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ORIGINAL PAPER

On a state-space modelling for functional data

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Abstract The objective of this paper is to derive a state-space model for several continuous-time processes, by applying the Karhunen–Loève expansion, and then to apply the Kalman filter equations. The accuracy of the models on the basis of deterministic or random inputs is studied by means of simulation on two well-known processes.

Keywords Karhunen–Loève expansion · State-space models · Kalman filter · CAR(1) · Random binary signal

1 Introduction

The problem of working with continuous-time processes has a great interest in several applied areas, such as Economics, Astrophysics, etc. In fact, on the last years there has been an extensive research going on this field, which has resulted in numerous publications. So, Sinha (2000) gives an overall view of indirect and direct methods that have been developed for identification of continuous-time system from samples of input–output data, Johansson et al. (2001) and Wang and Zhang (2001) discuss about identification of system and Larsson and Söderström (2002) and Valderrama et al. (2003) analyze unevenly sample data in continuous system. The most obvious reason is that many processes

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in real life are inherently continuous in time. The main approach is based on replacing the differentiation operator with some approximations and deriving a discrete-time linear model.

Given a second-order and quadratic-mean continuous stochastic process it is possible to represent it as a denumerable series, known as the Karhunen–Loève orthogonal expansion. In the present paper, we use such expansions for writing some continuous-time processes in term of a state-space model (SSM), and we apply the Kalman filtering method obtaining estimations of the process by computing the available noisy observations. So, in the following Section we will develop two applications to continuous-time processes, CAR(1) and random binary signal, and we will suppose both deterministic and random inputs. With deterministic inputs we will consider that the initial state is Gaussian. Finally, a discussion of the results and conclusions is presented in the last section.

2 SSM built from continuous-time processes

2.1 SSM built from CAR(1)

A continuous-time AR(1) process, denoted as CAR(1), is defined as a stationary solution of the first-order stochastic differential equation

$$X(t) + aX(t) = \sigma B(t) + b, \tag{1}$$

where $\{B(t)\}$ is a standard Brownian motion, and a, b and σ are parameters (Harvey 1990). The derivative of the Brownian motion with respect to t does not exist in the usual sense, so Eq. 1 is interpreted as an Itô differential equation,

$$\mathrm{d}X(t) + aX(t)\,\mathrm{d}t = \sigma \mathrm{d}B(t) + b\,\mathrm{d}t, \quad t > 0,$$

with $\mathrm{d}X(t)$ and $\mathrm{d}B(t)$ denoting the increments of X and B in the time interval $(t, t+\mathrm{d}t)$ and $X(0)$ a random variable with finite variance, independent of $\{B(t)\}$. The solution of this equation can be written as

$$X(t) = \mathrm{e}^{-at}X(0) + \frac{b}{a}(1 - \mathrm{e}^{-at}) + \mathrm{e}^{-at}I(t),$$

where $I(t) = \sigma \int_0^t \mathrm{e}^{au} dB(u)$ is an Itô integral. Necessary and sufficient conditions for the process $\{X(t)\}$, with $E[X(t)] = \frac{b}{a}$ and $Cov[X(t+h), X(t)] = \frac{\sigma^2}{2a}\mathrm{e}^{-ah}$, to be stationary are

$$a > 0, \quad E[X(0)] = \frac{b}{a} \quad \text{and} V[X(0)] = \frac{\sigma^2}{2a}.$$

If $a > 0$ and $X(0)$ is $N(\frac{b}{a}, \frac{\sigma^2}{2a})$, then the CAR(1) process will also be Gaussian and strictly stationary.

If the observations are equally spaced, the joint density is exactly the same as the joint density of observations of the discrete-time Gaussian AR(1) process with this form

$$Y_n(t) - \frac{b}{a} = \mathrm{e}^{-a}\left(Y_{n-1}(t) - \frac{b}{a}\right) + Z_n(t), \quad \{Z(t)\} \sim WN\left(0, \frac{\sigma^2}{2a}\right).$$

This shows that the observations at discrete-time of the CAR(1) process are equal to discrete-time AR(1) process with coefficient e^{-a}.
Let us now consider the CAR(1) with zero-mean and covariance function given by $C(t,s) = \frac{\sigma^2}{2a}\mathrm{e}^{-a|t-s|}$.
To obtain the Karhunen–Loève series, further details can be seen in Valderrama et al. (2000), we need to solve the integral equation

$$\lambda\varphi(t) = \int_0^T \frac{\sigma^2}{2a}\mathrm{e}^{-a|t-s|}\varphi(s)\,\mathrm{d}s \quad \forall t \in [0,T].$$

Differentiating twice we have

$$\lambda\ddot{\varphi}(t) = \frac{\sigma^2 a}{2}\int_0^T \mathrm{e}^{-a|t-s|}\varphi(s)\,\mathrm{d}s - \sigma^2\varphi(t) \quad \forall t \in [0,T],$$

but the first term on the right-hand side is just $a^2\lambda\varphi(t)$. Therefore, for $\lambda \neq 0$,

$$\ddot{\varphi}(t) = \left[a^2 - \frac{\sigma^2}{\lambda}\right]\varphi(t) \quad \forall t \in [0,T].$$

