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Abstract : In certain applications, for instance biomechanics, turbulence, finance, or Internet traffic, it seems suitable to model the data by a generalization of a fractional Brownian motion for which the Hurst parameter H is depending on the frequency as a piece-wise constant function. These processes are called multiscale fractional Brownian motions. In this contribution, we provide a statistical study of the multiscale fractional Brownian motions. We develop a method based on wavelet analysis. By using this method, we find initially the frequency changes, then we estimate the different parameters and afterwards we test the goodness-of-fit. Lastly, we give the numerical algorithm. Biomechanical data are then studied with these new tools.

Keywords: Biomechanics; Detection of change; Goodness-of-fit test; Fractional Brownian motion; Semi-parametric estimation; Wavelet analysis.

1 Introduction

Fractional Brownian Motion (F.B.M.) was introduced in 1940 by Kolmogorov as a way to generate Gaussian "spirals" in a Hilbert space. But the seminal paper of Mandelbrot and Van Ness (1968) emphasizes the relevance of F.B.M. to model natural phenomena: hydrology, finance... Formally, a fractional Brownian motion $B_H = (B_H(t),\, t \in I\!R_+)$ could be defined as a real centered Gaussian process with stationary increments such that $B_H(0) = 0$ and $E\,|B_H(s) - B_H(t)|^2 = \sigma^2\,|t-s|^{2H}$, for all pair $(s,t) \in I\!R_+ \times I\!R_+$ where $H \in]0,1[$ and $\sigma > 0$. This process is characterized by two parameters : the Hurst index H and the scale parameter σ. We lay the emphasis on the fact that the same parameter H is linked to different properties of the F.B.M. as the smoothness of the sample paths, the long range dependence of its increments and the self-similarity.

During the decades 1970's and 1980's, the statistical study of F.B.M. was developed, to look at for instance the historical notes in Samorodnitsky & Taqqu (1994), [30, chap.14] and the references therein. Modelling by a F.B.M. became more and more widespread during the last decade (traffic Internet, turbulence, image processing...). Nevertheless, in many applications the real data does not fit exactly F.B.M. Thus, the F.B.M. must be regarded only as an ideal mathematical model. Therefore, various generalizations of F.B.M. have been proposed these last years to fill the gap between the mathematical modelling and real data. In one hand, Gaussian processes where the Hurst parameter H has been replaced by a function depending on the time were studied, see for instance Peltier and Lévy Vehel (1996), Benassi, Jaffard and Roux (1997), Ayache and Lévy Vehel (1999). However, this dependence of time implies the loss of the stationarity of the increments. In other hand, non Gaussian processes, mainly α stable $(0 < \alpha < 2)$ infinite variance processes, were considered, see for example the study of telecom processes in Pipiras and Taqqu (2002).

Here, we are concerned with Gaussian processes having stationary increments and a Hurst index changing with the frequencies. To our knowledge, these kinds of processes were introduced implicitly in biomechanics by Collins and de Luca (1993), in finance by Rogers (1997) and Cheridito (2003) and explicitly by Benassi and Deguy (1999) for image analysis or image synthesis. In any case, the probabilistic properties of these processes have not been thoroughly established and no rigorous statistical studies have been done. Both Collins and de Luca (1993) and Benassi and Deguy (1999) propose a model with two different Hurst indices corresponding respectively to the high and the low frequencies separated by one change point at the frequency ω_c. They use the log variogram to estimate these two Hurst indices. Indeed, in this case, the log variogram considered as a function of the logarithm of the scale presents two asymptotic directions with slopes being twice the Hurst index at low (respectively high) frequencies. The change point ω_c is then estimated as the abscise of the intersection of the these two straight lines. Numerically, this method is not robust. Moreover it could not be adapted in the case of more than one change point. Let us stress that it is not a question of a theoretical refinement, but one that corresponds precisely to the true situations. Indeed, in applications, we consider only finite frequency bands, therefore we should use a statistical method based on the information included in finite frequency bands. Wavelet analysis seems the tool *had hoc*, when the Fourier transform of the associated wavelet is compactly supported.

For these reasons, we put forward in Bardet and Bertrand (2003) a model of generalized F.B.M. including the cases with more than one frequency change point. We called it (M_K) multiscale fractional Brownian motion where K denote the number of frequency change points. More precisely, a (M_K) multiscale fractional Brownian motion is a Gaussian process with stationary increments where the Hurst parameter H is replaced by a piecewise constant function of the frequency $\xi \mapsto H(\xi)$ in the harmonizable representation, see Formula (3) below. The main probabilistic properties of this model were studied in Bardet and Bertrand (2003). In this work, we treat the statistical study of the multiscale F.B.M. and we focus on its application to biomechanics.

The remainder of the paper is organized as follows: in Section 2, we describe the biomechanical data and the corresponding statistical problem. In section 3, we recall the initial definition of the partial Brownian motion and its principal probabilistic properties. Then, we show that the variogram method is not suitable for the estimation of the various parameters of a (M_K)-F.B.M. We then develop a statistical estimation framework, based on wavelet analysis. We investigate the discretization of the wavelet coefficient and we state a functional Central Limit Theorem for the empirical wavelet coefficients. In Section 4, we first estimate the different frequency change points and Hurst parameters. Then, we propose a goodness of fit test and derive an estimator of the number of frequency changes. The numerical algorithm is detailed at the end of this section. Finally, in Section 5, the biomechanical data are studied with the tools developed in Section 4. The proof of the results of Sections 3 and 4 are given in appendix.

2 The Biomechanical Problem

One of the motivations of this work is to model biomechanical data corresponding to the regulation of the upright position of the human being. By using a force platform, the position of the center of pressure (C.O.P.) during quiet postural stance is determined. This position is usually measured at a frequency of 100 Hz for the one minute period, which yields a data set of 6000 observations. The experimental conditions are formed to the standards of the Association Française de Posturologie (AFP), for instance the feet position (angle and clearance), the open or closed eyes.

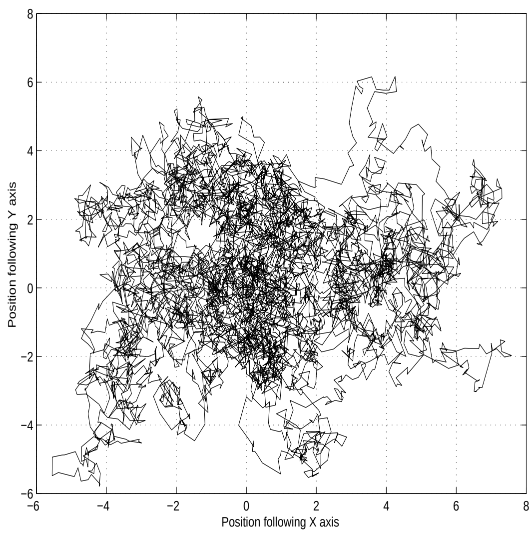

Figure 1 : An example [1] of the trajectory of the C.O.P. during 60s at 100Hz (in mm)

The X axis of the platform corresponds to the fore-aft direction and the Y axis corresponds to the medio-lateral direction. During the 1970's, these data were analyzed as a set of points, *i.e.* without taking into account their temporal order. During the following decade some studies considered them as a process, and Collins and de Luca (1993) introduced the use of F.B.M. to model these data. In fact, they used a generalization of F.B.M. More precisely, let the position X_i of the C.O.P. be observed at times $t_i = i\Delta$ for $i = 1, \ldots, N$ ($\Delta = 0.01\ s$). The study of Collins and de Luca is based on the empirical variogram

$$V_N(\delta) = \frac{1}{(N-\delta)} \sum_{i=1}^{N-\delta} \left(X_{(i+\delta)\Delta} - X_{i\Delta} \right)^2 \qquad (1)$$

[1] these experimental data were realized by A. Mouzat and are used in [13].

where $\delta \in \mathbb{N}^*$. For a F.B.M., we have $E\,V_N(\delta) = \sigma^2\,\Delta^{2H} \times \delta^{2H}$ and after plotting the log-log graph of the variogram as a function of the time lag , i.e. $(\log\delta, \log V_N(\delta))$, a linear regression provides the slope $2H$. Typically, one gets the following type of figure (see Figure 2). It is considered by Collins and de Luca to be a "F.B.M." with two regimes : with slope $2H_0$ *(short term)* and with slope $2H_1$ *(long term)* separated by a critical time lag δ_c and these parameters are estimated graphically :

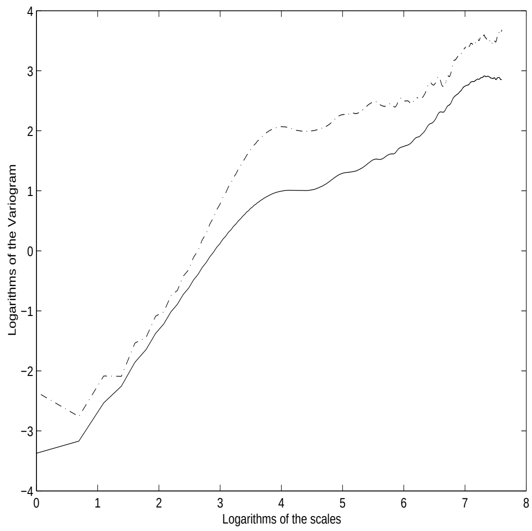

Figure 2 : An example of the log-log graph of the variogram for the previous trajectories X (-.) and Y (-).

They found $H_0 > 0.5$, $H_1 < 0.5$ and a critical time lag $\delta_c \simeq 1\ s$. These results were interpreted as corresponding to two different kinds of regulation of the human stance : in the *long term* $H_1 < 0.5$ and the process is anti-persistent, in the *short term* $H_0 > 0.5$ and the process is persistent. This method was employed several times in biomechanics under the various experimental conditions (opened eyes versus closed eyes, different feet angles,...). But, a lack of mathematical models and of statistical studies has made impossible to obtain confidence intervals on the two slopes $2H_0$, $2H_1$ and the critical time lag δ_c.

3 The multiscale fractional Brownian motion and its statistical study based on wavelet analysis

3.1 Description of the model

A fractional Brownian motion $B_H = \{B_H(t),\, t \in \mathbb{R}\}$ of parameters $(H,\,\sigma)$ is a real centered Gaussian process with stationary increments and $E\,|B_H(s) - B_H(t)|^2 = \sigma^2\,|t-s|^{2H}$, for all $(s,t) \in \mathbb{R}^2$ where $H \in]0,1[$ and $\sigma > 0$. The fractional Brownian motion (F.B.M.) has been proposed by Kolmogorov (1940) who defined it by the harmonizable representation :

$$B_H(t) = \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H+1/2}}\,\overline{\widehat{W}}(d\xi), \qquad \text{for all} \;\; t \in \mathbb{R}, \tag{2}$$

where $W(dx)$ is a Brownian measure and $\widehat{W}(d\xi)$ its Fourier transform (namely for any function $f \in L^2(\mathbb{R})$ one has almost surely, $\int_{\mathbb{R}} f(x)W(dx) = \int_{\mathbb{R}} \widehat{f}(\xi)\,\widehat{W}(d\xi)$, with the convention that $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi\,x} f(x)\,dx$ when $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$). We refer to Samorodnitsky and Taqqu (1994) for the question of the equivalence of the different representations of the F.B.M. From the harmonizable representation, a natural generalization is the multiscale fractional Brownian motion with a Hurst index depending on the frequency. More precisely, we define :

Definition 3.1 *For $K \in I\!N$, a (M_K)-multiscale fractional Brownian motion $X = \{X(t), t \in I\!R\}$ (simplify by (M_K)-F.B.M.) is a process such as*

$$X(t) = 2\sum_{j=0}^{K} \int_{\omega_j}^{\omega_{j+1}} \sigma_j \frac{(e^{it\xi} - 1\ )}{|\xi|^{H_j+1/2}} \widehat{\overline{W}}(d\xi) \qquad \text{for all } \ t \in I\!R \tag{3}$$

with $\omega_0 = 0 < \omega_1 < \cdots < \omega_K < \omega_{K+1} = \infty$ by convention, $\sigma_i >$ and $H_i \in]0,1[$ for $i \in \{0, 1, \cdots, K\}$.

The (M_K)-F.B.M. was notably introduced in order to relax the self-similarity property of F.B.M. Indeed, the self-similarity is a form of invariance with respect to changes of time scale [27] and it links the behavior to the high frequencies with the behavior to the low frequencies. In Bardet and Bertrand (2003), the main properties of these processes are provided : X is a Gaussian centered process with stationary increments, its trajectories are a.s. of Hölder regularity α, for every $0 \leq \alpha < H_K$ and its increments form a long-memory process (except if the different parameters satisfy a particular relationship, *i.e.*, if its spectral density is a continuous function with $0 < H_i < 1/2$ for $i = 0, 1, \cdots, K$).

3.2 The question of the choice of the estimator

In the remainder of this paper, we suggest a statistical study of such a model based on wavelet analysis. In this subsection, we explain the reason of this choice.

To begin with, we will describe the statistical framework precisely. Let $X = \{X(t), t \in I\!R_+\}$ be a (M_K)-F.B.M. defined by (3). We observe one path of the process X on the interval $[0, T_N]$ at the discrete times $t_i = i \cdot \Delta_N$ for $i = 1, \ldots, N$ with $T_N = N \cdot \Delta_N$. Therefore,

$$(X(\Delta_N), X(2\Delta_N), \ldots, X(N\Delta_N)) \quad \text{ is known,}$$

and we consider the asymptotic $N \to \infty$, $\Delta_N \to 0$ and $T_N \to \infty$. We want to estimate the parameters of the (M_K)-F.B.M. that are $(H_0, H_1, \ldots, H_K)$, $(\sigma_0, \sigma_1, \ldots, \sigma_K)$ and $(\omega_1, \ldots, \omega_K)$.

Even if the model is defined as a parametric one, we prefer to use a semi-parametric statistics based on the wavelet analysis. This choice is justified by the following reasons. First, the spectral density of X is not continuous in the general case. Thus, one cannot use the classical results on the consistency of the maximum likelihood or Whittle maximum likelihood estimators for long memory processes (see Fox and Taqqu, 1986, Dahlhaus, 1989 or Giraitis and Surgailis, 1990). Moreover, this is not a classical time series parametric estimation : indeed, we consider $(X(\Delta_N), X(2\Delta_N), \ldots, X(N\Delta_N))$ instead of $(X(1), X(2), \ldots, X(N))$ and therefore this is also an estimation problem of the parameters of a continuous stochastic process. Secondly, the following semi-parametric statistics are more robust than a parametric one if the model is misspecified. Consider the example where the function $H(\xi)$ is a not exactly a piece-wise constant function, but instead a constant function on several intervals and some unknown function on the other intervals. In this case, a parametric estimator could not work while the semi-parametric method based on the wavelet analysis will remain efficient.

Another semi-parametric method was developed from the seminal paper of Istas and Lang (1997). This method of estimation is derived from the variogram and provides good results in the case of F.B.M. (see Bardet, 2000) or of multifractional F.B.M. (see Benassi *et al.*, 1998). However, one faces difficulties in identifying the model (M_K)-F.B.M. with this kind of method. Indeed, one can easily satisfy that for $\delta > 0$:

$$\mathcal{V}(\delta) = E\ (X(t+\delta) - X(t))^2 \quad = \quad 4\sum_{j=0}^{K} \delta^{2H_j}\, \sigma_j^2 \int_{\delta\omega_j}^{\delta\omega_{j+1}} \frac{(1-\cos v)}{v^{2H_j+1}}\, dv. \tag{4}$$

The principle of the variogram's method ensues from the writing of $\log\Big(\mathcal{V}(\delta)\Big)$ as an affine function of $\log\delta$. For a (M_K)-F.B.M., with $C(H_i)=\displaystyle\int_0^\infty \frac{(1-\cos v)}{v^{2H_i+1}}\,dv$ for $i=0,1,\ldots,K$, two cases could provide such a relation :

1. for $\delta\to\infty$, $\log\Big(\mathcal{V}(\delta)\Big)=2H_0\cdot\log\delta+\log\big(4\cdot\sigma_0^2\cdot C(H_0)\big)+O(\delta^{-2H_0})$;

2. for $\delta\to 0$, $\log\Big(\mathcal{V}(\delta)\Big)=2H_K\cdot\log\delta+\log\big(4\cdot\sigma_K^2\cdot C(H_K)\big)+O(\delta^{2-2H_K})$

(the proof of such expansions is in the proof of Lemma A.1). In those cases, if one can show that there is a convergent estimator $V_N(\delta)$ of $\mathcal{V}(\delta)$, then a log-log regression of $\log\Big(V_N(\delta)\Big)$ onto $\log\delta$ could provide an estimation of the different parameters. Nevertheless, such a method would have a lot of drawbacks. On one hand, the estimation of "intermediate" parameters $(H_j)_{1\leq j\leq K-1}$ and $(\sigma_j^2)_{1\leq j\leq K-1}$ requires very specific asymptotic properties between all the frequency changes $(\omega_j)_{1\leq j\leq K-1}$. This implies a lack of generality of the methods based on the variogram. Moreover, concretely, the frequency changes are fixed and one obtains rough approximation instead of asymptotic properties. For instance, numerical simulations show that in some cases the log-log plot of the variogram does not exhibit any intermediate linear part. On the other hand, when the model is misspecified the variogram model could lead to inadequate results. For example the following picture gives the case of a (M_2)-F.B.M. where the variogram method would detect only one frequency change and could not precisely estimate its value. Finally, the variogram's method could perhaps be applied in the two first previous situations 1. and 2., *i.e.* for the estimation of (H_0,σ_0^2) or (H_K,σ_K^2) with δ will have to be a function of N (number of data). But this choice of function will depend on the unknown parameters H_0 or H_K for obtaining central limit theorems for $\log\Big(V_N(\delta)\Big)$... (see the same kind of problem in Abry *et al.*, 2002).

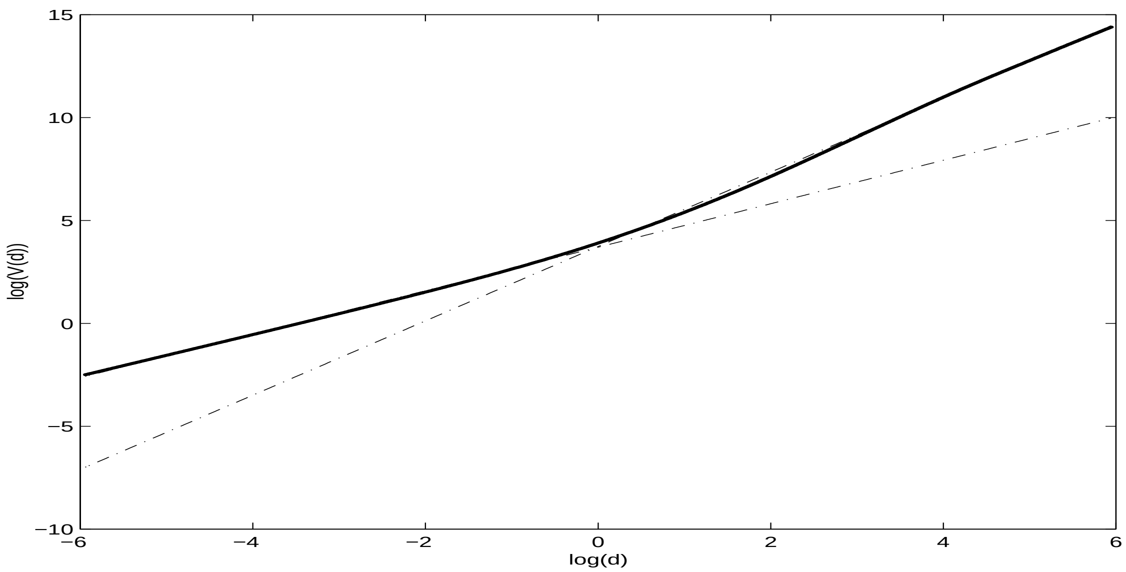

Figure 3: An example of a theoretical variogram for a (M_2)-f.B.m, with $H_0=0.9$, $H_1=0.2$, $H_2=0.5$, and $\sigma_0=\sigma_1=\sigma_2=5$ and $\omega_1=0.05$, $\omega_2=0.5$ (in solid, the theoretical variogram, in dot-dashed, its theoretical asymptotes for $\delta\to0$ and $\delta\to\infty$).

