On the additive theory of prime numbers II

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Abstract

The undecidability of the additive theory of primes (with identity) as well as the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n denotes the $(n+1)$-th prime, are open questions. As a possible approach, we extend the latter theory by adding some extra function. In this direction we show the undecidability of the existential part of the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, where r_n is the remainder of p_n divided by n in the euclidian division.

Résumé

L'indécidabilité de la théorie additive des nombres premiers ainsi que de la théorie $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, où p_n désigne le $(n+1)$-ième premier, sont deux questions ouvertes. Nous étendons cette dernière théorie en lui ajoutant une fonction supplémentaire et nous montrons l'indécidabilité de la théorie $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, où r_n désigne le reste de p_n de la division euclidienne de p_n par n, et même de sa seule partie existentielle.

Introduction - The additive theory of primes contains longtime open classical conjectures of Number Theory, as famous Goldbach's binary conjecture or twin primes conjecture, and so on. Some authors provided [BJW,BM,LM] conditional proofs (through Schinzel's Hypothesis [SS]) of the undecidability of the additive theory of primes $\mathrm{Th}(\mathbb{N}, +, \mathbb{P})$, where $\mathbb{P}$ is the set of all primes. Weakening the problem by strengthenning this theory, we introduced [CRV] the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n is the $(n+1)$-th prime, and posed the problem of its (un)decidability. As usual for a language containing a function symbol, we suppose it contains identity. Note that $\mathbb{P}$ is existentially definable within $\langle \mathbb{N}, n \mapsto p_n \rangle$, hence $\mathrm{Th}(\mathbb{N}, +, \mathbb{P})$ is a subtheory of $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$. At the moment, the undecidability of the latter theory is still an open question, and our approach in [CRV] was to consider several approximations of the function $n \mapsto p_n$ as, for instance, $n \lfloor \log n \rfloor$ and on this way we showed the undecidability of theories $\mathrm{Th}(\mathbb{N}, +, nf(n))$ for a family of functions f including $\lfloor \log \rfloor$ mentioned above. Another approach consists of extending the language $\{+, n \mapsto p_n\}$ to $\{+, n \mapsto p_n,\ n \mapsto r_n\}$, where r_n is the remainder of p_n divided by n. The main result of this paper is the following:

Theorem 1 *Multiplication is existentially $\langle \mathbb{N}, +, \mathrm{n} \mapsto \mathrm{p_n},\ \mathrm{n} \mapsto \mathrm{r_n} \rangle$-definable at first-order.*

This leads to the following (without use of conjectures) result:

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Corollary 1 $\mathrm{Th}_{\exists}(\mathbb{N},+,\mathrm{n}\mapsto \mathrm{p_n},\ \mathrm{n}\mapsto \mathrm{r_n})$ *is undecidable.*

Remark Actually all positive integer constants are existentially $\{+,\mathbb{P}\}$-definable in the following manner:

$$\begin{array}{lcl} x=0 & \Leftrightarrow & x+x=x; \\ x=1 & \Leftrightarrow & \exists y(y=x+x\wedge y\in\mathbb{P}); \\ x=2 & \Leftrightarrow & \exists y(y=1\wedge x=y+y); \\ & \vdots & \\ x=n+1 & \Leftrightarrow & \exists y\exists z(y=n\wedge z=1\wedge x=y+z). \end{array}$$

As we mentioned above, $\mathbb{P}$ is existentially definable within the language $\{+,n\mapsto p_n\}$, hence all positive integer constants are also existentially $\{+,n\mapsto p_n\}$-definable. Note, that $n\left\lfloor\frac{p_n}{n}\right\rfloor=p_n-r_n$. We intend to define (section 2, see Lemma 3) $\left\lfloor\frac{p_n}{n}\right\rfloor$ from $+$ and $n\left\lfloor\frac{p_n}{n}\right\rfloor$. Then the strategy will be to define multiplication through the function $n\mapsto cn^2$ (where c is a fixed constant), which is to be proved $\{+,\left\lfloor\frac{p_n}{n}\right\rfloor,n\left\lfloor\frac{p_n}{n}\right\rfloor\}$-definable. Consequently, multiplication will be existentially $\{+,n\mapsto p_n,\ n\mapsto r_n\}$-definable at first-order.

