Self-inductance coefficient for toroidal thin conductors

Youcef Amirat, Rachid Touzani

Self–inductance coefficient for toroidal thin conductors

Y. Amirat and R. Touzani

Laboratoire de Mathématiques Appliquées, UMR CNRS 6620
Université Blaise Pascal (Clermont–Ferrand 2)
63177 Aubière cedex, France

Abstract

We consider the inductance coefficient for a thin toroidal inductor whose thickness depends on a small parameter $\varepsilon > 0$. An explicit form of the singular part of the corresponding potential u^ε is given. This allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance coefficient L^ε. We establish some estimates of the deviation $u^\varepsilon - u$ and of the error of approximation of the inductance. The main result shows that L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \to 0$.

Key words: Asymptotic behaviour, self inductance, eddy currents, thin domain

1 Introduction

In electrotechnical engineering, eddy current devices often involve thick conductors in which a magnetic field is induced, and circuits made of thin wires or coils, as inductors, connected to a power source generator. Mathematical modelling of such devices has then to take into account the simultaneous presence of thick conductors and thin inductors. For a two–dimensional configuration where the magnetic field has only one nonvanishing component, it was shown that the eddy current equation has the Kirchhoff circuit equation as a limit problem, as the thickness of the inductor tends to zero, see [1]. For the three–dimensional case, eddy current models require the use of a relevant quantity that is the self–inductance of the inductor, see [2], [3]. This number has to be evaluated *a priori* as a part of problem data. It is the purpose of the present

Email addresses: Youcef.Amirat@math.univ-bpclermont.fr (Y. Amirat), Rachid.Touzani@math.univ-bpclermont.fr (R. Touzani).

paper to study the asymptotic behaviour of this number when the thickness of the inductor goes to zero.

Let us consider a toroidal domain of $\mathbb{R}^3$, denoted by Ω_ε, whose thickness depends on a small parameter $\varepsilon > 0$. The geometry of Ω_ε will be described in the next section. We denote by Γ_ε the boundary of Ω_ε, by n_ε the outward unit normal to Γ_ε, and by Ω'_ε the complement of its closure, that is $\Omega'_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}_\varepsilon$. We denote by Σ a cut in the domain Ω'_ε, that is, Σ is a smooth orientable surface such that, for any $\varepsilon > 0$, $\Omega'_\varepsilon \setminus \Sigma$ is simply connected.

Let now $\boldsymbol{h}^\varepsilon$ denote the time–harmonic and complex–valued magnetic field. Neglecting the displacement currents, it follows from Maxwell's equations that

$$\mathbf{curl}\, \boldsymbol{h}^\varepsilon = 0, \ \operatorname{div} \boldsymbol{h}^\varepsilon = 0 \quad \text{in } \Omega'_\varepsilon.$$

Then, by a result in [4], p. 265, $\boldsymbol{h}^\varepsilon$ may be written in the form

$$\boldsymbol{h}^\varepsilon_{|\Omega'_\varepsilon} = \nabla \varphi^\varepsilon + I^\varepsilon \nabla u^\varepsilon, \tag{1}$$

where I^ε is a complex number, $\varphi^\varepsilon \in W^1(\Omega'_\varepsilon)$ and satisfies

$$\Delta \varphi^\varepsilon = 0 \qquad \text{in } \Omega'_\varepsilon,$$

and u^ε is solution of :

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0. & \end{cases} \tag{2}$$

Here $W^1(\Omega'_\varepsilon)$ is the Sobolev space

$$W^1(\Omega'_\varepsilon) = \left\{ v; \ \rho v \in L^2(\Omega'_\varepsilon), \ \nabla v \in \boldsymbol{L}^2(\Omega'_\varepsilon) \right\},$$

equipped with the norm

$$\|v\|_{W^1(\Omega'_\varepsilon)} = \left(\|\rho v\|^2_{L^2(\Omega'_\varepsilon)} + \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \right)^{\frac{1}{2}}, \tag{3}$$

where $\boldsymbol{L}^p(\Omega'_\varepsilon)$ denotes the space $L^p(\Omega'_\varepsilon)^3$ and ρ is the weight function $\rho(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{-\frac{1}{2}}$. Let us note here, see [4], pp. 649–651, that

$$|v|_{W^1(\Omega'_\varepsilon)} = \left(\int_{\Omega'_\varepsilon} |\nabla v|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}}$$

is a norm on $W^1(\Omega'_\varepsilon)$, equivalent to (3). In (2), n is the unit normal on Σ, and $[u^\varepsilon]_\Sigma$ (resp. $\left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma$) denotes the jump of u^ε (resp. $\dfrac{\partial u^\varepsilon}{\partial n}$) across Σ.

