Mathematical Model of Scheduler with Semi-Markov Input and Bandwidth Sharing Discipline

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***Abstract*—In this paper, we consider single server queueing system with multiple semi-Markov inputs and buffers. Each request of the flows brings to the system some random amount of information. According to the bandwidth sharing discipline, each buffer has its own part of the throughput and the server transmits the information from buffers simultaneously. The aim of the current research is to derive the probability distribution of the amount of information in single buffer.**

***Index Terms*—queueing system, semi-Markov flow, bandwidth sharing discipline**

I. Introduction

Stochastic models of schedulers are divided by two groups: loss models and delay models. In both groups server (transmission node) deals with multiple inputs. Loss models are used to solve the problem of bandwidth allocation in cellular networks [1], [2]. In this paper, we focus on delay models, which reflect the structure and performance of systems with buffer(s) [3], [4], [5].

The majority of mathematical models of transmission nodes use Poisson process for modeling packets arrivals [6], [7].

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On the other hand, models with sophisticated input processes allow to study the system mostly in terms of means [3]. Moreover, the generalization of the traffic model usually leads to the fact that analytical results can be obtained only for an exponential distribution of the service time.

In the current paper, we consider the queueing system with multiple inputs and buffers as a model of scheduler. The input process is semi-Markov flow of packets arrivals. We take into account the size of packets and model it as arbitrary distributed variables. Bandwidth sharing discipline allows to transmit the information from all buffers at the same time. For this end, the scheduler allocates the virtual communication channel for each buffer with constant rate of transmission as a part of the throughput of the system. Thus, the service time linearly depends on the packet length and is also arbitrary distributed variable. We use asymptotic analysis method to derive the approximation of the probability distribution of the amount of information in the buffers under the limit condition of high load.

The rest of the paper is organized as follows. In section 2, we build mathematical model of the system with multiple inputs and derive preliminary formulas. Section 3 is devoted to the analysis of processes accompanying the system state, which in our case is the amount of information in the buffers. In section 4, we obtain asymptotic characteristic function of the amount of information in the buffers and derive the approximation of its cumulative distribution function. Section 5 contains the numerical example. Section 6 is dedicated to the concluding remarks.

II. Model Description and Preliminaries

We consider single server queueing system with N buffers and inputs as a mathematical model of scheduler. Each request of n-th flow brings to the system random amount of information, whose cumulative distribution function is $B_n(x)$. We assume that the service in the system has the meaning of the transmission of the information through the communication channel.

We consider queueing system with bandwidth sharing service discipline, which allows servicing all buffers at the same time. Values v_n characterize the part of throughput, which is allotted to the service of n-th input. These values define the speed of data transmission as follows: the amount of transmitted information is $v_n\Delta t$ per time Δt for n-th input flow.

Let $S_n(t)$ denote the total amount of information collected in n-th buffer until moment t. Assuming that inputs are independent and values v_n are constant and do not depend on the system state, we obtain that the total amounts of information in different buffers $S_n(t)$ are independent processes. Therefore, we can make a decomposition of the system with N inputs to the system with single input. Without loss of generality, we consider the system with only first input denoting $S_1(t) = S(t)$, $B_1(x) = B(x)$. In this part of the study, we consider the traffic model in a form of semi-Markov flow defined by semi-Markov matrix $\mathbf{A}(x)$. Elements $A_{k\nu}(x)$ of the matrix $\mathbf{A}(x)$ are given by

$$A_{k\nu}(x) = P\{\xi(n+1) = \nu, \tau(n+1) < x | \xi(n) = k\}. \quad (1)$$

We take into account that

$$\mathbf{P} = \mathbf{A}(\infty), \quad (2)$$

where $\mathbf{P}$ is a matrix of transition probabilities of embedded Markov chain $\xi(n)$. Moments t_n of arrivals in semi-Markov flow are given by formulas $t_{n+1} = t_n + \tau(n+1)$. In further investigation, we use semi-Markov random process $k(t)$, which is defined as

$$k(t) = \xi(n+1), \text{if } t_n < t \le t_{n+1} = t_n + \tau(n+1). \quad (3)$$

Value v_1 is determined as $v_1 = v/\rho$ and value v is given by the equality $v = \lambda b_1$, where λ is the intensity of input flow, b_1 is the first raw moment of the distribution $B(x)$ and value ρ determines the system load. We note that ρ satisfies the stability condition $\rho < 1$.