We deduce from the definition of the model and the previous discussion that a wavelet analysis could be an interesting semi-parametric method for estimating the parameters of a (M_K)-F.B.M. Indeed, such a method is based on the change of scales (or frequencies). Therefore, as it is developed below, a wavelet analysis is able to detect the different spectral domain of self-similarity and then estimate the different parameters of the model.

3.3 A statistical study based on wavelet analysis

This method has been introduced by Flandrin (1992) and was developed by Abry *et al.* (2002) and Bardet *et al.* (2000). We also use in the following similar results on wavelet analysis for (M_K)-F.B.M. obtained in Bardet and Bertrand (2003). Let ψ be a wavelet satisfying the following assumption :

Assumption (A1): $\psi : \mathbb{R} \mapsto \mathbb{R}$ *is a* $\mathcal{C}^\infty$ *function satisfying :*

- *for all* $m \in \mathbb{R}$, $\displaystyle\int_{\mathbb{R}} |t^m \psi(t)|\, dt < \infty$;
- *its Fourier transform* $\widehat{\psi}(\xi)$ *is an even function compactly supported on* $[-\beta, -\alpha] \cup [\alpha, \beta]$ *with* $0 < \alpha < \beta$.

We stress these conditions are sufficiently mild and are satisfied in particular by the Lemarié-Meyer *"mother"* wavelet. The admissibility property, i.e. $\displaystyle\int_{\mathbb{R}} \psi(t)dt = 0$, is a consequence of the second one and more generally, for all $m \in \mathbb{N}$,

$$\int_{\mathbb{R}} t^m \psi(t)dt = 0. \tag{5}$$

Note that it is not necessary to choose ψ to be a *"mother"* wavelet associated to a multiresolution analysis of $\mathbb{L}^2(\mathbb{R})$. The whole theory can be developed without resorting to this assumption. The choice of ψ is then very large.

Let $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ and denote $\lambda = (a, b)$. Then define the family of functions ψ_λ by $\psi_\lambda(t) = \dfrac{1}{\sqrt{a}} \psi\left(\dfrac{t}{a} - b\right)$. Parameters a and b are so-called the scale and the shift of the wavelet transform. Let us underline that we consider a continuous wavelet transform. Let $d_X(a, b)$ be the wavelet coefficient of the process X for the scale a and the shift b, with

$$d_X(a, b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t}{a} - b) X(t) dt =< \psi_\lambda, X >_{L^2(\mathbb{R})} .$$

If ψ satisfies Assumption (A1) and X is a (M_K)-F.B.M., the family of wavelet coefficients verifies the following properties (see Bardet and Bertrand, 2003) :

1. for $a > 0$, $(d_X(a, b))_{b \in \mathbb{R}}$ is a stationary centered Gaussian process such as :
$$E\left(d_X^2(a, .)\right) = \mathcal{I}_1(a) = a \int_{\mathbb{R}} |\widehat{\psi}(au)|^2 \ \cdot \rho^{-2}(u)\, du. \tag{6}$$
2. for all $i = 0, 1, \cdots, K$, if the scale a is such as $[\frac{\alpha}{a}, \frac{\beta}{a}] \subset [\omega_i, \omega_{i+1}]$, then
$$E\left(d_X^2(a, .)\right) = a^{2H_i + 1} \cdot \sigma_i^2 \cdot K_{H_i}(\psi), \quad \text{with } K_H(\psi) = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(u)\right|^2}{|u|^{2H+1}} du. \tag{7}$$

Property (7) means that the logarithm of the variance of the wavelet coefficient is an affine function of the logarithm of the scale with slope $2H_i + 1$ and intercept $\log \sigma_i^2 + \log K_{H_i}(\psi)$. This property is the key tool for estimating the parameters of X. Indeed, if we consider a convergent estimator of $\log\left(E\left(d_X^2(a, .)\right)\right)$, it provides a linear model in $\log a$ and $\log \sigma_i^2$. Before specifying such an estimator, let us stress that one only observes a discretized path $(X(0), X(\Delta_N), \ldots, X(N\Delta_N))$ instead of a continuous-time path.

As a consequence, for $a > 0$ and $N \in \mathbb{N}^*$, a natural estimator is the logarithm of the empirical variance of the wavelet coefficient, that is $\log I_N(a)$ where :

$$I_N(a) = \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} d_X^2(a, k\Delta_N), \tag{8}$$

with :

- $r \in]0, 1/3[$;
- $m_N = [r(N/a)]$ and $M_N = [(1-r)(N/a)]$ where $[x]$ is the integer part of $x \in \mathbb{R}$;
- $D_N(a) = \{m_N, m_N + 1, \ldots, M_N\}$ and $|D_N(a)|$ is the cardinal of the set $D_N(a)$.

For $0 < a_{min} < a_{max}$, a functional central limit theorem for $(\log I_N(a))_{a_{min} \le a \le a_{max}}$ can be established (see a similar proof in Bardet and Bertrand, 2003) :

Proposition 3.1 *Let X be a (M_K)-F.B.M., $0 < a_{min} < a_{max}$ and ψ satisfy Assumption (A1). Then :*

$$\sqrt{N\Delta_N}\left(\log I_N(a) - \log \mathcal{I}_1(a)\right)_{a_{min} \le a \le a_{max}} \xrightarrow[N \to \infty]{\mathcal{D}} (Z(a))_{a_{min} \le a \le a_{max}} \tag{9}$$

with $(Z(a))$ a centered Gaussian process such as for $(a_1, a_2) \in [a_{min}, a_{max}]^2$,

$$cov(Z(a_1), Z(a_2)) = \frac{2a_1\, a_2}{(1-2r)\, \mathcal{I}_1(a_1)\, \mathcal{I}_1(a_2)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du. \tag{10}$$

Then, if we specify the locations of the change points in terms of scales, *i.e.* frequencies, we obtain the following:

Corollary 3.1 *Let $i \in \{0, 1, \cdots, K\}$ and assume that $\dfrac{\beta}{\alpha} \le \dfrac{\omega_{i+1}}{\omega_i}$. Then,*

$$\sqrt{N\Delta_N}\left(\log I_N(1/f) + (2H_i+1)\log f - \log \sigma_i^2 - \log K_{H_i}(\psi)\right)_{\omega_i/\alpha \le f \le \omega_{i+1}/\beta} \xrightarrow[N \to \infty]{\mathcal{D}} (Z(1/f))_{\omega_i/\alpha \le f \le \omega_{i+1}/\beta} \tag{11}$$

with the centered Gaussian process $(Z(.))$ such as for $(f_1, f_2) \in [\dfrac{\omega_i}{\alpha}, \dfrac{\omega_{i+1}}{\beta}]^2$,

$$cov(Z(1/f_1), Z(1/f_2)) = \frac{2\,(f_1\, f_2)^{2H_i}}{(1-2r)\, K^2_{H_i}(\psi)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\overline{\widehat{\psi}}(\xi/f_1)\widehat{\psi}(\xi/f_2)}{|\xi|^{2H_i+1}} e^{-iu\xi} d\xi \right)^2 du. \tag{12}$$

For Δ_N small enough, this result shows that all parameters H_i and σ_i^2 could be estimated by using a linear regression of $\log I_N(1/f_j)$ versus $\log f_j$, when the frequencies ω_i are known. Moreover, this central limit theorem shows that a graph of $(\log f, \log I_N(1/f))$ for $f > 0$ exhibits different areas of asymptotic linearity : it suggests the procedure of the following section to estimate and test the frequency changes (see for instance figures 4 or 6).

3.4 The discretization problem

In the applications, we only observe a finite time series $(X(0), X(\Delta_N), \cdots, X((N-1) \times \Delta_N))$ and we must derived the empirical wavelet coefficients from this time series. Since the process X has almost a continuous path but with a regularity $\alpha_X < 1$ almost surely, we should use the Riemann sum. Thus, for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ we define the empirical wavelet coefficient by

$$e_X(a, b) = \frac{\Delta_N}{\sqrt{a}} \sum_{p=0}^{N-1} \psi\left(\frac{p\Delta_N}{a} - b\right) \times X(p\Delta_N) \tag{13}$$

and the discretized estimator by

$$J_N(a) = \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} e_X^2(a, k\Delta_N). \tag{14}$$

We also define for every $k \in D_N(a)$ the error

$$\varepsilon_N(a, k) = e_X(a, k\Delta_N) - d_X(a, k\Delta_N). \tag{15}$$

Now, it is possible to provide the functional central limit theorem for $(\log J_N(a))_{a_{min} \le a \le a_{max}}$ computed from $(X(0), X(\Delta_N), \cdots, X(N\Delta_N))$:

Theorem 3.1 *Under assumptions of Proposition 3.1 and with Δ_N such as $N\Delta_N \to \infty$ and $N(\Delta_N)^2 \to 0$ when $N \to \infty$. Then, with the same process Z than in (9),*

$$\sqrt{N\Delta_N}\left(\log J_N(a) - \log \mathcal{I}_1(a)\right)_{a_{min}\le a\le a_{max}} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(a))_{a_{min}\le a\le a_{max}}. \tag{16}$$

As a particular case, for $i \in \{0,1,\cdots,K\}$ and if $\dfrac{\beta}{\alpha} \le \dfrac{\omega_{i+1}}{\omega_i}$, then

$$\sqrt{N\Delta_N}\left(\log J_N(1/f) + (2H_i+1)\log f - \log \sigma_i^2 - \log K_{H_i}(\psi)\right)_{\omega_i/\alpha\le f\le \omega_{i+1}/\beta} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(1/f))_{\omega_i/\alpha\le f\le\omega_{i+1}/\beta}. \tag{17}$$

The convergence rate of the central limit theorem (16) is $\sqrt{N\Delta_N}$. Thus, the discretization problem implies that the maximum convergence rate is $o(N^{1/4})$ from the previous conditions on Δ_N.

4 Identification of the parameters

First, let us describe the method on a heuristic level. From Proposition 3.1, Formula (17), we have

$$\log J_N(1/f) = -(2H_i+1)\times\log(f) + \log\left(\sigma_i^2\right) + \log\left(K_{H_i}(\psi)\right) + \varepsilon_f^{(N)}, \tag{18}$$

for the frequencies f which satisfy the condition

$$\log(\omega_i) - \log(\alpha) \le \log(f) \le \log(\omega_{i+1}) - \log(\beta). \tag{19}$$

Moreover we have $(N\Delta_N)^{1/2}\left(\varepsilon_{f_j}^{(N)}\right)_{1\le j\le m} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(1/f_j))_{1\le j\le m}$. Formula (18) and condition (19) mean that for $\log(f) \in [\log(\omega_i) - \log(\alpha), \log(\omega_{i+1}) - \log(\beta)]$, we have a linear regression of $\log J_N(1/f)$ onto $\log(f)$ with slope $-(2H_i+1)$ and intercept
$\log\sigma_i^2 + \log K_{H_i}(\psi)$ and for $\log(f) \in [\log(\omega_{i+1}) - \log(\alpha), \log(\omega_{i+2}) - \log(\beta)]$ a linear regression with slope $-(2H_{i+1}+1)$ and intercept $\log\sigma_{i+1}^2 + \log K_{H_{i+1}}(\psi)$. This is a problem of detection of abrupt change on the parameters of a linear regression, but with a transition zone for $\log(f) \in]\log(\omega_{i+1}) - \log(\beta),\ \log(\omega_{i+1}) - \log(\alpha)[$.

Remark 4.1 *Condition (19) implies that $\omega_{i+1} > \dfrac{\beta}{\alpha}\times\omega_i$. Therefore we could only detect the frequency changes sufficiently spaced. For instance, if we choose the Lemarié-Meyer wavelet, we get $\beta/\alpha = 4$ which leads to the condition $\omega_{i+1} > 4\times\omega_i$.*

In this section, we describe the estimation of the parameters and a goodness of fit test. Both of them are based on the following assumption :

Assumption (B_K) **:** *The process X is a (M_K)-multiscale fractional Brownian motion. This process is characterized by the parameters Ω^*, H^* and σ^* where $\Omega^* = (\omega_1^*,\cdots,\omega_K^*)$ with $H^* = (H_0^*, H_1^*,\ldots,H_K^*)$ and $\sigma^* = (\sigma_0^*,\sigma_1^*,\ldots,\sigma_K^*)$. Moreover the following conditions are fulfilled*

- $\omega_{i+1}^* > \dfrac{\beta}{\alpha}\times\omega_i^*$ *for $i = 1,\cdots,K-1$;*
- $\min\limits_{0\le i\le (K-1)}\left\{\left(H_{i+1}^* - H_i^*\right)^2 + \left(\sigma_{i+1}^* - \sigma_i^*\right)^2\right\} > 0$ *and*
- *there exists a compact set $\mathcal{K} \subset]0,1[\times]0,\infty[$ such as $(H_i^*,\sigma_i^*) \in \mathcal{K}$ for all $i = 0,1,\cdots,K$.*

4.1 Estimation of the parameters

Let X be a (M_K)-F.B.M. satisfying the assumption (B_K) with K a known integer number. We observe one path of the process at N discrete times, that $(X(0), X(\Delta_N), \cdots, X(N\Delta_N))$. Let $[f_{min}, f_{max}]$, with $0 < f_{min} < f_{max}$, be the chosen frequency band (see section 5, for an example). We discretize a (slightly modified) frequency band and compute the wavelet coefficients at the frequencies $(f_k)_{0 \leq k \leq a_N}$ where

$$f_k = \frac{f_{min}}{\beta} (q_N)^k \qquad \text{for } k = 0, \cdots, a_N, \quad q_N = \left(\frac{f_{max}}{f_{min}} \frac{\beta}{\alpha} \right)^{1/a_N} \qquad \text{and} \quad a_N = N\Delta_N.$$

For notational convenience , we assume here that $N\Delta_N$ is an integer number. By definition, we have $f_0 = f_{min}/\beta$ and $f_{a_N} = f_{max}/\alpha$, then, using the wavelet coefficients at the frequencies $(f_k)_{0 \leq k \leq a_N}$, we could detect all frequency changes (ω_i^*) included in the band $]f_{min}, f_{max}[$. To simplify the notations, we use the following assumption :

Assumption (C) : $\omega_i^* \in]f_{min}, f_{max}[$ *for all* $i = 1, \ldots, K$.

In this framework, the estimation of the different parameters of X becomes a problem of linear regression with a known number of changes; thus, we follow the same method as in Bai (1994), Bai and Perron (1998), Lavielle (1999) or Lavielle and Moulines (2000) and define the estimated parameters $(\widehat{T}^{(N)}, \widehat{\Lambda}^{(N)})$ as the couple of vectors which minimize the quadratic criterion :

$$Q^{(N)}(T, \Lambda) = \sum_{j=0}^{K+1} \sum_{i=1+t_j}^{t_{j+1}-\tau_N} |Y_i - X_i \lambda_j|^2 \text{, and thus}$$

$$(\widehat{T}^{(N)}, \widehat{\Lambda}^{(N)}) = \text{Argmin} \left\{ Q^{(N)}(T, \Lambda);\ T \in \mathcal{A}_K^{(N)}, \Lambda \in \mathcal{B}_K \right\}$$

with

- $Y_i = \log\left(J_N(1/f_i)\right)$, $X_i = (\log f_i, 1)$ for $i = 0, \cdots, a_N$;
- $\tau_N = \left[\frac{\log(\beta/\alpha)}{\log q_N} \right]$, where $[x]$ is the integer part of x.
- $T = (t_0, t_1, \cdots, t_{K+1}) \in \mathcal{A}_K^{(N)}$ where

$$\mathcal{A}_K^{(N)} = \left\{ (t_0, \cdots, t_{K+1}) \in I\!N^{K+2};\ t_0 = 0, t_{K+1} = a_N + \tau_N, t_{j+1} - t_j > \tau_N \text{ for } j = 0, \cdots, K \right\};$$

- $\Lambda = (\lambda_0, \cdots, \lambda_K) \in \mathcal{B}_K$ where $\lambda_j = \begin{pmatrix} -(2H_j + 1) \\ \log \sigma_j^2 + \log K_{H_j}(\psi) \end{pmatrix}$ and then

$$\mathcal{B}_K = \left\{ (\lambda_0, \cdots, \lambda_K) \ \text{ with } \ (H_j, \sigma_j^2) \in \mathcal{K} \ \text{ for all } \ j \in \{0, 1, \cdots, K\} \right\}.$$

The integer τ_N corresponds to the number of frequencies in the transition zones and $\log f_{i+\tau_N} = \log f_i + \log(\beta/\alpha)$. Obviously, for $j = 0, \cdots, K$, the vector $\widehat{\lambda}_j^{(N)}$ provides the estimators $\widehat{H}_j^{(N)}$ of H_j^* and $\widehat{\sigma}_j^{(N)}$ of σ_j^* by the relation $\widehat{\lambda}_j^{(N)} = \begin{pmatrix} -(2\widehat{H}_j^{(N)} + 1) \\ \log\left((\widehat{\sigma}_j^{(N)})^2 \right) + \log K_{\widehat{H}_j^{(N)}}(\psi) \end{pmatrix}$. For a given $T \in \mathcal{A}_K^{(N)}$, each $\widehat{\lambda}_j^{(N)}$ is obtained from a linear regression of (Y_i) onto (X_i) for $i = t_j + 1, \cdots, t_{j+1} - \tau_N$. Thus, with $\widehat{T} = (\widehat{t}_j)_{0 \leq j \leq K+1}$ obtained from the minimization in T of $Q^{(N)}(T, \widehat{\Lambda})$, we define the different estimators of the change frequencies as

$$\widehat{\omega}_j^{(N)} = \alpha\, f_{\widehat{t}_j^{(N)}} = \alpha \cdot \frac{f_{min}}{\beta} \left(\frac{f_{max}}{f_{min}} \frac{\beta}{\alpha} \right)^{\frac{\widehat{t}_j^{(N)}}{a_N}} \quad \text{for } j = 1, \cdots, K. \tag{20}$$

We have the following convergence :

Proposition 4.1 *Let X satisfy Assumptions (C) and (B_K) with a known K, $(X_{\Delta_N},\cdots,X_{N\Delta_N})$ be a discretized path, and ψ satisfy Assumption (A1). Let Δ_N be such as $N\Delta_N\to\infty$ and $N(\Delta_N)^2\to 0$ when $N\to\infty$. Assume that $(\widehat{H}_i^{(N)},\widehat{\sigma}_i^{(N)})\in\mathcal{K}$ for all $i=0,\cdots,K$. Then for all $\varepsilon>0$, there exists $0<C<\infty$ such as for all large N,*

$$\mathbb{P}\left((N\Delta_N)^{1/4}\left|\widehat{\omega}_j^{(N)}-\omega_j^*\right|\geq C\right)\leq\varepsilon\quad\text{for } j=1,\cdots,K. \tag{21}$$

Remark 4.2 *The proof of this proposition shows a more general result,* i.e. *for $(p,q)\in[3/4,1]\times[0,1]$, for $\varepsilon>0$, there exists $C>0$ such as*

$$\mathbb{P}\left(a_N^{1-p}\left|\widehat{\omega}_j^{(N)}-\omega_j^*\right|\geq C\right)\leq\varepsilon\quad\text{for } j=1,\cdots,K$$

with $a_N=(N\Delta_N)^q$. For numerical considerations and convergence rate of the following estimators of the parameters, we are going to fix now on $p=3/4$ and $q=1$ and then $a_N=N\Delta_N$.