Remark. In the previous paper [CRV] we consider continuous real strictly increasing functions and their inverses. Since we work with integer parts we have to introduce pseudo-inverses of discrete functions. For such a discrete unbounded function f from $\mathbb{N}$ into $\mathbb{N}$, we define its pseudo-inverse f^{-1} from $\mathbb{N}$ into $\mathbb{N}$ by $f^{-1}(n)=\mu m[f(m+1)>n]$, where μ means "the smallest ... such that". Due to the unboundness of f such an f^{-1} is correctly defined.

1) Some preliminary results in Number Theory

Contrarily to what happens with log, the behavior of $\left\lfloor\frac{p_n}{n}\right\rfloor$ is *a priori* irregular but we shall prove it is not too much chaotic. Namely, we prove:

Proposition 1 *Let us denote the mapping* $n\mapsto\left\lfloor\frac{p_n}{n}\right\rfloor$ *by* f.
1) For $m>n$, *we have* $f(m)-f(n)\geq -1$;
2) For $n\geq 11$, *we have* $f^{-1}(n+1)-f^{-1}(n)>n$.

Proof 1) We use the following estimates for p_n ([RP], p. 249):

$$p_m\geq m\log m+m\log\log m-1.0072629m \quad \text{for } m\geq 2;$$
$$p_m\leq m\log m+m\log\log m-0.9385m \quad \text{for } m\geq 7022. \tag{1}$$

For $m>n\geq 7022$, we have $f(m)-f(n)=\left\lfloor\frac{p_m}{m}\right\rfloor-\left\lfloor\frac{p_n}{n}\right\rfloor$
$\geq\frac{p_m}{m}-\frac{p_n}{n}-1\geq log(\frac{m}{n})-log(\frac{logm}{logn})-0.9385+1.0072629-1.$
Hence $f(m)-f(n)\geq -1$ because the sum of the two first terms is positive as is the sum of terms three and four.
If $n<7022$, one may check the desired inequality by a direct computation.

2) Let m be $f^{-1}(n)$. By the very definition of f^{-1}, the equality $m=f^{-1}(n)$ is equivalent to the conjunction of the two following conditions:

$$\begin{cases} \left\lfloor\dfrac{p_{m+1}}{m+1}\right\rfloor\geq n+1; \\ \forall k\leq m\ \left\lfloor\dfrac{p_k}{k}\right\rfloor\leq n. \end{cases} \tag{2}$$

For $k \leq 7022$, the maximum of $\frac{p_k}{k}$ is attained for $k = 7012$ and equal to $\frac{p_{7012}}{7012} < 10.102824 < 11$. Consequently, we see that $m = f^{-1}(n) \geq f^{-1}(11) \geq 7022$ and this is the reason why in the hypothesis of Proposition 1, item 2) we assume $n \geq 11$.
To prove the inequality, it is sufficient to prove that for $k = m + n$ we have $\lfloor \frac{p_k}{k} \rfloor \leq n + 1$, or in other words,

$$\frac{p_k}{k} < n + 2. \tag{3}$$

Note that for $m \geq 7022$, we have by (2):

$$n + 1 \leq \left\lfloor \frac{p_{m+1}}{m+1} \right\rfloor + 1 \leq \frac{p_{m+1}}{m+1} + 1 \leq \log(m+1) + \log\log(m+1) - 0.07 < m.$$

