Medians of discrete sets according to a linear distance

Alain Daurat, Alberto Del Lungo, Maurice Nivat

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A. Daurat*, A. Del Lungo†, M. Nivat‡

Abstract

In this paper, we present some results concerning the median points of a discrete set according to a distance defined by means of two directions p and q. We describe a local characterization of the median points and show how these points can be determined from the projections of the discrete set along directions p and q. We prove that the discrete sets having some connectivity properties have at most four median points according to a linear distance, and if there are four median points they form a parallelogram. Finally, we show that the 4-connected sets which are convex along the diagonal directions contain their median points along these directions.

keywords: discrete set, linear distance, median point, projection, connected set, convexity.

1 Introduction

A *discrete set* is a non-empty finite subset of the integer lattice $\mathbb{Z}^2$. Let d be a distance on $\mathbb{Z}^2$. If $\mathcal{S}$ and P are a discrete set and a point P of $\mathbb{Z}^2$, we define:

$$D(P) = \sum_{Q \in \mathcal{S}} d(P, Q).$$

A point M of $\mathbb{Z}^2$ is said to be a *median point* of $\mathcal{S}$ according to d, if:

$$D(M) = \min_{Q \in \mathbb{Z}^2} D(Q).$$

In [7], the authors gave a simple method for determining the medians of a discrete set according to *Manhattan distance* d_1 (i.e., $d_1(P,Q) = |x_P - x_Q| + |y_P - y_Q|$ for each $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ belonging to $\mathbb{Z}^2$). The discrete set in Fig. 1 has two median points according to d_1. They showed how the median points can be determined by means of discrete set's projections along the horizontal and vertical directions. The *projection of* a discrete set $\mathcal{S}$ along direction $p(M) = ax_M + by_M = const$ is the function $rp : \mathbb{Z} \to \mathbb{N}$ defined by:

$$rp_i = \mathrm{card}(\{N \in \mathcal{S} \,|\, p(N) = i\}).$$

This, in turn, means that a projection of a discrete set $\mathcal{S}$ in a direction p is a function giving the number of points of $\mathcal{S}$ on each line parallel to this direction. For instance, Fig. 2 illustrates the projections of a discrete set along the directions $p(M) = x_M$ and $q(M) = y_M$. The projections along $p(M) = x_M$, $q(M) = y_M$ are said vertical and horizontal projections, respectively.

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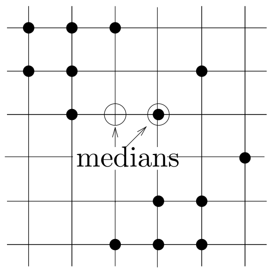


Figure 1: The median points of a discrete set according to Manhattan distance.

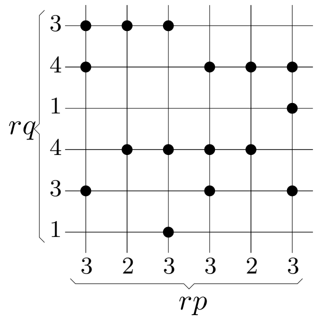


Figure 2: The projections of a discrete set along the horizontal and vertical directions.

The purpose of this paper is to extend the results on the median points given in [7] to the "linear" distances, that is a distance defined in the following way:

$$d(M,N) = \alpha|p(\overrightarrow{MN})| + \beta|q(\overrightarrow{MN})|$$

where α and β are two positive integer numbers, and p and q are two independent linear forms on $\mathbb{Q}^2$. We can assume that:

$$p(\vec{u}) = au_x + bu_y$$
$$q(\vec{u}) = cu_x + du_y$$

with $a, b, c, d \in \mathbb{Z}$, $ad-bc \neq 0$, $gcd(a,b) = 1$, $gcd(c,d) = 1$. Note that, if $a = 1, b = 0, c = 0, d = 1$ and $\alpha = \beta = 1$, then d is the Manhattan distance.

A priori if we want to verify if a point M is a median point of a discrete $\mathcal{S}$ according to a linear distance we have to compute the function D for all the points of $\mathbb{Z}^2$. In Section 3, we show that it is sufficient to compute the function D for only a few points around M. From this result it follows a simple method to find all the medians of $\mathcal{S}$ by means of its projections along directions p and q. In Section 4, we prove that the discrete sets verifying some connectivity properties have at most four median points, and if these sets have four median points they form a parallelogram. In [7] is proved that, if a 4-connected set which is convex along the horizontal and vertical directions contains its median points along these directions. The final Section 5 verifies if this property is true with two other directions. We illustrate that, a 4-connected discrete set has this property for the diagonal directions, while there exists a pair of directions for which it is not true.

2 Some definitions about discrete sets

Let $\mathcal{S}$ be a discrete set. A *column* (*row*) of $\mathcal{S}$ is the intersection of $\mathcal{S}$ with a line $x = k$ ($y = k$), $k \in \mathbb{Z}$. A *north-east diagonal* (*north-west diagonal*) of $\mathcal{S}$ is the intersection of $\mathcal{S}$ with a line $y = x + k$ ($y = -x + k$), $k \in \mathbb{Z}$.
A *4-neighbor* of a point $P = (x, y) \in \mathbb{Z}^2$ is a point $P^{'} = (x^{'}, y^{'})$ such that $|x - x^{'}| + |y - y^{'}| = 1$. A *8-neighbor* of $P = (x, y)$ is a point $P^{'} = (x^{'}, y^{'})$ such that $|x - x^{'}| \leqslant 1$ and $|y - y^{'}| \leqslant 1$ (see Fig. 3). Let P and Q be two points of $\mathcal{S}$. A *n-path* from P to Q (with $n = 4$ or 8) is a sequence

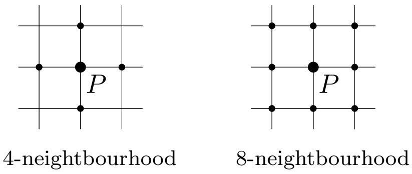


Figure 3: A point $P = (x, y) \in \mathbb{Z}^2$ and its neighbourhood

of points $P_0P_1 \ldots P_k$, where the elements of the sequence belong to $\mathcal{S}$, P_i is a n-neighbour of P_{i-1} for $1 \leqslant i \leqslant k$, $P_0 = P$ and $P_k = Q$. A discrete set $\mathcal{S}$ is n-connected if there is a n-path between any two points P and Q of $\mathcal{S}$ (see Fig. 4).