For $j=0,\cdots,K$, the natural estimates of H_j^* and σ_j^{2*} are given by the regression of (Y_i) onto $(\log f_i)$ for $i\in\{\widehat{t}_j^{(N)},\cdots,\widehat{t}_{j+1}^{(N)}-\tau_N\}$. But the probability that $[\widehat{t}_j^{(N)},\widehat{t}_{j+1}^{(N)}-\tau_N]\subset[t_j^*,t_{j+1}^*-\tau_N]$ does not increase fast enough to 1 as $N\to\infty$, in order to obtain a sufficiently fast convergence rate for these estimators. We address this difficulty as follows. We fix an integer number $m\geq 3$ and for $j=0,\cdots,K$, we consider $[\tilde{U}_j^{(N)},\tilde{V}_j^{(N)}]$ an interval strictly included in $[\widehat{t}_j^{(N)},\widehat{t}_{j+1}^{(N)}-\tau_N]$, such as

$$\tilde{U}_j^{(N)}=\widehat{t}_j^{(N)}+\left[\frac{\widehat{t}_{j+1}^{(N)}-\widehat{t}_j^{(N)}-\tau_N}{m+1}\right]\quad\text{and}\quad\tilde{V}_j^{(N)}=\widehat{t}_j^{(N)}+m\left[\frac{\widehat{t}_{j+1}^{(N)}-\widehat{t}_j^{(N)}-\tau_N}{m+1}\right]. \tag{22}$$

Then we estimate the parameters from a regression onto m points uniformly distributed in $[\tilde{U}_j^{(N)},\tilde{V}_j^{(N)}]$; it provides the following estimator $\tilde{\lambda}_j^{(N)}$ from a regression of (Y_i) onto (X_i) for
$i\in\{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}=\left\{\tilde{U}_j^{(N)}+(k-1)\left[\frac{\widehat{t}_{j+1}^{(N)}-\widehat{t}_j^{(N)}-\tau_N}{m+1}\right]\right\}_{1\leq k\leq m}$. By this way, define

$$\begin{aligned}\tilde{\lambda}_j^{(N)} &= \left(-(2\tilde{H}_j^{(N)}+1),\log\tilde{\sigma^2}_j^{(N)}+\log K_{\tilde{H}_j^{(N)}}(\psi)\right)'\\ &= \left((\tilde{X}_j^{(N)})'\tilde{X}_j^{(N)}\right)^{-1}(\tilde{X}_j^{(N)})'\tilde{Y}_j^{(N)}\quad\text{with}\quad\begin{cases}\tilde{X}_j^{(N)}=(\log f_i\ ,\ 1)_{i\in\{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}}\\ \tilde{Y}_j^{(N)}=(Y_i)_{i\in\{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}}\end{cases},\end{aligned}$$

and for all $k=1,\cdots,m$, define $g_0^*(k)=\dfrac{f_{min}}{\beta}\left(\dfrac{\omega_1^*}{f_{min}}\right)^{k/(m+1)}$, $g_K^*(k)=\dfrac{\omega_K^*}{\alpha}\left(\dfrac{f_{max}}{f_{min}}\right)^{k/(m+1)}$ and
$g_j^*(k)=\dfrac{\omega_j^*}{\alpha}\left(\dfrac{\alpha\omega_{j+1}^*}{\beta\omega_j^*}\right)^{k/(m+1)}$ for all $j\in\{1,\cdots,K-1\}$, .

We get the following central limit theorems for the corresponding estimators $(\tilde{H}_j^{(N)},\tilde{\sigma^2}_j^{(N)})$:

Proposition 4.2 *Under the same assumptions as in Proposition 4.1, for all $j=0,\cdots,K$,*

$$(N\Delta_N)^{1/2}\left(\tilde{\lambda}_j^{(N)}-\lambda_j^*\right)\quad\xrightarrow[N\to\infty]{\mathcal{D}}\quad\mathcal{N}(0,\Gamma_1^{\lambda_j^*}) \tag{23}$$

where $\Gamma_1^{\lambda_j^*}=\left(X_j^{*\prime}X_j^*\right)^{-1}X_j^*\Sigma_j^*X_j^{*\prime}\left(X_j^{*\prime}X_j^*\right)^{-1}$, with $X_j^*=\left(\log g_j^*(k)\ ,\ 1\right)_{1\leq k\leq m}$ and $\Sigma_j^*=(s_{kl}^{*j})_{1\leq k,l\leq m}$ the following matrix :

$$s_{kl}^{*j}=2\cdot\left(g_j^*(k)g_j^*(l)\right)^{2H_j^*}\cdot\frac{\displaystyle\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\overline{\widehat{\psi}}\left(\frac{\xi}{g_j^*(k)}\right)\widehat{\psi}\left(\frac{\xi}{g_j^*(l)}\right)|\xi|^{-(2H_j^*+1)}e^{-iu\xi}d\xi\right)^2du}{\left(\displaystyle\int_{\mathbb{R}}\left|\widehat{\psi}(u)\right|^2|u|^{-(2H_j^*+1)}du\right)^2}. \tag{24}$$

Remark 4.3 *Another possible choice would be to consider the regression for all the available frequencies in the interval $[\tilde{U}_j^{(N)},\tilde{V}_j^{(N)}]$. The number of considered frequencies increases then with the rate $a_N=N\Delta_N$. However, it does not improve significantly the convergence since the remainders of the regression are very strongly dependent.*

4.2 Goodness of fit test

It is also possible to estimate parameters H_j^* and σ_j^* from an feasible (or estimated) generalized least squares estimation (for more details, see Amemiya, chap. 6.3, 1985). Indeed, we can identify the asymptotic covariance matrix Σ_j^* for $j = 0, \cdots, K$: this matrix has the form $\Sigma_j^* = \Sigma(H_j^*, \omega_j^*, \omega_{j+1}^*)$ and, from the previous limit theorems, $\widehat{\Sigma}_j^{(N)} = \Sigma(\tilde{H}_j^{(N)}, \widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)})$ converges in probability to Σ_j^*. Thus, it is possible to construct an estimator $\underline{\lambda}_j^{(N)}$ of λ_j^* with a feasible generalized least squares (F.G.L.S.) regression *i.e.* by minimizing

$$\| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)}\lambda \|^2_{\widehat{\Sigma}_j^{(N)}} = (\tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)}\lambda)' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} (\tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)}\lambda).$$

First, we give asymptotic behavior of $\underline{\lambda}_j^{(N)} = \begin{cases} \left(-(2\underline{H}_j^{(N)}+1), \log \underline{\sigma}^{2}{}_j^{(N)} + \log K_{\underline{H}_j^{(N)}}(\psi)\right)' \\ \left((\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \tilde{X}_j^{(N)}\right)^{-1} (\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \tilde{Y}_j^{(N)} \end{cases}$.

Proposition 4.3 *Under the same assumptions as in Proposition 4.2, for all $j = 0, \cdots, K$,*

$$(N\Delta_N)^{1/2}\left(\underline{\lambda}_j^{(N)} - \lambda_j^*\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \Gamma_2^{\lambda_j^*}) \tag{25}$$

with $\Gamma_2^{\lambda_j^*} = \left(X_j^{*'} \left(\Sigma_j^*\right)^{-1} X_j^*\right)^{-1}$.

For $j = 0, \cdots, K$, the vectors $\tilde{Y}_j^{(N)}$ and $\tilde{X}_j^{(N)}\underline{\lambda}_j^{(N)}$ are two different estimators of the vector $\left(-(2H_j^*+1)\log f_i + \log \sigma_j^{2*} + \log K_{H_i^*}(\psi)\right)_{i\in\{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}}$. It suggests to define the following goodness of fit test. The test statistic $T_K^{(N)}$ is defined as the sum of the squared distances between these two estimators for all $K+1$ frequency ranges:

$$T_K^{(N)} = (N\Delta_N) \cdot \left(\sum_{j=0}^{K} \| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)}\underline{\lambda}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}}\right).$$

This distance is the F.G.L.S. distance between points $(\log f_i, Y_i)_{i\in\{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}}$ for $j = 0, \cdots, K$ and the $(K+1)$ F.G.L.S. regression lines. As a consequence, we get

Proposition 4.4 *Under assumptions of Proposition 4.1, we have*

$$T_K^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \chi^2((K+1)(m-2)). \tag{26}$$

Remark 4.4 *Proposition 4.4 may be explained with heuristic arguments. Remainders are turned white, thus it is only natural for the sum of the second regression remainder squares to asymptotically form a χ^2 process. The number of degrees of freedom is $(K+1)(m-2)$ because one loses two degrees of freedom after the twice estimation of the $(K+1)$ vectors λ_j^* (we also show that these vectors are asymptotically independent).*

4.3 Estimation of the number of frequency changes

Throughout the previous study, the number of frequency change, K, is assumed to be known. But the previous test provides a way for estimating K. In fact, it can be recursively done by beginning with $K = 0$ and continuing till the assumption "X is a (M_K)-F.B.M." is accepted. The following applications in biomechanics provide different examples of the power of discrimination of such a procedure. However, this estimation of the number of frequency changes must be carefully applied : from numerical and heuristic arguments, it does not seem reasonable to work with $K > 2$.

4.4 Estimation procedure and on the choice of parameters

Thus, for identifying a (M_K)-multiscale fractional (with K unknown) from a time series $(X_0, X_{\Delta_N}, \cdots, X_{N\Delta_N})$ we suggest the following procedure:

1. Begin with $K = 0$.
2. Choose a mother wavelet ψ (and thus α and β), a frequency band $[f_{min}, f_{max}]$ and m (see below for these different choices).
3. Compute the different frequencies $(f_i)_{0 \leq i \leq a_N}$.
4. Compute the vector $(Y_i)_{0 \leq i \leq a_N} = (\log J_N(1/f_i))_{0 \leq i \leq a_N}$.
5. Minimize $Q^{(N)}(T, \Lambda)$ and thus compute the different values of $\widehat{\omega}_j^{(N)}$ for $j = 1, \cdots, K$.
6. Compute the different regression moments $\{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}$ and then the estimators $\tilde{\lambda}_j^{(N)}$ (for $j = 0, \cdots, K$).
7. Compute the different matrices $\widehat{\Sigma}_j^{(N)}$ and then $\underline{\lambda}_j^{(N)}$ (for $j = 0, \cdots, K$).
8. Compute $T_K^{(N)}$ and compare its value to the 95%-quantile of a $\chi^2((K+1)(m-2))$. If the test is rejected then go back to step 2. with $K = K + 1$.

How to chose the function ψ and the parameters f_{min}, f_{max} and m ?

1. **Choice of ψ :** The mother wavelet ψ has to satisfy Assumptions (A1) but as we say previously it is not mandatory to associate this function to orthogonality properties. However, the Lemarié-Meyer wavelet is a natural choice with good numerical properties of asymptotic decreasing but a too large ratio β/α which implies a too large transition zone of frequencies. The function ψ can also be deduced from an arbitrary construction of its Fourier transform $\widehat{\psi}$; for instance, we propose $\widehat{\psi}_1(\lambda) = \exp\left(\dfrac{-1}{(|\lambda| - \alpha)(\beta - |\lambda|)}\right) \mathbf{1}_{\alpha \leq |\lambda| \leq \beta}$ and the function ψ_2 built from a translation of the Fourier transform of the Lemarié-Meyer function to $[-2\pi, -\pi] \cup [\pi, 2\pi]$ (thus the ratio is now $\beta/\alpha = 2$). The results obtained from those functions ψ_1 and ψ_2 are essentially the same than with the Lemarié-Meyer mother function, they appear more precise for the detection of frequency changes ω_j^* (because $\log \beta/\alpha$ and thus the transition band, could be as small as wanted) and less precise for the estimation of parameters H_j^* (because ψ_1 and ψ_2 are not concentrated as well around 0).
2. **Choice of f_{min} and f_{max} :** (we assume here that the frequencies are given in the inverse of $(X_1, X_2 \cdots)$ time unity). The choice of f_{min} and f_{max} is first driven by the selection of a frequency band inside which the process has to be studied; the inspected frequency band is then $[\dfrac{f_{min}}{\beta}, \dfrac{f_{min}}{\alpha}]$. Secondly, $N \times \dfrac{f_{min}}{\beta}$ should be large enough for computing $I_N(\dfrac{\beta}{f_{min}})$ in (8). Formally one only needs to have $N \times \dfrac{f_{min}}{\beta} \geq 1$ but numerically $N \times \dfrac{f_{min}}{\beta} \geq 10$ seems to be necessary to use correctly the central limit theorem. Finally, the discretization problem implies that f_{max} cannot be too large for providing a good estimation of $d_X(\dfrac{\alpha}{f_{max}}, k\Delta_N)$ by $e_X(\dfrac{\alpha}{f_{max}}, k\Delta_N)$. In practice $\dfrac{f_{max}}{\alpha} \leq \dfrac{1}{\Delta_N}$ appears as a minimal condition.
3. **Choice of m :** Formally, m could be chosen such as $3 \leq m < \min_j(t_{j+1}^* - \tau_N - t_j^*)$. Theoretically, the larger the m, the closer to 1 the power of the test. But numerical considerations imply that if m is too large then the different matrix $\widehat{\Sigma}_j^{(N)}$ are extremely correlated and the quality of the test is very dependent to the quality of the different estimations of $\widehat{\lambda}_j^*$. As a consequence, we chose $5 \leq m \leq 10$.

5 Numerical simulation and applications in Biomechanics

5.1 Simulations

1. Initially, we apply the estimators and tests to several simulated trajectories of classical F.B.M. (generated according to a Choleski decomposition) with different values of $H = 0.2,\ 0.4,\ 0.6$ and 0.8). The selected of values of different parameters are : $N = 6000$, $\Delta_N = 0.03$, $m = 5$, $f_{min} = 0.05$ and $f_{max} = 20$. There are 30 independent replications of each time series. The results are presented in the following table :

Theoretical values of H	0.2	0.4	0.6	0.8
Empirical mean of $\widehat{H}$	0.148	0.384	0.599	0.821
Standard deviation of $\widehat{H}$	0.034	0.031	0.041	0.048

The Figure 4 presents the log-log representation for one trajectory : the linearity is seeming. Moreover, Figure 5 exhibits a histogram of the distribution of the test statistic $T_0^{(N)}$ (in this case $K = 0$ and $30 \times 4 = 120$ independent realizations) compared to a χ^2-distribution with 3 degrees of freedom. The goodness-of-fit Kolmogorov-Smirnov test for $T_0^{(N)}$ to the $\chi^2(3)$ distribution is also accepted (with $D \simeq 0.091$ and $p - value \simeq 0.272$).

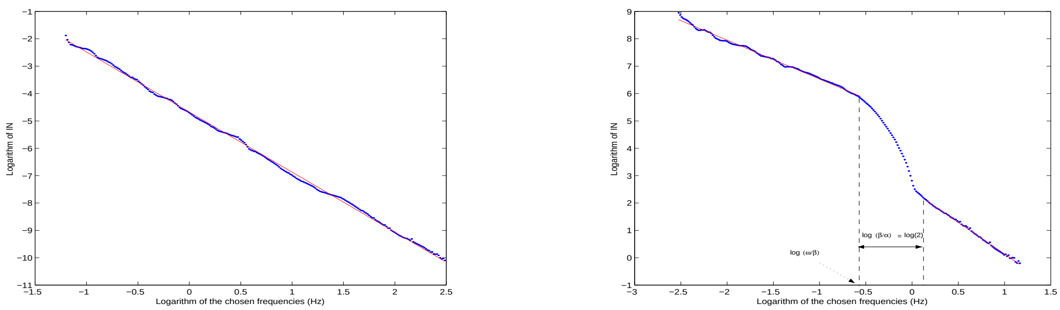

Figure 4 : The log-log representation for a trajectory of a (M_0)-FBM (left, with $H = 0.6$) and (M_1)-FBM (right)

2. Then, we apply to 30 independent replications trajectories of (M_1)-FBM (generated according to a Choleski decomposition with numerical approximations of the covariances) with $H_0 = 0.2$ and $\sigma_0^2 = 10$, $H_1 = 0.7$, and $\sigma_1^2 = 5$, and $\omega_1 = 5$. The results (with parameters : $N = 6000$, $\Delta_N = 0.03$, $m = 5$, $f_{min} = 0.8$ and $f_{max} = 16$) are the following :

Theoretical value	$H_0 = 0.2$	$H_1 = 0.7$	$\omega_1 = 5$
Empirical mean	0.197	0.693	5.18
Standard deviation	0.110	0.068	0.491

Figure 4 presents the log-log representation for one trajectory, with the 2 regression lines. The hypothesis of the modelling with a simple FBM (therefore with $K = 0$) is always rejected (in such a case, the model is misspecified and the statistic $T_0^{(N)}$ is then between 39.3 and 126.8, very different from the realizations of χ^2-distribution with 3 degrees of freedom). On the contrary, the hypothesis of the modelling with a (M_1)-FBM is always accepted and a histogram of the realizations of the test statistic $T_1^{(N)}$ is presented in Figure 5 (compared to a χ^2-distribution with 6 degrees of freedom). The goodness-of-fit Kolmogorov-Smirnov test for $T_1^{(N)}$ to the $\chi^2(6)$ distribution is also accepted (with $D \simeq 0.187$ and $p - value \simeq 0.059$).

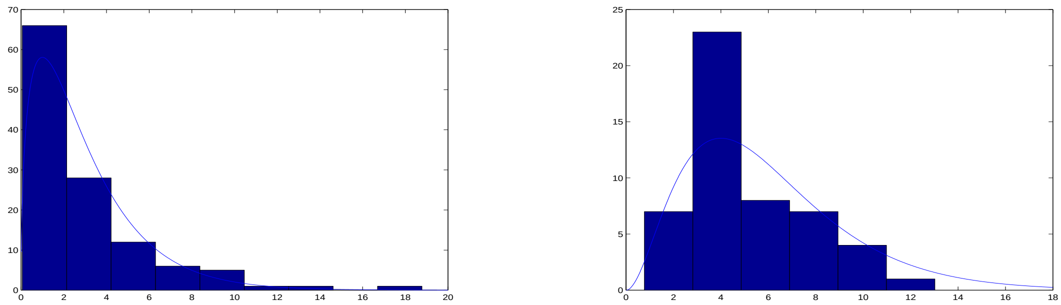

Figure 5 : The empirical distribution of $T_0^{(N)}$ and $T_1^{(N)}$ (respectively) compared to the corresponding χ^2 distribution in the cases of simulated trajectories of (M_0)-FBM (left) and (M_1)-FBM (right)

Conclusion of these simulations : the results are surprisingly good compared with the complexity of the method. The asymptotic distribution of the test statistics can be used for real data. However, the computation time is important (especially for the computation of the test statistic) : 3 hours are necessary for the treatment of each (M_1)-FBM replication.

5.2 Applications in Biomechanics

We apply our statistics to different trajectories (see the description in the Introduction) with the following parameters :

- $N = 6000$ and $\Delta_N = 0.03$;
- The mother wavelet is ψ_1 (with $\alpha = 5$ and $\beta = 10$).
- The choice of the frequency band is $f_{min} = 0.15$ and $f_{max} = 15$ which corresponds to a detection frequency band $[0.52\ ,\ 38.32]$ Hz (with mother wavelet ψ_1);
- $m = 5$.

First, we study the X-trajectories of one subject (fore-aft direction) for different feet position (0; 2; 10; 20 cm clearance and 0; 15; 30; 45^o angle). In all the cases, the test (with a type I error of 5%) rejects the hypothesis of a modelling with a simple (M_0)-FBM. But the modelling with a (M_1)-FBM is accepted by the test 12 times out of 16, with an empirical mean of $\widehat{\omega}_1 \simeq 3.5$ and a standard deviation of $\widehat{\omega}_1 \simeq 1$ (the different values of $\tilde{H}_0$ and $\tilde{H}_1$ are in $[0.9, 1]$ in the different cases).

For the different Y-trajectories (medio-lateral direction) of the same patient, the test rejects the hypothesis of a modelling with a simple (M_0)-FBM in all the case. The modelling with a (M_1)-FBM is accepted by the test 13 times out of 16, with an empirical mean of $\widehat{\omega}_1 \simeq 3.1$ and a standard deviation of $\widehat{\omega}_1 \simeq 1$ (the different values of $\tilde{H}_0$ and $\tilde{H}_1$ are in $[0.8, 1]$ in the different cases).

Figure 6 presents log-log plots of $J_N(f_k)$ versus f_k (i.e. $\log J_N(f_k)$ vs. $\log f_k$) of all the experiments, for X-trajectories (left) and Y-trajectories (right).