The values of λ that satisfy this equation have solution when $0 < \lambda < \sigma^2$ and are

$$\lambda_n = \frac{\sigma^2}{a^2 + b_n^2}, \quad n \in N,$$

where $\{b_n\}$ are solutions to the equation

$$\left[\tan(bt) + \frac{b}{a}\right]\left[\tan(bt) - \frac{a}{b}\right] = 0.$$

The values of b that satisfy this equation can be determined by approximative methods. Table 1 shows, by supposing $T = 2$, $a = 0.5$ and $\sigma = 1$, the approximations of b_n, the corresponding eigenvalues and the accumulated variance.

Table 1 Approximations of b_n, corresponding eigenvalues (λ_n) and accumulate variance, for $T = 2, \alpha = 0.5$ and $\sigma = 1$

n	b_n	Eigenvalues λ_n	Accumulated variance %
1	0.4302	2.2985	57.46
2	1.0144	0.7819	77.01
3	1.7128	0.3141	84.86
4	1.4566	0.1591	88.84
5	3.2187	0.0943	91.20
6	3.9893	0.0619	92.75
7	4.7647	0.0436	93.84
8	5.5428	0.0323	94.64

The total variance associated to the process is given by:

$$V = \sum_{i=1}^{\infty} \lambda_i = \int_{-T}^{T} C(t,t)\mathrm{d}t = \int_{-T}^{T} \frac{\sigma^2}{2a} \mathrm{e}^{-a|t-t|}\mathrm{d}t = \int_{-T}^{T} \frac{\sigma^2}{2a}\mathrm{d}t = \frac{\sigma^2}{2a}T.$$

The eigenfunctions are

$$\varphi_n(t) = \frac{\cos(b_n t)}{\sqrt{T}\left(1 + \frac{\sin(2b_n T)}{2b_n T}\right)^{\frac{1}{2}}}, \quad \text{for } n \text{ odd} \left(\tan(b_n t) = \frac{\alpha}{b_n}\right).$$

$$\varphi_n(t) = \frac{\sin(b_n t)}{\sqrt{T}\left(1 - \frac{\sin(2b_n T)}{2b_n T}\right)^{\frac{1}{2}}}, \quad \text{for } n \text{ even} \left(\tan(b_n t) = -\frac{b_n}{\alpha}\right)$$

By considering the state vector: $Z(t) = \left[X(t)\ \dot{X}(t) \right]^{\mathrm{T}}$ we can write the SSM as:

$$Z(t) = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} Z(t) + \begin{bmatrix} 0 \\ -\sigma^2 \sum_{i=1}^{q} \left(\frac{1}{\lambda_i}\right) \varphi_i(t)\, \xi_i \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} Z(t) + v(t),$$

where $w(t) = \ddot{\omega}(t) - \alpha^2 \omega(t)$ is a random residual and the variance–covariance of $v(t)$ is positive defined. Now it is possible to apply the Kalman filtering equations obtaining estimations of the process.

In order to show the results with deterministic and random inputs, just as Ruiz et al. (1995) consider, we have performed a computer simulation of a CAR(1) process. The simulation have been considered as a process corrupted by a white

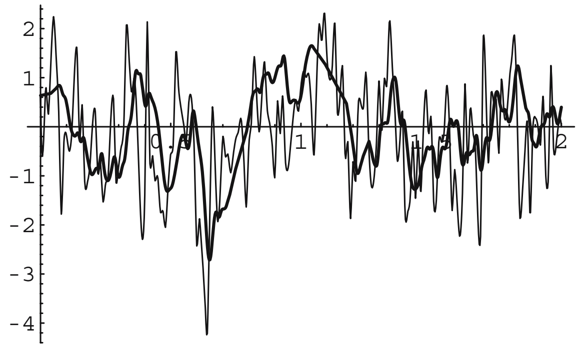

Fig. 1 Real *thin line* and filter *thick line* of the CAR(1) process with deterministic input

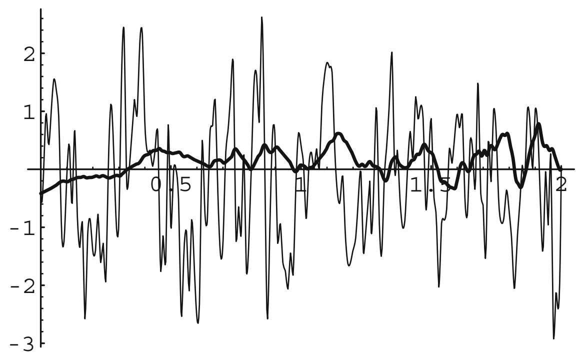

Fig. 2 Real *thin line* and filter *thick line* of the CAR(1) process with random input

noise and it has allowed to provide information to the Kalman filter in each step of the algorithm.