Consequently it is sufficient – and actually more convenient – to prove a somehow stronger result, namely the same inequality (3) but for $m \geq 7022$ and $m + 1 \leq k \leq 2m$.
From the second estimate of (1) we have, since $k \geq m \geq 7022$, the following inequalities:

$$\begin{aligned} \tfrac{p_k}{k} &< \log k + \log\log k - 0.9385 \\ &\leq \log 2m + \log\log 2m - 0.9385 \\ &= \log m + \log\log m + \log 2 + \log(1 + \tfrac{\log 2}{\log m}) - 0.9385; \end{aligned}$$

using the first estimate of (1) and $\frac{\log 2}{\log m} \leq \frac{\log 2}{\log 7022}$, we have:

$$\log m + \log\log m - 1.0072629 \leq \frac{p_m}{m};$$

consequently:

$$\frac{p_k}{k} \leq \frac{p_m}{m} + 0.07 + \log 2 + \log(1 + \frac{\log 2}{\log 7022}) \leq \frac{p_m}{m} + 1$$

by an easy computation and finally, due to (2), we obtain $\frac{p_k}{k} < n + 2$. □

Item 1) of previous proposition emphasizes on the fact that $f : n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is "almost" increasing and item 2) shows that the difference $f^{-1}(n+1) - f^{-1}(n)$ is big enough with respect to n. This suggests to introduce a new class of functions, containing f, for which we prove that the existential part of the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto nf(n))$ is undecidable.

2) The class $C(k, d, n_0)$ and some its properties

2.1) The class $C(k, d, n_0)$

Let $k \geq 0$ be a fixed nonnegative integer. We shall say f is k-*almost increasing* if and only if

$$\forall y \geq x[f(y) - f(x) \geq -k]. \tag{4}$$

In this sense 0-almost increasing means increasing (not necessarily strictly) and $n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is 1-almost increasing (due to Proposition 1).

Still looking at $n \mapsto \lfloor \frac{p_n}{n} \rfloor$, we intend to consider functions whose pseudo-inverse is defined and asymptotically increases quickly enough with respect to its argument. Let us say that f^{-1} has at least $(1/d)$-linear difference, if

$$\exists n_0 \in \mathbb{N} \forall n \geq n_0 [f^{-1}(n+1) - f^{-1}(n) > \frac{n}{d}]. \tag{5}$$

In fact, for $\lfloor \frac{p_n}{n} \rfloor$, the constant d is 1 and $n_0 = 11$, but results and proofs hold for an arbitrary (fixed) d.

Now let us definite the class $C(k, d, n_0)$ as the set of functions from $\mathbb{N}$ into $\mathbb{N}$ satisfying conditions (4) of being *k-almost increasing* and (5) of having its pseudo-inverse with an *at least* $(1/d)$*-linear difference.*

In order to prove FUNDAMENTAL LEMMA of section 3, whose Theorem 1 is a corollary, we show some properties of the class $C(k, d, n_0)$. Firstly, in section 2.2 we present in three lemmas these properties and comment them. Afterwards, in section 2.3, we prove them.

2.2) Properties of $C(k, d, n_0)$

Lemma 1 *For any function* $f \in C(k, d, n_0)$ *the following items hold:*

(i) For any $n \geq n_0$*, we have* $f^{-1}(n + d) - f^{-1}(n) > n$*;*

(ii) For any $n \geq n_0 + 1$*, the set* $\{x \in \mathbb{N} \mid f(x) = n\}$ *is nonempty;*

(iii) For any $n \geq n_0 + 1$*, the equality* $f(x) = n$ *implies*

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0 - 1)].$$

Lemma 2 *If* $f \in C(k, d, n_0)$ *and* $f(x) = n \geq n_0$*, then for any* c *such that* $1 \leq c \leq n$*, we have:*

$$-k \leq f(x + c) - f(x) \leq k + d. \tag{6}$$

Lemma 3 *For any* $f \in C(k, d, n_0)$*, let* $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$*.*
Consider $\tilde{f} : [x_0 + 1, +\infty[\cap\mathbb{N} \longrightarrow \mathbb{N}$ *with* $\tilde{f}(x) = f(x)$*. Then* $\tilde{f}$ *is existentially definable at first-order within* $\langle \mathbb{N}, +, 1, x \mapsto xf(x)\rangle$*.*