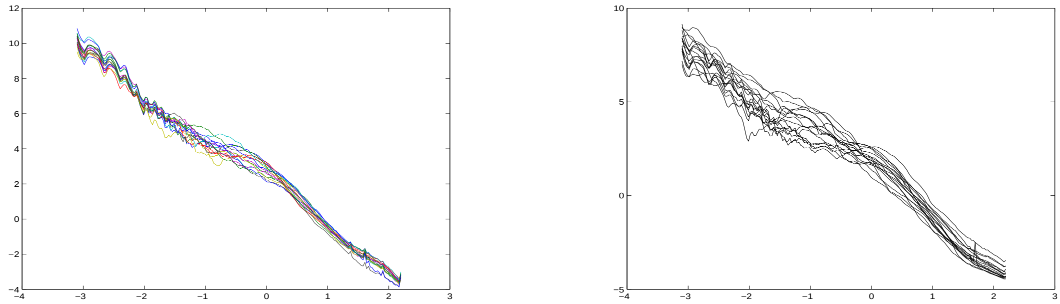

Figure 6 : The log-log representation of the 16 different X-trajectories (left) and Y-trajectories (right)

Conclusion of these applications to biomechanics data : all these results allow us to give new interpretations on the upright position. The behavior of X-trajectories and Y trajectories are very similar, for all the positions of the feet (the studied statistics do not seem to depend on the angle and clearance of the feet). The (M_1)-FBM models of these trajectories fit well, which suggests two different type of behavior for low and high frequencies. The frequency change is around 3 Hz, which corresponds to a physiological change : this could be interpreted for instance as the passage of a cerebral control of the stability by the inner ear to a muscular auto-stabilization. We return to [13] for a more detailed discussion of the biomechanical interpretations. Such an estimation of this frequency change would be very interesting for a better detection of certain pathologies and to help in their cure.

A Proofs

A.1 Proof of Theorem 3.1

First, we prove the following technical Lemma :

Lemma A.1 *Let X be a (M_K)-MBM. For $(t,t')\in \mathbb{R}^2$, and $(u,u')\in \mathbb{R}_+^2$, define :*

$$S(t,u,t',u') = E\Big[(X(t+u)-X(t))\cdot(X(t'+u')-X(t'))\Big]. \qquad (27)$$

1. *For all $(u,u',t,t')\in \mathbb{R}_+^2\times\mathbb{R}^2$, there exists a constant $C>0$ depending only on the parameters $(\omega_j)_j$, $(\sigma_j)_j$ and $(H_j)_j$ such that :*

$$\big|S(t,u,t',u')\big| \le C\cdot\big(u^{H_K}\cdot \mathbf{1}_{u\le 1}+u^{H_0}\cdot\mathbf{1}_{u>1}\big)\times\big(u'^{H_K}\cdot\mathbf{1}_{u'\le 1}+u'^{H_0}\cdot\mathbf{1}_{u'>1}\big)\,; \qquad (28)$$

2. *More precisely, if $\big(\max(u,u')\cdot\omega_K\big)<1$ and $\max(u,u')\le \frac{1}{4}\cdot|t'-t|$, there exists a constant $C>0$ depending only on the parameters $(\omega_j)_j$, $(\sigma_j)_j$ and $(H_j)_j$ such that :*

$$\big|S(t,2u,t',2u')\big|\le C\cdot\Big(u\cdot u'+\max(u,u')^4\Big)\left(\frac{1}{|t-t'+u'-u|}+\max_{i=0,1,\cdots,K}\left\{\frac{1}{|t-t'+u'-u|^{2-2H_i}}\right\}\right). \qquad (29)$$

Proof. 1/ First, the Cauchy-Schwarz inequality implies that

$$\big|S(t,u,t',u')\big|\le\sqrt{E\left[(X(t+a)-X(t))^2\right]}\times\sqrt{E\left[(X(t'+a')-X(t')^2)\right]}.$$

But, $E\left[(X(t+a)-X(t))\right]^2 = 4\sum_{j=0}^{K}\sigma_j^2\cdot a^{2H_j}\int_{a\omega_j}^{a\omega_{j+1}}\frac{(1-\cos v)}{v^{2H_j+1}}\,dv.$ Then, the following expansions :

$$\int_0^x \frac{(1-\cos v)}{v^{2H+1}}\,dv = \begin{cases} \dfrac{1}{2(2-2H)}\,x^{2-2H}+O(x^{4-2H}) & \text{for} \quad x\to 0;\\[2mm] C(H)-\dfrac{1}{2H}\,\dfrac{1}{x^{2H}}+O\big(\dfrac{1}{x^{2H+1}}\big) & \text{for} \quad x\to\infty.\end{cases}$$

with $C(H)=\displaystyle\int_0^\infty \frac{(1-\cos v)}{v^{2H+1}}\,dv$, imply that :

$$E\left[(X(t+u)-X(t))\right]^2=\left\{\begin{array}{ll}4\cdot\sigma_K^2\cdot C(H_K)\cdot u^{2H_K}+O(u^2) & \text{when } u\to 0;\\ \\ 4\cdot\sigma_0^2\cdot C(H_0)\cdot u^{2H_0}+O(1) & \text{when } u\to\infty;\end{array}\right. \tag{30}$$

that achieves the proof of the majoration (28).

2/ We turn now to the proof of the upper bound (29). To begin with, we remark that for all $(t,t',u,u')\in\mathbb{R}^4$, the following equalities are true :

$$\begin{aligned}S(t,2u,t',2u') &= \int_{\mathbb{R}}\frac{(e^{-i(t+2u)\xi}-e^{-it\xi})(e^{i(t'+2u')\xi}-e^{it'\xi})}{\rho^2(\xi)}\,d\xi\\ &= \int_{\mathbb{R}}\frac{(e^{-iu\xi}-e^{iu\xi})(e^{iu'\xi}-e^{-iu'\xi})}{\rho^2(\xi)}\,e^{i\xi(t'-t)+i\xi(u'-u)}\,d\xi\\ &= 8\int_0^\infty\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos\big(\xi(t'-t+u'-u)\big)}{\rho^2(\xi)}\,d\xi\\ &= 8\sum_{i=0}^K\sigma_i^2\int_{\omega_i}^{\omega_{i+1}}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos\big(\xi(t'-t+u'-u)\big)}{\xi^{2H_i+1}}\,d\xi.\end{aligned}\tag{31}$$

Then, we bound the different integrals.

• First, we threat the case $i=K$ that is when the upper limit of the integral is ∞. In this case, we can rewrite the integral between ω_K and ∞ as the difference of the integral between 0 and ∞ and the one between 0 and ω_K, that is $\displaystyle\int_{\omega_K}^\infty\gamma(\xi)\,d\xi=\int_0^\infty\gamma(\xi)\,d\xi-\int_0^{\omega_K}\gamma(\xi)\,d\xi$ where $\gamma(\xi)=\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos\big(\xi(t'-t+a'-a)\big)\,\xi^{-(2H_K+1)}$. The second integral of the right hand side can be bounded by the same argument than the terms of order $i=0$ in (31). The first one corresponds to the expression of the covariance of the increments of a F.B.M. B_{H_K} with Hurst parameter H_K and variance 1. Thus, for all $(t,t')\in\mathbb{R}^2$, $(u,u')\in\mathbb{R}_+^2$ such that $4\cdot\max(u,u')\le|t'-t|$, we get :

$$\begin{aligned}&\left|8\int_0^\infty\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos\big(\xi(t'-t+u'-u)\big)}{\xi^{2H_K+1}}\,d\xi\right|\\ &\quad= \Big|E\Big((B_{H_K}(t+2u)-B_{H_K}(t))\cdot(B_{H_K}(t'+2u')-B_{H_K}(t'))\Big)\Big|\\ &\quad= \frac{1}{2C^2(H_K)}\Big|\Big(|t-t'+2u|^{2H_K}-|t-t'+2u-2u'|^{2H_K}-|t-t'|^{2H_K}+|t'-t-2u'|^{2H_K}\Big)\Big|\\ &\quad\le D(H_K)\cdot\frac{u\cdot u'}{|t-t'+u'-u|^{2-2H_K}},\qquad\text{with } D(H_K)>0.\end{aligned}\tag{32}$$

• Next, we consider the integrals with a finite upper limit and a non-zero lower limit. This corresponds to $i=1,\ldots,K-1$. In these cases, for $b>0$, an integration by parts provides us

$$\begin{aligned}\int_{\omega_i}^{\omega_{i+1}}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos(b\xi)}{\xi^{2H_i+1}}\,d\xi&=\frac{1}{b}\Big(-\int_{\omega_i}^{\omega_{i+1}}\frac{u\cdot\cos(u\xi)\cdot\sin(u'\xi)+u'\cdot\cos(u'\xi)\cdot\sin(u\xi)}{\xi^{2H_i+1}}\cdot\sin(b\xi)\,d\xi\\ &+(2H_i+1)\int_{\omega_i}^{\omega_{i+1}}\frac{\sin(u\xi)\cdot\sin(u'\xi)}{\xi^{2H_i+2}}\cdot\sin(b\xi)\,d\xi+\left[\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\sin(b\xi)}{\xi^{2H_i+1}}\right]_{\omega_i}^{\omega_{i+1}}\Big).\end{aligned}\tag{33}$$

By using the majoration $|\sin(ux)|\le ux$, $|\sin(u'x)|\le u'x$, $|\cos(ux)|\le 1$, $|\cos(u'x)|\le 1$ and $|\sin(bx)|\le 1$ for $x\ge 0$, we deduce that for all $(u,u',b)\in\mathbb{R}_+^3$,

$$\Big|\int_{\omega_i}^{\omega_{i+1}}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos(b\xi)}{\xi^{2H_i+1}}\,d\xi\Big|\le C_i\cdot\frac{u\cdot u'}{b},\tag{34}$$

where $C_i > 0$ is a constant depending only on H_i, ω_i and ω_{i+1}.

• Finally, it remains to bound the two integrals with lower limit 0. We will show only how to bound $\int_0^{\omega_1} \frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos(b\xi)}{\xi^{2H_i+1}}\,d\xi$, since the other integral can be treated similarly. The integration by part formula (33) remains valid even when the lower limit is 0. Indeed, the integrand can be bounded by $C \times \xi^{1-2H_i}$ and $\int_0^1 \xi^{1-2H_i}\,d\xi < \infty$ as soon as $H_i < 1$. After this remark, we bound the three terms of the right hand side of (33).

i) From $|\sin(ux)| \leq ux$, $|\sin(u'x)| \leq u'x$ and $|\sin(bx)| \leq 1$ for $x \geq 0$, we deduce that for all $(u, u', b) \in I\!R_+^3$,

$$\Big|\Big[\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\sin(b\xi)}{\xi^{2H_0+1}}\Big]_0^{\omega_1}\Big| \quad\leq\quad \big(\omega_1^{1-2H_0}\big)\cdot u\cdot u'. \tag{35}$$

ii) For all $(\xi, \xi') \in [0, \omega_1]$, the power series expansion of $x \mapsto \sin(x)$ implies that

$$\sin(u\xi)\cdot\sin(u\xi) = u\cdot u'\cdot\xi^2 + \sum_{k=1}^{\infty}\Big(\sum_{j=0}^{k}\frac{u^{2j+1}\cdot(u')^{2(k-j)+1}}{(2j+1)!\cdot(2(k-j)+1)!}\Big)\cdot(-1)^k\xi^{2k+2}.$$

One can remark that

$$\begin{aligned}\sum_{j=0}^{k}\frac{u^{2j+1}\cdot(u')^{2(k-j)+1}}{(2j+1)!\cdot(2(k-j)+1)!} \quad&\leq\quad \max(u,u')^{2k+2}\sum_{j=0}^{k}\frac{1}{(2j+1)!\cdot(2(k-j)+1)!}\\ &\leq\quad \max(u,u')^{2k+2},\end{aligned}$$

because $\sum_{j\geq 0}\frac{1}{(2j+1)!} \leq 2$. As a consequence, when $(\max(u,u')\cdot\omega_1) < 1$ and $b > 0$, integration and summation can be interchanged and

$$\begin{aligned}\Big|\int_0^{\omega_1}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\sin(b\xi)}{\xi^{2H_0+2}}\,d\xi - u\cdot u'\int_0^{\omega_1}\frac{\sin(b\xi)}{\xi^{2H_0}}\,d\xi\Big| \quad&\leq\quad \sum_{k=1}^{\infty}\Big(\max(u,u')^{2k+2}\cdot\int_0^{\omega_1}\xi^{2k-2H_0}\,d\xi\Big)\\ &\leq\quad \frac{\max(u,u')^4\cdot\omega_1^{3-2H_0}}{1-(\max(u,u')\cdot\omega_1)^2}.\end{aligned}$$

But $\int_0^{\omega_1}\frac{\sin(b\xi)}{\xi^{2H_0}}\,d\xi = b^{2H_0-1}\int_0^{b\cdot\omega_1}\frac{\sin(\xi)}{\xi^{2H_0}}\,d\xi$. Denote $M(H) = \sup_{x\in I\!R_+}\Big|\int_0^x\frac{\sin(\xi)}{\xi^{2H}}\,d\xi\Big|$ for $0 < H < 1$. Thus

$$\Big|\int_0^{\omega_1}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\sin(b\xi)}{\xi^{2H_0+2}}\,d\xi\Big| \leq M(H_0)\cdot u\cdot u'\cdot b^{2H_0-1} + \frac{\omega_1^{3-2H_0}\cdot\max(u,u')^4}{1-(\max(u,u')\cdot\omega_1)^2}. \tag{36}$$

iii) Similarly for $(\xi, \xi') \in [0, \omega_1]$, we have

$$\cos(u\xi)\cdot\sin(u'\xi) = u'\cdot\xi + \sum_{k=1}^{\infty}\Big(\sum_{j=0}^{k}\frac{u^{2j}\cdot(u')^{2(k-j)+1}}{(2j)!\cdot(2(k-j)+1)!}\Big)\cdot(-1)^k\xi^{2k+1},$$

But for $k \geq 1$, $\Big(\sum_{j=0}^{k}\frac{u^{2j}\cdot(u')^{2(k-j)+1}}{(2j)!\cdot(2(k-j)+1)!}\Big) \leq \max(u,u')^{2k+1}$. As a consequence, when $(\max(u,u')\cdot\omega_1) < 1$ and $b > 0$, integration and summation can be interchanged and we get

$$\Big|\int_0^{\omega_1}\frac{u\cdot\cos(u\xi)\cdot\sin(u'\xi)\cdot\sin(b\xi)}{\xi^{2H_0+1}}\,d\xi - u\cdot u'\int_0^{\omega_1}\frac{\sin(b\xi)}{\xi^{2H_0}}\,d\xi\Big| \leq \frac{\max(u,u')^4\cdot\omega_1^{3-2H_0}}{1-(\max(u,u')\cdot\omega_1)^2}. \tag{37}$$

Therefore from (35), (36), (37) and (32), we deduce for (u, u') such that $\max(u,u')\cdot\omega_K < 1/2$:

$$\Big|\int_{\omega_K}^{\infty}\frac{\sin(u\xi)\cdot\sin(u'\xi)\cdot\cos(b\xi)}{\xi^{2H_K+1}}\,d\xi\Big| \leq (D(H_K)+3M(H_K))\cdot\frac{u\cdot u'}{b^{2-2H_K}} + 4\cdot\frac{\max(u,u')^4}{b}\cdot\omega_K^{3-2H_K} + \frac{u\cdot u'}{b}\cdot\omega_K^{1-2H_K}.$$

By combining the two previous bounds with (31) and (34), we deduce (29) and this finishes the proof. ■

The proof of Theorem 3.1 uses the two following lemmas:

Lemma A.2 *Under the same notations and assumptions as in Theorem 3.1, there exists two constants $C_1 > 0$ and $C_2 > 0$ depending only on r, a_{min} and a_{max} such that for all N*

$$i) \quad \sup_{a\in[a_{min},\,a_{max}]} \max_{k\in D_N(a)} E\,\varepsilon_N^2(a,k) \quad\leq\quad C_1\times\varphi(N) \tag{38}$$

$$ii) \quad \sup_{a_1,\,a_2\in[a_{min},\,a_{max}]} \max_{k\in D_N(a_1)\cap D_N(a_2)} E\,\left|\sqrt{a_1}\,\varepsilon_N(a_1,k)-\sqrt{a_2}\,\varepsilon_N(a_2,k)\right|^2 \quad\leq\quad C_2\times\varphi(N)\times|a_2-a_1|^2 \tag{39}$$

where $\varphi(N) = \Delta_N^2 + \Delta_N^{1+2H_K} + \Delta_N^2\log N + \Delta_N^{1+2\overline{H}} N^{-1+2\overline{H}} + (N\Delta_N)^{-2}$ *with* $\overline{H} = \max\{H_i\ ,\ i=0,\cdots,K\}$.
iii) Moreover $(N\,\Delta_N)\,\varphi(N)\to 0$ *when* $N\to\infty$.

Proof. The error $\varepsilon_N(a,k)$ contains three different terms, the first one corresponds to the replacement of the integral onto the interval $[0,T_N]$ by its Riemann sum, the second and the third ones correspond to the replacement of the integral onto $I\!R$ by the integral onto the interval $[0,T_N]$ where $T_N = N\Delta_N$. More precisely, we have

$$\varepsilon_N(a,k) \quad=\quad \frac{1}{\sqrt{a}}\times\left(\varepsilon_{1,N}(a,k)+\varepsilon_{2,N}(a,k)+\varepsilon_{3,N}(a,k)\right) \tag{40}$$

with

$$\begin{aligned}
\varepsilon_{1,N}(a,k) &= \int_0^{T_N}\psi(\frac{t}{a}-k\Delta_N)\,X(t)\,dt \;-\; \Delta_N\sum_{p=0}^{N-1}\psi(\frac{p\Delta_N}{a}-k\Delta_N)\,X(p\Delta_N),\\
\varepsilon_{2,N}(a,k) &= \int_{T_N}^{\infty}\psi(\frac{t}{a}-k\Delta_N)\,X(t)\,dt,\\
\varepsilon_{3,N}(a,k) &= \int_{-\infty}^{0}\psi(\frac{t}{a}-k\Delta_N)\,X(t)\,dt.
\end{aligned}$$

By using $(x+y+z)^2\leq 3\,(x^2+y^2+z^2)$ for all real numbers x, y, z, we deduce

$$E\,\varepsilon_N^2(a,k) \quad\leq\quad \left(\frac{3}{a}\right)\times\left(\sum_{i=1}^{3}E\,\varepsilon_{i,N}^2(a,k)\right). \tag{41}$$

We now bound the different terms $E\,\varepsilon_{i,N}^2(a,k)$ for $i=1,2,3$:

(1) Bound of $E\,\varepsilon_{1,N}^2(a,k)$.