In (1), the number I^ε can be interpreted as the total current flowing in the inductor, see [3].

The inductance coefficient is then defined by the expression

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}.$$

Our goal is to study the asymptotic behaviour of u^ε and L^ε as ε goes to zero. We first give an explicit form of the singular part of the potential u^ε which allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance L^ε. We then prove that the deviation $\|u^\varepsilon - u\|_{W^1(\Omega'_\varepsilon)}$ and the error of approximation of L^ε are of order $O(\varepsilon^{\frac{5}{6}-\eta})$ for every $\eta > 0$. Finally we show that the inductance coefficient L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \to 0$, and we thus recover the result stated (without proof) in [5], p. 137.

Let us outline the organization of this paper. In Section 2 we specify the geometry of the inductor, assuming that it is a toroidal neighborhood of a closed curve, the internal radius of the torus being proportional to a small positive number ε. Section 3 states the main result and gives the main steps in its proof. Let us note here that an extended version of this paper with detailed proofs can be consulted in [6].

2 Geometry of the domain

We consider a toroidal domain, with a small cross section. This domain may be defined as a tubular neighborhood of a closed curve. Let γ denote a closed Jordan arc of class $\mathcal{C}^3$ in $\mathbb{R}^3$, with a parametric representation defined by a function $\boldsymbol{g} : [0,1] \to \mathbb{R}^3$ satisfying

$$\boldsymbol{g}(0) = \boldsymbol{g}(1), \;\; \boldsymbol{g}'(0) = \boldsymbol{g}'(1), \;\; |\boldsymbol{g}'(s)| \ge C_0 > 0. \tag{4}$$

For each $s \in (0,1]$ we denote by $(\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s))$ the Serret–Frénet coordinates at the point $\boldsymbol{g}(s)$, *i.e.*, $\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s)$ are respectively the unit tangent vector to γ, the principal normal and the binormal, given by

$$\boldsymbol{t} = \frac{\boldsymbol{g}'}{|\boldsymbol{g}'|}, \;\; \boldsymbol{\nu} = \frac{\boldsymbol{t}'}{|\boldsymbol{t}'|}, \;\; \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{\nu},$$

and by κ and τ respectively the curvature and the torsion of the arc γ.

Let $\widehat{\Omega} = (0,1)^2 \times (0, 2\pi)$ and let δ denote a positive number to be chosen in a convenient way. We define, for any ε, $0 \le \varepsilon < \delta$, the mapping $\boldsymbol{F}_\varepsilon : \widehat{\Omega} \to \mathbb{R}^3$ by

$$\boldsymbol{F}_\varepsilon(s, \xi, \theta) = \boldsymbol{g}(s) + r_\varepsilon(\xi)(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s)),$$

where $r_\varepsilon(\xi) = (\delta - \varepsilon)\xi + \varepsilon$. The jacobian of $\boldsymbol{F}_\varepsilon$ is therefore given by

$$J_\varepsilon(s, \xi, \theta) = (\delta - \varepsilon) a_\varepsilon(s, \xi, \theta) r_\varepsilon(\xi),$$

where

$$a_\varepsilon(s, \xi, \theta) = |\boldsymbol{g}'(s)| - r_\varepsilon(\xi)\kappa(s)\cos\theta.$$

According to (4), if δ is chosen such that

$$\delta|\kappa(s)| < |\boldsymbol{g}'(s)|, \qquad 0 \le s \le 1,$$

then

$$0 < C_1 \le a_\varepsilon \le C_2,$$

and the mapping $\boldsymbol{F}_\varepsilon$ is a $\mathcal{C}^1$–diffeomorphism from $\widehat{\Omega}$ into $\Lambda_\varepsilon^\delta = \boldsymbol{F}_\varepsilon(\widehat{\Omega})$.

Here and in the sequel, the quantities $C, C_1, C_2, \ldots$ denote generic positive numbers that do not depend on ε.

We now set, for any $0 < \varepsilon < \delta$,

$$\Omega_\delta = \Lambda_0^\delta = \boldsymbol{F}_0(\widehat{\Omega}),\ \Omega'_\delta = \mathbb{R}^3 \setminus \overline{\Omega}_\delta,\ \Omega'_\varepsilon = \mathrm{Int}(\overline{\Omega}'_\delta \cup \overline{\Lambda}_\varepsilon^\delta),\ \Omega_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}'_\varepsilon.$$

For technical reasons, we choose in the sequel $0 < \varepsilon \le \frac{\delta}{2}$.