Remarks 1) Item (i) of Lemma 1 provides a linear lower bound of values of f^{-1} when difference of arguments is the parameter d of the considered class.
Item (ii) of the same lemma insure that f is asymptotically onto, and item (iii) gives a quadradic lower bound for solutions of the equation $f(x) = n$, that we need in section 3.

2) Actually, as the reader will see within the proof, Lemma 1 does not use condition (4) of being k-almost increasing.

3) Lemma 2 provides asymptotical bounds for the difference of two values of f with arguments taken in a short interval with respect to the values of these arguments. Refering to the previous Lemma 1 we see that n is at most $\sqrt{2dx + n_0^2} + 2$.

4) Lemma 3 generalizes the situation of the main result of the previous paper [CRV] of the same authors when $\lfloor \log n \rfloor$ was "extracted", *i.e.* defined, from $+$ and $n \lfloor \log n \rfloor$.

2.3) Proofs of the three Lemmas

Proof of Lemma 1 (i) By condition (5):

$$\begin{aligned} f^{-1}(n+d) - f^{-1}(n) &= [f^{-1}(n+d) - f^{-1}(n+d-1)] \\ &\quad +[f^{-1}(n+d-1) - f^{-1}(n+d-2)] \\ &\quad +\ldots \\ &\quad +[f^{-1}(n+1) - f^{-1}(n)] \\ &> \tfrac{n+d-1}{d} + \tfrac{n+d-2}{d} + \ldots + \tfrac{n}{d} \\ &> n. \end{aligned}$$

(ii) If there was no x such that $f(x) = n$, we would have $f^{-1}(n) = f^{-1}(n-1)$. But $f^{-1}(n) > f^{-1}(n-1)$ according to condition (5).

(iii) By definition of $f^{-1)}$, we have: $x > f^{-1}(n-1)$.
As in (i), we have:

$$\begin{array}{rcl} f^{-1}(n-1) - f^{-1}(n_0) & = & [f^{-1}(n-1) - f^{-1}(n-2)] \\ & & + \dots \\ & & +[f^{-1}(n_0+1) - f^{-1}(n_0)] \\ & > & \frac{n-2}{d} + \frac{n_0}{d} + \dots + \frac{n}{d} \\ & = & \frac{(n-2)(n-1) - n_0(n_0+1)}{2d}. \end{array}$$

and the result. □

Proof of Lemma 2 The left-hand side of the inequality is an immediate consequence of the very definition of a k-almost increasing function. For proving the right-hand side, note that, using k-almost increasing property of f together with $f(x) = n$, we obtain:

$$\max_{y \leq x} f(y) \leq f(x) + k = n + k,$$

so that $f^{-1}(n+k) \geq x$, by the definition of f^{-1}. By previous Lemma 1, item (i) and the latter inequality, we have:

$$f^{-1}(n+k+d) > f^{-1}(n+k) + n + k \geq x + n + k \geq x + n \geq x + c$$

since $1 \leq c \leq n$. Using again the definition of f^{-1}, we see that $f(x+c) \leq n+k+d = f(x) + k + d$ and we are done. □

Proof of Lemma 3 To define $\tilde{f}$ within the structure $\langle \mathbb{N}, +, x \mapsto xf(x) \rangle$ we shall make use of the inequality:

$$0 \leq f(x) < x$$

together with the remainder of $f(x)$ modulo $x+1$, which we must define in the considered structure.

Fact 1.- $f(x) < x$.