We have the decomposition $\varepsilon_{1,N}(a,k) \;=\; I_{1,N}(a,k) \;+\; I_{2,N}(a,k)$, where

$$\begin{aligned}
I_{1,N}(a,k) &= \sum_{p=0}^{N-1}\int_{p\Delta_N}^{(p+1)\Delta_N}\psi\Big(\frac{t}{a}-k\Delta_N\Big)\Big(X(t)-X(p\Delta_N)\Big)\,dt\\
\text{and}\qquad I_{2,N}(a,k) &= \sum_{p=0}^{N-1}\int_{p\Delta_N}^{(p+1)\Delta_N}\left(\psi\Big(\frac{t}{a}-k\Delta_N\Big)-\psi\Big(\frac{p\Delta_N}{a}-k\Delta_N\Big)\right)X(p\Delta_N)\,dt.
\end{aligned}$$

Then, the inequality $(x+y)^2\leq 2(x^2+y^2)$ which is valid for all $(x,y)\in I\!R^2$, implies

$$E\,\varepsilon_{1,N}^2(a,k)\leq 2\,E\left(I_{1,N}^2(a,k)\right)+2\,E\left(I_{2,N}^2(a,k)\right).$$

On one hand, we have

$$\begin{aligned}
E\left(I_{1,N}^2(a,k)\right) &= \sum_{p=0}^{N-1}\sum_{p'=0}^{N-1}\int_{p\Delta_N}^{(p+1)\Delta_N}\int_{p'\Delta_N}^{(p'+1)\Delta_N}dtdt'\psi\Big(\frac{t}{a}-k\Delta_N\Big)\psi\Big(\frac{t'}{a}-k\Delta_N\Big)E\Big((X(t)-X(p\Delta_N))(X(t')-X(p'\Delta_N))\Big)\\
&= \sum_{p=0}^{N-1}\sum_{p'=0}^{N-1}\int_0^{\Delta_N}\int_0^{\Delta_N}dudu'\psi\Big(\frac{u+p\Delta_N}{a}-k\Delta_N\Big)\psi\Big(\frac{u'+p'\Delta_N}{a}-k\Delta_N\Big)S(p\Delta_N,u,p'\Delta_N,u').
\end{aligned}$$

Afterwards, we use Lemma A.1 to bound the terms $S(p\Delta_N, u, p'\Delta_N, u')$, where we use different types of bounds depending wether (p,p') is in the vicinity of the diagonal or not. Namely, when $|p-p'| \leq 3$ we use the upper bound (28), and otherwise we use the one in (29). Observe that the assumptions of Lemma A.1 are satisfied for large enough N since $N\Delta_N \to \infty$, as $N \to \infty$. Thus, when N is large enough, we get

$$\begin{aligned}
E\left(I^2_{1,N}(a,k)\right) &\leq \sum_{\substack{p=0,\,p'=0\\|p-p'|\leq 3}}^{N-1} \int_0^{\Delta_N}\!\!\int_0^{\Delta_N} du du' \left|\psi\Big(\frac{u+p\Delta_N}{a}-k\Delta_N\Big)\right|\left|\psi\Big(\frac{u'+p'\Delta_N}{a}-k\Delta_N\Big)\right| C\cdot(u\cdot u')^{H_K} + \\
&+ \sum_{\substack{p=0,\,p'=0\\|p-p'|\geq 4}}^{N-1} \int_0^{\Delta_N}\!\!\int_0^{\Delta_N} du du' \left|\psi\Big(\frac{u+p\Delta_N}{a}-k\Delta_N\Big)\psi\Big(\frac{u'+p'\Delta_N}{a}-k\Delta_N\Big)\right| \times \cdots \\
&\qquad \cdots \times C\cdot\Delta_N^2\left(\frac{1}{(|p'-p|-1)\Delta_N}+\max_{i=0,\cdots,K}\left\{\frac{1}{((|p'-p|-1)\Delta_N)^{2-2H_i}}\right\}\right) \\
&\leq C\cdot\Delta_N^{2+2H_K} \sum_{\substack{p=0,\,p'=0\\|p-p'|\leq 3}}^{N-1} \sup_{\theta\in(0,1)}\left|\psi\left(\Big(\frac{\theta+p}{a}-k\Big)\Delta_N\right)\right| \times \sup_{\theta'\in(0,1)}\left|\psi\left(\Big(\frac{\theta'+p'}{a}-k\Big)\Delta_N\right)\right| \\
&\quad + C\cdot\Delta_N^4 \sum_{\substack{p=0,\,p'=0\\|p-p'|\geq 4}}^{N-1} \sup_{\theta\in(0,1)}\left|\psi\left(\Big(\frac{\theta+p}{a}-k\Big)\Delta_N\right)\right| \times \sup_{\theta'\in(0,1)}\left|\psi\left(\Big(\frac{\theta'+p'}{a}-k\Big)\Delta_N\right)\right| \times\cdots \\
&\qquad \cdots\times\left(\frac{1}{(|p'-p|-1)\Delta_N}+\max_{i=0,\cdots,K}\left\{\frac{1}{((|p'-p|-1)\Delta_N)^{2-2H_i}}\right\}\right).
\end{aligned}$$

However, according to Assumption (A1), for every integer $m \in I\!N^*$ there exists a constant $C>0$ such that for all $x \in I\!R$, $|\psi(x)| \leq C\cdot(1+|x|)^{-m}$. In particular, $\sup_{\theta'\in(0,1)}\left|\psi\left(\Big(\frac{\theta'+p}{a}-k\Big)\Delta_N\right)\right|$ is bounded. Therefore,

$$\begin{aligned}
E\left(I^2_{1,N}(a,k)\right) &\leq C\cdot\Delta_N^{2+2H_K}\sum_{p=0}^{N-1}\sup_{\theta\in(0,1)}\Big(1+|\theta+p-ak|\,\Delta_N/a\Big)^{-m} \\
&+ C\cdot\Delta_N^4\sum_{p=0}^{N-1}\sup_{\theta\in(0,1)}\Big(1+|\theta+p-ak|\,\Delta_N/a\Big)^{-m}\sum_{q=3}^{N}\left(\frac{1}{q\Delta_N}+\max_{i=0,\cdots,K}\left\{\frac{1}{(q\Delta_N)^{2-2H_i}}\right\}\right).
\end{aligned}$$

But

$$\begin{aligned}
\Delta_N\sum_{p=0}^{N-1}\sup_{\theta\in(0,1)}\left(1+|\theta+p-ak|\,\frac{\Delta_N}{a}\right)^{-m} &\leq C\,\Delta_N\sum_{\ell=-\infty}^{\infty}\left(1+\frac{|\ell\Delta_N|}{a}\right)^{-m} \leq C\int_{-\infty}^{\infty}\left(1+\frac{|x|}{a}\right)^{-m}dx \\
&= C\,|a|\int_{-\infty}^{\infty}(1+|y|)^{-m}\,dx \quad \leq C\,|a_{max}|.
\end{aligned}$$

Let us denote $\overline{H} = \max\{H_i\ ,\ i=0,\cdots,K\}$ and $\underline{H} = \min\{H_i\ ,\ i=0,\cdots,K\}$. We deduce

$$\begin{aligned}
E\left(I^2_{1,N}(a,k)\right) &\leq C\cdot\Delta_N^{1+2H_K}+C\cdot\Delta_N^3\sum_{q=3}^{N}\left(\frac{1}{q\Delta_N}+\max_{i=0,\cdots,K}\left\{\frac{1}{(q\Delta_N)^{2-2H_i}}\right\}\right) \\
&\leq C\left(\Delta_N^{1+2H_K}+\Delta_N^2\,\log N+\Delta_N^3\Big(\frac{1}{\Delta_N^{2-2\underline{H}}}\sum_{q=3}^{\Delta_N^{-1}}\frac{1}{q^{2-2\underline{H}}}+\frac{1}{\Delta_N^{2-2\overline{H}}}\sum_{q=\Delta_N^{-1}}^{N}\frac{1}{q^{2-2\overline{H}}}\Big)\right) \\
&\leq C\left(\Delta_N^{1+2H_K}+\Delta_N^2\,\log N+\Delta_N^2+\Delta_N^{1+2\overline{H}}N^{-1+2\overline{H}}\right). \qquad (42)
\end{aligned}$$

On the other hand, by using Lemma A.1, formula (27), we get

$$\begin{aligned}
E\left(I^2_{2,N}(a,k)\right) &= \sum_{p=0}^{N-1}\sum_{p'=0}^{N-1}\int_{p\Delta_N}^{(p+1)\Delta_N}\int_{p'\Delta_N}^{(p'+1)\Delta_N} du du' \Big(\psi\Big(\frac{u}{a}-k\Delta_N\Big)-\psi\Big(\frac{p\Delta_N}{a}-k\Delta_N\Big)\Big) \\
&\qquad \times \Big(\psi\Big(\frac{u'}{a}-k\Delta_N\Big)-\psi\Big(\frac{p'\Delta_N}{a}-k\Delta_N\Big)\Big)\, S(0,p\Delta_N,0,p'\Delta_N) \\
&\le C\left(\sum_{p=0}^{N-1}\int_0^{\Delta_N} du \Big|\psi\Big(\frac{u+p\Delta_N}{a}-k\Delta_N\Big)-\psi\Big(\frac{p\Delta_N}{a}-k\Delta_N\Big)\Big| \times \big|(p\Delta_N)^{H_0}+(p\Delta_N)^{H_K}\big|\right)^2 \\
&\le C\left(\sum_{p=0}^{N-1}\int_0^{\Delta_N} du\cdot\frac{u}{a}\cdot\sup_{t\in[0,\Delta_N]}\Big|\psi'\Big(\frac{t+p\Delta_N}{a}-k\Delta_N\Big)\Big| \times \big|(p\Delta_N)^{H_0}+(p\Delta_N)^{H_K}\big|\right)^2 \\
&\le C\Delta_N^4\left(\sum_{p=0}^{N-1}\sup_{t\in[0,\Delta_N]}\Big|\psi'\Big(\frac{t+p\Delta_N}{a}-k\Delta_N\Big)\Big| \times \big|(p\Delta_N)^{H_0}+(p\Delta_N)^{H_K}\big|\right)^2.
\end{aligned}$$

But Assumption (A1) implies that for $m=4$, there exists a constant $C>0$ such that for all $x\in I\!\!R$, $|\psi'(x)|\le C\cdot(1+|x|)^{-m}$. We deduce

$$\begin{aligned}
&\Delta_N\sum_{p=0}^{N-1}\sup_{t\in[0,\Delta_N]}\Big|\psi'\Big(\frac{t+p\Delta_N}{a}-k\Delta_N\Big)\Big| \times \big|(p\Delta_N)^{H_0}+(p\Delta_N)^{H_K}\big| \\
&\qquad \le C\Delta_N\sum_{p=-\infty}^{\infty}\frac{1}{\Big(1+|p\Delta_N|\Big)^m}\cdot\big|(p\Delta_N)^{H_0}+(p\Delta_N)^{H_K}\big| \le C\cdot\int_{-\infty}^{\infty}\frac{|x|^{H_0}+|x|^{H_K}}{(1+|x|)^m}\,dx < \infty.
\end{aligned}$$

Therefore,

$$E\left(I^2_{2,N}(a,k)\right) \le C\Delta_N^2. \tag{43}$$

(2) Bound of $E\,\varepsilon^2_{2,N}(a,k)$.
By using Lemma A.1 and Cauchy-Schwartz inequality, we deduce that for N large enough,

$$\begin{aligned}
E\,\varepsilon^2_{2,N}(a,k)) &= \int_{T_N}^{\infty}\int_{T_N}^{\infty}\psi\left(\frac{u}{a}-k\Delta_N\right)\psi\left(\frac{u'}{a}-k\Delta_N\right)\,S(0,u,0,u')\,du\,du' \\
&\le C\left(\int_{T_N}^{\infty}\left(1+\Big|\frac{u}{a}-k\Delta_N\Big|\right)^{-m}u^{2H_0}du\right)^2.
\end{aligned}$$

On one hand, $k\in D_N(a)$ implies that $k\le[(1-r)N/a]$. On the other hand, $u\ge T_N=N\cdot\Delta_N$. Therefore, we have$\Big(1+\Big|\frac{u}{a}-k\Delta_N\Big|\Big)\ge\left(u-(1-r)N\Delta_N\right)/a$. This implies that for $m\ge 4$ and N large enough,

$$\begin{aligned}
\int_{T_N}^{\infty}\left(1+\Big|\frac{u}{a}-k\Delta_N\Big|\right)^{-m}u^{2H_0}du &\le \int_{T_N}^{\infty}\left(\frac{u}{a}-(1-r)\frac{N\Delta_N}{a}\right)^{-m}u^{2H_0}du \\
&= \frac{a^m}{(N\Delta_N)^{m-2H_0-1}}\int_r^{\infty}\frac{(v+1-r)^{2H_0}}{v^m}\,dv,
\end{aligned}$$

by making the change of variable $u=(N\Delta_N)\,(v+1-r)$. Consequently,

$$E\,\varepsilon^2_{2,N}(a,k)\le C\cdot\frac{1}{(N\Delta_N)^2}. \tag{44}$$

(3) Bound of $E\,\varepsilon^2_{3,N}(a,k)$.

By using the same kind of argument than in (2), one obtains that for N large enough

$$\begin{aligned} E\,\varepsilon_{3,N}^2(a,k)) &= \int_\infty^0\int_\infty^0 \psi\left(\frac{u}{a}-k\Delta_N\right)\,\psi\left(\frac{u'}{a}-k\Delta_N\right)\,S(0,u,0,u')\,du\,du' \\ &\le C\left(\int_\infty^0\left(1+\left|\frac{u}{a}-k\Delta_N\right|\right)^{-m}(|u|^{2H_K}+|u|^{2H_0})du\right)^2 \\ &\le C\left(\int_{r/2\cdot N\Delta_N}^\infty \frac{1+(v-r/2\cdot N\Delta_N)^2}{v^m}\,dv\right)^2 \end{aligned}$$

As a consequence, for $m\geq 4$ and N large enough,

$$E\,\varepsilon_{3,N}^2(a,k)\leq C\cdot\frac{1}{(N\Delta_N)^2}. \tag{45}$$

Finally, from (42), (43), (44) and (45), we deduce that (38) holds. This finishes the proof of the point i). Since $N\Delta_N\to\infty$ and $N\Delta_N^2\to 0$, $(N\Delta_N)\varphi(N)$ converges to 0 when $N\to\infty$. This proves the point iii). To complete the proof of Lemma A.2, it remains to proves the point ii). We deduce from the decomposition (40) that

$$E\,\left|\sqrt{a_1}\varepsilon_N(a_1,k)-\sqrt{a_2}\varepsilon_N(a_2,k)\right|^2 \;\le\; 3\sum_{i=1}^3 E\,\left|\varepsilon_{i,N}(a_2,k)-\varepsilon_{i,N}(a_1,k)\right|^2 \tag{46}$$

The same calculations than the ones used to prove the point i) provide the upper bound on the terms $E\,\left|\varepsilon_{i,N}(a_2,k)-\varepsilon_{i,N}(a_1,k)\right|^2$. Indeed, consider for instance the terms with $\varepsilon_{2,N}$, then by using Taylor formula, for every pair (a_1,a_2) with $a_{min}\leq a_1<a_2\leq a_{max}$ there exists a real number $\theta\in(a_1,\,a_2)$ such that

$$\varepsilon_{2,N}(a_2,k)-\varepsilon_{2,N}(a_1,k) \;=\; (a_2-a_1)\times\int_{T_N}^\infty\left(\frac{-t}{\theta^2}\right)\psi'\left(\frac{t}{\theta}-k\Delta_N\right)\,X(t)\,dt$$

Next, by using the same kind of arguments than for the bound of $E\,\varepsilon_{2,N}^2(a,k)$ in point *i)*, we get that for every integer $m>4$, every a_1,a_2 in $[a_{min},\,a_{max}]$ and $k\in D_N(a_1)\cap D_N(a_2)$

$$\begin{aligned} E\,\left|\varepsilon_{2,N}(a_2,k)-\varepsilon_{2,N}(a_1,k)\right|^2 &\le \frac{C}{a_{min}^4}\,\left|a_2-a_1\right|^2\left(\int_{T_N}^\infty\left(1+\left|\frac{u}{\theta}-k\Delta_N\right|\right)^{-m}u^{1+2H_0}du\right)^2 \\ &\le C\,\left|a_2-a_1\right|^2\,(N\Delta_N)^{-2}. \end{aligned}$$

We deduce similarly that

$$E\,\left|\varepsilon_{3,N}(a_2,k)-\varepsilon_{3,N}(a_1,k)\right|^2 \;\le\; C\,\left|a_2-a_1\right|^2\,(N\Delta_N)^{-2}.$$

At this point, it remains to show

$$E\,\left|\varepsilon_{1,N}(a_2,k)-\varepsilon_{1,N}(a_1,k)\right|^2 \;\le\; C\,\left|a_2-a_1\right|^2\,\varphi(N) \tag{47}$$

to finish the proof of item ii). But, we have the decomposition

$$E\,\left|\varepsilon_{1,N}(a_2,k)-\varepsilon_{1,N}(a_1,k)\right|^2 \;\le\; 2\,E\,\left|I_{1,N}(a_2,k)-I_{1,N}(a_1,k)\right|^2+2\,E\,\left|I_{2,N}(a_2,k)-I_{2,N}(a_1,k)\right|^2$$

However, Taylor Formula implies the existence of two real numbers θ_1, $\theta_2\in(a_1,\,a_2)$ such that

$$I_{i,N}(a_2,k)-I_{1,N}(a_1,k)=(a_2-a_1)\cdot\widetilde{I}_{i,N}(\theta_i,k)\qquad\text{for } i=1 \text{ or } 2$$

where $\widetilde{I}_{i,N}(a,k)$ is obtained by replacing into the expression of $I_{i,N}(a,k)$ the map $\psi\Big(\dfrac{t}{a}-k\Delta_N\Big)$ by the map $\left(\dfrac{-t}{a^2}\right)\times\psi'\Big(\dfrac{t}{a}-k\Delta_N\Big)$ and $\psi\Big(\dfrac{p\Delta_N}{a}-k\Delta_N\Big)$ by $\left(\dfrac{-p\Delta_N}{a^2}\right)\times\psi'\Big(\dfrac{p\Delta_N}{a}-k\Delta_N\Big)$. So,

$$E\,\left|\varepsilon_{1,N}(a_2,k)-\varepsilon_{1,N}(a_1,k)\right|^2 \;\le\; C\,\left|a_2-a_1\right|^2\times\left\{E\,\widetilde{I}_{1,N}^2(\theta_1,k)\,+\,E\,\widetilde{I}_{2,N}^2(\theta_2,k)\right\}$$

Since the map $t \mapsto \Big(\dfrac{t}{a}\Big) \times \psi'\Big(\dfrac{t}{a} - k\Delta_N\Big)$ is still continuously differentiable and fast decreasing, one can lead same calculations that in the bound of $E\, I^2_{2,N}(a,k)$. We finally get $E\, \widetilde{I}^2_{2,N}(\theta_2, k) + E\, \widetilde{I}^2_{2,N}(\theta_2, k) \le C\, \varphi(N)$. This implies (47) and completes the proof of Lemma A.2. ■

Lemma A.3 *Under the same assumptions as in Theorem 3.1, there exists a positive constant $C > 0$ such that for every real number $a > 0$ and $N \in \mathbb{N}^*$, we have $E\, |I_N(a) - J_N(a)| \le C \times \varphi(N)^{1/2}$.*

Proof. Since the variables $d = d(a, k\Delta_N)$ and $e = e(a, k\Delta_N)$ are Gaussian, the variables $d^2 - e^2$ have finite second order moment and Jensen's inequality implies

$$\begin{aligned} E\, |I_N(a) - J_N(a)| &= E\, \Big| \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} \big(d^2(a, k\Delta_N) - e^2(a, k\Delta_N)\big) \Big| \\ &\le \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} \sqrt{E\, \big(d^2(a, k\Delta_N) - e^2(a, k\Delta_N)\big)^2} \end{aligned}$$

Then we derive an upper bound for the expectations $E\, \big(d^2(a, k\Delta_N) - e^2(a, k\Delta_N)\big)^2$. Indeed, d and e are jointly Gaussian variables with zero means. One has

$$E\, (d^2 - e^2)^2 = E\, (d - e)^2 (d + e)^2 =: E\, \varepsilon^2 Z^2,$$

where $\varepsilon = d - e$ and $Z = d + e$ are also jointly Gaussian and have mean zero. By using that $Z = \sigma_2\, {\sigma_1}^{-1} \rho\, \varepsilon + \xi$, where $\sigma_1^2 = E\, \varepsilon^2$, $\sigma_2^2 = E\, Z^2$, $\rho = corr(\varepsilon, Z)$ and where ξ is independent of ε and Gaussian, one can show that

$$\begin{aligned} E\, \varepsilon^2 Z^2 &= (E\, \varepsilon^4) \frac{\sigma_2^2 \rho^2}{\sigma_1^2} + E\, \varepsilon^2\, E\, \xi^2 = 3\sigma_1^2 \sigma_2^2 \rho^2 + \sigma_1^2 \sigma_2^2 (1 - \rho^2) \le 3\sigma_1^2 \sigma_2^2 \\ &= 3\, E\, \varepsilon^2 \times E\, (d + e)^2. \end{aligned}$$