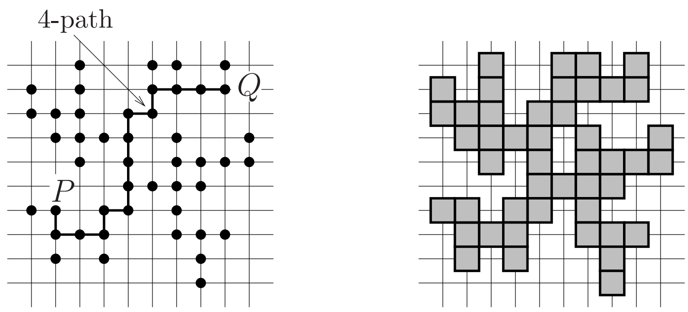
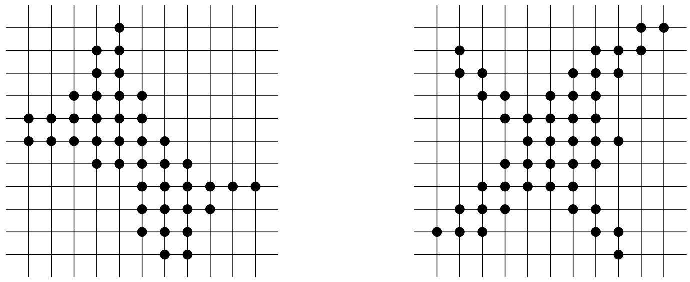


Figure 4: A 4-connected discrete set and its associated polyomino.

A discrete set can be represented by a set of *cells* (unitary squares), as shown in Fig. 4. The cells set corresponding to a 4-connected discrete set is called *polyomino*. They are well-known combinatorial objects [9, 10] and are related to many different problems, such as: tiling [2, 5], enumeration [3, 13] and discrete tomography [1, 8]. In [11], the authors studied the problem of determining the medians of a polyomino, by considering a polyomino as a graph and by using the classical metric on the graph. In [4], the authors study medians of a rectilinear polygon according to a distance. But by the definition used in this article the median point is always in the set, it is not the case for us.

A discrete set is *convex* if its columns and rows are 4-connected. A discrete set is *diagonally convex* if its north-east and north-west diagonals are 8-connected. Fig. 5 shows two examples of 4-connected discrete sets having the two different kinds of convexity.



(a) (b)

Figure 5: a) A convex set. b) A diagonally convex set.

3 Properties of the medians

3.1 Preliminaries

Let us denote the line whose equation is $p(M) = i$ by U_i, and the line whose equation is $q(M) = j$ by V_j. The intersection of U_i and V_j is not always in $\mathbb{Z}^2$ (see Fig. 6). The following

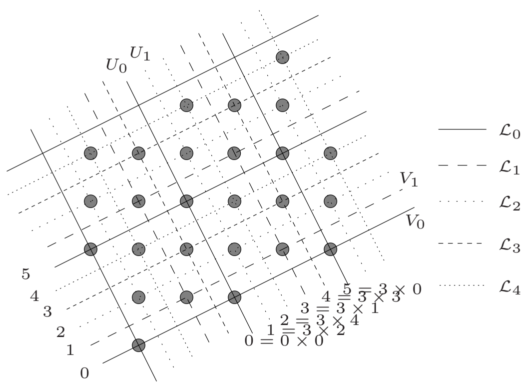


Figure 6: The integer lattice $\mathbb{Z}^2$ and the lines parallel to $p = 2x + y$ and $q = x - 2y$.

Proposition gives a condition to determine if the intersection of these lines is in $\mathbb{Z}^2$.

Proposition 3.1 *There exists k coprime with $\delta = |ad - bc|$ such that for all $i, j \in \mathbb{Z}$ ($U_i \cap V_j \in \mathbb{Z}^2$ if and only if $j \equiv ki$ $[\delta]$).*

Proof : The intersection $U_i \cap V_j \in \mathbb{Z}^2$ if and only if there exists a point $(x, y) \in \mathbb{Z}^2$ such that $ax + by = i$ and $cx + dy = j$. Therefore, if and only if there is $(X, Y) \in \mathbb{Z}^2$ such that:

$$a(uX - bY) + b(vX + aY) = i, \qquad c(uX - bY) + d(vX + aY) = j.$$

From the definition of linear distance it follows a and b are coprime and so there are u and v such that $ua + vb = 1$. Consequently, $U_i \cap V_j \in \mathbb{Z}^2$ if and only if there is $(X, Y) \in \mathbb{Z}^2$ such that:

$$X = i, \qquad (cu + dv)X + (ad - bc)Y = j,$$

that is, if and only if:

$$\frac{j - (cu + dv)i}{ad - bc} \in \mathbb{Z}.$$

From this condition we deduce that $U_i \cap V_j \in \mathbb{Z}^2$ if and only if $j \equiv ki\ [\delta]$, where $k = ((cu + dv)\mathrm{sign}(ad - bc)) \mod \delta$. □

In the following sections, the integers δ and k denote $|ad - bc|$ and $((cu + dv)\mathrm{sign}(ad - bc)) \mod |ad - bc|$, respectively. Proposition 3.1 brings the following definition:

Definition 1 *A subset $\mathcal{L} \subset \mathbb{Z}^2$ is a p-q-lattice if there is an integer $l \in \{0, 1, \ldots \delta - 1\}$ such that:*

$$\mathcal{L} = \mathcal{L}_l = \{M \in \mathbb{Z}^2 \mid q(M) \equiv kp(M) \equiv l\ [\delta]\}.$$

From this definition it follows that $\mathbb{Z}^2$ is the union of δ disjoint p-q lattices. For example, Fig. 6 shows that $\mathbb{Z}^2$ is the union of five disjoint p-q lattices: $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$. The interest of this definition is that each p-q lattice $\mathcal{L}_l$ is like $\mathbb{Z}^2$ with $p = x$ and $q = y$. This, in turn, means that given two points P and Q of $\mathcal{L}_l$, the intersection of the line along p and passing through P with the line along q and passing through Q belongs to $\mathcal{L}_l$.

3.2 Local characterization

A priori if we want to verify if a point M is a median point we have to compute the function D for all the points of $\mathbb{Z}^2$. In this section, we are going to show that it is sufficient to compute the function D for only a few points around M.

Definition 2 *The neighbor of zero (denoted $\mathcal{N}$) is the set:*

$$\mathcal{N} = \{\vec{u} \mid \vec{u} \neq \vec{0},\ -\delta < p(\vec{u}) < \delta,\ -\delta < q(\vec{u}) < \delta\} \cup \{\vec{u} \mid (p(\vec{u}), q(\vec{u})) \in \{(\pm\delta, 0), (0, \pm\delta)\}\}$$

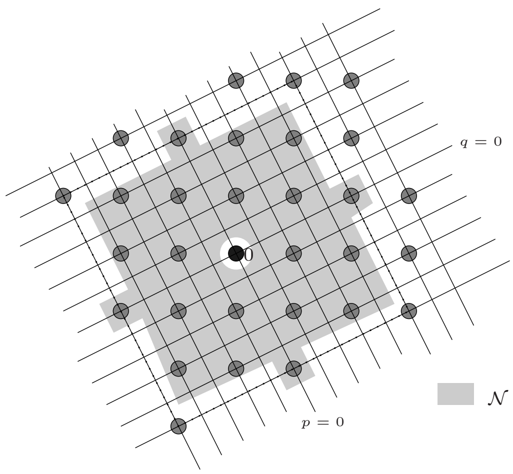


Figure 7: The neighbor of zero with the directions $p = 2x + y$ and $q = x - 2y$.