By the definition of f^{-1}, we have $f(x_0+1) > k+2+4d+n_0^2$ and by the k-almost increasing property we deduce, for $x \geq x_0 + 1$,

$$n = f(x) \geq f(x_0+1) - k > 2 + 4d + n_0^2. \tag{7}$$

Hence $\frac{n-2}{2d} > 2$.
From (7), we obtain $n > n_0 + 1$ so that by Lemma 1, item (iii), we have:

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0-1)],$$

hence:

$$x > 2(n-1) - \frac{n_0(n_0-1)}{2d} > 2(n-1) - n_0^2 = n + (n-2-n_0^2) > n = f(x). \;\square\square$$

Fact2.- *We have*:

$$f(x) \equiv (x+1)f(x+1) - xf(x) \pmod{x+1}. \tag{8}$$

It is sufficient to note that $(x+1)f(x+1) - xf(x) = f(x) + (x+1)[f(x+1) - f(x)]$. □□

We are still unable to define general congruences, fortunately here the difference $|f(x+1) - f(x)|$ is bounded, namely,

$$|f(x+1) - f(x)| \leq k + d, \tag{9}$$

due to Lemme 2, with $c = 1$. This suggests to introduce the notion of a restricted congruence, namely, for a, b, m in $\mathbb{N}$ and some fixed integer c, we define $a \equiv_c b (\bmod\, m)$ by:

$$\bigvee_{h=0}^{c} \{[a = b + \underbrace{m + \cdots + m}_{\text{h times}}] \vee [b = a + \underbrace{m + \cdots + m}_{\text{h times}}]\}.$$

Obviously, the first-order latter formula is expressible within the structure $\langle \mathbb{N}, + \rangle$, since c is fixed. The congruence (8) and inequality (9) provide together the following restricted congruence:

$$f(x) \equiv_{k+d} (x+1)f(x+1) - xf(x) (\bmod\, x+1),$$

which is a definition of $f(x)$ within $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$ since $1 \leq f(x) < x$. Finally, we provide explicitely an existential first-order definition of f, namely:

$$[x > x_0 \wedge y = f(x)] \leftrightarrow$$

$$\{x > x_0 \wedge y \leq x \wedge \bigvee_{h=0}^{k+d} [(y + xf(x) = (x+1)f(x+1) + \underbrace{(x+1) + \cdots + (x+1)}_{\text{h times}})$$

$$\vee ((x+1)f(x+1) = y + xf(x) + \underbrace{(x+1) + \cdots + (x+1)}_{\text{h times}})]\}.$$

3) Fundamental Lemma and the proof of the Main Theorem

In order to prove the undecidability of Th$(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$, we prove a more general result, namely:

Lemma 4 (Fundamental Lemma) *For any* $f \in C(k, d, n_0)$ [see §2], *multiplication is existentially* $\{+,\ 1,\ x \mapsto xf(x)\}$*-definable at first-order.*

As shown by Y. Matiyasevich, the existential true theory of numbers is exactly the set of arithmetical relations, which are definable by diophantine equations. Therefore the negative solution of the 10-th Hilbert's problem [MY] implies the following corollary.

Corollary 2 *The existential theory* Th$_\exists(\mathbb{N},\ +,\ 1,\ x \mapsto xf(x))$ *is undecidable.*

Proof of Lemma 4 It suffices to show that for some constants c and n_1 the function $n \mapsto cn^2$ from $[n_1, +\infty[\cap \mathbb{N}$ into $\mathbb{N}$ is $\{+,\ 1,\ x \mapsto xf(x)\}$-definable. More precisely, we shall take $c = 5d$ and $n_1 = 2 + 5d + n_0^2$. Consider $n \geq n_1$. Since $n_1 > n_0 + 1$, we can apply Lemma 1, item (ii), proving there exists x such that $f(x) = 5dn$. Let x_0 be the same as in Lemma 3, namely $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$. Let us show $x > x_0$. Otherwise $x \leq x_0$, so that by the k-almost increasing property $f(x) \leq f(x_0) - k$, implying, by the definitions of f^{-1} and x_0,