But $(d + e)^2 = (2d - \varepsilon)^2 \le 8\, d^2 + 2\varepsilon^2$, therefore

$$\begin{aligned} E\, |I_N(a) - J_N(a)| &\le \frac{\sqrt{6}}{|D_N(a)|} \sum_{k \in D_N(a)} \sqrt{E\, \varepsilon_N^2(a,k)} \times \sqrt{E\, \left[4\, d^2(a, k\Delta_N) + \varepsilon_N^2(a,k)\right]} \\ &\le \frac{\sqrt{6}}{|D_N(a)|} \Big\{ \sum_{k \in D_N(a)} E\, \varepsilon_N^2(a,k) \Big\}^{1/2} \times \Big\{ \sum_{k \in D_N(a)} E\, \left[4\, d^2(a, k\Delta_N) + \varepsilon_N^2(a,k)\right] \Big\}^{1/2} \\ &\le \sqrt{6} \Big\{ \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} E\, \varepsilon_N^2(a,k) \Big\}^{1/2} \times \Big\{ 4\, \mathcal{I}_1(a) + \frac{1}{|D_N(a)|} \sum_{k \in D_N(a)} E\, \varepsilon_N^2(a,k) \Big\}^{1/2}, \end{aligned}$$

where the two last inequalities follow from Cauchy-Schwartz inequality and $E\, d^2(a, k\Delta_N) = \mathcal{I}_1(a)$ for every integer k. Since $\widehat{\psi}$ is compactly supported, then $\sup_{a \in [a_{min},\, a_{max}]} |\mathcal{I}_1(a)| < \infty$. By combining this remark with Lemma A.2 i), this provides $E\, |I_N(a) - J_N(a)| \le C \times \varphi(N)^{1/2}$ and finishes the proof of the lemma. ■

Now, the following proof of Theorem 3.1 can be established :

Proof. [Theorem 3.1] From Lemma A.3 combined with Lemma A.2 iii), we deduce

$$\lim_{N \to \infty} (N\, \Delta_N)^{1/2} E\, |I_N(a) - J_N(a)| = 0. \tag{48}$$

Combined with (9), this implies the convergence of the finite-dimensional distribution in (16). Indeed, it suffices to show that

$$\sqrt{N\,\Delta_N}\,\left(\log J_N(a) - \log I_N(a)\right) \xrightarrow[N\to\infty]{\mathcal{P}} 0. \tag{49}$$

Let $\varepsilon > 0$. By using the inequality $|\log(x) - \log(y)| \le 2\,|x/y - 1|$, valid for all $|x/y-1| \le 1/2$, $x, y > 0$ one can show that

$$\begin{aligned}
&I\!\!P\left(\sqrt{N\,\Delta_N}\;|\log J_N(a) - \log I_N(a)| \ge \varepsilon\right)\\
&\qquad \le I\!\!P\left(2\;|J_N(a)/I_N(a) - 1| \ge \frac{\varepsilon}{\sqrt{N\,\Delta_N}}\right) + I\!\!P\left(|J_N(a)/I_N(a) - 1| > \frac{1}{2}\right)\\
&\qquad \le 2\,I\!\!P\left(|J_N(a) - I_N(a)| \ge \frac{\varepsilon\, I_N(a)}{2\sqrt{N\,\Delta_N}}\right) \qquad (50)\\
&\qquad \le 2\,I\!\!P\left(|J_N(a) - I_N(a)| \le \frac{\varepsilon\,\mathcal{I}_1(a)}{4\sqrt{N\,\Delta_N}} \text{ and } |J_N(a) - I_N(a)| \ge \frac{\varepsilon\, I_N(a)}{2\sqrt{N\,\Delta_N}}\right)\\
&\qquad\qquad + 2\,I\!\!P\left(|J_N(a) - I_N(a)| \ge \frac{\varepsilon\,\mathcal{I}_1(a)}{4\sqrt{N\,\Delta_N}}\right)\\
&\qquad \le 2\,I\!\!P\left(|J_N(a) - I_N(a)| \ge \frac{\varepsilon\,\mathcal{I}_1(a)}{4\sqrt{N\,\Delta_N}}\right) + 2\,I\!\!P\left(I_N(a) \le \frac{\mathcal{I}_1(a)}{2}\right). \qquad (51)
\end{aligned}$$

The second inequality in (50) is valid for all N such that $\varepsilon/\sqrt{N\Delta_N} \le 1/2$, that is, for all sufficiently large N. The second term in the right-hand side of (51) vanishes, as $N \to \infty$, because $I_N(a) \xrightarrow[N\to\infty]{\mathcal{P}} \mathcal{I}_1(a)$. By using the Markov inequality, one can bound above the first term in the right-hand side of (51) by

$$\frac{8\sqrt{N\,\Delta_N}}{\varepsilon\,\mathcal{I}_1(a)}\,E\;|J_N(a) - I_N(a)|\,.$$

Thus, from (48), one obtains Relation (49), which completes the proof of the convergence of the finite distributions. To finish with the proof of Theorem 3.1, we have to show the tightness of the sequence $(L_N(a))_{a_{min}\le a\le a_{max}}$ where $L_N(a) = \sqrt{N\,\Delta_N}\left(J_N(a) - \mathcal{I}_1(a)\right)$. Observe one has the decomposition $L_N(a) = L_{1,N}(a) + L_{2,N}(a)$ with $\begin{cases} L_{1,N}(a) &= \sqrt{N\,\Delta_N}\left(I_N(a) - \mathcal{I}_1(a)\right)\\ L_{2,N}(a) &= \sqrt{N\,\Delta_N}\left(J_N(a) - I_N(a)\right)\end{cases}$. In [8], one have proved the tightness and the weak convergence of $(L_{1,N}(a))_{a_{min}\le a\le a_{max}}$ in Skorokhod topology on the space of càd-làg functions on $[a_{min}, a_{max}]$. From the other hand, (48) implies that $L_{2,N}(a) \xrightarrow[N\to\infty]{\mathcal{D}} 0$ for all $a \in [a_{min}, a_{max}]$. Note that the limit process is null, thus it is obviously continuous. Then, provided one have shown the tightness of $(L_{2,N}(a))_{a_{min}\le a\le a_{max}}$, one can deduce the tightness of $(L_N(a))_{a_{min}\le a\le a_{max}}$, see for instance Jacod and Shyriaev, Cor 3.33, p. 317. Next, one deduce the weak convergence of $L_N(a)$ to $Z(a)$ in the Skorokod topology on the space of càd-làg functions on $[a_{min}, a_{max}]$. The last step is the proof of the tightness of $(L_{2,N}(a))_{a_{min}\le a\le a_{max}}$. Following Ikeda and Watanabe, Th.4.3 p. 18, it suffices to show the existence of a positive constant M_2 such that for all $a_1,\, a_2 \in [a_{min},\, a_{max}]$

$$E\;\left(L_{2,N}(a_2) - L_{2,N}(a_1)\right) \le M_2\;|a_2 - a_1|^2\,. \tag{52}$$

Nowever, from (8) and (14), we get

$$J_N(a) - I_N(a) \;=\; |D_N(a)|^{-1} \sum_{k\in D_N(a)} \left(e^2(a,k) - d^2(a,k)\right).$$

Therefore, for $a_1 < a_2$, we have

$$\begin{aligned} L_{2,N}(a_2) - L_{2,N}(a_1) &= \Big(N\,\Delta_N\Big)^{1/2} \times \Bigg\{ \sum_{k=[rN/a_2]}^{[rN/a_1]} |D_N(a_2)|^{-1} \big(e^2(a_2,k) - d^2(a_2,k)\big) \\ &+ \sum_{k=[rN/a_1]}^{[(1-r)N/a_2]} \Big[|D_N(a_2)|^{-1} \big(e^2(a_2,k) - d^2(a_2,k)\big) - |D_N(a_1)|^{-1} \big(e^2(a_1,k) - d^2(a_1,k)\big)\Big] \\ &+ \sum_{k=[(1-r)N/a_1]}^{[(1-r)N/a_2]} |D_N(a_1)|^{-1} \big(e^2(a_1,k) - d^2(a_1,k)\big) \Bigg\} \end{aligned}$$

Then, one remarks that for any finite family I of random variables $(X_i)_{i\in I}$ with finite variance we have $E\left(\sum_{i\in I} X_i\right)^2 = \sum_{(i,j)\in I^2} E\,(X_i\,X_j) \leq \sum_{(i,j)\in I^2} \sqrt{E\,X_i^2} \times \sqrt{E\,X_j^2} = \left(\sum_{i\in I}\sqrt{E\,X_i^2}\right)^2$ which combined with $(x+y+z)^2 \leq 3\,(x^2+y^2+z^2)$ implies

$$E\,\left|L_{2,N}(a_2) - L_{2,N}(a_1)\right|^2 \leq C\,(N\,\Delta_N) \times \big(S_1^2 + S_2^2 + S_3^2\big)$$

where

$$\begin{aligned} S_1 &= |D_N(a_2)|^{-1} \sum_{k=[rN/a_2]}^{[rN/a_1]} \sqrt{E\,\left(e^2(a_2,k) - d^2(a_2,k)\right)^2}, \\ S_2 &= |D_N(a_1)|^{-1} \sum_{k=[(1-r)N/a_1]}^{[(1-r)N/a_2]} \sqrt{E\,\left(e^2(a_1,k) - d^2(a_1,k)\right)^2}, \\ S_3 &= \sum_{k=[rN/a_1]}^{[(1-r)N/a_2]} \sqrt{E\,\Big[|D_N(a_2)|^{-1} \left(e^2(a_2,k) - d^2(a_2,k)\right) - |D_N(a_1)|^{-1} \left(e^2(a_1,k) - d^2(a_1,k)\right)\Big]^2}. \end{aligned}$$

From (15), we get $e^2(a,k) - d^2(a,k) = \varepsilon^2(a,k) + 2\,\varepsilon(a,k)\,d(a,k)$. Moreover, the random variables $X = \varepsilon(a,k)$ or $X = d(a,k)$ are centred Gaussian random variables, thus we have $\sqrt{E\,(X^4)} = \sqrt{3}\,E\,\left(X^2\right)$. Then, by combining this remark with Cauchy-Schwarz inequality and Lemma A.2, we deduce

$$\begin{aligned} E\,\left(e^2(a,k) - d^2(a,k)\right)^2 &\leq C\,\left\{E\,\varepsilon^4(a,k) + E\,\varepsilon^2(a,k)\,d^2(a,k)\right\} \\ &\leq C\,\left\{E\,\varepsilon^4(a,k) + \sqrt{E\,\varepsilon^4(a,k)} \times \sqrt{E\,d^4(a,k)}\right\} \\ &\leq C\,\left\{\Big(E\,\varepsilon^2(a,k)\Big)^2 + E\,\varepsilon^2(a,k) \times E\,d^2(a,k)\right\} \\ &\leq C \times \varphi(N) \times \Big\{\mathcal{I}_1(a) + \varphi(N)\Big\}. \end{aligned}$$

Afterwards

$$\begin{aligned} (N\,\Delta_N)\;S_1^2 &\leq C\,(N\,\Delta_N) \times \varphi(N) \times \Big\{\mathcal{I}_1(a) + \varphi(N)\Big\} \times |D_N(a_2)|^{-2} \times \Big\{[rN/a_1] - [rN/a_2]\Big\}^2 \\ &\leq C\,(N\,\Delta_N) \times \varphi(N) \times |a_2 - a_1|^2. \end{aligned} \qquad (53)$$

since $|D_N(a_2)| \sim (1-2r)N/a$ as N goes to ∞ and $\mathcal{I}_1(a)$ is bounded. The same calculations provide

$$(N\,\Delta_N)\;S_2^2 \leq C\,(N\,\Delta_N) \times \varphi(N) \times |a_2 - a_1|^2. \qquad (54)$$

Next, we derive the upper bound for S_3^2. Let us stress that the functions $a \mapsto N\,|D_N(a)|^{-1}$ converges uniformly to $a(1-2r)^{-1}$ when N goes to ∞. Thus one can replace S_3^2 by $\widetilde{S}_3^2$ where

$$\widetilde{S}_3 = \frac{1}{(1-2r)N} \sum_{k=[rN/a_1]}^{[(1-r)N/a_2]} \sqrt{E\,\left[a_2\,\left(e^2(a_2,k) - d^2(a_2,k)\right) - a_1\,\left(e^2(a_1,k) - d^2(a_1,k)\right)\right]^2}.$$

Then, by using (15), we get the following expansion of the term f_k define below

$$\begin{aligned} f_k &:= a_2\,\big(e^2(a_2,k)-d^2(a_2,k)\big)-a_1\,\big(e^2(a_1,k)-d^2(a_1,k)\big)\\ &= \Big(\sqrt{a_2}\,\varepsilon(a_2,k)-\sqrt{a_1}\,\varepsilon(a_1,k)\Big)\times\Big(\sqrt{a_2}\,\varepsilon(a_2,k)+\sqrt{a_1}\,\varepsilon(a_1,k)\Big)\\ &\quad +2\sqrt{a_2}\,d(a_2,k)\times\Big(\sqrt{a_2}\,\varepsilon(a_2,k)-\sqrt{a_1}\,\varepsilon(a_1,k)\Big)+\sqrt{a_1}\,\varepsilon(a_1,k)\times\Big(\sqrt{a_2}\,d(a_2,k)-\sqrt{a_1}\,d(a_1,k)\Big) \end{aligned}$$

We lay the emphasize on the fact that all the random variables in the above formula are Gaussian centred variables. But for two Gaussian centred random variables, say X and Y, we get $E\,\left(X^2Y^2\right)\leq\sqrt{E\,X^4}\times\sqrt{E\,Y^4}=3\left(E\,X^2\right)\times\left(E\,Y^2\right)$. By combining this remark with Lemma A.2, one obtains

$$\begin{aligned} E\,f_k^2 &\leq C\,\Big\{E\Big(\sqrt{a_2}\,\varepsilon(a_2,k)-\sqrt{a_1}\,\varepsilon(a_1,k)\Big)^2\times E\Big(\sqrt{a_2}\,\varepsilon(a_2,k)+\sqrt{a_1}\,\varepsilon(a_1,k)\Big)^2\\ &\quad + a_2\,E\,d^2(a_2,k)\times E\Big(\sqrt{a_2}\,\varepsilon(a_2,k)-\sqrt{a_1}\,\varepsilon(a_1,k)\Big)^2+a_1\,E\,\varepsilon^2(a_1,k)\times E\Big(\sqrt{a_2}\,d(a_2,k)-\sqrt{a_1}\,d(a_1,k)\Big)^2\Big\}\\ &\leq C\,\varphi(N)\,\left|a_2-a_1\right|^2\,\left\{a_1\,\mathcal{I}_1(a_1)+(a_1+a_2)\,\varphi(N)\right\}+C\,\varphi(N)\,E\Big(\sqrt{a_2}\,d(a_2,k)-\sqrt{a_1}\,d(a_1,k)\Big)^2. \end{aligned}$$

But Taylor Formula implies the existence of a real numbers $\theta_t\in(a_1,\,a_2)$ such that

$$\sqrt{a_2}\,d(a_2,k)-\sqrt{a_1}\,d(a_1,k) \;=\; (a_2-a_1)\int_{I\!R}\Big(\frac{-t}{\theta_t^2}\Big)\,\psi'\Big(\frac{t}{\theta_t}-k\Delta_N\Big)\,X(t)\,dt$$

and after $E\Big(\sqrt{a_2}\,d(a_2,k)-\sqrt{a_1}\,d(a_1,k)\Big)^2\leq C\left|a_2-a_1\right|^2$. Indeed, one observe that since $\theta_t\in(a_{min},a_{max})$, one haves $1/\theta_t^2\leq 1/a_{min}^2$. This implies

$$\begin{aligned} E\Big(\sqrt{a_2}d(a_2,k)-\sqrt{a_1}d(a_1,k)\Big)^2 &= |a_2-a_1|^2\int_{I\!R}\int_{I\!R}\frac{uv}{\theta_u^2\theta_v^2}\cdot\psi'\Big(\frac{u}{\theta}-k\Delta_N\Big)\,\psi'\Big(\frac{v}{\theta}-k\Delta_N\Big)\,S(0,u,0,v)\,du\,dv\\ &\leq\frac{|a_2-a_1|^2}{a_{min}^4}\left(\int_{I\!R}\sup_{\theta\in(a_{min},a_{max})}\Big|\psi'\Big(\frac{u}{\theta}-k\Delta_N\Big)\Big|\,\left(|u|^{1+H_K}\mathbf{1}_{u\leq 1}+|u|^{1+H_0}\right)\,du\right)^2. \end{aligned}$$

On the other hand, the fast decreasing of the function ψ' insures

$$\int_{I\!R}\sup_{\theta\in(a_{min},\,a_{max})}\Big|\psi'\Big(\frac{u}{\theta}-k\Delta_N\Big)\Big|\,\left(|u|^{1+H_K}\mathbf{1}_{u\leq 1}+|u|^{1+H_0}\right)\,du<\infty.$$

Therefore, since a_1, a_2, $\mathcal{I}_1(a)$ are bounded and $\varphi(N)\to 0$ as N goes to ∞, we have $E\,f_k^2\,\leq\,C\,\varphi(N)\,\left|a_2-a_1\right|^2$. This leads to

$$\begin{aligned} (N\,\Delta_N)\cdot\widetilde{S}_3^2 &\leq C\,N^{-1}\cdot(N\,\Delta_N)\times\varphi(N)\,\left|a_2-a_1\right|^2\Big([(1-r)N/a_2]-[rN/a_1]\Big)\\ &\leq C\cdot(N\,\Delta_N)\times\varphi(N)\,\left|a_2-a_1\right|^2. \end{aligned}$$

Eventually, combined with (53, 54), one obtains

$$E\,\left|L_{2,N}(a_2)-L_{2,N}(a_1)\right|^2\;\leq\;C\cdot(N\,\Delta_N)\times\varphi(N)\,\left|a_2-a_1\right|^2.$$

But, Lemma A.2 iii) implies that $(N\,\Delta_N)\times\varphi(N)$ converges to 0 when N converges to ∞, therefore we deduce (52). This finishes the proof of the tightness of the sequence $(L_N(a))_{a_{min}\leq a\leq a_{max}}$. Now, the functional Delta method (see for instance Van der Vaart, chapter 20, p. 297), provide a central limit theorem for $\log(I_N(.))-\log(\mathcal{I}_1(.))$, because the function log(.) is a Hadamard-differentiable function on the space of càd-làg function on $[a_{min},a_{max}]$; this completes the proof of Theorem 3.1. ■

A.2 Proofs of section 4

Proof. [**Proposition 4.1**] **We lay the emphasize on the fact that, in this proof, we generalize the choice of the frequencies by considering** $a_N = (N\Delta_N)^q$, **with** $q > 0$.

For a given N, denote $T^* = (t_0^* = 0, t_1^*, \cdots, t_K^*, t_{K+1}^* = a_N)$ such as :

$$f_{t_j^*} < \frac{\omega_j^*}{\alpha} \leq f_{t_j^*+1}, \quad \text{for all} \quad j = 1, \dots, K$$

and for $T = (0, t_1, \cdots, t_K, a_N) \in \mathcal{A}_K^{(N)}$, we denote $Z_i^{(N)} = \sqrt{N\Delta_N}(Y_i - \log \mathcal{I}_1(1/f_i))$,
$Y_{]t_j,t_{j+1}]} = (Y_{t_j+1}, \cdots, Y_{t_{j+1}-\tau_N})'$, $X_{]t_j,t_{j+1}]} = (\log f_{t_j+i}, 1)_{1\leq i\leq(t_{j+1}-t_j)}$, $Z^{(N)}_{]t_j,t_{j+1}]} = (Z^{(N)}_{t_j+1}, \cdots, Z^{(N)}_{t_{j+1}-\tau_N})'$.

First step : We would like to prove : $\widehat{\omega}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_j^*$ for all $j = 1, \dots, K$.