Let $z(t)$ denote the residual time of the next arrival in semi-Markov flow, $S(t)$ denote the total amount of information accumulated in the buffer until moment t. The notation for the probability distribution of three-dimensional process $\{k(t), S(t), z(t)\}$ is given by

$$P_k(s, z, t) = P\{k(t) = k, S(t) < s, z(t) < z\}. \quad (4)$$

For the analysis of the model, we derive the balance equality

$$P_k\left(s - \frac{v}{\rho}, z - \Delta t, t + \Delta t\right) = \\ = P_k(s, z, t) - P_k(s, \Delta t, t) + \\ + \sum_{v=1}^{K} \int_0^s P_\nu(s - x, \Delta t, t) dB(x) A_{\nu k}(z) + o(\Delta t),$$

from which we obtain the equation for steady state probability distribution $P_k(s, z)$

$$v\frac{\partial P_k(s,z)}{\partial s} + \rho\frac{\partial P_k(s,z)}{\partial z} - \rho\frac{\partial P_k(s,0)}{\partial z} + \\ + \rho\sum_{\nu=1}^{K} \int_0^s \frac{\partial P_\nu(s-x,0)}{\partial z} dB(x) A_{\nu k}(z) = 0, \quad (5)$$

where $\dfrac{\partial P_k(s,0)}{\partial z} = \dfrac{\partial P_k(s,z)}{\partial z}\Big|_{z=0}$.

Denoting partial characteristic functions $H_k(u, z)$ and characteristic function $\beta(u)$

$$H_k(u,z) = \int_0^\infty e^{jus} dP_k(s,z), \quad \beta(u) = \int_0^\infty e^{jux} dB(x), \quad (6)$$

we transform (6) to the equation for the partial characteristic functions. Since the derivative $\dfrac{\partial P_k(s,z)}{\partial s}$ has the gap in point $s = 0$, we introduce the Fourier transforms of it as follows:

$$\int_0^\infty e^{jus} d\frac{\partial P_k(s,z)}{\partial s} = ju\left\{\int_0^{+0} e^{jus} dP_k(s,z) - H_k(u,z)\right\}.$$

We denote probabilities

$$\int_0^{+0} e^{jus} dP_k(s,z) = P_k(+0,z) = V_k(z)$$

of that the buffer is empty, process $k(t)$ is in k-th state and the value of process $z(t)$ is less than z. We also denote vector characteristic function

$$\mathbf{H}(u,z) = \{H_1(u,z), H_2(u,z), ..., H_K(u,z)\},$$

vector

$$\mathbf{V}(z) = \{V_1(z), V_2(z), ..., V_K(z)\},$$

identity matrix $\mathbf{I}$ and vector of ones $\mathbf{e}$. We rewrite (6) together with the additional equation obtained taking the limit by $z \to \infty$ as follows:

$$\rho\frac{\partial \mathbf{H}(u,z)}{\partial z} + \rho\frac{\partial \mathbf{H}(u,0)}{\partial z}(\beta(u)\mathbf{A}(z) - \mathbf{I}) - juv\mathbf{H}(u,z) = = -juv\mathbf{V}(z),$$

$$\rho(1-\beta(u))\frac{\partial \mathbf{H}(u,0)}{\partial z}\mathbf{e} + juv\mathbf{H}(u)\mathbf{e} = juv\mathbf{V}\mathbf{e}, \qquad (7)$$

where $\mathbf{H}(u) = \mathbf{H}(u,\infty)$, $\mathbf{V} = \mathbf{V}(\infty)$.

III. Steady State Probability Distribution of Semi-Markov Process and Residual Time of Arrival

Denoting $\mathbf{H}(0,z) = \mathbf{R}(z)$, where $\mathbf{R}(z)$ is a stationary probability distribution of process $\{k(t), z(t)\}$, we set $u = 0$ in the first equation of system (7) and obtain

$$\mathbf{R}'(z) = \mathbf{R}'(0)(\mathbf{I} - \mathbf{A}(z)). \qquad (8)$$

The solution of (8) can be written as follows:

$$\mathbf{R}(z) = \int_0^z \mathbf{R}'(0)(\mathbf{I} - \mathbf{A}(x))dx. \qquad (9)$$

In (9), the integrand goes to zero as $z \to \infty$, then we can write

$$\mathbf{R}'(0) = \mathbf{R}'(0)\mathbf{P}.$$

The last linear system of algebraic equations for elements of vector $\mathbf{R}'(0)$ is the same as the system for steady state probability distribution $\mathbf{r}$ of the embedded Markov chain

$$\mathbf{r} = \mathbf{r}\mathbf{P}, \quad \mathbf{r}\mathbf{e} = 1. \qquad (10)$$

Thus, we present vector $\mathbf{R}'(0)$ as

$$\mathbf{R}'(0) = C\mathbf{r}. \qquad (11)$$

In (9), we take the limit by $z \to \infty$ and obtain

$$\mathbf{R} = C\mathbf{r}\mathbf{A}_1, \qquad (12)$$

where $\mathbf{R}$ is a vector of steady state probability distribution of semi-Markov process $k(t)$ and matrix $\mathbf{A}_1$ is determined as $\mathbf{A}_1 = \int_0^\infty (\mathbf{P} - \mathbf{A}(x))dx$. We obtain constant C from the normalization condition

$$C = \frac{1}{\mathbf{r}\mathbf{A}_1\mathbf{e}}.$$

Thus, we present (11) as

$$\mathbf{R}'(0) = \frac{\mathbf{r}}{\mathbf{r}\mathbf{A}_1\mathbf{e}},$$

which we substitute into (9) and obtain

$$\mathbf{R}(z) = \frac{\mathbf{r}}{\mathbf{r}\mathbf{A}_1\mathbf{e}} \int_0^z (\mathbf{P} - \mathbf{A}(x))dx. \qquad (13)$$

IV. Asymptotic Analysis of the Model

To derive the approximation of cumulative distribution function $F(s) = P\{S(t) < s\}$, we apply asymptotic analysis method to system (7) under limit condition of high load $\rho \to 1$.

Theorem. Asymptotic characteristic function of the amount of information in the buffer in considered queueing system with semi-Markov flow under the limit condition of high load has the following form:

$$h(u) = \frac{(1-\rho)\gamma}{(1-\rho)\gamma - ju}, \qquad (14)$$

where

$$\gamma = \frac{2b_1(\mathbf{r}\mathbf{A}_1\mathbf{e})^{-1}}{b_2(\mathbf{r}\mathbf{A}_1\mathbf{e})^{-1} + 2b_1\mathbf{g}'(0)\mathbf{e}}, \qquad (15)$$

b_1 and b_2 are the first and second raw moments of the amount of information in one request of the flow. Vector $\mathbf{g}'(0)$ is the solution of inhomogeneous system of equations

$$\mathbf{g}'(0)(\mathbf{I} - \mathbf{P}) = b_1(\mathbf{r}\mathbf{A}_1\mathbf{e})^{-1}(\mathbf{r} - \mathbf{R}),$$

$$\mathbf{g}'(0)\mathbf{A}_1\mathbf{e} = \frac{b_1}{2}\frac{\mathbf{r}\mathbf{A}_2\mathbf{e}}{(\mathbf{r}\mathbf{A}_1\mathbf{e})^2} - b_1, \qquad (16)$$

where matrix $\mathbf{A}_2$ is determined by

$$\mathbf{A}_2 = \int_0^\infty x^2 d\mathbf{A}(x).$$

Asymptotic probability distribution in the limit by $\rho \to 1$ of the amount of information in the buffer is exponential with parameter $(1-\rho)\gamma$, then the approximation can be written as follows:

$$F(s) = P\{S(t) < s\} \approx 1 - \rho e^{-(1-\rho)\gamma s}, \qquad (17)$$

where γ is given by (15).

V. Numerical Example

We set semi-Markov matrix as follows:

$$\mathbf{A}(x) = \mathbf{P} \circ \mathbf{G}(x),$$

where $\mathbf{P}$ is the transition matrix of the embedded Markov chain $\xi(n)$ and $\mathbf{G}(x)$ is the matrix of conditional distributions of the process $\tau(n)$, operation $\circ$ is Hadamard product of matrices.

Matrix $\mathbf{P}$ is presented as follows:

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.05 \\ 0.8 & 0.2 \end{bmatrix}.$$

The elements of matrix $\mathbf{G}(x)$ are gamma distribution functions with shape parameters $\alpha_{11} = 0.001$, $\alpha_{12} = 0.05$, $\alpha_{21} = 0.1$, $\alpha_{22} = 1.5$ and scale parameter $\beta = 1$. We assume that the amount of information in one packet is deterministic and equals to $b_1 = 1.255$.

Figures 1-4 show the probability distribution of the amount of information in a single buffer of considered system obtained via simulation (solid line) compared with asymptotic results (dash line) for system load values $\rho = 0.5$, $\rho = 0.7$, $\rho = 0.9$ and $\rho = 0.95$.

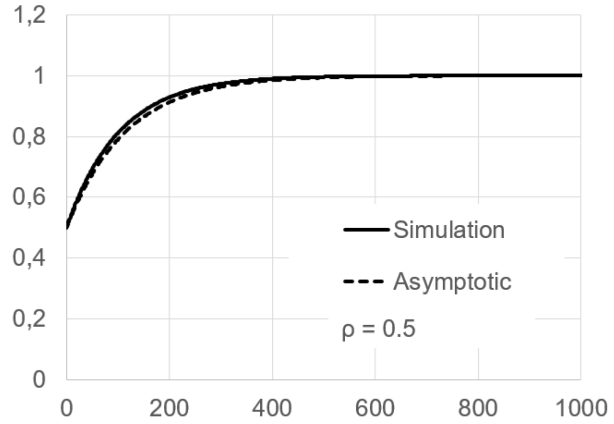

Fig. 1. The probability distribution of the amount of information in the buffer and its asymptotic approximation for system load $\rho = 0.5$

Table 1 contains the values of Kolmogorov distance $\Delta = \max\limits_{0 \le s < \infty} \left| F_{sim}(s) - F(s) \right|$ between empirical distribution function obtained via simulation $F_{sim}(s)$ and asymptotic cumulative distribution function $F(s)$ of the amount of information in the buffer given by (17).

TABLE I
Kolmogorov distance between empirical distribution function of the amount of information in the buffer and its asymptotic approximation

	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$	$\rho = 0.95$
Δ	0.0212	0.0188	0.0145	0.0141

As we can see, the asymptotic results accurately approximate prelimit distribution function of the amount of information in the buffer. The approximation is accurate enough even for small values of system load ρ.

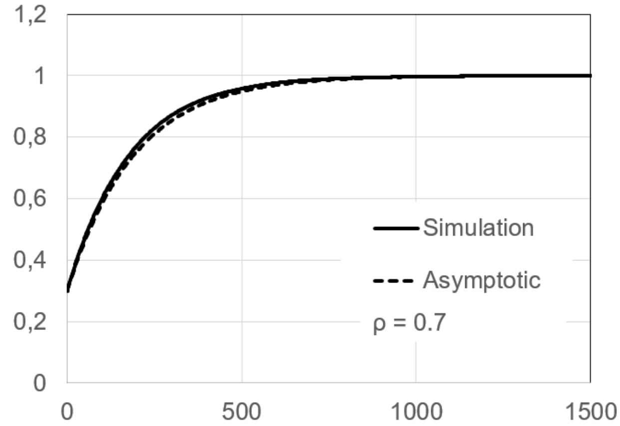

Fig. 2. The probability distribution of the amount of information in the buffer and its asymptotic approximation for system load $\rho = 0.7$

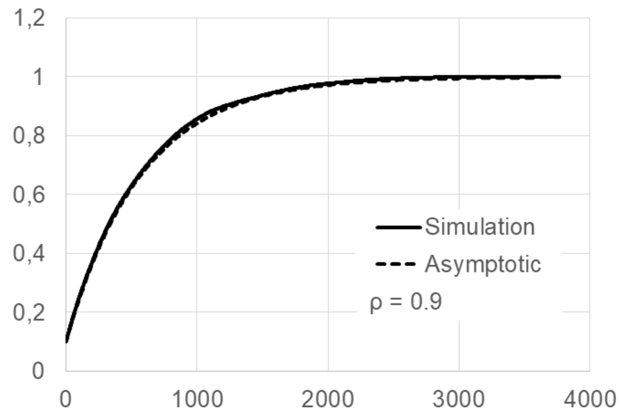

Fig. 3. The probability distribution of the amount of information in the buffer and its asymptotic approximation for system load $\rho = 0.9$

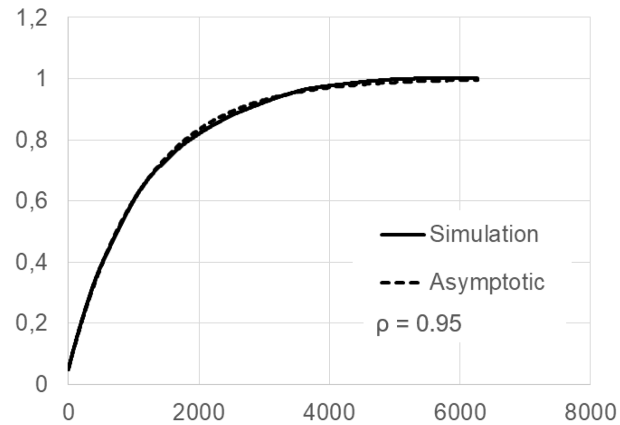

Fig. 4. The probability distribution of the amount of information in the buffer and its asymptotic approximation for system load $\rho = 0.95$

VI. Conclusion

We have considered the mathematical model of scheduler with multiple semi-Markov inputs, which perform in following mode: the scheduler allocates the bandwidth between buffers and sets to each of them the virtual communication channel. For the amount of information in each buffer of the system, we have obtained the limiting distribution under the limit condition of high load.

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