$$f(x) \leq 2 + 4d + n_0^2 + k - k < n_1 < 5dn_1 \leq 5dn = f(x),$$

which is impossible.
Note that $5dn$ is $\{+\}$-definable as the sum of $5d$ terms equal to n (d is a fixed constant).
Now thanks to Lemma 3, an x such that $f(x)=5dn$ is $\{+,1,x\mapsto xf(x)\}$-definable.
On the other hand:

$$(x+n)f(x+n)-xf(x)=(x+n)[f(x+n)-f(x)]+nf(x)=(x+n)[f(x+n)-f(x)]+5dn^2.$$

By Lemma 2 applied to $c=n$, we have $|f(x+n)-f(x)|\leq k+d$, so that:

$$5dn^2 \equiv_{k+d} (x+n)f(x+n)-xf(x)(\bmod x+n). \tag{10}$$

According to Lemma 1 and item (iii) since $f(x)=5dn$ and $5dn>n_1>n_0+1$ the inequalities $n\geq n_1>n_0^2$ and:

$$\begin{aligned} x+n &> \frac{(5dn-1)(5dn-2)}{2d}-\frac{n_0(n_0-1)}{2d}+n \\ &> \frac{25d^2n^2-15nd}{2d}>5dn^2 \end{aligned} \tag{11}$$

hold.
Using (10) and (11), a similar argument as in Lemma 3 shows that the function $n\mapsto 5dn^2=cn^2$ with domain $[n_1,+\infty[\cap\mathbb{N}$ is existentially $\{+,1,x\mapsto xf(x)\}$-definable. By a routine argument, multiplication is clearly existentially $\{+,1,x\mapsto xf(x)\}$-definable. □

Proof of the Main-Theorem We remind the reader that 1 was existentially $\{+,\ \mathbb{P}\}$ and $\{+,\ n\mapsto p_n\}$-defined in the introduction.
We also noted that $n\left\lfloor\frac{p_n}{n}\right\rfloor=p_n-r_n$ and $n\mapsto n\left\lfloor\frac{p_n}{n}\right\rfloor$ belongs to $C(1,1,11)$, the latter due to Proposition 1, §1. Then Fundamental Lemma can be applied and multiplication is existentially $\{+,n\mapsto p_n,n\mapsto r_n\}$-definable. □

Conclusion: Our main result is absolute in the sense that does not depend on any conjecture. In order to shed more light on the considered theories $\mathrm{Th}_\exists(\mathbb{N},+,\mathbb{P})$ and $\mathrm{Th}_\exists(\mathbb{N},n\mapsto p_n,n\mapsto r_n)$, we recall a conditional result of A. Woods. Let us recall that Dickson's conjecture [DL] claims that if $a_1,a_2,\ldots a_n,b_1,b_2,\ldots b_n$ are integers with all $a_i>0$ and

$$\forall y\neq 1\exists x[y\not|\prod_{1\leq i\leq n}(a_ix+b_i)]$$

then there exist infinitely many x such that a_ix+b_i are primes for all i. Let us call *DC* this conjecture, then A. Woods proved [WA]:
If DC is true then the existential theory $\mathrm{Th}_\exists(\mathbb{N},+,\mathbb{P})$ *is decidable.*
Now, the question is to know whether there is a gap between $\mathrm{Th}_\exists(\mathbb{N},+,n\mapsto p_n,n\mapsto r_n)$ or whether they are exactly the same. In the case of equality between these two theories, *DC* is false (and hence Schinzel's hypothesis on primes, whose *DC* is the linear case, is also false).

Open problem: *Is* $\mathrm{Th}_\exists(\mathbb{N},+,\mathbb{P})$ *equal to* $\mathrm{Th}_\exists(\mathbb{N},+,n\mapsto p_n,n\mapsto r_n)$?

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