to answer this question, we have examined the maximal clique lattice of a cocomparability graph and have presented a characterization theorem of such lattices. This characterization has lead to new algorithms for finding a maximal interval subgraph of a cocomparability graph and for finding the set of simplicial vertices in a cocomparability graph. Both of these algorithms roughly follow maximal chains of this lattice and in the maximal interval subgraph case uses a new graph search, LocalMNS. In [12] some other interesting applications of this framework have been developed; for example to compute a minimal clique separator decomposition of a cocomparability graph in linear time.

Our work raises a number of algorithmic questions:

- Does there exist a LocalMNS that can be implemented in linear time when used as a +sweep on cocomp ordering σ? Can techniques similar to those used in [25] help?
- Are there other polynomial time solvable interval graph problems that are amenable to the ChainClique approach?

Similarly our work raises a number of structural questions:

- Given our characterization theorem of maximal clique lattices of a cocomparability graph, a natural question is to study the structure imposed on cocomparability graphs by restrictions of the lattice structure.
- Can anything of interest be found about the clique structure of AT-free graphs, the natural generalization of cocomparability graphs?
- In section 3.3, we have exhibited some relationships shared by cocomparability graphs and interval graphs and the importance of graph searches in cocomparability graphs. But, we still have not managed to give a full answer to the question of why some interval graph algorithms can be "lifted" to work on cocomparability graphs. Does there exist some generic greedoid structure for cocomparability graphs that explains why these greedy algorithms work? So far we have no good answer for this question.

Acknowledgements: DGC wishes to thank the Natural Sciences and Engineering Research Council (NSERC) of Canada for financial support of this research.

References

[1] Behrendt B. Maximal antichains in partially ordered sets. *Ars Combinatoria*, 25C:149–157, 1988.

[2] G. Birkhoff. Lattice theory. *American Mathematical Society*, 25(3), 1967.

[3] Andreas Brandstädt, Van B. Le, and Jeremy P. Spinrad. *Graph Classes, a survey.* SIAM Monographs on Discrete Mathematics and Applications, 1999.

[4] N. Caspard, B. Leclerc, and B. Monjardet. *Finite Ordered Sets Concepts, Results and Uses*. Cambridge University Press, 2012.

[5] Derek G. Corneil, Barnaby Dalton, and Michel Habib. LDFS-based certifying algorithm for the minimum path cover problem on cocomparability graphs. *SIAM J. Comput.*, 42(3):792–807, 2013.

[6] Derek G. Corneil, Jérémie Dusart, Michel Habib, and E. Köhler. On the power of graph searching for cocomparability graphs. *SIAM J. Discrete Math.*, 30(1):569–591, 2016.

[7] Derek G. Corneil, Stephan Olariu, and Lorna Stewart. Linear time algorithms for dominating pairs in asteroidal triple-free graphs. *SIAM J. Comput.*, 28(4):1284–1297, 1999.

[8] Derek G. Corneil, Stephan Olariu, and Lorna Stewart. The LBFS structure and recognition of interval graphs. *SIAM J. Discrete Math.*, 23(4):1905–1953, 2009.

[9] Christophe Crespelle and Ioan Todinca. An $O(n^2)$-time algorithm for the minimal interval completion problem. *Theor. Comput. Sci.*, 494:75–85, 2013.

[10] Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order (2. ed.)*. Cambridge University Press, 2002.

[11] P.M. Dearing, D.R. Shier, and D.D. Warner. Maximal chordal subgraphs. *Discrete Applied Mathematics*, 20(3):181 – 190, 1988.

[12] Jérémie Dusart. *Graph Searches with Applications to Cocomparability Graphs*. PhD thesis, Université Paris Diderot, June 2014.

[13] Jérémie Dusart and Michel Habib. A new LBFS-based algorithm for cocomparability graph recognition. *to appear in Discrete Applied Mathematics*, 2016.

[14] Ben Dushnik and E. W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63(3):pp. 600–610, 1941.

[15] P.C. Fishburn. *Interval orders and interval graphs*. Wiley, 1985.

[16] P.C. Gilmore and A.J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canad. J. Math.*, 16:539–548, 1964.

[17] Martin Charles Golumbic. *Algorithmic Graph Theory and Perfect Graphs (Annals of Discrete Mathematics, Vol 57)*. North-Holland Publishing Co., Amsterdam, The Netherlands, The Netherlands, 2004.

[18] G. Grätzer. *General Lattice Theory*. Birkhäuser, 1968.

[19] Michel Habib, Ross M. McConnell, Christophe Paul, and Laurent Viennot. Lex-BFS and partition refinement, with applications to transitive orientation, interval graph recognition and consecutive ones testing. *Theor. Comput. Sci.*, 234(1-2):59–84, 2000.

[20] Michel Habib and Rolf H. Möhring. Treewidth of cocomparability graphs and a new order-theoretic parameter. *Order*, 11(3):47–60, February 1991.

[21] Michel Habib, Michel Morvan, Maurice Pouzet, and Jean-Xavier Rampon. Extensions intervallaires minimales. In *Compte Rendu à l'Académie des Sciences Paris, présenté en septembre 91, par le Pr. G. Choquet*, volume 313, pages 893–898, 1991.

[22] Michel Habib, Michel Morvan, Maurice Pouzet, and Jean-Xavier Rampon. Incidence structures, coding and lattice of maximal antichains. Technical Report 92-079, LIRMM, 1992.

[23] Pinar Heggernes, Karol Suchan, Ioan Todinca, and Yngve Villanger. Minimal interval completions. In Gerth Stølting Brodal and Stefano Leonardi, editors, *ESA*, volume 3669 of *Lecture Notes in Computer Science*, pages 403–414. Springer, 2005.

[24] Ján Jakubík. Maximal antichains in a partially ordered set. *Czechoslovak Mathematical Journal*, 41(1):75–84, 1991.

[25] Ekkehard Köhler and Lalla Mouatadid. Linear time LexDFS on cocomparability graphs. In *Algorithm Theory - SWAT 2014 - 14th Scandinavian Symposium and Workshops, Copenhagen, Denmark, July 2-4, 2014. Proceedings*, pages 319–330, 2014.

[26] Dieter Kratsch and Lorna Stewart. Domination on cocomparability graphs. *SIAM J. Discrete Math.*, 6(3):400–417, 1993.

[27] C. Lekkerkerker and J. Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae*, 51(1):45–64, 1962.

[28] George Markowsky. The factorization and representation of lattices. *Transactions of the American Mathematical Society*, 203:185–200, 1975.

[29] George Markowsky. Primes, irreductibles and extremal lattices. *Order*, 9:265–290, 1992.

[30] Ross M. McConnell and Jeremy Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201(1-3):189–241, 1999.

[31] Daniel Meister. Recognition and computation of minimal triangulations for AT-free claw-free and co-comparability graphs. *Discrete Applied Mathematics*, 146(3):193–218, 2005.

[32] George B. Mertzios and Derek G. Corneil. A simple polynomial algorithm for the longest path problem on cocomparability graphs. *SIAM J. Discrete Math.*, 26(3):940–963, 2012.

[33] Rolf H. Möhring. Algorithmic aspects of comparability graphs and interval graphs. In Ivan Rival, editor, *Graphs and Order*, volume 147 of *NATO ASI Series*, pages 41–101. Springer Netherlands, 1985.

[34] Rolf H. Möhring. Triangulating graphs without asteroidal triples. *Discrete Appl. Math.*, 64(3):281–287, February 1996.

[35] Andreas Parra. *Structural and algorithmic aspects of chordal graph embeddings.* PhD thesis, Technische Universitt Berlin, 1996.

[36] K. Reuter. The jump number and the lattice of maximal antichains. *Discrete Mathematics*, 88:289–307, 1991.

[37] B. S. W. Schröder. *Ordered sets, an introduction.* Birkhäuser, 2002.

[38] William T. Trotter. Combinatorial aspects of interval orders and interval graphs. *Electronic Notes in Discrete Mathematics*, 2:153, 1999.