Given a function v on $\Lambda_\varepsilon^\delta$, we define the function $\widehat{v}$ on $\widehat{\Omega}$ by $\widehat{v} = v \circ \boldsymbol{F}_\varepsilon$. If $v \in L^p(\Lambda_\varepsilon^\delta)$, $1 \le p \le \infty$, then $\widehat{v} \in L^p(\widehat{\Omega})$ and we have

$$\int_{\Lambda_\varepsilon^\delta} v\, d\boldsymbol{x} = \int_{\widehat{\Omega}} \widehat{v}\,(\delta - \varepsilon)\, a_\varepsilon r_\varepsilon\, d\widehat{\boldsymbol{x}}.$$

Moreover, for u and v in $H^1(\Lambda_\varepsilon^\delta)$, we have

$$\begin{aligned}\int_{\Lambda_\varepsilon^\delta} \nabla u.\nabla v\, d\boldsymbol{x} = (\delta - \varepsilon) \int_{\widehat{\Omega}} \Bigg(& \frac{r_\varepsilon}{a_\varepsilon} \frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial s} + \frac{r_\varepsilon a_\varepsilon}{(\delta - \varepsilon)^2} \frac{\partial \widehat{u}}{\partial \xi} \frac{\partial \widehat{v}}{\partial \xi} \\ & + \left(\frac{a_\varepsilon}{r_\varepsilon} + \frac{\tau^2 r_\varepsilon}{a_\varepsilon} \right) \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial \theta} \\ & - \frac{r_\varepsilon \tau}{a_\varepsilon} \left(\frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial \theta} + \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial s} \right) \Bigg) d\widehat{\boldsymbol{x}}.\end{aligned}$$

We also define the set $\widehat{\Gamma} = (0,1) \times (0, 2\pi)$ and the mapping $\boldsymbol{G}_\varepsilon : \widehat{\Gamma} \to \mathbb{R}^3$ by

$$\boldsymbol{G}_\varepsilon(s, \theta) = \boldsymbol{g}(s) + \varepsilon(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s)).$$

The boundary of Ω'_ε is then represented by $\Gamma_\varepsilon = \overline{\boldsymbol{G}_\varepsilon(\widehat{\Gamma})}$. If $w \in L^2(\Gamma_\varepsilon)$, we define $\widehat{w} \in L^2(\widehat{\Gamma})$ by $\widehat{w} = w \circ \boldsymbol{G}_\varepsilon$, and we have

$$\int_{\Gamma_\varepsilon} w \, d\sigma = \int_{\widehat{\Gamma}} \widehat{w} \, \varepsilon(|\boldsymbol{g}'| - \varepsilon\kappa\cos\theta) \, d\widehat{\sigma}.$$

Clearly, Ω_ε and its complement Ω'_ε are connected domains but they are not simply connected. To define a cut in Ω'_ε, we denote by Σ_0 the set $\boldsymbol{F}_0((0,1)^2 \times \{0\})$ and $\partial\Sigma_0 = \boldsymbol{F}_0((0,1) \times \{1\} \times \{0\})$. Let Σ' denote a smooth simple surface that has $\partial\Sigma_0$ as a boundary and such that the surface $\Sigma = \Sigma' \cup \Sigma_0$ is oriented and of class $\mathcal{C}^1$ (cf. [7]). We denote by Σ^+ (*resp.* Σ^-) the oriented surface with positive (*resp.* negative) orientation, and by n the unit normal on Σ directed from Σ^+ to Σ^-. If $w \in W^1(\mathbb{R}^3 \setminus \Sigma)$, we denote by $[w]_\Sigma$ the jump of w across Σ in the direction of n, *i.e.*

$$[w]_\Sigma = w_{|\Sigma^+} - w_{|\Sigma^-}.$$

3 Formulation of the problem and statement of the result

We consider the boundary value problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0, & \end{cases} \tag{5}$$

where n_ε denotes the unit normal on Γ_ε pointing outward Ω'_ε and $\boldsymbol{n}$ is the unit normal on Σ oriented from Σ^+ toward Σ^-. The inductance coefficient is defined by

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}. \tag{6}$$

We want to describe the asymptotic behaviour of u^ε and L^ε as $\varepsilon \to 0$.

We first exhibit a function that has the same singularity as might be expected for the solution of Problem (5) (as $\varepsilon \to 0$). Let us define

$$\widehat{v}(s,\xi,\theta) = \frac{\theta}{2\pi}\,\widehat{\varphi}(\xi), \qquad (s,\xi,\theta) \in \widehat{\Omega},$$

where $\widehat{\varphi} \in \mathcal{C}^2(\mathbb{R})$ and such that

$$\widehat{\varphi}(\xi) = 1 \text{ for } 0 \le \xi \le \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } \xi \ge \frac{3}{4}.$$

We then define $v : \mathbb{R}^3 \to \mathbb{R}$ by :

$$v(\boldsymbol{x}) = \begin{cases} \widehat{v}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \Omega_\delta, \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta. \end{cases}$$

Let us also define

$$\begin{aligned} \widehat{f}(s,\xi,\theta) = \frac{1}{2\pi a_0} & \left(\frac{\kappa \sin\theta}{\delta\xi} - \frac{\tau^2\,\delta\,\xi\,\kappa\,\sin\theta}{a_0^2} - \frac{\partial}{\partial s}\left(\frac{\tau}{a_0}\right)\right)\widehat{\varphi} \\ & + \frac{\theta}{2\pi\,a_0\,\delta^2\,\xi}(2a_0 - |\boldsymbol{g}'|)\,\widehat{\varphi}' + \frac{\theta}{2\pi\,\delta^2}\,\widehat{\varphi}'', \qquad (s,\xi,\theta) \in \widehat{\Omega}, \end{aligned}$$

$$f(\boldsymbol{x}) = \begin{cases} \widehat{f}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \Omega_\delta, \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta, \end{cases}$$

$$\varphi(\boldsymbol{x}) = \begin{cases} \widehat{\varphi}(\xi) & \text{if } \boldsymbol{x} \in \Omega_\delta, \text{ with } (s,\xi,\theta) = \boldsymbol{F}_0^{-1}(\boldsymbol{x}), \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta. \end{cases}$$

By straightforward calculations, we see that function v is solution of

$$\begin{cases} \Delta v = f & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [v]_\Sigma = \varphi, \\ \left[\dfrac{\partial v}{\partial n}\right]_\Sigma = 0. \end{cases} \tag{7}$$

Moreover, it satisfies

$$\frac{\partial v}{\partial n_\varepsilon} = 0 \qquad \text{on } \Gamma_\varepsilon.$$

Furthermore, we have for any $1 \le p < 2$,

$$f \in L^p(\mathbb{R}^3),\ v \in L^\infty(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3 \setminus \Sigma).$$

We note here that $v \notin H^1(\mathbb{R}^3 \setminus \Sigma)$. However, for any ε, $v \in H^1(\Omega'_\varepsilon \setminus \Sigma)$.

Let us now set $w^\varepsilon = u^\varepsilon - v$. We have by subtracting (7) from (5),

$$\begin{cases} -\Delta w^\varepsilon = f & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial w^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [w^\varepsilon]_\Sigma = 1 - \varphi, \\ \left[\dfrac{\partial w^\varepsilon}{\partial n}\right]_\Sigma = 0. \end{cases} \tag{8}$$

We note here that Problem (8) differs from (5) by the value of the jump of the solution across Σ and by the presence of a right-hand side f. However, we notice that $1-\varphi$ vanishes in a neighborhood of $\partial\Sigma$ and then, for Problem (8), the jump of w^ε vanishes in a neighborhood of $\partial\Sigma$.

Now, to study the asymptotic behaviour of w^ε and L^ε as $\varepsilon \to 0$ we consider the following decomposition. Let w_1 denote the solution of

$$\begin{cases} \Delta w_1 = 0 & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [w_1]_\Sigma = 1 - \varphi, & \\ \left[\dfrac{\partial w_1}{\partial n}\right]_\Sigma = 0, & \\ w_1(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{9}$$

Using [4], p. 654, and the fact that $1 - \varphi$ vanishes in a neighborhood of $\partial\Sigma$, we see that Problem (9) has a unique solution in $W^1(\mathbb{R}^3 \setminus \Sigma)$ given by

$$w_1(\boldsymbol{x}) = \frac{1}{4\pi} \int_\Sigma (1 - \varphi(\boldsymbol{y})) \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^3} \, d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Sigma.$$

Then we write $w^\varepsilon = w_1 + w_2^\varepsilon$, where the function w_2^ε is the unique solution, in $W^1(\Omega'_\varepsilon)$, of the exterior Neumann problem

$$\begin{cases} -\Delta w_2^\varepsilon = f & \text{in } \Omega'_\varepsilon, \\ \dfrac{\partial w_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ w_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{10}$$

Finally, let w_2 denote the unique solution in $W^1(\mathbb{R}^3)$ of

$$\begin{cases} -\Delta w_2 = f & \text{in } \mathbb{R}^3, \\ w_2(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to \infty. \end{cases} \tag{11}$$

As it is classical (see [8] for instance) the function w_2 is given by

$$w_2(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

Summarizing the decomposition process of the solution to Problem (5), we have

$$u^\varepsilon = v + w_1 + w_2^\varepsilon \qquad \text{in } \Omega'_\varepsilon \setminus \Sigma,$$

where v, w_1 and w_2^ε are solutions of (7), (9) and (10) respectively.

We now state our main result.

Theorem 3.1 *Let u^ε be the solution of Problem* (5) *and let L^ε be the inductance coefficient defined by* (6). *Let u be the function defined in $\mathbb{R}^3 \setminus \Sigma$ by $u = v + w_1 + w_2$, where v, w_1 and w_2 are solutions of* (7), (9) *and* (11) *respectively. Then, for every $\eta > 0$,*

$$\begin{aligned}
&\|u - u^\varepsilon\|_{W^1(\Omega'_\varepsilon)} = O(\varepsilon^{\frac{5}{6}-\eta}),\\
&L^\varepsilon = -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' - \int_{\mathbb{R}^3} f(w_1+w_2)\,d\boldsymbol{x}\\
&\qquad + \int_\Sigma (1-\varphi)\left(\frac{\partial w_1}{\partial n} + \frac{\partial w_2}{\partial n} + 2\frac{\partial v}{\partial n}\right) d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}),
\end{aligned}$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}}\left(a_0\xi\theta^2(\widehat{\varphi}')^2 + \frac{\delta^2\xi\tau^2}{a_0}\widehat{\varphi}^2\right) d\widehat{\boldsymbol{x}} + \ell_\gamma\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{2\pi\xi}\,d\xi.$$

The next section is devoted to the proof of this result.

3.1 Proof of error estimate

Let $\widetilde{w}_2^\varepsilon = w_2^\varepsilon - w_2$. Clearly $\widetilde{w}_2^\varepsilon = u^\varepsilon - u$, $\widetilde{w}_2^\varepsilon \in W^1(\Omega'_\varepsilon)$ and it satisfies

$$\begin{cases}
\Delta\widetilde{w}_2^\varepsilon = 0 & \text{in } \Omega'_\varepsilon,\\
\dfrac{\partial\widetilde{w}_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} - \dfrac{\partial w_2}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon,\\
\widetilde{w}_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to +\infty.
\end{cases} \tag{12}$$

To estimate the solution of Problem (12), we need the following result.

Lemma 3.1 *There is a constant C, independent of ε, such that :*

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}\,|\ln\varepsilon|^{\frac{1}{2}}\,\|\psi\|_{W^1(\Omega'_\varepsilon)} \quad \text{for all } \psi \in W^1(\Omega'_\varepsilon), \tag{13}$$

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\left(\varepsilon^{\frac{1}{2}}\,\|\psi\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{4}{3}-\frac{2}{p}}\,\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)}\right)$$
$$\text{for all } \psi \in W^{1,p}(\Omega'_\varepsilon) \text{ with compact support, } \frac{3}{2} < p < 2. \tag{14}$$

For the proof we refer to [6].

Using the variational formulation associated with (12), Cauchy–Schwarz inequality and Estimate (13), we deduce

$$
\begin{aligned}
\int_{\Omega'_\varepsilon} |\nabla \widetilde{w}_2^\varepsilon|^2 \, d\boldsymbol{x} &= \int_{\Gamma_\varepsilon} \left(\frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right) \widetilde{w}_2^\varepsilon \, d\sigma \\
&\leq \left\| \frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \|\widetilde{w}_2^\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\
&\leq C \, \varepsilon^{\frac{1}{2}} \, |\ln \varepsilon|^{\frac{1}{2}} \left(\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} + \left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \right) \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}.
\end{aligned}
\tag{15}
$$

Using the integral representation of w_1, we easily check that

$$
\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^\infty(\Gamma_\varepsilon)} \leq C.
$$