Denote $Q_*^{(N)} = Q^{(N)}(t^*, \widehat{\Lambda}(t^*))$ where $\widehat{\Lambda}(t^*)$ is obtained from a linear regression of (Y_i) on $(\log f_i)$ for $i = t_j^* + 1, \cdots, t_{j+1}^* - \tau_N$. Let $\varepsilon > 0$ and $\| T - T' \|_\infty = \max_{j\in\{1,\cdots,K\}} |t_j - t_j'|$ for $T = (0, t_1, \cdots, t_K, a_N) \in \mathcal{A}_K^{(N)}$ and $T' = (0, t_1', \cdots, t_K', a_N) \in \mathcal{A}_K^{(N)}$. Then, we get,

$$I\!\!P\left(\| \widehat{T} - t^* \|_\infty \geq \varepsilon a_N\right) \leq I\!\!P\left(\min_{T\in V_{\varepsilon a_N}} Q^{(N)}(T, \widehat{\Lambda}(T)) \leq Q_*^{(N)}\right),$$

where $V_{\varepsilon a_N} = \left\{T \in \mathcal{A}_K^{(N)}, \| T - t^* \|_\infty \geq \varepsilon a_N\right\}$. We want to show that for all $T \in V_{\varepsilon a_N}$, $Q_*^{(N)} = o(Q^{(N)}(T, \widehat{\Lambda}(T)))$. In fact,

$$\begin{aligned} Q_*^{(N)} &= \frac{1}{N\Delta_N} \sum_{j=0}^{K+1} (Z^{(N)}_{]t_j^*,t_{j+1}^*]})' \left[Id - X_{]t_j^*,t_{j+1}^*]} \left(X'_{]t_j^*,t_{j+1}^*]} X_{]t_j^*,t_{j+1}^*]}\right)^{-1} X'_{]t_j^*,t_{j+1}^*]}\right] Z^{(N)}_{]t_j^*,t_{j+1}^*]} \\ &\leq \frac{1}{N\Delta_N} \sum_{j=0}^{K+1} (Z^{(N)}_{]t_j^*,t_{j+1}^*]})' Z^{(N)}_{]t_j^*,t_{j+1}^*]} \\ &\leq \frac{1}{N\Delta_N} (Z^{(N)}_{[1,a_N]})' Z^{(N)}_{[1,a_N]}. \end{aligned}$$

From Proposition 3.1, we deduce

$$\frac{1}{a_N} (Z^{(N)}_{[1,a_N]})' Z^{(N)}_{[1,a_N]} \xrightarrow[N\to\infty]{\mathcal{D}} I_Z = \int_0^1 Z^2 \left(\frac{\beta}{f_{min}} \left(\frac{\alpha f_{min}}{\beta f_{max}}\right)^u\right) du, \tag{55}$$

which is a positive and $I\!\!L^\infty$ random variable because Z is a continuous Gaussian process. Afterward, for a sequence $(\psi_k)_k \in I\!\!R^{I\!\!N}$ and a sequence of random variables $(\xi_k)_{k\in I\!\!N}$, we will write $\xi_N = O_P(\psi_N)$ as $N \to \infty$, if for all $\varepsilon > 0$, there exists $c > 0$, such as ,

$$P\Big(|\xi_N| \leq c \cdot \psi_N\Big) \geq 1 - \varepsilon,$$

for all sufficiently large N. Here, we obtain :

$$Q_*^{(N)} = O_P\left(\frac{a_N}{N\Delta_N}\right). \tag{56}$$

Now, let $T \in V_{\varepsilon a_N}$, we want a lower bound of $Q^{(N)}(T, \widehat{\Lambda}(T))$. We use the following decomposition

$$\begin{aligned} Q^{(N)}(T, \widehat{\Lambda}(T)) &= \sum_{j=0}^{K+1} \sum_{i=t_j+1}^{t_{j+1}-\tau_N} [Y_i - \log \mathcal{I}_1(1/f_i)]^2 + \left[X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)\right]^2 + \\ &\qquad 2\,[Y_i - \log \mathcal{I}_1(1/f_i)] \times \left[X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)\right] \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

Then :

1. Since $Q_1 = \dfrac{1}{N\Delta_N} \sum_{j=0}^{K+1} (Z^{(N)}_{]t_j,t_{j+1}]})' Z^{(N)}_{]t_j,t_{j+1}]}$, as previously we get
$$Q_1 = O_P\left(\frac{a_N}{N\Delta_N}\right). \tag{57}$$

2. Let $\underline{\tau} = \left(\log\left(\dfrac{\beta f_{max}}{\alpha f_{min}}\right)\right)^{-1} \min_{j=1,\cdots,K} \left\{\log\left(\dfrac{\alpha\omega^*_{j+1}}{\beta\omega^*_j}\right)\right\}$. Then, for all $j \in \{0,1,\cdots,K\}$, $t^*_{j+1} - \tau_N \geq t^*_j + \underline{\tau} a_N$. Since $T \in V_{\varepsilon a_N}$, we have $\eta = \min\{\varepsilon, \underline{\tau}, \log(\beta/\alpha)\} > 0$ and there exists an integer $j \in \{0,\cdots,K+1\}$ for which there are no estimated abrupt change in the interval $[t^*_j - \eta a_N, t^*_j]$ or $[t^*_j - \tau_N, t^*_j - \tau_N + \eta a_N]$. Thus there exists $k \in \{0,\cdots,K+1\}$ satisfying $[t^*_j - \eta a_N, t^*_j] \subset [t_k, t_{k+1} - \tau_N]$ (we follow here a similar proof than Bai and Perron in Lemma 2, p 69) and
$$\begin{aligned} Q_2 &\geq \sum_{i=t^*_j-\eta a_N+1}^{t^*_j} |X_i \widehat{\lambda}_k - \log \mathcal{I}_1(1/f_i)|^2 \\ &\geq \sum_{i=t^*_j-\eta a_N+1}^{t^*_j} \left| A(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) + \frac{i}{a_N} \cdot B(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) - g\left(\frac{i}{a_N}\right)\right|^2, \end{aligned} \tag{58}$$
with :
   - $A(H,\sigma) = \log\Big(\sigma^2 \cdot K_H(\psi)\Big) - (2H+1)\cdot \log\left(\dfrac{f_{min}}{\beta}\right)$ for all $(H,\sigma) \in \mathcal{K}$;
   - $B(H,\sigma) = -(2H+1)\cdot\log\left(\dfrac{\beta f_{max}}{\alpha f_{min}}\right)$ for all $(H,\sigma)\in\mathcal{K}$;
   - $g\left(\dfrac{i}{a_N}\right) = \log\Big(\mathcal{I}_1(1/f_i)\Big) = \log\left(\mathcal{I}_1\left(\dfrac{\beta}{f_{min}}\left(\dfrac{\beta f_{max}}{\alpha f_{min}}\right)^{-i/a_N}\right)\right)$.

   Since for all $(H,\sigma)\in\mathcal{K}$, the function $x \mapsto L_{(H,\sigma)}(x) = \Big(A(H,\sigma) + x\cdot B(H,\sigma) - g(x)\Big)^2$ is an infinitely differentiable function on $I\!R$, we know from the theory of Riemann sums that :
$$u_N(H,\sigma) = \frac{1}{a_N}\sum_{i=t^*_j-\eta a_N+1}^{t^*_j} \left|A(H,\sigma) + \frac{i}{a_N}\cdot B(H,\sigma) - g\left(\frac{i}{a_N}\right)\right|^2$$
$$\xrightarrow[N\to\infty]{} u(H,\sigma) = \int_{s^*_j-\eta}^{s^*_j} \left(A(H,\sigma) + x\cdot B(H,\sigma) - g\left(x\right)\right)^2 dx,$$
   with $s^*_j = \log\left(\dfrac{\omega^*_j}{f_{min}}\right)\left(\log\left(\dfrac{\alpha f_{max}}{\beta f_{min}}\right)\right)^{-1} = \lim_{N\to\infty} \dfrac{t^*_j}{a_N}$. Moreover, the sequence $(u_N(H,\sigma))_N$ converges uniformly to $u(H,\sigma)$ because for N large enough
$$\begin{aligned} \sup_{(H,\sigma)\in\mathcal{K}} |u_N(H,\sigma) - u(H,\sigma)| &\leq \left(\frac{1}{a_N^2} + \eta\left|s^*_j - \frac{t^*_j}{a_N}\right|\right)\cdot \sup_{(H,\sigma)\in\mathcal{K}}\left\{\sup_{0\leq x\leq (s^*_K+1)}\left|\frac{\partial L_{(H,\sigma)}}{\partial x}(x)\right|\right\} \\ &\xrightarrow[N\to\infty]{} 0, \end{aligned}$$
   since $\mathcal{K}$ is a compact set of $[0,1]\times]0,\infty[$ and thus $\sup_{(H,\sigma)\in\mathcal{K}}\left\{\sup_{0\leq x\leq (s^*_K+1)}\left|\dfrac{\partial L_{(H,\sigma)}}{\partial x}(x)\right|\right\} < \infty$. As a consequence, from (58) and since we assumed that $(\widehat{H}^{(N)}_i, \widehat{\sigma}^{(N)}_i)\in\mathcal{K}$ for all $i=0,\cdots,K$, for some sufficiently small, fixed $\xi>0$ and for all sufficiently large N,
$$Q_2 \geq a_N\left(\int_{s^*_j-\eta}^{s^*_j}\Big(A(\widehat{H}^{(N)}_k,\widehat{\sigma}^{(N)}_k) + x\cdot B(\widehat{H}^{(N)}_k,\widehat{\sigma}^{(N)}_k) - g\left(x\right)\Big)^2 dx - \xi\right). \tag{59}$$

But it is impossible that there exists $(a,b) \in I\!\!R^2$ such as $g(x)=a+b\cdot x$ for all $x\in[s_j^*-\eta,s_j^*]$, *i.e.*, $\mathcal{I}_1\Big(c_1\cdot e^{c_2\cdot x}\Big)=e^a\cdot e^{b\cdot x}$ for all $x\in[s_j^*-\eta,s_j^*]$ with $c_1=\dfrac{\beta}{f_{min}}$, $c_2=\log\left(\dfrac{\alpha f_{min}}{\beta f_{max}}\right)$, which can also be written as :

$$\mathcal{I}_1(x)=a_1\cdot x^{b_1} \quad \text{for all} \quad x\in[\alpha/\omega_j^*,\alpha/\omega_j^*+\eta'], \tag{60}$$

with $\eta'>0$ and $(a_1,b_1)\in I\!\!R^2$. Indeed, assume now (60) is true. But, for all $x\in[\alpha/\omega_j^*,\alpha/\omega_j^*+\eta']$,

$$\mathcal{I}_1(x)=2\left(\sigma_{j-1}^{*2}\cdot x^{2H_{j-1}^*+1}\int_\alpha^{x\cdot\omega_j^*}\frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du+\sigma_j^{*2}\cdot x^{2H_j^*+1}\int_{x\cdot\omega_j^*}^{\beta}\frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du\right).$$

Then $\dfrac{\partial^n\mathcal{I}_1}{\partial x^n}(\alpha/\omega_j^*)=a_1\cdot\dfrac{\partial^n x^{b_1}}{\partial x^n}(\alpha/\omega_j^*)$ for $n=0,1$, which implies that $b_1=(2H_j^*+1)$ and $a_1=2\sigma_j^{*2}K_{H_j^*}(\psi)$ (here, we use the equality $\widehat{\psi}(\alpha)=0$). Thus, for all $x\in[\alpha/\omega_j^*,\alpha/\omega_j^*+\eta']$,

$$\begin{aligned}\sigma_{j-1}^{*2}\cdot x^{2H_{j-1}^*+1}\int_\alpha^{x\cdot\omega_j^*}\frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du \quad &= \quad \sigma_j^{*2}\cdot x^{2H_j^*+1}\int_\alpha^{x\cdot\omega_j^*}\frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du,\\ \Longrightarrow \int_{\alpha/x}^{\omega_j^*}|\widehat{\psi}(x\cdot y)|^2&\left(\frac{\sigma_{j-1}^{*2}}{y^{2H_{j-1}^*+1}}-\frac{\sigma_j^{*2}}{y^{2H_j^*+1}}\right)dy=0,\end{aligned}$$

and hence $\left\{\begin{array}{l}\sigma_{j-1}^{*2}=\sigma_j^{*2}\\ H_{j-1}^*=H_j^*\end{array}\right.$. But this condition is impossible from Assumption (B_K) and consequently there is no $(a,b)\in I\!\!R^2$ such as $g(x)=a+b\cdot x$ for all $x\in[s_j^*-\eta,s_j^*]$.

The function g belongs to the Hilbert space $I\!\!L^2([s_j^*-\eta,s_j^*];dx)$. Since $\mathcal{L}=\{A+B\cdot x,\ x\in[s_j^*-\eta,s_j^*],\ (A,B)\in I\!\!R^2\}$ is a closed linear subspace of $I\!\!L^2([s_j^*-\eta,s_j^*];dx)$, there exits a distance between g and $\mathcal{L}$ in $I\!\!L^2([s_j^*-\eta,s_j^*];dx)$, *i.e.* there exists $(\tilde{A},\tilde{B})\in I\!\!R^2$ such as

$$\int_{s_j^*-\eta}^{s_j^*}\Big(\tilde{A}+\tilde{B}\cdot x-g(x)\Big)^2\,dx=\inf_{(A,B)\in I\!\!R^2}\int_{s_j^*-\eta}^{s_j^*}\big(A+B\cdot x-g(x)\big)^2\,dx=C>0,$$

because $g\notin\mathcal{L}$. Then, by choosing ξ such as $0<\xi<C/2$, the inequality (59) implies :

$$Q_2\geq\frac{C}{2}\cdot a_N \tag{61}$$

for all sufficiently large N, with C a real positive number only depending on η, s_j^*, H_{j-1}^*, H_j^*, σ_{j-1}^*, σ_j^* and ψ.

3. The previous evaluations of Q_1 and Q_2 provide an upper bound of Q_3.We get

$$\begin{aligned}Q_3 \quad &\leq \quad 2\,(Q_1)^{1/2}\left(\sum_{k=0}^{K+1}\sum_{i=t_k+1}^{t_{k+1}-\tau_N}(X_i\widehat{\lambda}_k-\log\mathcal{I}_1(1/f_i))^2\right)^{1/2}\\ &\leq \quad 2\,(Q_1)^{1/2}\times\left(a_N\cdot\sup_{f_{min}\leq f\leq f_{max}}\Big\{2\sup_{\lambda\in\mathcal{K}}\{(\log f,1)\cdot\lambda)^2+2\log^2\mathcal{I}_1(1/f)\Big\}\right)^{1/2},\\ &= \quad O_P\left(\frac{a_N}{\sqrt{N\Delta_N}}\right).\end{aligned} \tag{62}$$

We deduce from (57), (61) and (62) that $Q_1=o(Q_2)$ and $Q_3=o(Q_2)$, which implies $I\!\!P\left(\min_{T\in V_{\varepsilon a_N}}Q^{(N)}(T,\widehat{\Lambda}(T))\geq\dfrac{C}{4}\cdot a_N\right)\underset{N\to\infty}{\longrightarrow}1$ and thus

$$\lim_{N\to\infty}I\!\!P\left(\|\,\widehat{T}-T^*\,\|_\infty\geq\varepsilon a_N\right)=0 \quad\Longrightarrow\quad \widehat{\omega}_i^{(N)}\ \underset{N\to\infty}{\overset{\mathcal{P}}{\longrightarrow}}\ \omega_i^*.$$

Second step : For $j = 1, \cdots, K$, we want to prove that if $3/4 \le p \le 1$ and $0 \le q \le 1$, for all $\varepsilon > 0$, there exists $0 < C < \infty$ such as for sufficiently large N, $\mathbb{P}\left(a_N^{1-p}\left|\widehat{\omega}_j^{(N)} - \omega_j^*\right| \ge C\right) \le \varepsilon$.

Mutatis mutandis, we follow the same method as in the proof of the convergence in probability. Now, let $0 < p < 1$, $0 < \eta = \dfrac{1}{2}\min\{\underline{\tau}, \log(\beta/\alpha)\}$ and consider $\min_{T \in W^\eta_{Ca_N^p}} Q^{(N)}(T, \widehat{\Lambda}(T))$ with

$$W^\eta_{Ca_N^p} = \left\{T \in \mathcal{A}_K^{(N)}, Ca_N^p \le \| T - t^* \|_\infty \le \eta a_N\right\}.$$

Then, as previously, for $T \in W^\eta_{Ca_N^p}$ and N large enough, it exists $j \in \{1, \cdots, K\}$ such as

$$t_j + Ca_N^p \le t_j^* < t_{j+1} - \tau_N \tag{63}$$

(the following proof is valid even if one considers the alternative $t_j^* \le t_j - Ca_N^p$). Then

$$\begin{aligned} Q^{(N)}(T, \widehat{\Lambda}(T)) \quad &\ge \sum_{i=t_j^*+1}^{t_{j+1}-\tau_N} (Y_i - \log \mathcal{I}_1(1/f_i))^2 + (X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i))^2 + \\ &\qquad +2(Y_i - \log \mathcal{I}_1(1/f_i))(X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)) \\ &\ge Q_1' + Q_2' + Q_3'. \end{aligned}$$

1. First, we have again,

$$Q_1' = O_P\left(\frac{a_N}{N\Delta_N}\right). \tag{64}$$

2. Secondly, $Q_2' = \displaystyle\sum_{i=t_j^*+1}^{t_{j+1}-\tau_N} (X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i))^2$. But we know $\log \mathcal{I}_1(1/f_i) = X_i\lambda_j^*$ for $i \in \{t_j^*+1, \cdots, t_{j+1} - \tau_N\}$. Moreover, for $a_i = 1/f_i$, $i \in \{t_j + 1, \cdots, t_j^*\}$ and N large enough, $a_i \simeq \alpha/\omega_i^*$, and

$$\mathcal{I}_1(a_i) = \mathcal{I}_1\left(\frac{\alpha}{\omega_j^*}\right) + \left(a_i - \frac{\alpha}{\omega_j^*}\right)\mathcal{I}_1'\left(\frac{\alpha}{\omega_j^*}\right) + O\left(a_i - \frac{\alpha}{\omega_j^*}\right)^2.$$

But $\mathcal{I}_1(a_i) = 2\left(\sigma_{j-1}^{*2} a_i^{2H_{j-1}^*+1}\displaystyle\int_\alpha^{a_i\omega_j^*}\frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du + \sigma_j^{*2}a_i^{2H_j^*+1}\int_{a_i\omega_j^*}^\beta \frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du\right)$ and $\mathcal{I}_1'\left(\dfrac{\alpha}{\omega_j^*}\right) = 2\sigma_j^{*2}K_{H_j^*}(\psi)(2H_j^*+1)\left(\dfrac{\alpha}{\omega_j^*}\right)^{2H_j^*}$; thus for $i \in \{t_j+1, \cdots, t_j^*\}$,

$$\log \mathcal{I}_1(1/f_i) = X_i\lambda_j^* + \left[(2H_j^*+1)\frac{f_{min}}{\beta}\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)\right]\cdot\left(\frac{t_j^*-i}{a_N}\right) + O\left(\frac{t_j^*-i}{a_N}\right)^2. \tag{65}$$

Then, with $\widehat{\lambda}_j = (\widehat{a}_j, \widehat{b}_j)'$, one gets for $i \in \{t_j^*+1, \cdots, t_{j+1} - \tau_N\}$,

$$\left(X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)\right) = (\log f_i - \overline{logf})(\widehat{a}_j - a_j^*) + \overline{Z}, \tag{66}$$

$\overline{XXX}$ indicates the empirical mean of XXX between $t_j + 1$ and $t_{j+1} - \tau_N$. Thus,

$$Q_2' \quad \ge \sum_{i=t_j^*+1}^{t_{j+1}-\tau_N}\left(\left(\log f_i - \overline{\log f}\right)(\widehat{a}_j - a_j^*) + \frac{1}{\sqrt{N\Delta_N}}\overline{Z}\right)^2. \tag{67}$$

We also have :

$$\begin{aligned}\widehat{a}_j &= \frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Y_i-\overline{Y}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}\\ &= \frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)+\frac{1}{\sqrt{N\Delta_N}}Z_i^{(N)}-\overline{\log\mathcal{I}_1}-\frac{1}{\sqrt{N\Delta_N}}\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2},\end{aligned}$$

and thus,

$$\widehat{a}_j-a_j^*=\frac{\displaystyle\sum_{i=t_j+1}^{t_j^*}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)-X_i'\lambda_j^*\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2} + \frac{1}{\sqrt{N\Delta_N}}\frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Z_i^{(N)}-\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}. \quad (68)$$

From the definition of $(\log f_i)$,

$$\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2\simeq\left[\frac{1}{12}\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)\right](t_{j+1}-\tau_N-t_j)=O(a_N). \quad (69)$$

Expansions (69) and (65) imply there exist two constants $C_1>0$ and $C_2>0$ such as for N large enough :

$$C_1\left(\frac{t_j^*-t_j}{a_N}\right)^2\leq\left|\frac{\displaystyle\sum_{i=t_j+1}^{t_j^*}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)-X_i'\lambda_j^*\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}\right|\leq C_2\left(\frac{t_j^*-t_j}{a_N}\right)^2.$$

Moreover

$$\frac{1}{\sqrt{N\Delta_N}}\frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Z_i^{(N)}-\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}=O_P\left(\frac{1}{\sqrt{N\Delta_N}}\right).$$

Thus, we deduce from (68) that :

$$C_1\left(\frac{t_j^*-t_j}{a_N}\right)^2+O_P\left(\frac{1}{\sqrt{N\Delta_N}}\right)\leq\left|\widehat{a}_j-a_j^*\right|.$$

As a consequence, for (p,q) such as $4q(1-p)\leq 1$ (for instance, $p=3/4$ and $q=1$), then $\left(\frac{t_j^*-t_j}{a_N}\right)^2\cdot\sqrt{N\Delta_N}\geq C^2$, and thus for all $\varepsilon>0$, for N sufficiently large, we can chose $C>0$ such as :

$$\mathbb{P}\left(\frac{C_1^2}{2}\left(\log f_i-\overline{\log f}\right)^2\left(\frac{t_j^*-t_j}{a_N}\right)^4\leq\left(\left(\log f_i-\overline{\log f}\right)\left(\widehat{a}_j-a_j^*\right)\right)^2\right)\geq 1-\varepsilon. \tag{70}$$

Now, from (67), (70) and with $\mathbb{P}(t_{j+1}-\tau_N-t_j^*\geq\frac{\eta}{2}a_N)\underset{N\to\infty}{\longrightarrow}1$, for $(p,q)\in[3/4,1]\times[0,1]$, for all $\varepsilon>0$, for N sufficiently large, we can also chose $C>0$ such as :

$$\begin{aligned}\mathbb{P}\left(\frac{C_1^2}{4}\left(\frac{t_j^*-t_j}{a_N}\right)^4\cdot\sum_{i=t_j^*+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2\leq Q_2'\right)&\geq 1-\varepsilon,\\ \Longrightarrow\quad\mathbb{P}\left(C^4\cdot C_2\cdot a_N^{4p-3}\leq Q_2'\right)&\geq 1-\varepsilon,\end{aligned} \tag{71}$$

with $C_2>0$ a real number not depending on C, N and ε.

