Approximate X-rays reconstruction of special lattice sets

Sara Brunetti, Alain Daurat, Alberto Del Lungo

▶ To cite this version:

HAL Id: hal-00023034

https://hal.archives-ouvertes.fr/hal-00023034

Submitted on 19 Apr 2006

Approximate X-rays reconstruction of special lattice sets [*]

Sara Brunetti[†]   Alain Daurat[‡]   Alberto Del Lungo[§]

Abstract

Sometimes the inaccuracy of the measurements of the X-rays can give rise to an inconsistent reconstruction problem. In this paper we address the problem of reconstructing special lattice sets in $\mathbb{Z}^2$ from their approximate X-rays in a finite number of prescribed lattice directions. The class of "strongly Q-convex sets" is taken into consideration and a polynomial time algorithm for reconstructing members of that class with line sums having possibly some bounded differences with the given X-ray values is provided. In particular, when these differences are zero, the algorithm exactly reconstructs any set. As a result, this algorithm can also be used to reconstruct convex subsets of $\mathbb{Z}^2$ from their exact X-rays in a finite set of suitable prescribed lattice directions.

keywords: algorithms, combinatorial problems, discrete tomography, lattice sets, convexity, X-rays.

1 Introduction

The problem of reconstructing two-dimensional lattice sets from their X-rays has been studied in discrete mathematics and applied in several areas. This problem is the basic reconstruction problem in discrete tomography [10] and it has various interesting applications in image processing [16], statistical data security [12], biplane angiography [14], graph theory [1], and reconstructing crystals from images taken by a transmission electron microscope [13, 15]. In practice, the inaccuracy of the measurements of the X-rays acts in such a way that only approximate X-rays are available. For example, in medical applications an organ is reconstructed by measuring the attenuation of the intensity of the rays crossing the organ in several directions. In this paper we study the problem of reconstructing special lattice sets in $\mathbb{Z}^2$ from their approximate X-rays in a finite number of prescribed lattice directions.

In most practical applications we have some a priori properties about the sets to be reconstructed : in the angiography of the coronary arteries the form of a section of artery is similar to an ellipse. The algorithms can take advantage of this information to reconstruct the set. Mathematically, these properties can be described in terms of a subclass of subsets of $\mathbb{Z}^2$ to which the solution must belong. For instance, there are polynomial time algorithms to reconstruct hv-convex polyominoes (i.e., two-dimensional lattice subsets which are 4-connected and convex in the horizontal and vertical directions) from their X-rays in horizontal and vertical directions [3, 6]. The class of convex lattice subsets (i.e., finite subsets F with $F = \mathbb{Z}^2 \cap \operatorname{conv} F$) is another well known and studied class in discrete tomography. Gardner and Gritzmann [9] proved that the X-rays in four suitable or any seven prescribed mutually non parallel lattice directions uniquely determine all the convex lattice subsets. The complexity of the reconstruction problem on this class is an open problem raised by Gritzmann during the workshop: *Discrete Tomography: Algorithms and Complexity* (1997).

In this paper, we study the problem of reconstructing "Q-convex sets" from their X-rays in a finite number of prescribed directions. The Q-convexity is a weak convexity property linked to a finite number of directions and the class of Q-convex sets contains all the convex lattice subsets. In detail, we address the problem of reconstructing Q-convex sets from their "approximate" X-rays and we provide a polynomial time algorithm to solve the problem. This means that, the algorithm decides whether or not there is a Q-convex set whose X-rays all lie within prescribed bounds. If there is at least one Q-convex set having X-rays lying within these bounds, the algorithm reconstructs one of them in polynomial time.

Boufkhad et al. [4] studied the problem of reconstructing hv-convex polyominoes from their "approximate" horizontal and vertical X-rays. We show that our algorithm solves this problem in polynomial time. The algorithm can be used for reconstructing Q-convex sets from their "exact" X-rays. A greedy algorithm for solving this problem has been defined in [5], and our new approach is faster than this algorithm for a number of directions equal to two and three.

We point out that recently, it is proved in [7] that the uniqueness results of Gardner and Gritzmann can be extended to the class of Q-convex sets. From this uniqueness result for Q-convex sets it follows that our algorithm can be used for reconstructing convex lattice subsets from their exact X-rays in some sets of four prescribed lattice directions, or in any set of seven prescribed mutually nonparallel lattice directions. This means that, our algorithm and the one defined in [5] solve Gritzmann's problem for these special sets of directions.

2 A reconstruction algorithm for two X-rays

In this section, we are going to define an algorithm for reconstructing Q-convex sets from their approximate X-rays in two directions. The basic idea of the algorithm is to determine a polynomial transformation of our reconstruction problem to the 2-Satisfiability problem which can be solved in linear time [2]. A similar approach has been described in [3, 6].

2.1 Definitions and preliminaries

Let p and q be two independent linear forms on $\mathbb{Q}^2$. We can assume that: $p(x,y) = ax + by$ and $q(x,y) = cx + dy$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc \neq 0$, $gcd(a,b) = 1$, $gcd(c,d) = 1$. Assuming that $M = (x_M, y_M)$, we denote $p(x_M, y_M)$ by $p(M)$. The *X-ray* of a lattice set F along direction $p(M) = const$ is the function $X_p F : \mathbb{Z} \to \mathbb{N}_0$ defined by:

$$X_p F(i) = \text{card}(\{N \in F \mid p(N) = i\}).$$

This, in turn, means that an X-ray of a lattice set F in a direction p is a function giving the number of points of F on each line parallel to this direction (see Fig. 2(a)). We define four zones around a point M of $\mathbb{Z}^2$ (see Fig. 1) as follows:

$$Z_0(M) = \{N \in \mathbb{Z}^2 \mid p(N) \leq p(M) \text{ and } q(N) \leq q(M)\},$$
$$Z_1(M) = \{N \in \mathbb{Z}^2 \mid p(N) \geq p(M) \text{ and } q(N) \leq q(M)\},$$
$$Z_2(M) = \{N \in \mathbb{Z}^2 \mid p(N) \geq p(M) \text{ and } q(N) \geq q(M)\},$$
$$Z_3(M) = \{N \in \mathbb{Z}^2 \mid p(N) \leq p(M) \text{ and } q(N) \geq q(M)\}.$$

We can now introduce the definition of Q-convex set around two directions.

Definition 2.1 *A lattice set F is* Q-convex *around p and q if and only if for each $M \notin F$ there exists k such that $Z_k(M) \cap F = \emptyset$ for $k \in \{0, 1, 2, 3\}$.*
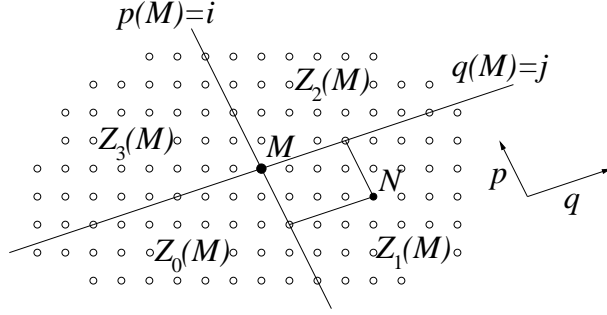
Figure 1: The four zones around a point M of $\mathbb{Z}^2$.

By the definition, if there is at least one point of F in every zone $Z_0(M)$, $Z_1(M)$, $Z_2(M)$ and $Z_3(M)$, the point M has to belong to F. Fig. 2(a) shows some examples of Q-convex sets. We point out that, from the definition it follows that a Q-convex set around p and q is a discrete set which is convex along p and q. A discrete set F is convex along p if for each pair of points (M, N) of F such that $p(M) = p(N)$, the discrete segment $[MN] \cap \mathbb{Z}^2$ is contained in F. Let us now take the following problem into consideration.
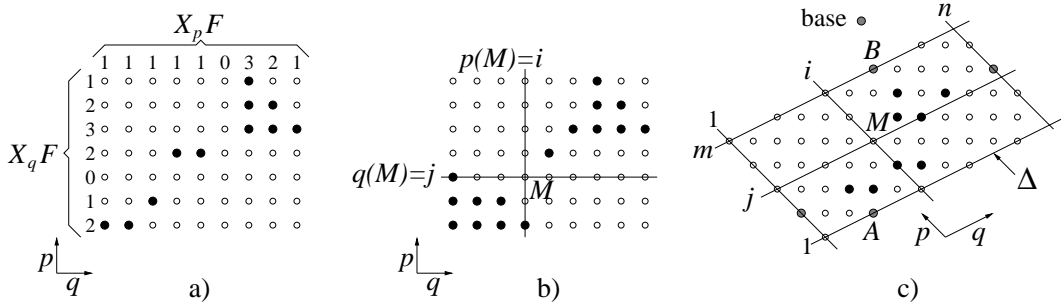

Figure 2: a) A Q-convex set around $p(x,y) = x$ and $q(x,y) = y$. b) A set which is not Q-convex set around $p(x,y) = x$ and $q(x,y) = y$. c) A Q-convex set around $p(x,y) = x + y$ and $q(x,y) = x - 2y$.

Problem 2.1 Approximate Consistency with two directions
Instance: *four vectors* $P = (p_1, \ldots, p_n)$, $P' = (p'_1, \ldots, p'_n)$, $Q = (q_1, \ldots, q_m)$, $Q' = (q'_1, \ldots, q'_m)$ *whose elements are no-negative integer numbers and* p_1, p_n, p'_1, p'_n, q_1, q_m, q'_1, q'_m *are positive integer numbers.*
Question: *is there a Q-convex set F around p and q such that* $p_i \leq X_p F(i) \leq p'_i$ *for* $i = 1, \ldots, n$ *and* $q_j \leq X_q F(j) \leq q'_j$ *for* $j = 1, \ldots, m$ *?*

The problem is to decide whether or not there is a Q-convex set around p and q whose X-rays in these two directions lie within prescribed bounds. If $P = P'$ and $Q = Q'$, we have the *Exact Consistency* problem with two directions.
In the following subsections, we determine a polynomial transformation of Problem 2.1 to the 2-Satisfiability problem (2SAT).

2.2 Q-convexity

The intersection of the p-line $p(M) = i$ with the q-line $q(M) = j$ is not always in $\mathbb{Z}^2$. It is easy to prove that the point M intersection of $p(M) = i$ with $q(M) = j$ belongs to $\mathbb{Z}^2$ if and only if $j \equiv \kappa i \pmod{\delta}$, where: $\delta = |ad - bc|$, $\kappa = (cu + dv)\text{sign}(ad - bc) \pmod{\delta}$ and $au + bv = 1$ (see Fig. 3(a) and [8]).
Without any loss of generality, we can assume that a Q-convex set F around p and q whose X-rays are such that: $p_i \leq X_p F(i) \leq p'_i$ and $q_j \leq X_q F(j) \leq q'_j$ for all i, j, is contained in the lattice parallelogram:

$$\Delta = \{N \in \mathbb{Z}^2 \mid 1 \leq p(N) \leq n \text{ and } 1 \leq q(N) \leq m\}.$$

3

We denote the point $M \in \Delta$ intersection of $p(M) = i$ with $q(M) = j$ by $M = (i, j)$ (see Fig. 2(c)). Let $K = \{0, 1, 2, 3\}$. We associate four boolean variables $V_k(M)$, with $k \in K$, at every point $M \in \Delta$ (i.e., one variable for each zone $Z_k(M)$). The idea of the algorithm is to build a 2SAT formula APPROX on the variables $(V_k(M))_{k \in K, M \in \Delta}$ so that there is a solution F of Problem 2.1 if and only if APPROX is satisfiable. If there is an evaluation V of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfying APPROX, the corresponding lattice set F solving Problem 2.1 is defined by function Φ as follows:

$$F = \Phi(V) \text{ iff } F = \{M \in \Delta \mid \overline{V_k(M)} \text{ is true, } \forall k \in K\},$$

where $\overline{V_k(M)}$ is true if and only if $V_k(M)$ is false. Conversely, if F is a subset of Δ solving Problem 2.1, the corresponding evaluation V of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfying APPROX is defined by function Ψ as follows:

$$V = \Psi(F) \text{ iff } V_k(M) = \text{``} Z_k(M) \cap F = \varnothing\text{''}, \text{ with } k \in K, \ M \in \Delta.$$

We assume that all literals outside Δ are true. The boolean formula APPROX is made up of three sets of clauses expressing: the *Q-convexity* (QCONV), a *lower bound* (LB) and an *upper bound* (UB) on the X-rays. The Q-convexity can be expressed with the boolean variables by the formulas:

$$\mathcal{Z}_0 = \bigwedge_{M=(i,j)\in\Delta} \left(\begin{array}{c} (V_0(i,j) \Rightarrow V_0(i-\delta,j)) \wedge (V_0(i,j) \Rightarrow V_0(i,j-\delta)) \\ \wedge \bigwedge_{\substack{0<u<\delta \\ 0<v<\delta \\ v\equiv\kappa u (\mathrm{mod}\delta)}} \left(V_0(i,j) \Rightarrow V_0(i-u,j-v) \right) \end{array} \right)$$

$$\mathcal{Z}_1 = \bigwedge_{M=(i,j)\in\Delta} \left(\begin{array}{c} (V_1(i,j) \Rightarrow V_1(i+\delta,j)) \wedge (V_1(i,j) \Rightarrow V_1(i,j-\delta)) \\ \wedge \bigwedge_{\substack{0<u<\delta \\ 0<v<\delta \\ v\equiv\kappa (\mathrm{mod}\delta)}} \left(V_1(i,j) \Rightarrow V_1(i+u,j-v) \right) \end{array} \right)$$

$$\mathcal{Z}_2 = \bigwedge_{M=(i,j)\in\Delta} \left(\begin{array}{c} \left(V_2(i,j) \Rightarrow V_2(i+\delta,j) \right) \wedge (V_2(i,j) \Rightarrow V_2(i,j+\delta)) \\ \wedge \bigwedge_{\substack{0<u<\delta \\ 0<v<\delta \\ v\equiv\kappa u (\mathrm{mod}\delta)}} \left(V_2(i,j) \Rightarrow V_2(i+u,j+v) \right) \end{array} \right)$$

$$\mathcal{Z}_3 = \bigwedge_{M=(i,j)\in\Delta} \left(\begin{array}{c} (V_3(i,j) \Rightarrow V_3(i-\delta,j)) \wedge (V_3(i,j) \Rightarrow V_3(i,j+\delta)) \\ \wedge \bigwedge_{\substack{0<u<\delta \\ 0<v<\delta \\ v\equiv\kappa u (\mathrm{mod}\delta)}} \left(V_3(i,j) \Rightarrow V_3(i-u,j+v) \right) \end{array} \right)$$

The points in the grey zone around $M = (i, j)$ in Fig. 3(a) are the points of Δ used in $\mathcal{Z}_0$, $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ (i.e., the points $(i \pm \delta, j \pm \delta)$, $(i \pm u, j \pm v)$ with $0 < u < \delta, 0 < v < \delta$). Let us denote $\mathcal{Z}_0 \wedge \mathcal{Z}_1 \wedge \mathcal{Z}_2 \wedge \mathcal{Z}_3$ by

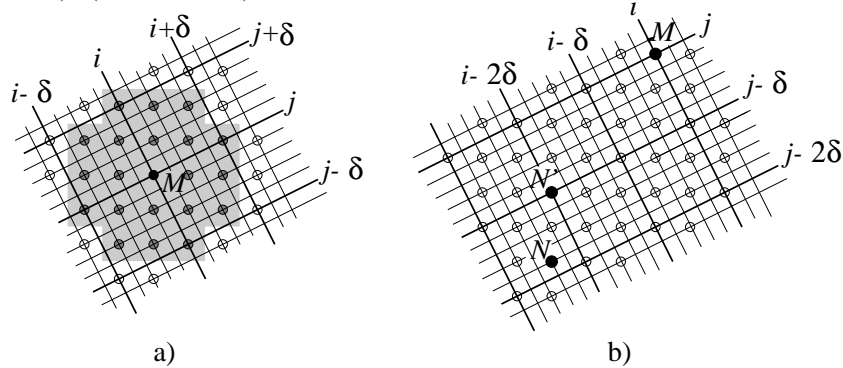
a)   b)

Figure 3: The points around M used in $\mathcal{Z}_0$, $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ with $p(x,y) = 2x + y$ and $q(x,y) = x - 2y$ QCONV.

4

Lemma 2.1 *Let V be an evaluation of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfying QCONV. If $M \in \Delta$ and $V_k(M)$ is true, then $V_k(N)$ is true for all $N \in Z_k(M)$.*

Proof. Assume that $k = 0$ and let $M = (i_M, j_M)$. We have: $V_0(i, j) \Rightarrow V_0(i - \delta, j)$ and $V_0(i, j) \Rightarrow V_0(i, j - \delta)$ are satisfied, for all i, j. Therefore, by induction we can prove that $V_0(i_M - k\delta, j_M - l\delta)$ is true, for all $k, l \in \mathbb{N}$. Let N be a point of $Z_0(M)$. Let N' be the point of Δ such that:

$$\frac{p(M) - p(N')}{\delta} = \left\lfloor \frac{p(M) - p(N)}{\delta} \right\rfloor, \qquad \frac{q(M) - q(N')}{\delta} = \left\lfloor \frac{q(M) - q(N)}{\delta} \right\rfloor$$

where $\lfloor x \rfloor$ designs the largest integer not greater than x.

By the previous statement we have $V_0(N')$ is true (see Fig. 3(b)) and, since the formula $V_0(N') \Rightarrow V_0(N)$ is in $\mathcal{Z}_0$, we finally obtain $V_0(N)$ is true. We proceed in the same way for k equal to 1,2 and 3. $\square$

Thus, we can characterize the Q-convexity by means of the formula QCONV.

Lemma 2.2   ● *For any set $F \subset \Delta$ the evaluation $V = \Psi(F)$ of the boolean variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies the formula QCONV.*

    ● *If an evaluation V of the boolean variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies the formula QCONV, then $F = \Phi(V)$ is Q-convex around p and q.*

Proof.

- Assume that $V = \Psi(F)$ does not satisfy QCONV. By this assumption, there exists k such that at least a clause of $\mathcal{Z}_k$ is not satisfied. Then $V_k(M)$ and $\overline{V_k(N)}$ are true, where $N \in Z_k(M)$. As a consequence, $Z_k(M) \cap F = \varnothing$, $Z_k(N) \cap F \neq \varnothing$ and $Z_k(N) \subset Z_k(M)$. We got a contradiction and so $V = \Psi(F)$ satisfies QCONV.

- The second statement is just a consequence of Lemma 2.1. If $M \notin F$, by the definition of Φ there exists k such that $V_k(M)$ is true. Therefore, by Lemma 2.1, we have $V_k(N)$ for each $N \in Z_k(M)$. Consequently, $Z_k(M) \cap F = \varnothing$.

$\square$

2.3 A lower bound

Now we want to express that X-ray values of a lattice set in the direction p are greater than some prescribed integers. Let us take the line $p(M) = i$ into consideration. Let

$$min_q^i = \min\{j | (i, j) \in \Delta\} \text{ and } max_q^i = \max\{j | (i, j) \in \Delta\}.$$

Notice that, if $\delta \neq 1$ these numbers are not always equal to 1 and m. We define the formula $\mathrm{LB}(p, i, l)$ in the following way:

$$\mathrm{LB}(p, i, l) = TRUE \qquad\qquad\qquad\qquad \textbf{if } l = 0$$

$$\mathrm{LB}(p, i, l) = \left(\begin{array}{c} \bigwedge_{\substack{1 \leq j \leq m - \delta l \\ j \equiv \kappa i \pmod{\delta}}} (L_1(j) \wedge L_2(j) \wedge L_3(j) \wedge L_4(j)) \\ \wedge L_1' \wedge L_2' \wedge L_1'' \wedge L_2'' \end{array} \right) \qquad \textbf{otherwise}$$

5

where

$$L_1(j) = V_0(i,j) \Rightarrow \overline{V_2(i,j+\delta l)}$$
$$L_2(j) = V_0(i,j) \Rightarrow \overline{V_3(i,j+\delta l)}$$
$$L_3(j) = V_1(i,j) \Rightarrow \overline{V_2(i,j+\delta l)}$$
$$L_4(j) = V_1(i,j) \Rightarrow \overline{V_3(i,j+\delta l)}$$
$$L_1' = \overline{V_2(i,min_q^i + \delta(l-1))}$$
$$L_2' = \overline{V_3(i,min_q^i + \delta(l-1))}$$
$$L_1'' = \overline{V_0(i,max_q^i - \delta(l-1))}$$
$$L_2'' = \overline{V_1(i,max_q^i - \delta(l-1))}$$

Lemma 2.3 • *If a lattice set F is Q-convex around p and q and its X-ray along p is such that $X_pF(i) \geq l$, then the evaluation $V = \Psi(F)$ of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies $QCONV \wedge \mathrm{LB}(p,i,l)$.*

• *If an evaluation V of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies $QCONV \wedge \mathrm{LB}(p,i,l)$, then the X-ray of $F = \Phi(V)$ along p is such that $X_pF(i) \geq l$.*

Proof.

• The case $l = 0$ is trivial, so we assume $l \geq 1$. If F is Q-convex around p and q, by Lemma 2.2 $V = \Psi(F)$ satisfies $QCONV$. Thus, we only have to show that $V = \Psi(F)$ satisfies $\mathrm{LB}(p,i,l)$ when the X-ray of F is such that $X_pF(i) \geq l$. Assume that $\mathrm{LB}(p,i,l)$ is not satisfiable; in this case at least one clause of the formula is not satisfied:

  – If $\overline{L_1(j)}$ is true, then $V_0(i,j) \wedge V_2(i,j+\delta l)$ is true. Since V satisfies $QCONV$, by Lemma 2.1 the set $\{j \mid (i,j) \in F\}$ has to be contained in $]j, j+\delta l[$. Therefore $X_pF(i) < l$, and by contradiction, the thesis follows.

  – If $\overline{L_1'}$ is true, then $V_2(i,min_q^i + \delta(l-1))$ is true. Since V satisfies $QCONV$, by Lemma 2.1 the set $\{j \mid (i,j) \in F\}$ has to be contained in $[min_q^i, min_q^i + \delta(l-1)[$. Therefore $X_pF(i) < l$, and by contradiction, the thesis follows.

  The proof is similar for the other cases.

• By Lemma 2.1 and L_1', L_2', we deduce $\overline{V_2(i,j)} \wedge \overline{V_3(i,j)}$ is true, for each $min_q^i \leq j \leq min_q^i + (l-1)\delta$. Assuming that $\overline{V_0(i,min_q^i)}$ and $\overline{V_1(i,min_q^i)}$ are true, always by Lemma 2.1, $\overline{V_0(i,j)} \wedge \overline{V_1(i,j)}$ is true for each $min_q^i \leq j \leq max_q^i$. Since $F = \Phi(V)$, we have $(i,j) \in F$ for each $min_q^i \leq j \leq min_q^i + \delta(l-1)$, namely, $X_pF(i) \geq l$.
  Now we assume that $V_0(i,min_q^i)$ or $V_1(i,min_q^i)$ is true. From the formulas L_2' and L_3' and Lemma 2.1, we deduce that there exists $min_q^i < j' \leq max_q^i - \delta(l-1)$ such that:

$$j' = \min(\{j \ : \ \overline{V_0(i,j)} \text{ and } \overline{V_1(i,j)}\})$$

  From $L_1(j') \wedge L_2(j') \wedge L_3(j') \wedge L_4(j')$ and $V_0(i,j') \vee V_1(i,j')$, it follows that $\overline{V_2(i,j'+\delta(l-1))}$ and $\overline{V_3(i,j'+\delta(l-1))}$ are true, and so $(i,j'+\delta(l-1)) \in F$. Moreover, Lemma 2.1 ensures that: $\overline{V_2(i,j)}$ and $\overline{V_3(i,j)}$ are true for each $j < j' + \delta(l-1)$, and $\overline{V_0(i,j)}$ and $\overline{V_1(i,j)}$ are true for each $j \geq j'$. Therefore, $\overline{V_k(i,j)}$ is true for all k and $j' \leq j \leq j' + \delta(l-1)$, and this means that $(i,j) \in F$ for each $j' \leq j \leq j' + \delta(l-1)$. Consequently, $X_pF(i) \geq l$.

6

□

We define the formula $\text{LB}(q, j, l)$ for the lines in the direction q in a similar way.

2.4 An upper bound

Now we want to express that X-ray values of a lattice set in the direction q are smaller than some prescribed integers. For this upper bound, we need to fix two points A and B such that $p(A) = 1$ and $p(B) = n$. We call *bases* these two points (see Fig. 2(c)). Let us take the line $q(M) = j$ into consideration. We introduce the formula:

$$\text{UB}(q, j, l, A, B) = \text{IN}(A) \wedge \text{IN}(B) \wedge \bigwedge_{\substack{1 \leq i \leq n - \delta l \\ j \equiv \kappa i \pmod{\delta}}} U(i)$$

where : $\text{IN}(M) = \overline{V_0(M)} \wedge \overline{V_1(M)} \wedge \overline{V_2(M)} \wedge \overline{V_3(M)}$ and

a) If $j \leq \min\{q(A), q(B)\}$, $U(i) = \overline{V_0(i, j)} \Rightarrow V_1(i + \delta l, j)$

b) If $q(A) \leq j \leq q(B)$ $U(i) = \overline{V_3(i, j)} \Rightarrow V_1(i + \delta l, j)$

c) If $q(B) \leq j \leq q(A)$ $U(i) = \overline{V_0(i, j)} \Rightarrow V_2(i + \delta l, j)$

b) If $j \geq \max\{q(A), q(B)\}$ $U(i) = \overline{V_3(i, j)} \Rightarrow V_2(i + \delta l, j)$

Lemma 2.4   • *If a lattice set F containing the bases A, B is Q-convex around p and q, and its X-ray along q is such that $X_q F(j) \leq l$, then the evaluation $V = \Psi(F)$ of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies $QCONV \wedge \text{UB}(q, j, l, A, B)$.*

• *If an evaluation V of the boolean variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies $QCONV \wedge \text{UB}(q, j, l, A, B)$, then $F = \Phi(V)$ contains the bases A, B, and its X-ray along q is such that $X_q F(j) \leq l$.*

Proof.

• Since the chosen bases A and B belong to F, at least two variables among the four ones associated to any $(i, j) \in \Delta$ are false. By Lemma 2.2, $V = \Psi(F)$ satisfies QCONV. Assume that $V = \Psi(F)$ does not satisfy $\text{UB}(q, j, l, A, B)$ and $j \leq \min\{q(A), q(B)\}$ (i.e., case (a)). By this assumption, there exists i' such that $\overline{V_0(i', j)}$ and $\overline{V_1(i' + \delta l, j)}$ are true, and so from Lemma 2.1 it follows that $\overline{V_0(i, j)}$ and $\overline{V_1(i, j)}$ are true for each $i' \leq i \leq i' + \delta l$. Moreover, $V_2(i, j)$ and $V_3(i, j)$ are false for all i because of bases A and B. Then $\overline{V_k(i, j)}$ is true for each k and $i' \leq i \leq i' + \delta l$, that is $(i, j) \in F$ for each $i' \leq i \leq i' + \delta l$ contradicting the thesis.
We proceed in the same way for the cases (b), (c) and (d).

• Since V satisfies $\text{IN}(A) \wedge \text{IN}(B)$, the bases A and B belong to F. Assuming that $j \leq \min\{q(A), q(B)\}$ (i.e., case (a)), by Lemma 2.1 and $\text{IN}(A) \wedge \text{IN}(B)$, the variables $V_2(i, j)$ and $V_3(i, j)$ are false for all i. If we have $V_0(i, j)$ is true for all i, then F has no point in the line $q(M) = j$ and $X_q F(j) = 0 \leq l$. So, we can suppose that there is i such that $\overline{V_0(i, j)}$ is true. By Lemma 2.1, there exists i' such that $\overline{V_0(i', j)}$ and $V_0(i, j)$ are true for $i < i'$. Therefore, $(i, j) \notin F$ for all $i < i'$. Since V satisfies $\text{UB}(q, j, l, A, B)$ and $\overline{V_0(i, j)}$ is true for $i \geq i'$, we have that $V_1(i, j)$ is true for all $i \geq i' + \delta l$, and so $\{i \mid (i, j) \in F\} \subset [i', i' + \delta l[$. Consequently, $X_q F(j) \leq l$.
We proceed in the same way for the cases (b), (c) and (d).

□

We define the formula $\text{UB}(p, i, l, A, B)$ for the lines in the direction p in a similar way .

2.5 The reconstruction algorithm

Let (P, P', Q, Q') be an instance of Problem 2.1. We fix four bases A, B, C, D such that $p(A) = 1$, $p(B) = n$, $q(C) = 1$, $q(D) = m$ and then we build the formula:

$$\text{APPROX}(P, P', Q, Q', A, B, C, D) = QCONV \wedge$$
$$\bigwedge_{1 \leq i \leq n} \big(\text{LB}(p, i, p_i) \wedge \text{UB}(p, i, p'_i, C, D)\big) \wedge \bigwedge_{1 \leq j \leq m} \big(\text{LB}(q, j, q_j) \wedge \text{UB}(q, j, q'_j, A, B)\big)$$

As a consequence of Lemmas 2.2, 2.3 and 2.4, we get:

Theorem 2.1 $\text{APPROX}(P, P', Q, Q', A, B, C, D)$ *is satisfiable if and only if there is a Q-convex set F around p and q containing the bases A, B, C, D and having X-rays along p and q such that $p_i \leq X_p F(i) \leq p'_i$, for $i = 1, \dots, n$, and $q_j \leq X_q F(j) \leq q'_j$, for $j = 1, \dots, m$.*

Since $\text{APPROX}(P, P', Q, Q', A, B, C, D)$ is a boolean formula in conjunctive normal form with at most two literals in each clause, from Theorem 2.1 we have a transformation of Problem 2.1 to 2SAT problem. The algorithm chooses four bases A, B, C, D, and builds $\text{APPROX}(P, P', Q, Q', A, B, C, D)$. Each formula $\text{APPROX}(P, P', Q, Q', A, B, C, D)$ has size $O(mn)$ and can be constructed in $O(mn)$ time. This is a 2SAT formula and so it can be solved in $O(mn)$ time (see [2]). If the formula is satisfiable and V is the evaluation of the boolean variables, then $F = \Phi(V)$ is solution of Problem 2.1. On the contrary, the reconstruction attempt fails and the algorithm chooses a different position of the four bases A, B, C, D, and repeats the procedure. The number of reconstruction attempts is bounded by the number of different positions of the four bases A, B, C, D, and this is at most $m^2 n^2$. Consequently:

Corollary 2.1 *Problem 2.1 can be solved in $O(m^3 n^3)$ time.*

Remark 2.1 An 8-connected hv-convex set is a Q-convex set around $p(x, y) = x$ and $q(x, y) = y$ with at least one point in each row and column. This class of lattice sets is a well-know generalization of the class of hv-convex polyominoes [3, 6] which are 4-connected and convex in horizontal and vertical directions. Boufkhad et al. [4] studied the "approximate consistency with two directions" problem on the class of hv-convex polyominoes. In detail, given a pair of vectors $V = (v_1, \dots, v_n)$ and $H = (h_1, \dots, h_m)$, they want to reconstruct an hv-convex polyomino whose X-rays along vertical and horizontal directions are such that: $|X_p F(i) - v_i| \leq 1$ for $i = 1, \dots, n$ and $|X_q F(j) - h_j| \leq 1$ for $j = 1, \dots, m$ (i.e., $P = V - 1, P' = V + 1, Q = H - 1$ and, $Q' = H + 1$). Our algorithm solves this problem in polynomial time on the classes of 8-connected hv-convex sets and hv-convex polyominoes (with an extra condition on the boolean variables). We point out that the goal of Boufkhad et al. is to solve the corresponding optimization problem. They want to reconstruct hv-convex polyominoes from these "approximate" horizontal and vertical X-rays and such that the sum of the absolute differences is minimum. By means of a SAT solver, the authors defined a heuristic algorithm for solving this problem. We do not know at this time if our algorithm can be used for solving this optimization problem in polynomial time.

Remark 2.2 Mirsky [11] proved that "approximate consistency with two directions" problem on the class of all lattice sets can be solved in polynomial time.

2.5.1 The exact reconstruction.

If $P = P'$ and $Q = Q'$, Problem 2.1 becomes the *Exact Consistency* problem with two directions on the Q-convex sets around p and q. Notice that, $\sum_{i=1}^{n} p_i = \sum_{j=1}^{n} q_j$ is a necessary condition. This problem has been studied in [5] and the authors propose a greedy algorithm whose computational cost is $O(m^2 n^2 (m +$

$n) \min\{m^2, n^2\}$). We are going to show an algorithm which is faster than the algorithm defined in [5]. Let A, B be two bases such that $p(A) = 1$ and $p(B) = n$. We build the following formula:

$$\text{EXACT}(P, Q, A, B) = QCONV \bigwedge_{1 \le i \le n} \text{LB}(p, i, p_i) \bigwedge_{1 \le j \le m} \text{UB}(q, j, q_j, A, B).$$

Proposition 2.1 $\text{EXACT}(P, Q, A, B)$ *is satisfiable if and only if there is a Q-convex set F around p and q containing A, B and having X-rays along p and q such that $X_p F(i) = p_i$, for $i = 1, \dots, n$, and $X_q F(j) = q_j$, for $j = 1, \dots, m$.*

Proof.

- Assume that $\text{EXACT}(P, Q, A, B)$ is satisfied by an evaluation V of $(V_k(M))_{k \in K, M \in \Delta}$. By Lemmas 2.2, 2.3 and 2.4, the set $F = \Phi(V)$ satisfies the conditions: $A, B \in F$, $X_p F(i) \ge p_i$ and $X_q F(j) \le q_j$ for all i, j. Then,

$$\sum_{j=1}^{m} q_j \ge \sum_{j=1}^{m} X_q F(j) = \sum_{i=1}^{n} X_p F(i) \ge \sum_{i=1}^{m} p_i$$

  and, since $\sum_{j=1}^{m} q_j = \sum_{i=1}^{n} p_i$, $X_p F(i) = p_i$ and $X_q F(j) = q_j$ for all i, j.

- If F is Q-convex around p and q and satisfies $A, B \in F$, $X_p F(i) = p_i$, and $X_q F(j) = q_j$, by Lemmas 2.2, 2.3 and 2.4, the evaluation $V = \Psi(F)$ satisfies $\text{EXACT}(P, Q, A, B)$.

$\square$

The number of reconstruction attempts for the exact consistency problem is bounded by the number of different positions of the two bases A, B, and this is at most $\min\{m^2, n^2\}$. Consequently:

Corollary 2.2 *Exact Consistency problem is solved in $O(mn \min\{m^2, n^2\})$ time.*

3 More than two directions

We now outline an algorithm for reconstructing Q-convex sets from their X-rays in more than two directions. Let us introduce a definition of Q-convex set around more than two directions. Let U be a set of d directions $\{\vec{u}_h = (a_h, b_h)_{h=1}^{d}\}$ (i.e., pairs of coprime integers, with $b_h \ge 0$). The linear form corresponding to vector $\vec{u}_h = (a_h, b_h)$ is $u_h(x, y) = b_h x - a_h y$. Given two directions $u_i, u_j \in U$, we define four zones $Z_k^{(i,j)}(M)$ around every $M \in \mathbb{Z}^2$ as in the previous section. Therefore, there are $2d(d - 1)$ zones for each $M \in \mathbb{Z}^2$ and we are going to select $2d$ of these zones. A point M of a line in direction u_h splits it into the following two semi-lines having origin in M:

$$s_h^+(M) = \{N \in \mathbb{Z}^2 \mid u_h(N) = u_h(M) \text{ and } \vec{u}_h \cdot \overrightarrow{ON} \ge \vec{u}_h \cdot \overrightarrow{OM}\}$$
$$s_h^-(M) = \{N \in \mathbb{Z}^2 \mid u_h(N) = u_h(M) \text{ and } \vec{u}_h \cdot \overrightarrow{ON} \le \vec{u}_h \cdot \overrightarrow{OM}\}$$

where "$\cdot$" denotes the scalar product of two vectors and O is any origin point.

Definition 3.1 *An almost-semi-plane (or ASP) along U is a zone $Z_k^{(i,j)}(M)$ such that for each direction u_h of U only one of the two semi-lines $s_h^+(M), s_h^-(M)$ is contained in $Z_k^{(i,j)}(M)$.*
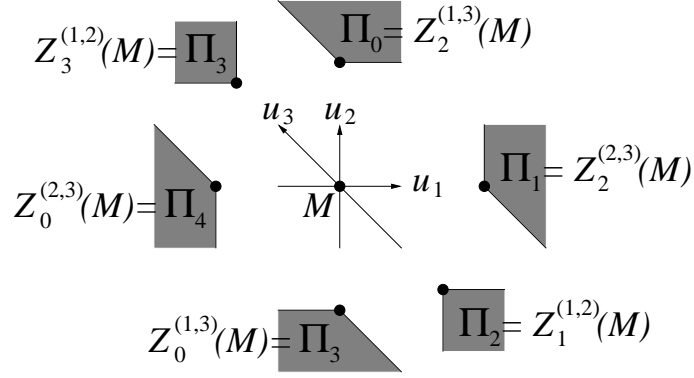
Figure 4: The six ASP around M, with $U = \{u_1, u_2, u_3\}$, $u_1 = y$, $u_2 = x$, $u_3 = x + y$.

Let $\Pi_0(M)$ be the ASP containing $s_h^+(M)$ for each $h = 1, \ldots, d$. We denote the other almost-semi-planes encountered clockwise around M from $\Pi_0(M)$ by $\Pi_1(M), \ldots, \Pi_{2d-1}(M)$. For example, let $U = \{u_1, u_2, u_3\}$, with $u_1 = y$, $u_2 = x$, $u_3 = x + y$. The six ASP around a point M are: $\Pi_0(M) = Z_2^{(1,3)}(M), \Pi_1(M) = Z_2^{(2,3)}(M), \Pi_2(M) = Z_1^{(1,2)}(M), \Pi_3(M) = Z_0^{(1,3)}(M), \Pi_4(M) = Z_0^{(2,3)}(M)$ and $\Pi_5(M) = Z_3^{(1,2)}(M)$ (see Fig. 4). Now we can generalize the Q-convexity to any set of directions:

Definition 3.2 *A lattice set F is* strongly Q-convex *around U if and only if for each $M \notin F$ there exists an ASP $\Pi_k(M)$ around M such that $F \cap \Pi_k(M) = \varnothing$.*

Let us consider the approximate consistency problem on this class.

Problem 3.1 Approximate Consistency with more than two directions
Instance: *$2d$ vectors $P_1 = (p_{1,1}, \ldots, p_{1,n_1}), P_1' = (p_{1,1}', \ldots, p_{1,n_1}'), \ldots, P_d = (p_{d,1}, \ldots, p_{d,n_d})$ and $P_d' = (p_{d,1}', \ldots, p_{d,n_d}')$ whose elements are no-negative integer numbers and $p_{1,1}, p_{1,n_1}, p_{1,1}', p_{1,n_1}', \ldots, p_{d,1}, p_{d,n_d}, p_{d,1}', p_{d,n_d}'$ are positive integer numbers.*
Question: *is there a strongly Q-convex set F around U such that:*
$p_{h,i} \leq X_{u_h} F(i) \leq p_{h,i}'$ *for $i = 1, \ldots, n_h$ and $h = 1, \ldots, d$?*

A Q-convex set F around U whose X-rays are such that: $p_{h,i} \leq X_{u_h} F(i) \leq p_{h,i}'$ for all h, i, is contained in the lattice polygon:
$$\Delta = \{N \in \mathbb{Z}^2 \mid 1 \leq u_h(N) \leq n_h \ \forall 1 \leq h \leq d\}.$$

We use the strategy of the previous section, by replacing the zone $Z_k(M)$ with $\Pi_k(M)$. Assuming that $K = \{0, 1, \ldots, 2d - 1\}$, we associate $2d$ boolean variables $V_k(M)$, with $k \in K$, at every point M in Δ (i.e., one variable for each $\Pi_k(M)$ around M). We build a 2SAT formula APPROX on the variables $(V_k(M))_{k \in K, M \in \Delta}$, so that there is a solution F of Problem 3.1 if and only if APPROX is satisfiable. If there is an evaluation V of $(V_k(M))_{k \in K, M \in \Delta}$ satisfying APPROX, the corresponding set F solving Problem 3.1 is $F = \Phi(V)$, with Φ defined as in the previous section. Conversely, if F is a subset of Δ solving Problem 2.1, the corresponding evaluation V of $(V_k(M))_{k \in K, M \in \Delta}$ satisfying APPROX is $V = \Psi(F)$, with Ψ defined as in the previous section. Since every ASP $\Pi_k(M)$ is equal to a zone $Z_h^{(i,j)}(M)$ defined by directions u_i, u_j, it is easy to generalize the formulas to contexts having n directions. The generalization SQCONV of QCONV is such that SQCONV$= \mathcal{Z}_0 \wedge \ldots \wedge \mathcal{Z}_{2d-1}$, where $\mathcal{Z}_k$ corresponds to $\Pi_k(M)$ (i.e., $Z_h^{(i,j)}(M)$). Thus, we have the following extensions of Lemma 2.2:

Lemma 3.1   • *For any set $F \subset \Delta$ the evaluation $V = \Psi(F)$ of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies SQCONV.*

10

- *If an evaluation V of the variables $(V_k(M))_{k \in K, M \in \Delta}$ satisfies SQCONV, then $F = \Phi(V)$ is strongly $\mathbb{Q}$-convex around U.*

A lower bound of l on the X-ray of F along the line $u_h(M) = i$ is expressed by a 2SAT formula. Let us take the line $u_h(M) = i$ into consideration. Analogously to the case of two directions, by definition, min_h^i and max_h^i allows us to determine two points A_h^i and B_h^i of Δ lying on $u_h = i$ such that $u_h(A_h^i) = min_h^i$ and $u_h(B_h^i) = max_h^i$. A point M of a line in direction u_h splits it into the two semi-lines $s_h^+(M)$ and $s_h^-(M)$. Hence, for each direction u_h we can split the set of indices $K = \{0, 1, \ldots, 2d - 1\}$ into two subsets:

$$K_h^+ = \{t \in K \mid s_h^+(M) \subset \Pi_t(M)\}$$
$$K_h^- = \{t \in K \mid s_h^-(M) \subset \Pi_t(M)\}.$$

From the definition of ASP it follows that, $K = K_h^+ \cup K_h^-$ and $K_h^+ \cap K_h^- = \emptyset$. The required boolean formula can be constructed by proceeding as in the two-directions case. More precisely, the formula $\mathrm{SLB}(u_h, i, l)$ is defined in the following way.

$$\mathrm{SLB}(u_h, i, l) = TRUE \qquad \qquad \textbf{if } l = 0$$

$$\mathrm{SLB}(u_h, i, l) = \left(\bigwedge_{\substack{M \in I}} \bigwedge_{\substack{k \in K_h^+ \\ k' \in K_h^-}} L_{k,k'}(M) \right) \wedge \left(\bigwedge_{k \in K_h^+} L_k' \right) \wedge \left(\bigwedge_{k \in K_h^-} L_k'' \right) \quad \textbf{otherwise}$$

where

$$I = \{M \in \Delta \mid u_h(M) = i \text{ and } M + l\overrightarrow{u_h} \in \Delta\}$$
$$L_{k,k'}(M) = (V_k(M) \Rightarrow \overline{V_{k'}(M + l\overrightarrow{u_h})})$$
$$L_k' = \overline{V_k(A_h^i + (l-1)\overrightarrow{u_h})}$$
$$L_k'' = \overline{V_k(B_h^i - (l-1)\overrightarrow{u_h})}$$

Finally, we fix $d - 1$ pairs of bases for each directions, in order to give an upper bound of l on the X-ray of F in the direction u_h. The chosen bases correspond to the other $d - 1$ directions. The upper bound is expressed by the formula $\mathrm{SUB}(u_h, i, l, A_1, B_1, \ldots, A_{h-1}, B_{h-1}, A_{h+1}, B_{h+1}, \ldots, A_d, B_d)$, where A_j, B_j are the bases of the direction u_j. This formula is a generalization of the upper bound formula for the two-directions case. At first we have to prove the following lemma:

Lemma 3.2 *For any $M \in \Delta$ such that $u_h(M) = i$ there exists at most one $k \in K_h^+$ such that for any $k' \in K_h^+ \setminus \{k\}$ the ASP $\Pi_{k'}(M)$ contains one of the points $A_1, B_1, \ldots, A_{h-1}, B_{h-1}, A_{h+1}, B_{h+1}, \ldots, A_d, B_d$.*

Proof. Let M_{max} be the point of $\mathbb{Q}^2$ such that:

- $u_h(M_{max}) = i$

- it maximizes $\overrightarrow{ON} \cdot \overrightarrow{u_h}$ for each N such that $u_h(N) = i$ in the continuous polygon $\Delta_c = \{N \in \mathbb{R}^2 \mid 1 \leq u_h(N) \leq n_h \ \forall 1 \leq h \leq d\}$.

For any $k' \in K_h^+$ we have $\Pi_{k'}(M) \supset \Pi_{k'}(M_{max})$ so it is sufficient to prove this property for $M = M_{max}$.

M_{max} is on an edge e of Δ_c which has a direction $u_{h_1} \neq u_h$. We have a base C fixed on this edge. We can suppose that $C \in s_h^+(M_{max})$ without loss of generality. Consider the edge e' consecutive to $s_h^-(M_{max})$ in Δ_c. This edge is in direction $u_{h_2} \in U$.

- Suppose that $h_2 \neq h$. Let D be the base fixed on this edge. There exists $k \in K_h^+$ and $r \in \{0, 1, 2, 3\}$ such that $Z_r^{(h_1, h_2)}(M_{max})$ is an ASP. Let $\Pi_k(M_{max}) = Z_r^{(h_1, h_2)}(M_{max})$. We can see that for any $k' \in K_h^+ \setminus \{k\}$ the ASP $\Pi_{k'}(M)$ the ASP $\Pi_{k'}(M)$ contains C or D (see Fig. 5).
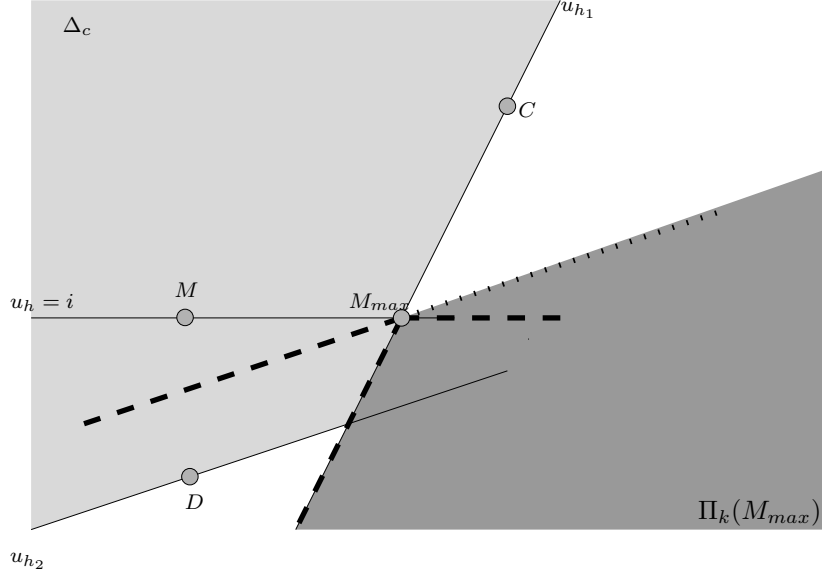

Figure 5: Why only $\Pi_k(M_{max})$ does not contain neither C nor D ?

- If $h_2 = h$ then $\Pi_k(M_{max})$ is $Z_r^{(h_1, h)}(M_{max})$ which does not contain C and the semi-line $s_h^-(M_{max}) \setminus M_{max}$.

$\square$

Let k be as in the statement of the lemma: we denote $W_h^+(M) = V_k(M)$.
We can define $W_h^-(M)$ on the same way. Therefore the upper bound is expressed by the following 2SAT formula:

$$\text{SUB}(u_h, i, l, A_1, B_1, \ldots, A_{h-1}, B_{h-1}, A_{h+1}, B_{h+1}, \ldots, A_d, B_d) =$$
$$\text{IN}(A_1) \wedge \ldots \wedge \text{IN}(B_d) \wedge \bigwedge_{M \in I} U(M)$$

where:

$$I = \{M \in \Delta \mid u_h(M) = i \text{ and } M + l\overrightarrow{u_h} \in \Delta\}$$
$$\text{IN}(M) = \bigwedge_{k \in K} \overline{V_k(M)}$$
$$U(M) = \left(\overline{W_h^-(M)} \Rightarrow W_h^+(M + l\overrightarrow{u_h}) \right).$$

Let $(P_1, P_1', \ldots, P_d, P_d')$ be an instance of Problem 3.1. We fix $2d$ bases $A_1, B_1, \ldots, A_d, B_d$ and then we build the following boolean formula:

$$\text{APPROX}(P_1, P_1', \ldots, P_d, P_d', A_1, B_1, \ldots, A_d, B_d) = SQCONV \wedge$$
$$\bigwedge_{\substack{1 \le h \le d \\ 1 \le i \le n_h}} \text{LB}(u_h, i, p_i) \wedge \text{UB}(p, i, p_i', A_1, B_1, \ldots, A_{h-1}, B_{h-1}, A_{h+1}, B_{h+1}, \ldots, A_d, B_d)$$

We deduce that:

Theorem 3.1 APPROX$(P_1, P_1' \dots, A_d, B_d)$ *is satisfiable if and only if there is a strongly Q-convex set F around U containing the bases $A_1, B_1, \dots, A_d, B_d$ and having X-rays along u_h such that $p_{h,i} \leq X_{u_h} F(i) \leq p'_{h,i}$, for $i = 1, \dots, n_h$ and $h = 1, \dots, d$.*

The algorithm chooses d pair of bases, and builds the 2SAT expression APPROX. Assuming that $n = \max\{n_1, \dots, n_d\}$, we have that each formula APPROX has size $O(n^2)$ and can be constructed in $O(n^2)$ time. The number of reconstruction attempts is bounded by the number of different positions of the $2d$ bases, and this is at most n^{2d}. Consequently:

Corollary 3.1 *Problem 3.1 can be solved in $O(n^{2d+2})$ time.*

If $P_h = P_h'$ for each $1 \leq h \leq d$, Problem 3.1 become the exact consistency problem with more than two directions. In this case, we have to choose $d - 1$ pair of bases for the upper bound and we complexity of algorithm for solving this problem is $O(n^{2d})$. The convex lattice sets are special Q-convex sets, and so by uniqueness results for Q-convex sets proved in [7], the algorithm can be used for reconstructing convex lattice subsets from their exact X-rays in some sets of four suitable lattice directions, or in any set of seven prescribed mutually nonparallel lattice directions. This means that the two algorithms solve Gritzmann's problem for these special sets of directions.

References

[1] R. P. Anstee, Invariant sets of arcs in network flow problems, *Discrete Applied Mathematics*, **13** (1986) 1-7.

[2] B. Aspvall, M.F. Plass and R.E. Tarjan, A linear-time algorithm for testing the truth of certain quantified Boolean formulas, *Information Processing Letters*, **8** 3 (1979) 121-123.

[3] E. Barcucci, A. Del Lungo, M. Nivat and R. Pinzani, Reconstructing convex polyominoes from their horizontal and vertical projections, *Theoretical Computer Science*, **155** (1996) 321-347.

[4] Y. Boufkhad, O. Dubois and M. Nivat, Reconstructing (h,v)-convex bidimensional patterns of objects from approximate horizontal and vertical projections, to appear in *Theoretical Computer Science*.

[5] S. Brunetti, A. Daurat, Reconstruction of Discrete Sets From Two or More Projections in Any Direction, *Proc. of the Seventh International Workshop on Combinatorial Image Analysis* (IWCIA 2000), Caen, (2000) 241-258.

[6] M. Chrobak, C. Dürr, Reconstructing hv-Convex Polyominoes from Orthogonal Projections, *Information Processing Letters*, **69** 6 (1999) 283-289.

[7] A. Daurat, Uniqueness of the reconstruction of Q-convex from their projections, preprint.

[8] A. Daurat, A. Del Lungo and M. Nivat, The medians of discrete sets according to some linear metrics, *Discrete & Computational Geometry*, **23** (2000) 465-483.

[9] R. J. Gardner and P. Gritzmann, Discrete tomography: determination of finite sets by X-rays, *Trans. Amer. Math. Soc.*, **349** (1997) 2271-2295.

[10] A. Kuba and G.T. Herman, Discrete Tomography: A Historical Overview, in *Discrete Tomography: Foundations, Algorithms and Applications*, editors G.T. Herman and A. Kuba, Birkhauser, Boston, MA, USA, (1999) 3-34.

[11] L. Mirski, Combinatorial theorems and integral matrices, *Journal of Combinatorial Theory*, **5** (1968) 30-44.

[12] R. W. Irving and M. R. Jerrum, Three-dimensional statistical data security problems, *SIAM Journal of Computing*, **23** (1994) 170-184.

[13] C. Kisielowski, P. Schwander, F. H. Baumann, M. Seibt, Y. Kim and A. Ourmazd, An approach to quantitative high-resolution transmission electron microscopy of crystalline materials, *Ultramicroscopy*, **58** (1995) 131-155.

[14] G. P. M. Prause and D. G. W. Onnasch, Binary reconstruction of the heart chambers from biplane angiographic image sequence, *IEEE Transactions Medical Imaging*, **15** (1996) 532-559.

[15] P. Schwander, C. Kisielowski, M. Seibt, F. H. Baumann, Y. Kim and A. Ourmazd, Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy, *Physical Review Letters* **71**, 4150-4153 (1993).

[16] A. R. Shliferstein and Y. T. Chien, Switching components and the ambiguity problem in the reconstruction of pictures from their projections, *Pattern Recognition*, **10** (1978) 327-340.