Therefore

$$
\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \leq C \, (\operatorname{meas} \Gamma_\varepsilon)^{\frac{1}{2}} \leq C_1 \varepsilon^{\frac{1}{2}}. \tag{16}
$$

To estimate $\dfrac{\partial w_2}{\partial n_\varepsilon}$, we use standard regularity results for elliptic problems, see [9], p. 343, to deduce, since $f \in L^p(\mathbb{R}^3)$ for $p < 2$, that $w_2 \in W^{2,p}_{\text{loc}}(\mathbb{R}^3)$. Then we apply Estimate (14) to the function $u = \dfrac{\partial w_2}{\partial x_i}$, $1 \leq i \leq 3$ with $p = 2 - \eta$, $0 < \eta < \frac{1}{2}$,

$$
\left\| \frac{\partial w_2}{\partial x_i} \right\|_{L^2(\Gamma_\varepsilon)} \leq C \left(\varepsilon^{\frac{1}{2}} \left\| \frac{\partial w_2}{\partial x_i} \right\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}} \left\| \frac{\partial}{\partial x_i} \nabla w_2 \right\|_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)} \right).
$$

Since both norms on the right–hand side of the above inequality are uniformly bounded and since the outward unit normal n_ε is uniformly bounded we obtain

$$
\left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \leq C \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}}. \tag{17}
$$

Substituting (16) and (17) into (15) and using the inequality $|\ln \varepsilon| \leq C\varepsilon^{-2\eta}$, we get

$$
\int_{\Omega'_\varepsilon} |\nabla \widetilde{w}_2^\varepsilon|^2 \, d\boldsymbol{x} \leq C_1 \, \varepsilon^{\frac{5}{6} - \frac{\eta}{2-\eta} - \eta} \, \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}.
$$

Therefore

$$
\|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \leq C_2 \varepsilon^{\frac{5}{6} - \eta} \qquad \text{for all } \eta > 0.
$$

□

Using the decomposition $u^\varepsilon = v + w^\varepsilon = v + w_1 + w_2^\varepsilon$ it follows

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} + \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla w^\varepsilon|^2\,d\boldsymbol{x} + 2\int_{\Omega'_\varepsilon\setminus\Sigma} \nabla v\cdot\nabla w^\varepsilon\,d\boldsymbol{x}.$$

Using Green's formulae, we can write L^ε in the form

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w^\varepsilon\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} + 2\frac{\partial v}{\partial n}\right)d\sigma. \qquad (18)$$

Using the previous estimate for w_2^ε and some regularity results ($w_2 \in W^{2,p}(\Omega_\delta)$, $w_1 \in H^2(\Omega_{\frac{\delta}{2}})$), we can estimate each term in (18), to obtain for all $\eta > 0$,

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} - \int_{\mathbb{R}^3} f w\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\left(\frac{\partial w}{\partial n} + 2\frac{\partial v}{\partial n}\right)d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}),$$

where $w = w_1 + w_2$.

To complete the result, an explicit calculation yields

$$\int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} = -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' + O(\varepsilon),$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}} \left(a_0\xi\theta^2(\widehat{\varphi}')^2 + \frac{\delta^2\xi\tau^2}{a_0}\widehat{\varphi}^2\right)d\widehat{\boldsymbol{x}} + \frac{\ell_\gamma}{2\pi}\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{\xi}\,d\xi.$$

References

[1] R. Touzani, Analysis of an eddy current problem involving a thin inductor, Comp. Methods Appl. Mech. Eng., Vol. 131 (1996) 233–240.

[2] A. Bossavit, Electromagnétisme en vue de la modélisation, Springer–Verlag (1988).

[3] A. Bossavit, J.C. Vérité, The TRIFOU Code : Solving the 3–D eddy–current problem by using h as a state variable, IEEE Transactions on Magnetics, MAG–19, No. 6, (1983) 2465–2470.

[4] R. Dautray, J.L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Tome 2, Masson, Paris (1985).

[5] L. Landau, E. Lifshitz, Electrodynamics of Continuous Media, Pergamon, London (1960).

[6] Y. Amirat, R. Touzani, Asymptotic behaviour of the inductance coefficient for thin conductors, Preprint.

[7] E. Kreyszig, Differential Geometry, Dover Publications, New York (1991).

[8] J.C. Nedelec, Approximation des équations intégrales en mécanique et en physique, Centre de Mathématiques Appliquées, Ecole Polytechnique Palaiseau (1977).

[9] R. Dautray, J.L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Tome 1, Masson, Paris (1984).