3. Finally, from the classical bound of Q_3', we obtain,

$$Q_3'\leq 2\cdot\left(Q_2'\right)^{1/2}\cdot\left(Q_1'\right)^{1/2}.$$

But, following a similar method as previously, from (70 one can find a upper-bound for Q_2', *i.e.* for $(p,q)\in[3/4,1]\times[0,1]$, for all $\varepsilon>0$, for N sufficiently large, we can also chose $C>0$ such as :

$$\mathbb{P}\left(Q_2'\leq C^4\cdot C_3\cdot a_N^{4p-3}\right)\quad\geq\quad 1-\varepsilon,$$

with $C_3>0$ a real number not depending on C, N and ε. Thus, for $(p,q)\in[3/4,1]\times[0,1]$, for all $\varepsilon>0$, we can also chose $C>0$ such as :

$$\mathbb{P}\left(Q_3'\leq C^2\cdot C_4\cdot\frac{a_N^{2p-2}}{\sqrt{N\Delta_N}}\right)\quad\geq\quad 1-\varepsilon, \tag{72}$$

with $C_4>0$ a real number not depending on C and N.

Now, from (64), (71) and (72), one deduces that for $(p,q)\in[3/4,1]\times[0,1]$, for all $\varepsilon>0$, for N sufficiently large, we can chose $C>0$ sufficiently large such as :

$$\mathbb{P}\left(\min_{T\in W^{\eta}_{Ca_N^p}}Q^{(N)}(T,\widehat{\Lambda}(T))\geq C^4\cdot\frac{C_2}{2}\cdot a_N^{4p-3}\right)\geq 1-\varepsilon.$$

and thus like $Q_*^{(N)}=O_P\left(\frac{a_N}{N\Delta_N}\right)$ from (56),

$$\mathbb{P}\left(\min_{T\in W^{\eta}_{Ca_N^p}}Q^{(N)}(T,\widehat{\Lambda}(T))\leq Q_*^{(N)}\right)\leq\varepsilon,$$

that leads to $\mathbb{P}\left(a_N^{1-p}\left|\widehat{\omega}_j^{(N)}-\omega_j^*\right|\geq C\right)\leq\varepsilon$ for sufficiently large C and N. ■

Proof. [**Proposition 4.2**] From Proposition 4.1, we deduce that $\forall j=0,\cdots,K$,

$$\mathbb{P}\left([\tilde{U}_j^{(N)},\tilde{V}_j^{(N)}]\subset[t_j^*,t_{j+1}^*-\tau_N]\right)\underset{N\to\infty}{\longrightarrow}1.$$

Denote $A_j^{(N)}$ the event $[\tilde U_j^{(N)},\tilde V_j^{(N)}]\subset[t_j^*,t_{j+1}^*-\tau_N]$. Then, $\forall j=0,\cdots,K$ and $\forall(x,y)\in I\!R^2$,

$$\begin{aligned}&I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\\&\quad=I\!P\left(A_j^{(N)}\right)\times I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)+\\&\qquad+I\!P\left(\overline{A_j^{(N)}}\right)\times I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid\overline{A_j^{(N)}}\right).\end{aligned}$$

Now, since $I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid\overline{A_j^{(N)}}\right)\leq 1$ and $I\!P\left(\overline{A_j^{(N)}}\right)=1-I\!P\left(A_j^{(N)}\right)$, we obtain :

$$\begin{aligned}&I\!P\left(A_j^{(N)}\right)\cdot I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\\&\quad\leq I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\\&\qquad\leq I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)+1-I\!P\left(A_j^{(N)}\right).\qquad(73)\end{aligned}$$

Since $\widehat\omega_j^{(N)}\xrightarrow[N\to\infty]{\mathcal P}\omega_j^*$ and $\widehat\omega_{j+1}^{(N)}\xrightarrow[N\to\infty]{\mathcal P}\omega_{j+1}^*$, therefore $(\widehat\omega_j^{(N)},\widehat\omega_{j+1}^{(N)})\xrightarrow[N\to\infty]{\mathcal P}(\omega_j^*,\omega_{j+1}^*)$, we have $(f_k)_{k\in\{\tilde U_j^{(N)},\cdots,\tilde V_j^{(N)}\}}\xrightarrow[N\to\infty]{\mathcal P}(g_j^*(k))_{1\leq k\leq m}$ and $\tilde X_j^{(N)}\xrightarrow[N\to\infty]{\mathcal P}X_j^*$. Thus, from Proposition 3.1 and central limit theorem (17), for all $(x_k)_{1\leq k\leq m}\in I\!R^m$, we get

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde Y_j^{(N)}-\tilde X_j^{(N)}\lambda_j^*\right)\in\prod_{k=1}^m]-\infty,x_k]\ \mid A_j^{(N)}\right)\quad-\quad I\!P\left(\tilde Z_j\in\prod_{k=1}^m]-\infty,x_k]\ \mid A_j^{(N)}\right)\xrightarrow[N\to\infty]{}0,$$

with $\tilde Z_j\overset{\mathcal D}{\sim}\mathcal N_m(0,\Sigma_j^*)$ and $\Sigma_j^*=\left(\operatorname{cov}\left(Z\Big(\frac{1}{g_j^*(k)}\Big),Z\Big(\frac{1}{g_j^*(l)}\Big)\right)\right)_{1\leq k,l\leq m}$ (it explains the expression (24) of Σ_j^*). From the equality $\tilde\lambda_j^{(N)}=\left((\tilde X_j^{(N)})'\tilde X_j^{(N)}\right)^{-1}(\tilde X_j^{(N)})'\tilde Y_j^{(N)}$, we deduce that for all $(x,y)\in I\!R^2$, with $\tilde\xi_j\overset{\mathcal D}{\sim}\mathcal N_2(0,\Gamma_1^{\lambda_j^*})$ and $\Gamma_1^{\lambda_j^*}=\left(X_j^{*\prime}X_j^*\right)^{-1}X_j^*\Sigma_j^*X_j^{*\prime}\left(X_j^{*\prime}X_j^*\right)^{-1}$,

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)-I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\xrightarrow[N\to\infty]{}0.\qquad(74)$$

We also have :

$$\begin{aligned}&I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right)+I\!P\left(A_j^{(N)}\right)-1\leq\\&\quad\leq I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\leq\frac{I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right)}{I\!P\left(A_j^{(N)}\right)}.\qquad(75)\end{aligned}$$

Now, as $I\!P\left(A_j^{(N)}\right)\xrightarrow[N\to\infty]{}1$, from (73), (74) and (75), we deduce that for all $(x,y)\in I\!R^2$:

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\xrightarrow[N\to\infty]{}I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right),$$

that achieves the proof. ■

Proof. [**Proposition 4.3**] First, from the expression of each s_{kl} given in (24) and with $\mathcal M_m(I\!R)$ the set of real m-by-m matrix, the function $\Sigma:(H,u,v)\mapsto\Sigma(H,u,v)\in\mathcal M_m(I\!R)$ is a continuous (and therefore measurable) function of (H,u,v) for H in a compact set included in $]0,1[$ and $(u,v)\in]f_{min},f_{max}[^2$. For all $j=0,\cdots,K$, we have :

1. from Assumptions (B_K) and (C), $(\tilde{H}_j^{(N)}, \tilde{\sigma}_j^{(N)}) \in \mathcal{K}$ and $(\widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \in]f_{min}, f_{max}[^2$;
2. from (21) and (23), $\tilde{H}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} H_j^*$, $\widehat{\omega}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_j^*$, $\widehat{\omega}_{j+1}^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_{j+1}^*$ and therefore
$$(\tilde{H}_j^{(N)}, \widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \xrightarrow[N\to\infty]{\mathcal{P}} (H_j^*, \omega_j^*, \omega_{j+1}^*).$$

As a consequence, $\widehat{\Sigma}_j^{(N)} = \Sigma(\tilde{H}_j^{(N)}, \widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \xrightarrow[N\to\infty]{\mathcal{P}} \Sigma_j^*$, for all $j = 0, \cdots, K$, and since $\Sigma(H, u, v)$ is an invertible covariance matrix for all $(H, u, v) \in]0, 1[\times]f_{min}, f_{max}[^2$,
$$\left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \xrightarrow[N\to\infty]{\mathcal{P}} \left(\Sigma_j^*\right)^{-1}, \quad \text{for all} \quad j = 0, \cdots, K. \tag{76}$$

Secondly, denote $\begin{cases} \tilde{M}_j^{(N)} = \left((\tilde{X}_j^{(N)})' \left(\Sigma_j^*\right)^{-1} \tilde{X}_j^{(N)}\right)^{-1} (\tilde{X}_j^{(N)})' \left(\Sigma_j^*\right)^{-1} \\ \widehat{M}_j^{(N)} = \left((\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \tilde{X}_j^{(N)}\right)^{-1} (\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \end{cases}$.
The 2-by-m matrix $\tilde{M}_j^{(N)}$ verifies :
$$\tilde{\lambda}_j^{(N)} = \tilde{M}_j^{(N)} \tilde{Y}_j^{(N)} = \lambda_j^* + \frac{1}{\sqrt{N\Delta_N}} \tilde{M}_j^{(N)} \tilde{Z}_j^{(N)}$$

with $\tilde{Z}_j^{(N)} = \left(Z^{(N)}(1/f_i)\right)_{i\in\{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}}$ and $\tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \tilde{Z}_j = \left(Z(1/g_j^*(k))\right)_{1\le k\le m}$ from the central limit theorem (17). In the same way,
$$\underline{\lambda}_j^{(N)} = \widehat{M}_j^{(N)} \tilde{Y}_j^{(N)} = \lambda_j^* + \frac{1}{\sqrt{N\Delta_N}} \widehat{M}_j^{(N)} \tilde{Z}_j^{(N)}.$$

From (76), we obtain $\widehat{M}_j^{(N)} - \tilde{M}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} 0$, and thus,
$$\sqrt{N\Delta_N} \left(\underline{\lambda}_j^{(N)} - \lambda_j^*\right) - \tilde{M}_j^{(N)} \tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} 0,$$

with $\tilde{M}_j^{(N)} \tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_2(0, \Gamma_2^{\lambda_j^*})$ (the same covariance matrix as that obtained with a generalized least squares estimation), and this implies Proposition 4.3. ■

Proof. [**Proposition 4.4**] For each $j = 0, \cdots, K$, one first show that
$$N\Delta_N \cdot \| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)} \underline{\lambda}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{D}} \chi^2(m-2). \tag{77}$$

Indeed, $\| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)} \underline{\lambda}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} = \| \widehat{P}_{j\perp}^{(N)} \tilde{Y}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} = \frac{1}{N\Delta_N} \| \widehat{P}_{j\perp}^{(N)} \tilde{Z}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}}$ where $\widehat{P}_{j\perp}^{(N)} = I_m - \tilde{X}_j^{(N)} \widehat{M}_j^{(N)}$ is the matrix of the orthogonal projector in $\mathbb{R}^m$ on the orthogonal of V_j, where $V_j = \{\tilde{X}_j^{(N)} \lambda, \ \lambda \in \mathbb{R}^2\}$ is the 2-dimensional subspace of $\mathbb{R}^m$ generated by $\tilde{X}_j^{(N)}$ (here the notion of orthogonality is based on the inner product $< u, v >_{\widehat{\Sigma}_j^{(N)}} = u' \cdot \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \cdot v$ for $u, v \in \mathbb{R}^m$). From the previous proofs, we know :

- $\widehat{\Sigma}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \Sigma_j^*$, $\tilde{X}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} X_j^*$ and therefore $\widehat{P}_{j\perp}^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} P_{j\perp}^*$ where $P_{j\perp}^* = \left(I_m - X_j^* \left(X_j^{*'} (\Sigma_j^*)^{-1} X_j^*\right)^{-1} X_j^{*'} (\Sigma_j^*)^{-1}\right)$ is the matrix of an orthogonal projector on a $(m-2)$-dimensional subspace of $\mathbb{R}^m$;
- $< u, v >_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{P}} < u, v >_{\Sigma_j^*}$ for $u, v \in \mathbb{R}^m$;

- $\tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \tilde{Z}_j$ with $\tilde{Z}_j \overset{\mathcal{D}}{\sim} \mathcal{N}_m(0,\Sigma_j^*)$.

Consequently, $\| \widehat{P}_{j\perp}^{(N)} \tilde{Z}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{D}} \| P_{j\perp}^* \tilde{Z}_j \|^2_{\Sigma_j^*}$. From Cochran's Theorem, we know $\| P_{j\perp}^* \tilde{Z}_j \|^2_{\Sigma_j^*} \overset{\mathcal{D}}{\sim} \chi^2(m-2)$ and therefore (77) is proved.

Moreover, with the notations of Proposition 3.1, if $\log f \geq \log f' + \log \beta/\alpha$ then $\mathrm{cov}\,(Z(1/f), Z(1/f')) = 0$. But for all $(i,j) \in \{0,\cdots,K\}^2$, $i \neq j$, $\forall k \in \{\tilde{U}_i^{(N)},\cdots,\tilde{V}_i^{(N)}\}$ and $\forall k' \in \{\tilde{U}_j^{(N)},\cdots,\tilde{V}_j^{(N)}\}$, $|\log f_k - \log f_{k'}| \geq \log \beta/\alpha$. Thus, we deduce that the different $\underline{\lambda}_j^{(N)}$ are asymptotically Gaussian and independent. It provides the end of the proof of the Proposition 4.4. ■

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References

[1] Abry, P., Flandrin, P., Taqqu, M.S. and Veitch, D. (2002). Self-similarity and long-range dependence through the wavelet lens, in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser.

[2] Amemiya,T. (1985). *Advanced Econometrics.* Cambridge : Harvard University Press.

[3] Ayache, A. and Lévy Véhel, J. (1999): Generalized multifractional Brownian motion: definition and preliminary results, *in* M. Dekking, J. Lévy Véhel, E. Lutton & C. Tricot, eds, "Fractal theory an applications in engineering", Springer Verlag.

[4] Bai J. (1998). Least squares estimation of a shift in linear processes. *J. of Time Series Anal.* 5, p. 453-472.

[5] Bai J. and Perron P. (1998). Estimating and testing linear models with multiple structural changes. *Econometrica* 66, p. 47-78.

[6] Bardet J.M. (2000). Testing for the presence of self-similarity of Gaussian time series having stationary increments. *J. of Time Series Anal.* 21, p. 497-516.

[7] Bardet J.M. (2002). Statistical study of the wavelet analysis of fractional Brownian motion. *IEEE Trans. Inform. Theory.* 48, p. 991-999.

[8] Bardet J.M. and Bertrand, P. (2003). Definition, properties and wavelet analysis of multiscale fractional Brownian motion. Preprint LSP, Toulouse III.

[9] Bardet J.M., Lang G., Moulines E. and Soulier P. (2000). Wavelet estimator of long-range dependent processes. *Statistical Inference for Stochastic Processes* 3, p. 85-99.

[10] Benassi, A., Cohen, S. and Istas, J. (1998). Identifying the multifractional function of a Gaussian process. *Statistics and probability letters* 39, p. 337-345.

[11] Benassi, A. and Deguy, S. (1999). Multi-scale fractional Brownian motion : definition and identification. Preprint LAIC.

[12] Benassi, A., Jaffard, S. and Roux, D. (1997). Elliptic Gaussian random processes. *Rev. Mathemàtica Iberoamericana* 13 (1), p. 19-90.

[13] Bertrand, P., Bardet, J.M, Dabonneville, M., Mouzat, A. and Vaslin, P. (2001). Automatic determination of the different control mechanisms in upright position by a wavelet method, *IEEE Engineering in Medicine and Biology Society* p. 25-28.

[14] Billingsley, P. (1968). *Convergence of Probability Measures.* New-York. Wiley.

[15] Cheridito, P. (2003). Arbitrage in Fractional Brownian Motion Models. *Finance and Stochastics* 7, p. 533-553.

[16] Cohen, S. (2000). Champs localement auto-similaires. In *Fractals et loi d'échelles*, P.Abry, P. Goncalvès and J. Lévy-Vehel editors, Hermès, Paris

[17] Collins J.J. and de Luca C.J. (1993). Open-loop and closed-loop control of posture : A random walk analysis of center-of-pressure trajectories. *Experimental Brain Research* 9, p. 308-318.

[18] Dahlhaus R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* 17, p. 1749-1766

[19] Flandrin, P. (1992). Wavelet analysis and synthesis of fractional Brownian motion. *IEEE Trans. on Inform. Theory* 38, p. 910-917.

[20] Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent Gaussian time series. *Ann. Statist.* 14, p. 517-532.

[21] Giraitis L. and Surgailis D. (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its applications to the asymptotic normality of Whittle estimate. *Prob. Th. and Rel. Fields.* 86, p. 87-104.

[22] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, North Holland Publishing Co.

[23] Istas, J. and Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. Poincaré* 33, 407-436.

[24] Jacod, J. and Shiryaev, A.N. (1987). *Limit theorems for stochastic processes*, Springer Verlag.

[25] Lavielle, M. (1999). Detection of multiple changes in a sequence of dependent variables. *Stoc. Proc. Appl.* 83, p. 79-102.

[26] Lavielle, M. and Moulines, E. (2000). Least-squares estimation of an unknown number of shifts in a time series. *J. of Time Series Anal.* 21, p. 33-59.

[27] Mandelbrot, B. and Van Ness J. (1968). Fractional Brownian motion, fractional noises and applications. *SIAM review* 10, p.422-437.

[28] Peltier, R. and Lévy Véhel, J. (1995). Multifractional Brownian motion : definition and preliminary results, Technical Report 2645, INRIA, Le Chesnay, France.

[29] Rogers, L.C.G. (1997). Arbitrage with fractional Brownian motion. *Mathematical Finance* 7, p. 95-105.

[30] Samorodnitsky, G. and Taqqu M.S. (1994). *Stable non-Gaussian Random Processes*, Chapman and Hall.

[31] Van der Vaart, A. (1998). *Asymptotic statistics*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge.