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Abstract

We prove the Boundary Harnack Principle related to fractional powers of Laplacian for some natural regions in the two-dimensional Sierpiński carpet. This is a natual application of some more general approach based on the Ikeda-Watanabe formula.

Résumé

Nous présentons le principe de Harnack à la frontière pour des puissances fractionaires du laplacien dans les domaines naturels du tapis de Sierpiński 2-dimensionel. C'est un exemple très naturel d'un argument plus général basé sur la formule d'Ikeda-Watanabe.

1 Introduction

Analysis on the Sierpiński carpet (and on a class of similar sets) has been developing for over ten years (see [BB1], [BB2] and references therein). Barlow and Bass showed numerous results including e.g. the construction of the analogue for the Brownian motion, the estimates of its transition densities (the heat kernel) and the Harnack inequality. It is natural to refer to the corresponding generator as to the Laplacian, even though this is not known whether this Brownian motion is unique or not. In this paper we deal with a fractional power of this Laplacian defined by means of subordination procedure (see below). For this operator we give a proof of the Boundary Harnack Principle for some natural regions in the fractal.

In [BSS] ([BSS1]) the Boundary Harnack Principle was established for *cells* in the Sierpiński gasket (or, more generally, simple nested fractals). The proof in that case

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resembled the one for intervals in the real line. In particular, the Boundary Harnack Principle was a consequence of the (elliptic) Harnack inequality. This simplification was due to the *finite ramification* property of the Sierpiński gasket, i.e the fact it can be disconnected by taking away a finite number of points. In particular, the boundary of some natural regions (e.g. small triangles) is always a set with a finite number of elements. Certainly, the method of [BSS] can not be carried out to infinitely ramified fractals, such as the Sierpiński carpet.

In what follows we were influenced by [B] which solves the problem in the case of Lipschitz domains in $\mathbb{R}^N$. Our contribution is a different methodology in proofs which can be described as follows. We have no analytic tools and no exact formula for the Poisson kernel of the ball which are used in [B] (cf. e.g. Lemma 3 or Lemma 12 in that paper). Also, a related proof in [SW] uses theory of smooth functions on $\mathbb{R}^N$. Our aim is to present a more general approach relied on the Ikeda-Watanabe formula. The Sierpiński carpet makes a natural opportunity for application of this argument. Certainly, the latter depends on the geometric issues, it seems, however, not to be restricted to this particular fractal.

2 Preliminaries

We consider the (unbounded) Sierpiński carpet F which is defined as follows. Let $F_0 = [0,1]^2$. Let A be the interior of the *middle square* of the relative size 1/3, i.e. $A = (1/3, 2/3)^2$. Set $F_1 = F_0 \setminus A$. Then F_1 consists of eight closed squares of side 1/3. To obtain F_2 we apply subsequently the above subtraction procedure to these squares in F_1, and so on. Set

$$F_\infty = \bigcap_{n=0}^{\infty} F_n, \qquad F = \bigcup_{n=0}^{\infty} 3^n F_\infty.$$

We call F the (unbounded) Sierpiński carpet.

By a *natural cell* (or simply *cell*) we mean the intersection of F with a square of the form $[k3^{-n}, (k+1)3^{-n}] \times [m3^{-n}, (m+1)3^{-n}]$, $k, m, n \in \mathbb{N}$. The family of cells with sides 3^{-n} is denoted by $\mathcal{S}_n$.

In what follows D always denotes a *region* in F i.e. the interior of a sum of finite number of natural cells. Since a cell can be viewed as an union of cells of smaller size, we may and do assume that D consists of cells which have the same size and disjoint interiors. In other words, there exist $n_0, m_0 \in \mathbb{N}$, and $S_i \in \mathcal{S}_{m_0}$, $i = 1, 2, ..., n_0$ such that

$$D = \operatorname{int}\left(\bigcup_{i=1}^{n_0} S_i\right). \tag{1}$$

Note that the interior is taken with respect to the topology of F (inherited from $\mathbb{R}^2$) and since S_i are closed, any two adjacent cells always make a connected set. Moreover, the distance between any two disjoint cells in D is at least $R_1 = R_1(D) > 0$. Let $R_2 = 3^{-m_0}$ (i. e. R_2 is the side of cells in D). Set $R_0 = (1/3)\min(R_1, R_2)$, the number that describes *Lipschitz character* of D.

Notation and conventions. For $x \in F$ and $D \subseteq F$ we denote $\delta(x) = \operatorname{dist}(x, \partial D)$. For $A \subseteq F$ we write $A^c = F \setminus A$. By $B(x, r)$ we denote the Euclidean ball (with the center $x \in F$ and the radius $r > 0$) intersected with F. For $x, y \in F$, $|x - y|$ always means the Euclidean distance. Let $d = \dim(F)$ be the Hausdorff dimension of F. By μ we denote the d-dimensional Hausdorff measure restricted to F. In the sequel c (without subscripts) denotes a generic constant that depends only on F and α (see below) and may change its value from one instance to another. Constants are numbered consecutively within each proof. We write $f(x) \asymp g(x), \quad x \in F$, to indicate that there are constants $c_1, c_2 > 0$ (independent of x) such that $c_1 f(x) \le g(x) \le c_2 f(x)$ for all $x \in F$.

To introduce the fractional power of the Laplacian in our framework, we shortly recall the definition of the *α-stable process* from [S] (cf. also [K1],[FJ]). Let $q(u, x, y)$, $u > 0$, $x, y \in F$, denote transition density (with respect to μ) of the *fractional diffusion* ([Ba], [BB1]) on F. Set $\alpha \in (0, 2)$ and let $\eta_t(\cdot)$, $t > 0$, be a function on $\mathbb{R}^+$ characterized by its Laplace transform $\mathcal{L}(\eta_t(\cdot))(\lambda) = \exp(-t\lambda^{\alpha/2})$. (see [Be] or [BG] for more details and a probabilistic interpretation). For $t > 0$ and $x, y \in F$ we define

$$p(t, x, y) = \int_0^\infty q(u, x, y)\eta_t(u)du.$$

By the general theory $p(t, x, y)$ is a transition density of a Markov process called the subordinate process (see [BG, p. 18]), which we denote by $(X_t)_{t>0}$ and call *α-stable.* Its generator may be naturally labelled as the $\Delta^{\alpha/2}$.

To simplify the notation, for the rest of the paper we let $d_\alpha = d + \alpha d_w/2$, where d_w is, in general, a constant characteristic for the fractal. For the Sierpiński carpet $d_w \approx 2.097$.

For a Borel set $B \subseteq F$ we define *exit time* $\tau_B = \inf\{t \ge 0 : \quad X_t \notin B\}$. Let u be a Borel measurable function u on F, which is bounded from below (above). We say that u is *α-harmonic* in an open set $U \subseteq F$ if

$$u(x) = E^x u(X(\tau_B)), \qquad x \in B,$$

for every bounded open set B with the closure $\bar{B}$ contained in U. We say that u is *regular α-harmonic* in U if

$$u(x) = E^x u(X(\tau_U)), \qquad x \in U.$$

For a Borel subset $\Omega \subseteq F$ denote by ω_Ω^x the harmonic measure, i. e. $\omega_\Omega^x(E) = P^x[X_{\tau_\Omega} \in E]$.

We say that $\Omega \subseteq F$ has *the outer fatness property* (cf. [BSS]) if there are constants $c_1 = c_1(\Omega)$ and $r_0 = r_0(\Omega)$ such that

$$\mu(\Omega^c \cap B(x, r)) \ge c_1 r^d, \qquad x \in \partial\Omega,\ r \in (0, r_0). \tag{2}$$

We say that Ω has *the inner fatness property* if there exist constants $\theta = \theta(\Omega) \in (0, 1)$ and $r_0 = r_0(\Omega)$ such that for every $r \in (0, r_0)$ and $Q \in \partial\Omega$ there is a point $A = A_r(Q) \in \Omega \cap B(Q, r)$ such that

$$B(A, \theta r) \subseteq \Omega \cap B(Q, r). \tag{3}$$

Remark. Observe that (2) and (3) holds for a region D. It follows that the Carleson estimate given in Proposition 8.5 of [BSS] applies. For the sake of convenience we state it below (Lemma 2.1). Note that if D is a cell of size 3^{-k} (or a finite union of them) then it satisfies (3) with $r_0 = r_0(k)$ and θ which is an absolute constant, e. g. $\theta = 1/9$. We will use this fact without further mention dropping from the notation the dependence on θ.

Lemma 2.1. *Assume $\alpha < 2/d_w$. Let $\Omega \subseteq F$ be a set satisfying (3). There exist a constant $c_1 = c_1(\theta)$ such that for all $Q \in \partial\Omega$ and $r \in (0, r_0/2)$, and functions $u \ge 0$, regular α-harmonic in $\Omega \cap B(Q, 2r)$ and satisfying $u(x) = 0$ on $\Omega^c \cap B(Q, 2r)$, we have*

$$u(x) \le c_1 u(A), \qquad x \in \Omega \cap B(Q, r), \tag{4}$$

where A is given in (3)

It can be seen from the proof in [BSS] (cf. also [B, (3.29)]) that (4) holds for $x \in \Omega \cap B(Q, 5r/4)$, i.e. we have

$$u(x) \le c_1 u(A), \qquad x \in \Omega \cap B(Q, \frac{5r}{4}). \tag{5}$$

This fact will be invoked later.

Finally, we include the following remark which is due to Prof. Takashi Kumagai [K2]. The Harnack inequality that we apply here was proved in [BSS] for $\alpha \in (0, 2/d_w) \cup (2d/d_w, 2)$. However, observe that once we have transition density estimates ([BSS, Theorem 3.1]) then it is relatively easy to deduce the *tightness*, i.e. Proposition 4.1 of [CK] for all $\alpha \in (0,2)$. Actually, this result is contained in [BSS, Lemma 4.3] (note a different conventions: α in [CK] means $\alpha d_w/2$ from [BSS]). Using this and [CK, Lemma 4.7] one verifies Lemmas 4.9 - 4.13 of [CK]. Consequently, we can repeat the proof of the parabolic Harnack inequality [CK, Proposition 4.3]. This in turn gives our (elliptic) Harnack inequality for all $\alpha \in (0, 2)$.

Unfortunately, in the present paper we have to assume even stronger restrictions on α (see Lemma 3.4). However, we believe the restrictions are of the technical nature and once we have the Harnack inequality for $\alpha \in (0, 2)$, the boundary Harnack Principle holds for the same range of α.

3 Boundary Harnack Principle

The main result can be stated as follows.

Theorem 3.1 (Boundary Harnack Principle). *Let $\alpha < 2(d-1)/d_w$. Suppose that D is a region, $Q \in \partial D$ and $r \in (0, R_0/2)$. Then for any functions $u, v \ge 0$, positive regular α-harmonic in $D \cap B(Q, 2r)$ and with value 0 in $D^c \cap B(Q, 2r)$, and satisfying $u(A_r(Q)) = v(A_r(Q))$ we have*

$$c_o^{-1} v(x) \le u(x) \le c_o v(x), \qquad x \in D \cap B(Q, r/27),$$

where $c_o = c_o(D)$.

We start the proof by stating some lemmas. Their assertions have analogues in [B]. However, there are essential changes in the argument. This is required at least for a key step of comparison of the harmonic measure and the Green function for a region (Lemma 3.4). Moreover, the proofs we provide are more elementary in the sense they rely on basic properties of the process. In particular, we make use of Ikeda-Watanabe formula and the transition densities estimates (Proposition 6.1 and Theorem 3.1 in [BSS]). The price we pay at the moment is the restriction on α (see Lemma 3.4).

Lemma 3.2. *There exist $c_0 > 0$ such that for any D, all $Q \in \partial D$ and $r \in (0, R_0)$ we have*

$$\omega_D^x(B(Q, r)) \ge c_0, \qquad x \in B(Q, r) \cap D.$$

Proof. Fix $x \in B(Q, r) \cap D$. Recall that $y \to P_D(x, y)$ is the Poisson kernel for a region

D, i.e. the density of $\omega^x(\cdot)$. By [BSS, Proposition 6.4] and (2) we get

$$\begin{aligned}
\omega_D^x(B(Q,r)) &\geq P^x[X_{\tau_{B(x,\delta(x)/2)}} \in B(Q,r)\cap D^c] \\
&\geq \int_{B(Q,\delta(x))\cap D^c} P_{B(x,\delta(x)/2)}(x,y)d\mu(y) \\
&\geq c\delta(x)^{\alpha d_w/2}\int_{B(Q,\delta(x))\cap D^c} |x-y|^{-d_\alpha}d\mu(y) \\
&\geq c\delta(x)^{\alpha d_w/2}(2\delta(x))^{-d_\alpha}\mu(B(Q,\delta(x))\cap D^c) \\
&\geq c_0,
\end{aligned}$$

which completes the proof. □

Recall that for a region D, (2) and (3) hold with some constants R_0 and θ.

Lemma 3.3. *Let $\alpha < 2d/d_w$. There exists a constant c_1 such that for any region D, all $Q \in \partial D$, $r \in (0, R_0)$ and $x \in D \setminus B(Q,r)$ we have*

$$r^{d-\alpha d_w/2}G_D(x, A_{r/2}(Q)) \leq c_1\omega_D^x(B(Q,r)).$$

Proof. First we show

$$\omega_D^x(B(Q,r)) \geq cP^x[T_{B_y} < \tau_D], \tag{6}$$

where $y = A_{r/2}(Q)$ and $B_y = B(y, \frac{\theta r}{4})$. For $x \in D$ we have

$$\begin{aligned}
\omega_D^x(B(Q,r)) &\geq E^x[\mathbf{1}_{B(Q,r)}(X_{\tau_D}); T_{B_y} < \tau_D] \\
&= E^x[E^{X(T_{B_y})}[\mathbf{1}_{B(Q,r)}(X_{\tau_D})]; T_{B_y} < \tau_D] \\
&\geq \inf_{w\in B_y} E^w\mathbf{1}_{B(Q,r)}(X_{\tau_D})P^x[T_{B_y} < \tau_D] \\
&\geq \inf_{w\in B(Q,r)} \omega_D^w(B(Q,r))P^x[T_{B_y} < \tau_D] \\
&\geq c_0P^x[T_{B_y} < \tau_D],
\end{aligned}$$

where c_0 comes from Lemma 3.2.

Now fix $x \in D \setminus B(Q,r)$. We claim that there exist c_2 such that

$$c_2G_D(x,y)\delta(y)^{d-\alpha d_w/2} \leq P^x[T_{B_y} < \tau_D]. \tag{7}$$

To prove our claim observe that $G_D(x,\cdot)$ is α-harmonic on $D \setminus \{x\}$ (for $\alpha \neq 2d/d_w$, see e. g. [BSS]). Note that $B(y,\delta(y)) \subseteq B(y,r/2) \subseteq B(Q,r)$. Hence $x \notin B(y,\delta(y))$ and $\overline{B}(y,\delta(y)) \subseteq D \setminus \{x\}$. By the Harnack inequality for the ball $B(y,\delta(y))$ we get

$$c_3^{-1}G_D(x,z) \leq G_D(x,y) \leq c_3G_D(x,z), \qquad z \in B(y,\delta(y)/2). \tag{8}$$

Since $\theta r/2 < \delta(y)$ we have $B_y \subseteq B(y,\delta(y)/2)$ and hence, by (8) and the strong Markov property,

$$\begin{aligned}
G_D(x,y)\delta(y)^d &\le c\theta^{-d}G_D(x,y)\mu(B_y)\\
&\le c\int_{B_y} G_D(x,z)d\mu(z)\\
&= cG_D\mathbf{1}_{B_y}(x)\\
&= cE^x\left[\int_0^{\tau_D}\mathbf{1}_{B_y}(X_s)ds;\, T_{B_y}<\tau_D\right]\\
&= cE^x[E^{X(T_{B_y})}\left[\int_0^{\tau_D}\mathbf{1}_{B_y}(X_s)ds\right];\, T_{B_y}<\tau_D]\\
&\le cP^x[T_{B_y}<\tau_D]\sup_{w\in B_y}E^w[\int_0^{\tau_D}\mathbf{1}_{B_y}(X_s)ds].
\end{aligned}$$

It is easy to see that for $w\in B(y,s)$ we have

$$\int_{B(y,s)}\frac{d\mu(z)}{|w-z|^{d-\alpha d_w/2}}\le\int_{B(w,2s)}\frac{d\mu(z)}{|w-z|^{d-\alpha d_w/2}}\le cs^{\alpha d_w/2},\qquad s>0,$$

cf. [BSS, Lemma 2.1]. It follows that for $w\in B_y$ we have

$$\begin{aligned}
E^w\int_0^{\tau_D}\mathbf{1}_{B_y}(X_s)ds &\le \int_0^\infty E^w\mathbf{1}_{B_y}(X_s)ds\\
&= \int_{B_y}\int_0^\infty p(s,w,v)ds\,d\mu(v)\\
&\le c\int_{B_y}\frac{d\mu(v)}{|v-w|^{d-\alpha d_w/2}}\\
&\le c\left(\frac{\theta\delta(y)}{4}\right)^{\alpha d_w/2},
\end{aligned}$$

where the last but one inequality is justified by [BSS, Lemma 5.3]. Note that this is the only place where we used $\alpha<d_s$. The claim follows.

Since $\theta r/2\le\delta(y)\le r/2$ (i.e. $\delta(y)\asymp r$), (6) and (7) imply the assertion of the lemma. $\square$

Lemma 3.4. *If $\alpha<2(d-1)/d_w$ then there exists a constant c_1 such that for any D, all $Q\in\partial D$ and $r\in(0,R_0/2)$ we have*

$$\omega_D^x(B(Q,r))\le c_1r^{d-\alpha d_w/2}G_D(x,A_{r/2}(Q)),\qquad x\in D\setminus B(Q,2r).$$

Proof. Fix $x\in D\setminus B(Q,2r)$. It can be observed that the harmonic measure does not charge ∂D. Indeed, it is enough to adapt Lemma 6 of [B] with *outer cone property* replaced by (2). For the sake of reader's convenience we sketch the argument. Denote $\tau_x=\tau_{B(x,\delta(x)/3)}$. Then, by the strong Markov property,

$$\omega_D^x(\partial D)=P^x[X_{\tau_x}\in\partial D]+E^x[\omega_D^{X_{\tau_x}};\,X_{\tau_x}\in D]=:p_0(x)+r_0(x).$$

Define inductively

$$p_{k+1}(x)=E^x[p_k(X_{\tau_x});\,X_{\tau_x}\in D],$$

$$r_{k+1}(x) = E^x[r_k(X_{\tau_x});\, X_{\tau_x} \in D].$$

Then $r_k = p_{k+1} + r_{k+1}$, $k = 0, 1, .., $ and

$$\omega_D^x(\partial D) = p_0(x) + p_1(x) + ... + p_k(x) + r_k(x), \quad x \in D,\, k = 0, 1, ... \tag{9}$$

Let $x_0 \in \partial D$ be such that $|x_0 - x| = \delta(x)$. By [BSS, Proposition 6.4] and (2) we get

$$\begin{aligned} P^x[X_{\tau_x} \in D^c] &\geq P^x[X_{\tau_x} \in B(x_0, \delta(x)) \cap D^c] \\ &\geq c\delta(x)^{\alpha d_w/2} \int_{B(x_0,\delta(x))\cap D^c} \frac{d\mu(y)}{|x-y|^{d_\alpha}} \\ &\geq \frac{c\delta(x)^{\alpha d_w/2}}{(2\delta(x))^{d_\alpha}} \mu(B(x_0, \delta(x)) \cap D^c) \\ &\geq c_0, \end{aligned}$$

for each $x \in D$. Consequently,

$$\sup_{x\in D} r_{k+1}(x) \leq (1 - c_0) \sup_{x\in D} r_k(x) \leq (1 - c_0)^{k+1} \longrightarrow 0, \quad k \to \infty.$$

From (9) it follows that

$$\omega_D^x(\partial D) = \sum_{k=0}^{\infty} p_k(x).$$

Since μ does not charge ∂D we immediately get $p_k(x) = 0$, $x \in D$, $k = 0, 1, ..$ (see also the remark after Corollary 6.2 in [BSS]). This gives our claim.

Now, since $\omega_D^x(\partial D) = 0$, from the Ikeda-Watanabe formula (see also [BSS, (51)]) we have

$$\begin{aligned} \omega_D^x(B(Q,r)) &= \int_{B(Q,r)\cap D^c} P_D(x,y) d\mu(y) \\ &\asymp \int_{B(Q,r)\cap D^c} \int_D \frac{G_D(x,z)}{|z-y|^{d_\alpha}} d\mu(z) d\mu(y) \\ &= \left(\int_{D\setminus B(Q,\frac{5r}{4})} + \int_{D\cap B(Q,\frac{5r}{4})} \right) \left[G_D(x,z) \int_{B(Q,r)\cap D^c} \frac{d\mu(y)}{|z-y|^{d_\alpha}} \right] d\mu(z) \\ &= J_1 + J_2. \end{aligned}$$

First we deal with the integral J_1. Let $A_0 = A_{r/2}(Q)$. Then we have $|z - y| \geq r/4$ and so $|z - A_0| \leq |z - y| + |y - A_0| \leq |z - y| + (3/2)r \leq |z - y| + 6|z - y| = 7|z - y|$. It follows that

$$\int_{B(Q,r)\cap D^c} \frac{d\mu(y)}{|z-y|^{d_\alpha}} \leq \frac{c}{|z-A_0|^{d_\alpha}} \mu(B(Q,r)) \asymp \frac{cr^d}{|z-A_0|^{d_\alpha}}$$

and

$$J_1 \leq cr^d \int_{D\setminus B(Q,5r/4)} \frac{G_D(x,z)}{|z-A_0|^{d_\alpha}} d\mu(z). \tag{10}$$

Denote $B_0 = B(A_0, \theta r/2)$. For the Poisson kernel of the ball B_0 by [BSS, Proposition 6.4] we have

$$P_{B_0}(A_0, z) \geq c\frac{(\theta r/2)^{\alpha d_w/2}}{|z-A_0|^{d_\alpha}}, \quad z \in B_0^c.$$

By rearranging and putting this into (10) we obtain

$$J_1 \le c r^{d-\alpha d_w/2} \int_{B_0^c} P_{B_0}(A_0, z) G_D(x,z) d\mu(z).$$

Since $z \to G_D(x,z)$ is regular α-harmonic on B_0, the last integral does not exceed $G_D(x, A_0)$. Remark that the integral is not necessarily equal to $G_D(x, A_0)$, since we do not know whether the process hits the boundary of B_0; however, we do not need this fact and the equality. Finally,

$$J_1 \le c r^{d-\alpha d_w/2} G_D(x, A_{r/2}(Q)), \tag{11}$$

as desired.

To deal with the integral J_2 observe that

$$\int_{B(Q,r)\cap D^c} \frac{d\mu(y)}{|z-y|^{d_\alpha}} \le \int_{B(z,\delta(z))^c} \frac{d\mu(y)}{|z-y|^{d_\alpha}} \le c\delta(z)^{-\alpha d_w/2},$$

where the last inequality is justified by Lemma 2.1 of [BSS]. Since $z \mapsto G_D(x,z)$ is regular α-harmonic on $D \cap B(Q, 2r)$, from (5) it follows that

$$\begin{aligned} J_2 &\le c \int_{D\cap B(Q,5r/4)} G_D(x,z)\delta(z)^{-\alpha d_w/2} d\mu(z) \\ &\le c G_D(x, A_{5r/4}(Q)) \int_{D\cap B(Q,5r/4)} \delta(z)^{-\alpha d_w/2} d\mu(z). \end{aligned} \tag{12}$$

We have $|A_{5r/4} - A_0| \le |A_{5r/4} - Q| + |Q - A_0| \le 5r/4 + r/2 \le c(\theta r/2)$. By [BSS, Lemma 7.6] with $x_1 = A_{5r/4}$ and $x_2 = A_0 = A_{r/2}(Q)$ we obtain

$$G_D(x, A_{5r/4}(Q)) \le c G_D(x, A_{r/2}(Q)). \tag{13}$$

Now, it is enough to estimate

$$\int_{D\cap B(Q,5r/4)} \delta(z)^{-\alpha d_w/2} d\mu(z).$$

Let $k_o \in \mathbb{N}$ be such that $3^{-k_o-1} < 5r/4 \le 3^{-k_o}$. Then, clearly, $r \asymp 3^{-k_o}$. Let H_0 be the union of cells S that satisfy

(a) $S \in \mathcal{S}_{k_o}$,

(b) $S \subseteq \overline{D}$,

(c) $\partial S \cap \partial D \neq \emptyset$,

(d) $S \cap B(Q, 5r/4) \neq \emptyset$.

In other words H_0 is a covering of $D \cap B(Q, 5r/4)$ by *smallest* cells adjacent to ∂D. Define H_k, $k = 1, 2, ...$, in the same way as H_0 but with (a) replaced by $S \in \mathcal{S}_{k_o+k}$ and (d) replaced by $S \subseteq H_0$. Thus, H_k is a layer of cells of side 3^{-k-k_o} adjacent to $\partial D \cap \partial H_0$. Then, there is at most $h_k = 2.3^k + 1$ cells in H_k, $k = 1, 2, ...$ (this may happen when H_0

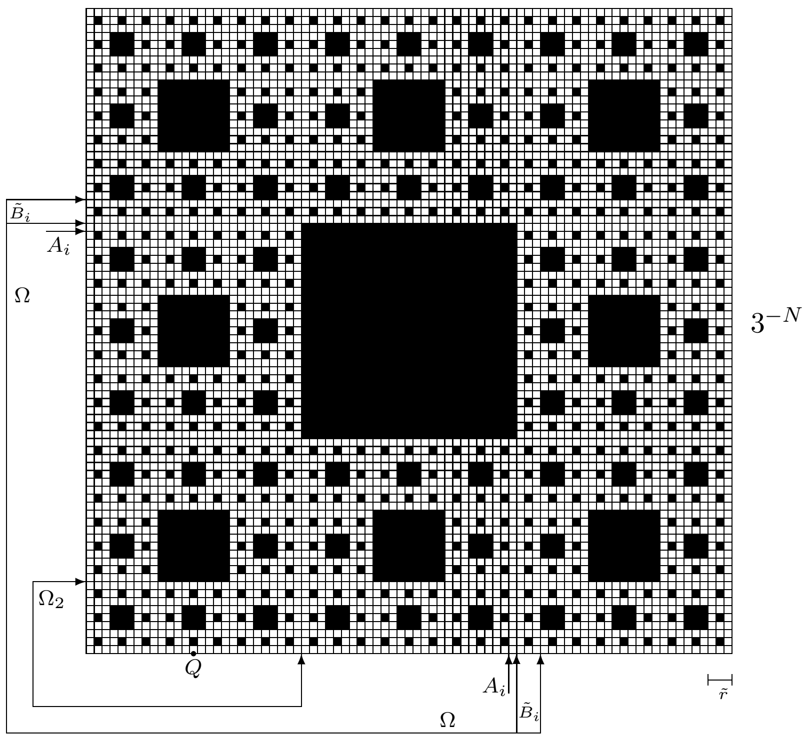
consists of three cells, i.e. $Q \in \delta D$ is a corner point). Let $R_k = H_k \setminus H_{k+1}$. Then $z \in R_k$ implies $\delta(z) \geq 3^{-(k_o+k+1)} \geq cr3^{-k}$. It follows that

$$\begin{aligned}
\int_{D\cap B(Q,5r/4)} \delta(z)^{-\alpha d_w/2} d\mu(z) &\leq \sum_{k=0}^{\infty} \int_{R_k} \delta(z)^{-\alpha d_w/2} d\mu(z) \qquad (14)\\
&\leq c\sum_{k=0}^{\infty} (3^{-k}r)^{-\alpha d_w/2} \mu(R_k) \\
&\leq cr^{-\alpha d_w/2} \sum_{k=0}^{\infty} 3^{k\alpha d_w/2}(3^{-k}r)^d h_k \\
&\leq cr^{d-\alpha d_w/2} \sum_{k=0}^{\infty} 3^{k(\alpha d_w/2-d+1)} \\
&\leq cr^{d-\alpha d_w/2},
\end{aligned}$$

provided $\alpha < 2(d-1)/d_w$. Combining (11), (12), (13) and (14) we get the assertion. □

Remark. In our particular case $2(d-1)/d_w \approx 0.851$.

Proof of Theorem 3.1. This is based on a general idea of the proof of Lemma 13 from [B]. Since the context is different, we present a version adapted to our needs. The argument goes the following way. First, we introduce the basic geometrical objects and notations. Then, the first step of the proof is to establish the comparability of the harmonic measures of the region Δ and of its propper subset B_1 (see below). This is given in (16) which is a key ingredient in the proof. Then we decompose the functions to be compared into two parts (17). In Steps 2 and 3 we prove the inequality for each of these parts: (19) and (24) respectively. Step 2 is the crucial one and it uses (16); Step 3 is covered by the Poisson kernel estimates and the (usual) Harnack inequality.



Let $N \in \mathbb{N}$ be such that $3^{-N} \le r < 3^{-N+1}$. For $Q \in \partial D$ let $S_i^\nu(Q)$, $i = 1, 2$, be cells from $\mathcal{S}_{N+i}$ such that $Q \in S_i^\nu(Q) \subseteq D$. There can be one, two or three such cells indexed by ν. Define

$$\Omega_i = \operatorname{int}\left(\bigcup_\nu S_i^\nu(Q)\right), \qquad i = 1, 2.$$

If the union above consists of the single $S_i^1(Q)$ then we set

$$\Omega_i = \operatorname{int}\left(S_i^1(Q) \cup \bigcup_{\nu=1}^{2} N_i^\nu(Q)\right), \qquad i = 1, 2.$$

where N_i^ν are the *neighbours* of $S_i^1(Q)$, i.e. cells satisfying

(i) $N_i^\nu \in \mathcal{S}_{N+i}$ and $N_i^\nu \subseteq \overline{D}$,

(ii) $\partial N_i^\nu \cap \partial D \cap \partial S_i^1(Q) \neq \emptyset$ (recall that cells are closed).

Finally, denote $\Omega = \Omega_1$.

Set $\tilde r = 3^{-N-3}$ and let $A \in \Omega$ be a point such that $\operatorname{dist}(A, D^c) = 3\tilde r$ and $\operatorname{dist}(A, \Omega_2) = \tilde r$ (clearly, A is not unique).

Remark. In the course of the proof it is convenient to identify A with $A_r(Q)$ from the hypothesis of our theorem. Note that there is no loss of generality; indeed, by [BSS, Lemma 7.6] we have $u(A_r(Q)) \asymp u(\tilde A_r(Q))$ for any harmonic function u satisfying hypothesis of Theorem 3.1 and points $A_r(Q)$, $\tilde A_r(Q)$ of the inner fatness property. Actually, this is the reason we can use our our definition of A and $A_r(Q)$ without determining uniquely the points.

Let $\tilde B_i \in \mathcal{S}_{N+3}$, $i = 1, 2, .., n_0(\Omega)$, are cells satisfying $\tilde B_i \subseteq \overline{D} \cap \Omega^c$ and $\partial \tilde B_i \cap \partial\Omega \neq \emptyset$. Since $18 \le n_0(\Omega) \le 54$, we drop the dependence n_0 on Ω without further mention. Set $\tilde B_1$ to be one of $\tilde B_i$ satisfying additionally $\operatorname{dist}(\tilde B_1, \partial D) \ge 8\tilde r$. Let S_i be the mid-point of the line segment $\partial\Omega \cap \partial \tilde B_i$; if the set consists of one point $\{x_o\}$ then let $S_i = x_o$(a *vertex point*). Let $B_i = B(S_i, \tilde r\sqrt{2})$ and

$$\Delta = \bigcup_i B_i \cap D \cap \Omega^c.$$

Let $A_i \in \Omega$, $i = 1, 2, ..n_0$, be the point such that $|A_i - S_i| = \operatorname{dist}(A_i, \delta(\Omega)) = \tilde r/3$, provided S_i is not a vertex point of Ω, and $|A_i - S_i| = \tilde r\sqrt{2}/3$ in the opposite case. $\operatorname{dist}(A_i, \delta(\Omega)) = \tilde r/3$. Since $\operatorname{dist}(\tilde B_1, \partial D) \ge 8\tilde r$ then there exists a cell, denoted by T, such that $T \in \mathcal{S}_{N+4}$, $T \subseteq D \setminus (\Omega \cup \Delta)$, $\operatorname{dist}(T, D^c) \ge 8\tilde r$ and $\operatorname{dist}(T, B_1) \le \tilde r$.

Step 1. Let $\theta = 1/9$. Then if $x \in B(A_i, \theta\tilde r\sqrt{2}/2)$ then $|x - S_i| \le |x - A_i| + |A_i - S_i| \le \tilde r\sqrt{2}/18 + \tilde r\sqrt{2}/3 \le \tilde r\sqrt{2}/2$, which yields $B(A_i, \theta\tilde r\sqrt{2}/2) \subseteq \Omega \cap B(S_i, \tilde r\sqrt{2}/2)$. In other words, A_i can be regarded as $A_{\tilde r\sqrt{2}/2}(S_i)$ in the inner fatness property (3) for Ω. It follows that by Lemmas 3.3 and 3.4 applied to Ω and B_i we get

$$(\tilde r\sqrt{2})^{d-\alpha d_w/2} G_\Omega(z, A_i) \asymp \omega_\Omega^z(B_i), \qquad z \in \Omega \setminus B(S_i, 2\tilde r\sqrt{2}).$$

For the rest of the proof fix $x \in \Omega_2$. Then $|x - S_i| \ge 6\tilde r$, $i = 1, 2, ..., n_0$, and hence

$$\tilde r^{d-\alpha d_w/2} G_\Omega(x, A_i) \asymp \omega_\Omega^x(B_i).$$

Recall $\mathrm{dist}(A, D^c) = 3\tilde{r}$. Since $\mathrm{dist}(A_i, \partial\Omega) = \tilde{r}/3$, $|A_i - A| \leq \mathrm{diam}(\Omega) \leq c(\tilde{r}/3)$ and $G_\Omega(x, \cdot)$ is regular α-harmonic in $B(A_i, \tilde{r}/3) \cup B(A, \tilde{r}/3)$, by Harnack inequality ([BSS, Lemma 7.6]) we obtain

$$G_\Omega(x, A_i) \asymp G_\Omega(x, A). \tag{15}$$

It follows that

$$\begin{aligned} \omega_\Omega^x(\Delta) &\leq \textstyle\sum_{i=1}^{n_0} \omega_\Omega^x(B_i) \\ &\asymp \tilde{r}^{d-\alpha d_w/2} \textstyle\sum_{i=0}^{n_0} G_\Omega(x, A_i) \\ &\asymp \tilde{r}^{d-\alpha d_w/2} G_\Omega(x, A_1) \\ &\asymp \omega_\Omega^x(B_1). \end{aligned} \tag{16}$$

STEP 2. Let u_1, u_2 be functions such that

$$u_1(y) = \begin{cases} u(y), & y \in \Delta, \\ 0, & y \in \Omega^c \setminus \Delta, \end{cases} \qquad u_2(y) = \begin{cases} 0, & y \in \Delta \\ u(y), & y \in \Omega^c \setminus \Delta, \end{cases} \tag{17}$$

and u_1 and u_2 are regular α-harmonic in Ω. Note that $u_1, u_2 \geq 0$ and $u_1 + u_2 = u$. Analogously we define v_1 and v_2.

By (4) and (16) we obtain

$$\begin{aligned} u_1(x) &= E^x[u(X_{\tau_\Omega});\ X_{\tau_\Omega} \in \Delta] \\ &\leq \sup\{u(z);\ z \in \Delta\}\omega_\Omega^x(\Delta) \\ &\leq cu(A)\omega_\Omega^x(\Delta) \\ &\leq cu(A)\omega_\Omega^x(B_1). \end{aligned} \tag{18}$$

Since $\mathrm{dist}(A \cup B_1, \partial D) \geq \tilde{r}$ and for $y \in B_1$ we have $\mathrm{dist}(A, y) \leq \mathrm{diam}(\Omega) + \mathrm{diam}(B_1) \leq c\tilde{r}$, from [BSS, Lemma 7.6] it follows that

$$v_1(y) = v(y) \geq cv(A), \qquad y \in B_1.$$

Consequently, we have

$$\begin{aligned} v_1(x) &= E^x[v(X_{\tau_\Omega});\ X_{\tau_\Omega} \in \Delta] \\ &\geq E^x[v(X_{\tau_\Omega});\ X_{\tau_\Omega} \in B_1] \\ &\geq cv(A)\omega_\Omega^x(B_1). \end{aligned}$$

Combinig this and (18) we get

$$u_1(x) \leq cv_1(x) \leq cv(x). \tag{19}$$

STEP 3. Now, let $K = \Omega \cup \Delta \cup (D^c \cap B(Q, 2r))$. Clearly, $\bigcup_i \tilde{B}_i \subseteq \Delta$. So if $z \in D \setminus (\Omega \cup \Delta)$ then $\mathrm{dist}(z, \Omega) \geq \tilde{r}$. Hence, for $z \in \Omega$ and $y \in K^c$ we have $|y - z| \asymp |y - Q|$. Therefore, by the Ikeda-Watanabe formula

$$\begin{aligned} u_2(x) &= \int_{K^c} P_\Omega(x, y) u(y) d\mu(y) \\ &\asymp \int_{K^c} \left(\int_\Omega G_\Omega(x, z) |z - y|^{-d_\alpha} d\mu(z) \right) u(y) d\mu(y) \\ &\asymp \int_{K^c} \left(\int_\Omega G_\Omega(x, z) d\mu(z) \right) u(y) |y - Q|^{-d_\alpha} d\mu(y) \\ &= E^x \tau_\Omega \int_{K^c} u(y) |y - Q|^{-d_\alpha} d\mu(y) \end{aligned}$$

From this and the analogous relation for v_2 it follows that

$$u_2(x)/u_2(A) \asymp E^x\tau_\Omega/E^A\tau_\Omega \asymp v_2(x)/v_2(A). \tag{20}$$

We claim that

$$v_2(A) \geq cv(A). \tag{21}$$

Indeed, recall that $T \cap \Delta = \emptyset$ and we have

$$v_2(A) \geq E^A[v(X_{\tau_\Omega});\ X_{\tau_\Omega} \in T] \geq \inf_{z\in T} v(z)\omega_\Omega^A(T). \tag{22}$$

Since $\mathrm{dist}(A\cup T, \partial D) \geq 3\tilde{r}$ and $\mathrm{dist}(A,T) \leq c\tilde{r}$, by the Harnack inequality we have

$$v(z) \asymp v(A), \qquad z \in T. \tag{23}$$

Moreover, $\mathrm{diam}(\Omega) \asymp \mathrm{diam}(T) \asymp \mathrm{dist}(\Omega, T) \asymp \tilde{r}$ yields $|y-z| \asymp \tilde{r}$, $y \in \Omega$, $z \in T$. Hence, by [BSS, Proposition 4.4]

$$\begin{aligned}
\omega_\Omega^A(T) &\asymp \int_T\int_\Omega \frac{G_\Omega(A,y)}{|y-z|^{d_\alpha}} d\mu(y)d\mu(z) \\
&\asymp \tilde{r}^{-d_\alpha}\int_T\int_\Omega G_\Omega(A,y)d\mu(y)d\mu(z) \\
&= \mu(T)\tilde{r}^{-d_\alpha}E^A\tau_\Omega \\
&\geq c\tilde{r}^{-\alpha d_w/2}E^A\tau_{B(A,\tilde{r})} = c_1,
\end{aligned}$$

where c_1 is independent of Ω, T, r, etc. Putting this and (23) into (22) we get our claim.

Denote the last quotient in (20) by q_o. Then, by (20), definition of u_2, the assumption $u(A) = v(A)$ and (21),

$$\begin{aligned}
u_2(x) \leq cq_ou_2(A) \leq cq_ou(A) = cq_ov(A) \\
\leq cq_ov_2(A) = cv_2(x) \qquad\qquad x \in \Omega_2.
\end{aligned} \tag{24}$$

Together with (19) and the symmetry this ends the proof. □

Remark. Although the proof relies on particular geometric properties of the Sierpiński carpet, we believe that this argument can be carried out to a slightly wider context, e.g. to generalized Sierpiński carpets.

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