We also denote $\mathcal{N}' = \{(p(\vec{u}), q(\vec{u})) \mid \vec{u} \in \mathcal{N}\}$. From Proposition 3.1 it follows that that:

$$\mathcal{N}' = \{(l, m) \in]-\delta, \delta[^2 \ \mid\ (l, m) \neq (0, 0),\ m \equiv kl\ [\delta]\} \cup \{(\pm\delta, 0), (0, \pm\delta)\},$$

and so:

$$\begin{aligned} \mathrm{card}(\mathcal{N}) = \mathrm{card}(\mathcal{N}') &= 4\mathrm{card}\{(l, m) \in]0, \delta[^2 \ \mid\ m \equiv kl\ [\delta]\} + 4 \\ &= 4(\delta - 1) + 4 \\ &= 4\delta \end{aligned}$$

For example, $\delta = 5$ for the lattice in Fig. 7 and $\text{card}(\mathcal{N}) = 20$. We point out that $\vec{0} \notin \mathcal{N}$. For any point $M \in \mathbb{Z}^2$ the set:

$$M + \mathcal{N} = \{M + \vec{u} \mid \vec{u} \in \mathcal{N}\}$$

is called neighbor of M. Let us now give the main Theorem of this section.

Theorem 3.2 *A point M is a median of a discrete set $\mathcal{S}$ if and only if:*

$$\forall \vec{u} \in \mathcal{N} \quad D(M + \vec{u}) \geqslant D(M). \tag{3.1}$$

Proof : From the definition of linear distance we deduce that:

$$D(M) = \alpha \sum_{Q \in \mathcal{S}} |p(\overrightarrow{MQ})| + \beta \sum_{Q \in \mathcal{S}} |q(\overrightarrow{MQ})|.$$

By setting:

$$DP_i = \alpha \sum_{i' \in \mathbb{Z}} |i - i'| rp_{i'} \ , \qquad \text{where } rp_{i'} = \text{card}(\{N \in \mathcal{S} \mid p(N) = i'\}),$$
$$DQ_j = \beta \sum_{j' \in \mathbb{Z}} |j - j'| rq_{j'} \ , \qquad \text{where } rq_{j'} = \text{card}(\{N \in \mathcal{S} \mid q(N) = j'\}),$$

it follows:

$$D(M) = DP_{p(M)} + DQ_{q(M)}.$$

We prove that the functions DP and DQ are decreasing then increasing functions. Let:

$$SP_i = \sum_{i' \leqslant i} rp_{i'} = \text{card}(\{M \in \mathcal{S} \mid p(M) \leqslant i\}),$$
$$SP'_i = \sum_{i' \geqslant i} rp_{i'} = \text{card}(\{M \in \mathcal{S} \mid p(M) \geqslant i\}),$$
$$SQ_j = \sum_{j' \leqslant j} rq_{j'} = \text{card}(\{M \in \mathcal{S} \mid q(M) \leqslant j\}),$$
$$SQ'_j = \sum_{j' \geqslant j} rq_{j'} = \text{card}(\{M \in \mathcal{S} \mid q(M) \geqslant j\}).$$

We have:

$$SP_i + SP'_{i+1} = SQ_j + SQ'_{j+1} = \text{card}(\mathcal{S}) \tag{3.2}$$

And :

$$\begin{aligned}
DP_{i+1} - DP_i &= \alpha \sum_{i' \in \mathbb{Z}} (|i + 1 - i'| - |i - i'|) rp_{i'} \\
&= \alpha \left(\sum_{i' \leqslant i} ((i + 1 - i') - (i - i')) rp_{i'} + \sum_{i' \geqslant i+1} ((i' - i - 1) - (i' - i)) rp_{i'} \right) \\
&= \alpha \left(\sum_{i' \leqslant i} rp_{i'} + \sum_{i' \geqslant i+1} -rp_{i'} \right) \\
&= \alpha (SP_i - SP'_{i+1})
\end{aligned}$$

So we have :

$$DP_{i+1} - DP_i = \alpha(SP_i - SP'_{i+1}) = \alpha(2SP_i - \text{card}(\mathcal{S})) \tag{3.3}$$

On the same way we have :

$$DQ_{j+1} - DQ_j = \beta(SQ_j - SQ'_{j+1}) = \beta(2SQ_j - \text{card}(\mathcal{S})). \tag{3.4}$$

Since α and β are positive integer number, we obtain

$$\begin{aligned} DP_{i+1} - 2DP_i + DP_{i-1} = 2\alpha r p_i \geqslant 0, \\ DQ_{j+1} - 2DQ_j + DQ_{j-1} = 2\beta r q_j \geqslant 0, \end{aligned} \tag{3.5}$$

that is, DP and DQ are convex functions (see Fig. 8).

Let us now prove the theorem. If M is a median point, it obviously verifies condition (3.1). Vice versa, let us assume that M verifies (3.1). We have to prove that M is a global minimum of D. Let N be a point of $\mathbb{Z}^2$. We are going to look for a point P which is in the neighbor of M and which verifies $D(N) \geqslant D(P)$.
Let $i = p(M)$, $j = q(M)$, $i' = p(N)$ and $j' = q(N)$. Let i'' and j'' be two integer numbers such that:

$$\begin{aligned} \text{sign}(i'' - i) = \text{sign}(i' - i) \quad & \text{sign}(j'' - j) = \text{sign}(j' - j), \\ i'' \equiv i'[\delta] \quad & j'' \equiv j'[\delta], \\ -\delta < i'' - i < \delta \quad & -\delta < j'' - j < \delta, \end{aligned}$$

(see Fig. 8). From Proposition 3.1 it follows $j' \equiv ki'$ $[\delta]$. Therefore, $j'' \equiv ki''$ $[\delta]$ and so there

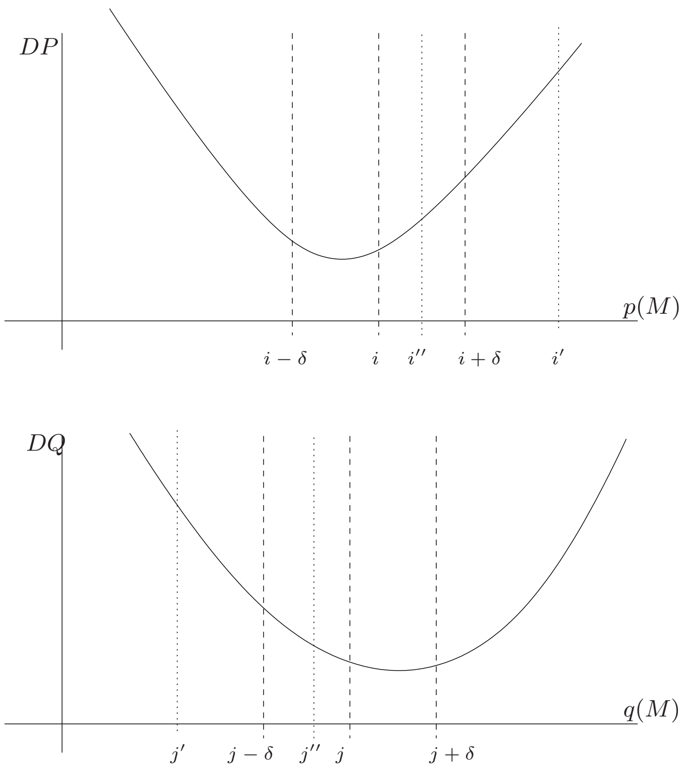


Figure 8: The functions DP and DQ.

is a point P such that $p(P) = i''$ and $q(P) = j''$. From the definitions of i'' and j'' we deduce that $P \in M + \mathcal{N}$ and so $DP_{i''} + DQ_{j''} = D(P) \geqslant D(M) = DP_i + DQ_j$.
We assume that $i' \geqslant i + \delta$. If $p(\vec{u}) = +\delta$ and $q(\vec{u}) = 0$, then $D(M + \vec{u}) = DP_{i+\delta} + DQ_j$. Since $(+\delta, 0) \in \mathcal{N}'$, from condition (3.1) we have $D(M) \leqslant D(M + \vec{u})$ and so $DP_i \leqslant DP_{i+\delta}$. The function DP is a decreasing then increasing function and so DP is increasing on $]i + \delta, +\infty[$. Consequently, $DP_i \leqslant DP_{i+\delta} \leqslant DP_{i'}$. From $\text{sign}(i'' - i) = \text{sign}(i' - i)$, $-\delta < i'' - i < \delta$ and $i' \geqslant i + \delta$, it follows that $i \leqslant i'' \leqslant i'$. Since function DP can be an increasing function, or a decreasing then increasing function on $[i, i']$, we have:

$$DP_{i''} \leqslant \max(DP_i, DP_{i'}) = DP_{i'}.$$

If $i' \leqslant i - \delta$, we can prove similarly that $DP_{i''} \leqslant DP_{i'}$. Finally, if $i' \in]i - \delta, i + \delta[$, we get $i' = i''$, that is $DP_{i''} = DP_{i'}$. So, we obtain $DP_{i'} \geqslant DP_{i''}$ for each position of i'.
By proceeding analogously we get $DQ_{j'} \geqslant DQ_{j''}$. Therefore:

$$D(N) = DP_{i'} + DQ_{j'} \geqslant DP_{i''} + DQ_{j''} \geqslant DP_i + DQ_j = D(M)$$

and so M is a global minimum of D. □

Remark 1 If we take $p = x$ and $q = y$ and $\alpha = \beta = 1$, then the distance d is the *Manhattan distance* d_1 which is defined by $d_1(M, N) = |x_M - x_N| + |y_M - y_N|$. In this case, the set $\mathcal{N}' = \{(\pm 1, 0), (0, \pm 1)\}$ and so the point M with $p(M) = i$ and $q(M) = j$ is a median of $\mathcal{S}$ if and only if

$$DP_i \leqslant DP_{i-1}, \quad DP_i \leqslant DP_{i+1}, \qquad DQ_j \leqslant DQ_{j-1}, \qquad DQ_j \leqslant DQ_{j+1}$$

which can be rewritten by the formulas (3.3) and (3.4) as follows:

$$SP_{i-1} \leqslant \frac{\text{card}(\mathcal{S})}{2} \leqslant SP_i, \quad SQ_{j-1} \leqslant \frac{\text{card}(\mathcal{S})}{2} \leqslant SQ_j. \tag{3.6}$$

The medians according to Manhattan distance have been studied in [7] and its Theorem 3.1 has been found.

Remark 2 If we take the diagonal directions $p = x + y$ and $q = x - y$ and $\alpha = \beta = 1$, then the distance d is the distance $2d_\infty$ where $d\infty$ is defined by $d_\infty(M, N) = \max(|x_M - x_N|, |y_M - y_N|)$. The set $\mathcal{N}'$ is $\{(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1)\}$ and from the formulas (3.3) and (3.4) it follows that: a point M is a median for this distance if and only if it verifies the eight inequalities:

$$\begin{aligned} SP_{i-2} + SP_{i-1} \leqslant \text{card}(\mathcal{S}) \leqslant SP_i + SP_{i+1} \\ SQ_{j-2} + SQ_{j-1} \leqslant \text{card}(\mathcal{S}) \leqslant SQ_j + SQ_{j+1} \\ SP_i + SQ_j \geqslant \text{card}(\mathcal{S}) \quad SP_i + SQ'_j \geqslant \text{card}(\mathcal{S}) \\ SP'_i + SQ_j \geqslant \text{card}(\mathcal{S}) \quad SP'_i + SQ'_j \geqslant \text{card}(\mathcal{S}) \end{aligned} \tag{3.7}$$

Theorem 3.2 gives a very fast algorithm to decide if a given point is a median point of $\mathcal{S}$. Moreover, we deduce a simple method to find all the medians of $\mathcal{S}$. From formulas (3.3) and (3.4) we deduce that condition (3.1) is equivalent to 4δ inequalities whose terms are the projections of $\mathcal{S}$ along directions p and q. For instance, if $p = x + y$ and $q = x - y$ and $\alpha = \beta = 1$, then condition (3.1) is equivalent to 8 inequalities (3.7) whose terms are SP_i and SQ_j (i.e., the partial sums of the projections of $\mathcal{S}$ along the diagonal directions). Therefore, if we want to determine all the median points of $\mathcal{S}$, we only have to determine the projections rp and rq of $\mathcal{S}$, evaluate the partial sums SP_i and SQ_j of these projections vectors, and determine the lines satisfying the inequalities on SP_i and SQ_j equivalent to condition (3.1).

Example 1 Let us take the discrete set in Fig. 9 and the diagonal directions $p = x + y$ and $q = x - y$ into consideration. We have:

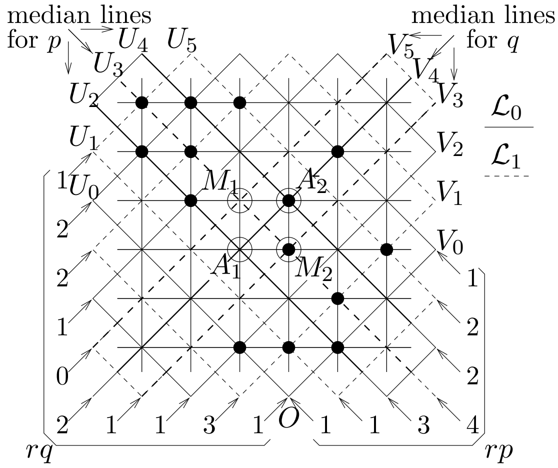


Figure 9: The median points of a discrete set according to d_∞ distance.

$$
\begin{aligned}
rp &= (1, 1, 3, 4, 2, 2, 1),\\
rq &= (1, 3, 1, 1, 2, 0, 1, 2, 2, 1),\\
SP &= (1, 2, 5, 9, 11, 13, 14),\\
SQ &= (1, 4, 5, 6, 8, 8, 9, 11, 13, 14).
\end{aligned}
$$

Moreover, $\text{card}(\mathcal{S}) = 14$, $SP'_i = 14 - SP_{i-1}$ and $SQ'_j = 14 - SQ_{j-1}$. There are only two points satisfying the inequalities (3.7): M_1 with $p(M_1) = 3$ and $q(M_1) = 3$

$$
\begin{aligned}
&7 = SP_1 + SP_2 \leqslant 14 \leqslant SP_3 + SP_4 = 20,\\
&9 = SQ_1 + SQ_2 \leqslant 14 \leqslant SQ_3 + SQ_4 = 14,\\
&15 = SP_3 + SQ_3 \geqslant 14 \quad 18 = SP_3 + SQ'_3 \geqslant 14\\
&15 = SP'_3 + SQ_3 \geqslant 14 \quad 18 = SP'_3 + SQ'_3 \geqslant 14,
\end{aligned}
$$

and M_2 with $p(M_2) = 3$ and $q(M_2) = 5$

$$
\begin{aligned}
&7 = SP_1 + SP_2 \leqslant 14 \leqslant SP_3 + SP_4 = 20,\\
&14 = SQ_3 + SQ_4 \leqslant 14 \leqslant SQ_5 + SQ_6 = 17,\\
&17 = SP_3 + SQ_5 \geqslant 14 \quad 15 = SP_3 + SQ'_5 \geqslant 14\\
&17 = SP'_3 + SQ_5 \geqslant 14 \quad 15 = SP'_3 + SQ'_5 \geqslant 14.
\end{aligned}
$$

Therefore, the medians point of this set according to d_∞ are M_1 and M_2.

In [7] the authors introduce the concept of median row and column of a discrete set $\mathcal{S}$, and show that the median points of $\mathcal{S}$ according to Manhattan distance are the intersection of median rows and columns (see Theorem 3.1 in [7]). We can generalize this concept:

- the line U_i, whose equation is: $p(M) = i$, is a *median line for* p of $\mathcal{S}$ if $DP_{i\pm\delta} \geqslant DP_i$,

- the line V_j, whose equation is: $q(M) = j$, is a *median line for* q of $\mathcal{S}$ if $DQ_{j\pm\delta} \geqslant DQ_j$.

If a point M is the intersection of a median line for p with a median line for q, then:

$$DP_{p(M)\pm\delta} \geqslant DP_{p(M)}, \quad DQ_{q(M)\pm\delta} \geqslant DQ_{q(M)}. \tag{3.8}$$

From Theorem 3.2 we deduce that M is a minimum of the function D on a p-q lattice $\mathcal{L}_l$. Since the number of p-q lattice is δ, we have at least δ of these minima. The median points are the smallest values of these minima points.
If $p = x$, $q = y$, there is only one p-q lattice and so the median points of $\mathcal{S}$ are the intersection of the median lines for p with the median lines for q. We note that the lines U_i and V_j are median lines for p and q if they verify inequalities (3.6).
If $p = x + y$, $q = x - y$ and $\alpha = \beta = 1$, there are two p-q lattice: $\mathcal{L}_0$ and $\mathcal{L}_1$. If a point is the intersection of a median line for p with a median line for q, then it is a minimum of D on $\mathcal{L}_0$ or $\mathcal{L}_1$. For instance, the discrete set in Fig. 9 has three median lines for p: U_2, U_3 and U_4, and three median line for q: V_3, V_4 and V_5. Therefore:

- the minima of D on $\mathcal{L}_0$ are $A_1 = U_2 \cap V_4$ and $A_2 = U_4 \cap V_4$,
- the minima of D on $\mathcal{L}_1$ are $M_1 = U_3 \cap V_3$ and $M_2 = U_3 \cap V_5$.

Since $D(A_1) = D(A_2) = 29$ and $D(M_1) = D(M_2) = 28$, the median points of this discrete sets are M_1 and M_2. We note that the lines U_i and V_j are median lines for $p = x + y$ and $q = x - y$ if they verify the first two conditions of (3.7). By means of the last two conditions of (3.7) we determine which point between A_i and M_i has the smallest values of D.

4 The number of median points

In this section, we are going to show that the discrete sets verifying some connectivity properties have at most four median points, and if there are four median points they form a parallelogram. Theorem 3.2 ensures that all the medians points are intersections of a median line for p with a median line for q. Moreover, this condition is not sufficient for a pair of directions different from $p = x$ and $q = y$ (see previous example). Therefore, the number of median points is equal or smaller than the product of the numbers of median lines along each direction.

Let us assume that $p = x$ and $q = y$. In this case, the median points of a discrete set $\mathcal{S}$ according to this distance are the intersection of the median lines for p with the median lines for q. So, the number of median points is exactly the product of the numbers of median lines along each direction. The line U_i is median lines for p if $SP_{i-1} \leqslant \text{card}(\mathcal{S})/2 \leqslant SP_i$. If $\mathcal{S}$ has at least one point in each vertical line, SP is a strictly increasing function, and so $\mathcal{S}$ has one or two consecutive median lines for p. Consequently, if $\mathcal{S}$ has at least one point in each vertical and horizontal line, then it has at most four medians points.

Example 2 Let us take the discrete set in Fig. 10 into consideration. It has at least one point in each vertical and horizontal line. Moreover, $rp = (3, 2, 3, 3, 2, 3)$ and $rq = (1, 3, 4, 1, 4, 3)$, hence: $\text{card}(\mathcal{S})/2 = 8$, and:

$$\begin{aligned} 5 = SP_1 \leqslant 8 \leqslant SP_2 = 8 \ \text{ and } \ SP_2 \leqslant 8 \leqslant SP_3 = 11, \\ 4 = SQ_1 \leqslant 8 \leqslant SQ_2 = 8 \ \text{ and } \ SQ_2 \leqslant 8 \leqslant SQ_3 = 9, \end{aligned}$$

that is, the lines U_2, U_3, and V_2, V_3 verify inequalities (3.6) and so they are median lines for p and q. Therefore, there are four median points: $M_1 = U_2 \cap V_3$, $M_2 = U_3 \cap V_3$, $M_3 = U_3 \cap V_2$, and $M_4 = U_2 \cap V_2$.

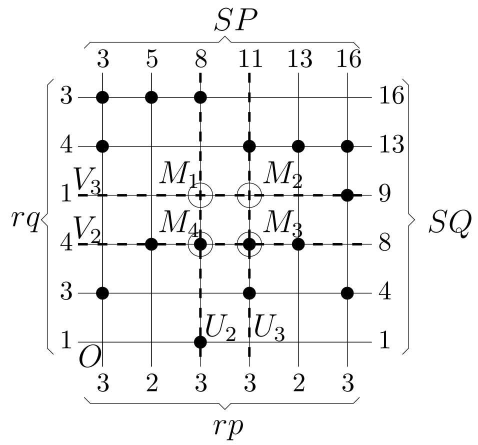


Figure 10: Median points and lines of a discrete set according to Manhattan distance.

In the general case it is not so simple. Take for example $p = x + y$ and $q = x - y$. The line U_i is a median line for p if $SP_{i-2} + SP_{i-1} \leqslant \mathrm{card}(\mathcal{S}) \leqslant SP_i + SP_{i+1}$. If $\mathcal{S}$ has at least one point in each north-west diagonal, SP is a strictly increasing function, and so $\mathcal{S}$ has two or three consecutive median lines for p. It follows that, if $\mathcal{S}$ has at least one point in each north-east and north-west diagonal, it has at most five medians points (see Fig. 11). We will see that, there

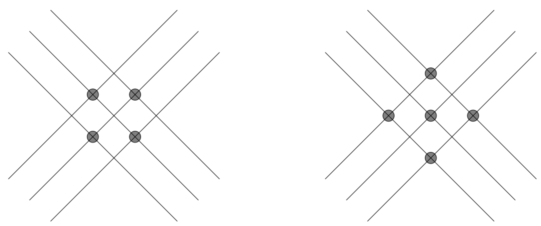


Figure 11: Intersection of 2×3 diagonal consecutive lines.

cannot be five median points because the function D is strictly smaller in the center point than in the four lateral points. We can generalize this result for any directions. At first, we have to define the notion of indivisibility.

Definition 3 *Let p be a linear form and $k \in \mathbb{N}$. A set $\mathcal{S}$ is k-p-indivisible if there are two integers p_1 and p_2 so that:*

$$\forall N \in \mathcal{S} \quad p_1 \leqslant p(N) \leqslant p_2$$

$$\forall i \in \mathbb{Z} \quad p_1 \leqslant i \leqslant p_2 - k + 1 \Longrightarrow \exists N \in \mathcal{S} \quad i \leqslant p(N) \leqslant i + k - 1$$

(see Fig 12). A discrete set is 1-p-indivisible for $p = x$ ($p = x + y$) if it has at least one point in each vertical (diagonal) line. Now, we can show the main result of this section:

Theorem 4.1 *If $\mathcal{S}$ is 1-p-indivisible and 1-q-indivisible discrete set, then $\mathcal{S}$ has at most four median points. Moreover, if $\mathcal{S}$ has exactly four median points they form a parallelogram.*

We introduce some Lemmas for proving this Theorem.

Lemma 4.2 *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function which verifies*

$$\forall x \in \mathbb{Z} \quad D^2 f(x) = f(x+1) - 2f(x) + f(x-1) \geqslant 0,$$

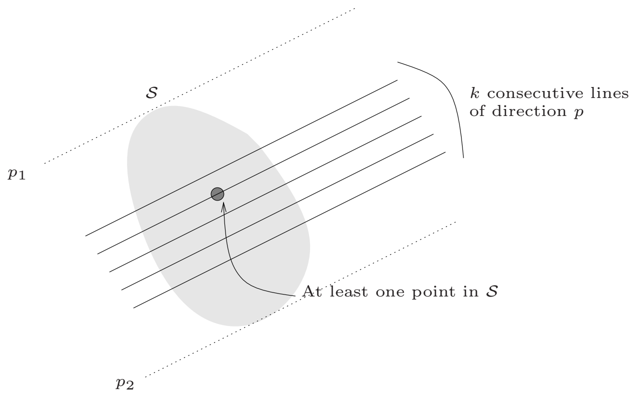


Figure 12: Definition of the k-p-indivisibility

and x_1, x_2, x_3 be three integers such that $x_1 \leqslant x_2 \leqslant x_3$. If $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \in \mathbb{Z}$, where:

$$\lambda_1 > 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 > 0,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 1,$$

assuming that there is an integer $x' \in]x_1, x_3[$ which verifies $D^2 f(x') > 0$, we obtain:

$$f(x) < \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3).$$

Proof : If $i < j$, we have:

$$\frac{f(j) - f(i)}{j - i} = f(i+1) - f(i) + \sum_{k=i+1}^{j-1} \frac{j-k}{j-i} D^2 f(k) \geqslant f(i+1) - f(i), \tag{4.1}$$

else if $j < i$:

$$\frac{f(i) - f(j)}{i - j} = f(i+1) - f(i) - \sum_{k=j+1}^{i} \frac{k-j}{i-j} D^2 f(k) \leqslant f(i+1) - f(i). \tag{4.2}$$

Assuming that $x \in]x_1, x_2]$, the equations (4.1) and (4.2) provide:

$$\frac{f(x) - f(x_1)}{x - x_1} \leqslant f(x+1) - f(x) \leqslant \frac{f(x_2) - f(x)}{x_2 - x}$$
$$\frac{f(x) - f(x_1)}{x - x_1} \leqslant f(x+1) - f(x) \leqslant \frac{f(x_3) - f(x)}{x_3 - x}$$

The existence of $x' \in]x_1, x_3[$ which verifies $D^2 f(x') > 0$ gives:

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_3) - f(x)}{x_3 - x},$$

and so:

$$(x_2 - x_1) f(x) \leqslant (x_2 - x) f(x_1) + (x - x_1) f(x_2), \tag{4.3}$$
$$(x_3 - x_1) f(x) < (x_3 - x) f(x_1) + (x - x_1) f(x_3). \tag{4.4}$$

The integer $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, where λ_1, λ_2, and λ_3 are three positive or null rational numbers whose sum is 1. By setting $\alpha = \lambda_2/\lambda_3$, we obtain:

$$\lambda_1 = \frac{1}{\beta}(\alpha x_2 + x_3 - (1+\alpha)x),$$
$$\lambda_2 = \frac{1}{\beta}(\alpha(x - x_1)),$$
$$\lambda_3 = \frac{1}{\beta}(x - x_1),$$

where $\beta = \alpha x_2 + x_3 - (1+\alpha)x_1$. By making $\alpha(4.3) + (4.4)$ we have:

$$(\alpha x_2 + x_3 - (1+\alpha)x_1)f(x) < (\alpha x_2 + x_3 - (1+\alpha)x)f(x_1) + \alpha(x - x_1)f(x_2) + (x - x_1)f(x_3)$$

which is exactly:

$$f(x) < \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3).$$

□

Remark 3 *This lemma is much easier in $\mathbb{R}$ (see for example [12]). Unfortunately, we cannot make the same proof here because the number $\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}$ is not in $\mathbb{Z}$ in general.*

From equations (3.5) it follows that we can apply this lemma to the functions DP and DQ, and since $D(M) = DP_{p(M)} + DQ_{q(M)}$, we get:

Lemma 4.3 *Let $\mathcal{S}$ be a 1-p-indivisible and 1-q-indivisible discrete set. Let A, B and C be three points of $\mathbb{Z}^2$ which are not all equal. Let $M = \lambda_A A + \lambda_B B + \lambda_C C \in \mathbb{Z}^2 \setminus \{A, B, C\}$, where λ_A, λ_B and λ_C are three positive rational numbers whose sum is 1. Then:*

$$D(M) < \lambda_A D(A) + \lambda_B D(B) + \lambda_C D(C)$$

Remark 4 *We point out that it is possible to assume that $\mathcal{S}$ is only $(2\det(p,q)-1)$-p-indivisible and 1-q-indivisible in this lemma. In fact, if we do not have $q(A) = q(B) = q(C)$ then $DQ(M) < \lambda_A DQ(A) + \lambda_B DQ(B) + \lambda_C DQ(C)$, that is sufficient to prove the statement of the lemma. If $q(A) = q(B) = q(C)$, $p(A) \leqslant p(B) \leqslant p(C)$ and $M \in]AC[\cap \mathbb{Z}^2$ then we have $p(C) - p(A) \geqslant 2\det(p,q)$. So, if $\mathcal{S}$ is $(2\det(p,q)-1)$-p-indivisible then $DP(M) < \lambda_A DP(A) + \lambda_B DP(B) + \lambda_C DP(C)$.*

The following lemma shows that there is no point in the interior of any triangle whose vertices are medians of a discrete sets.

Lemma 4.4 *Let $\mathcal{E}$ be a subset of $\mathbb{Z}^2$ which verifies the following property:*

$$\forall A, B, C \in \mathcal{E} \quad N \in \mathrm{conv}(A, B, C) \cap \mathbb{Z}^2 \Longrightarrow N \in \{A, B, C\}.$$

Then $\mathcal{E}$ has at most four points. Moreover, if $\mathcal{E}$ has four points, then it is a parallelogram.

Proof : Let A, B, C, M be four distinct points of $\mathcal{E}$. Suppose at first that:

$$M \notin \mathcal{I} = \{\alpha A + \beta B + \gamma C \mid \alpha, \beta, \gamma \in \mathbb{Z},\ \alpha + \beta + \gamma = 1\}.$$

We have that $\overrightarrow{AM} = u\overrightarrow{AB} + v\overrightarrow{AC}$, with u and v not both integers. We set M' so that $\overrightarrow{AM'} = u'\overrightarrow{AB} + v'\overrightarrow{AC}$, with $u' = u \mod 1$ and $v' = v \mod 1$.

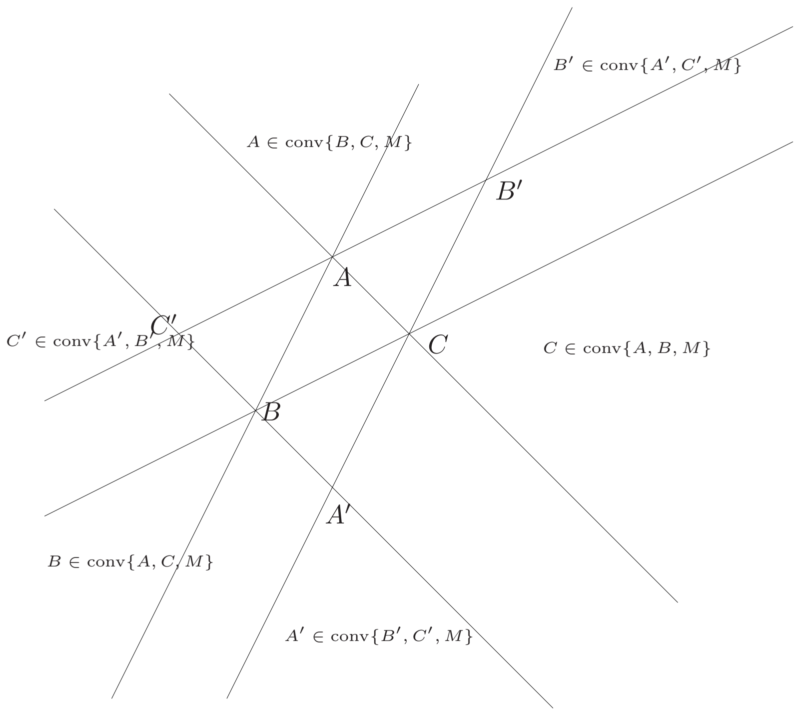


Figure 13: Where can the point M be put ?

If $u'+v' \leqslant 1$, then $M' \in \text{conv}(A, B, C)$. From the property of $\mathcal{E}$, we deduce that $M' \in \{A, B, C\}$ and so $M \in \mathcal{I}$. Therefore, we cannot have $u' + v' \leqslant 1$. We cannot have $u' + v' \geqslant 1$, because in this case $B + \overrightarrow{M'C} \in \text{conv}(A, B, C)$ and so $M \in \mathcal{I}$.
Consequently, we must have $M = \alpha A + \beta B + \gamma C$ with $\alpha, \beta, \gamma \in \mathbb{Z}$, $\alpha + \beta + \gamma = 1$. Assuming that: $A' = -A + B + C$, $B' = A - B + C$, $C' = A + B - C$. We are going to prove that $M \in \{A', B', C'\}$. Suppose that it is not the case and take the following different cases into consideration (see Fig. 13).

- If $\alpha \geqslant 1$, $\beta \leqslant 0$, $\gamma \leqslant 0$, then $A \in \text{conv}(B, C, M)$ and so $A \in \{B, C, M\}$ for the property of $\mathcal{E}$. Since A, B, C and M are four distinct point of $\mathcal{E}$ this is impossible.
- If $\alpha \leqslant 0$, $\beta \geqslant 1$, $\gamma \leqslant 0$, then $B \in \text{conv}(A, C, M)$ and this is impossible.
- If $\alpha \leqslant 0$, $\beta \leqslant 0$, $\gamma \geqslant 1$, then $C \in \text{conv}(A, B, M)$ and this is impossible.
- If $\alpha \leqslant -1$, $\beta \geqslant 1$, $\gamma \geqslant 1$, then $A' \in \text{conv}(B, C, M)$ and this is impossible.
- If $\alpha \geqslant 1$, $\beta \leqslant -1$, $\gamma \geqslant 1$, then $B' \in \text{conv}(A, C, M)$ and this is impossible.
- If $\alpha \geqslant 1$, $\beta \geqslant 1$, $\gamma \leqslant -1$, then $C' \in \text{conv}(A, B, M)$ and this is impossible.

Therefore, $M \in \{A', B', C'\}$ and so:

$$\{A, B, C\} \subset \mathcal{E} \subset \{A, B, C, A', B', C'\}$$

We cannot have more than one point in the set $\mathcal{E} \cap \{A', B', C'\}$ because the middle of any two of these points is in $\{A, B, C\}$ and again for the property of $\mathcal{E}$ this is impossible. So, if $\mathcal{E} \neq \{A, B, C\}$, it has got four points and it is a parallelogram. □

Finally, we can give the proof of Theorem 4.1.

Proof of Theorem 4.1 : Let $\mathcal{E}$ be the set of the median points of $\mathcal{S}$. If we take three distinct points of $A, B, C \in \mathcal{E}$ and a point $M \in \text{conv}(A, B, C)$, the point M must be in $\{A, B, C\}$. In fact, if $M \notin \{A, B, C\}$, from Lemma 4.3 we have $D(M) < \lambda_A D(A) + \lambda_B D(B) + \lambda_C D(C)$, where $\lambda_A, \lambda_B, \lambda_C$ are three positive rational numbers such that $\lambda_A + \lambda_B + \lambda_C = 1$. Since A, B and C are median points, we have $D(A) = D(B) = D(C)$, and so $D(M) < D(A)$. Consequently, A is not a median point. So $M \in \{A, B, C\}$ and we can apply the last Lemma which gives the statement of the Theorem. □

Remark 5 *We have proved that a strictly convex function from $\mathbb{Z}^2$ to $\mathbb{R}$ reaches its minimum in at most four points.*

Remark 6 *By the remark 4 and Theorem 4.1, if $\mathcal{S}$ is only $(2det(p, q) - 1)$-p-indivisible and 1-q-indivisible then we have also at most four points.*

From the definition of connected set we obtain:

Property 4.5 *Let $\mathcal{S}$ be a discrete set.*

- *If $\mathcal{S}$ is 8-connected and $p = x$ or $p = y$, then $\mathcal{S}$ is 1-p-indivisible.*
- *If $\mathcal{S}$ is 4-connected and $p = x + y$ or $p = x - y$, then $\mathcal{S}$ is 1-p-indivisible.*

Proof : Consider a point $A \in \mathcal{S}$ which minimizes p and a point B which maximizes p. Since $\mathcal{S}$ is k-connected ($k = 4$ for the first case and $k = 8$ for the second case), there is a k-path $\mathcal{C} \subset \mathcal{S}$ which connects the points A and B. The k-path $\mathcal{C}$ has a common point with all the lines whose direction is p and which are between A and B. Therefore, $\mathcal{S}$ is 1-p-indivisible. □

From Theorem 4.1 and Property 4.5, we obtain the following result:

Corollary 4.6 *Let $\mathcal{S}$ be a discrete set.*

- *If $\mathcal{S}$ is 8-connected then $\mathcal{S}$ has at most four median points for d_1.*
- *If $\mathcal{S}$ is 4-connected then $\mathcal{S}$ has at most four median points for d_∞.*

Moreover, if $\mathcal{S}$ has exactly four median points, they form a parallelogram.

5 Membership of the diagonal medians

In [7], the authors have proved that a 4-connected set which is convex along the horizontal and vertical directions contain its median points along these directions. We have a similar result for the median points according to d_∞.

Theorem 5.1 *If $\mathcal{S}$ is 4-connected diagonally convex set, it contains its median points according to the distance d_∞*

Proof : Let us assume that $\mathcal{S}$ is a subset of $\mathbb{Z}^2$ which satisfies the conditions of the theorem. Let M be a median of $\mathcal{S}$ for the distance d_∞. In this case, $p = x + y$ and $q = x - y$ and we define four regions around M (see Fig. 14) as follows:

$$R_0 = \{N \ / \ p(N) \leqslant p(M) \text{ and } q(N) \leqslant q(M)\}$$
$$R_1 = \{N \ / \ p(N) \geqslant p(M) \text{ and } q(N) \leqslant q(M)\}$$
$$R_2 = \{N \ / \ p(N) \geqslant p(M) \text{ and } q(N) \geqslant q(M)\}$$
$$R_3 = \{N \ / \ p(N) \leqslant p(M) \text{ and } q(N) \geqslant q(M)\}$$

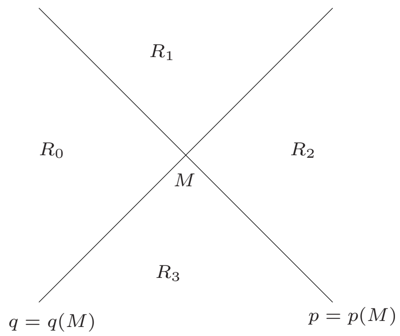


Figure 14: The four regions.

We point out that M belongs to each region. Since M is a median point, we have the equations (3.7):

$$\begin{array}{ll} SP_i + SQ_j \geqslant \mathrm{card}(\mathcal{S}) & SP_i + SQ'_j \geqslant \mathrm{card}(\mathcal{S}) \\ SP'_i + SQ_j \geqslant \mathrm{card}(\mathcal{S}) & SP'_i + SQ'_j \geqslant \mathrm{card}(\mathcal{S}) \end{array}$$

where $i = p(M)$ and $j = q(M)$.
Suppose that there is no point of $\mathcal{S}$ in R_0. Then, $M \notin \mathcal{S}$ and $SP_i + SQ_j = \mathrm{card}(\mathcal{S} \cap (R_1 \cup R_3))$. From the first inequality, it follows that $\mathrm{card}(\mathcal{S} \cap (R_1 \cup R_3)) \geqslant \mathrm{card}(\mathcal{S})$, and so $\mathcal{S} \subset R_1 \cup R_3$ which is impossible because $\mathcal{S}$ is 4-connected and $M \notin \mathcal{S}$. Therefore the set $\mathcal{S} \cap R_0$ is not empty. By proceeding in the same way, we can prove that the set $\mathcal{S} \cap R_i \neq \varnothing$ for all i.
Let B_i be the 4-border of the region R_i. This borders are unions of two diagonal semi-lines: $B_i = \Delta_i \cup \Delta_{i+1}$ (see Fig. 15). Since $\mathcal{S}$ is 4-connected and the sets $\mathcal{S} \cap R_i$ are not empty, there

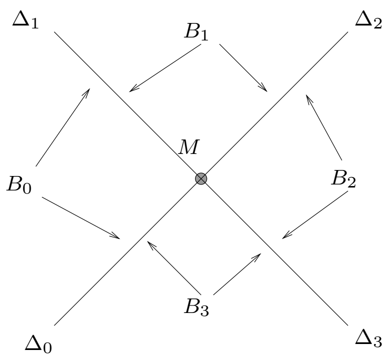


Figure 15: The four borders.

is at least a point of $\mathcal{S}$ in each border B_i. That is, $\mathcal{S} \cap \Delta_i \neq \varnothing$ or $\mathcal{S} \cap \Delta_{i+1} \neq \varnothing$.
Consider $\mathcal{S} \cap B_0$ and suppose that $\mathcal{S} \cap \Delta_0 \neq \varnothing$. If $\mathcal{S} \cap \Delta_2 \neq \varnothing$, then $M \in \mathcal{S}$ follows from diagonal convexity. Conversely, if $\mathcal{S} \cap \Delta_2 = \varnothing$, since $\mathcal{S} \cap B_1 \neq \varnothing$ and $B_1 = (\Delta_1 \cup \Delta_2)$, we have $\mathcal{S} \cap \Delta_1 \neq \varnothing$. Moreover, $\mathcal{S} \cap B_2 \neq \varnothing$ and $B_2 = (\Delta_2 \cup \Delta_3)$ and so $\mathcal{S} \cap \Delta_3 \neq \varnothing$. By diagonal convexity of $\mathcal{S}$, $M \in \mathcal{S}$.
Consequently, the median point M is in $\mathcal{S}$. □

We could expect that we have a similar membership property for any couple of directions, but unfortunately we cannot simply generalize the previous Theorem. For example, if we take $d(M,N) = |p(\overrightarrow{MN})| + |q(\overrightarrow{MN})|$ with $p = x$ and $q = x + 4y$, then the set of the figure 16 does

not contain all its medians along d.
In fact, if we want to have membership property Theorem, we must impose a "stronger" convexity to the sets (see [6]).

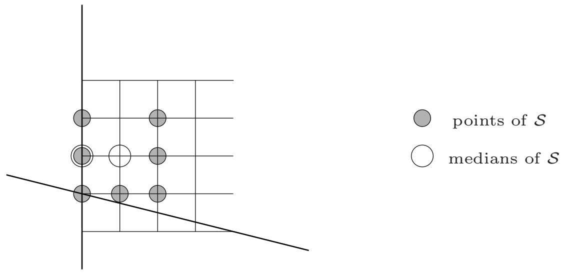


Figure 16: A set which does not contain its medians.

6 Acknowledgements

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