After that we have built the SSM associated to the first six random variables and the results of the filter for models with deterministic and random inputs, together with a real values are shown in Figs. 1 and 2. Mean square errors associated to these figures are 1.53687 and 1.40039, respectively.

The simple and partial autocorrelation functions of the innovations are shown in Fig. 3 with deterministic inputs and with random inputs in Fig. 4.

2.2 SSM built from random binary signal

Let us consider a process $\{\chi(t), t \geq 0\}$ given by:

$$\chi(t) = \begin{cases} 1 & \text{if success is obtained at the } n\text{th test} \\ -1 & \text{if not success is obtained at the } n\text{th test} \end{cases}$$

with $(n-1)T < t < nT$. Let us define another process $\{X(t) = \chi(t-e), t \geq 0\}$ where e is a random variable which is uniformly distributed in $[0, T]$ and independent of $\chi(t)$. The new process, known as random binary signal (Papoulis 1980), is a continuous time process defined in the interval $[0, T]$, with zero-mean and covariance function given by

$$C(t,s) = \begin{cases} 1 - \frac{|t-s|}{T} & \text{if } 0 \leq |t-s| \leq T \\ 0 & \text{in other case.} \end{cases} \tag{2}$$

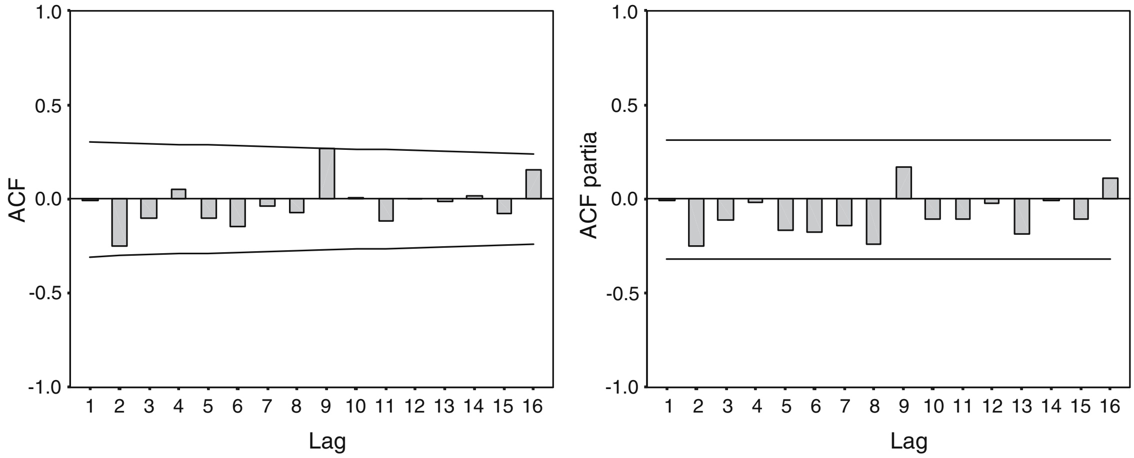

Fig. 3 Estimated simple and partial autocorrelation functions of the innovations for CAR(1) process with deterministic input

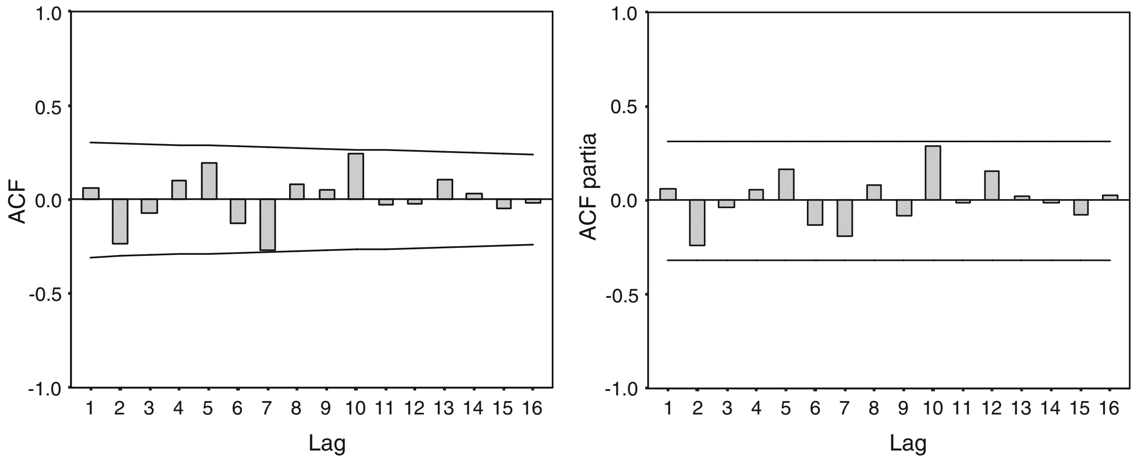

Fig. 4 Estimated simple and partial autocorrelation functions of the innovations for CAR(1) process with random input

To obtain the decomposition of Karhunen–Loève in the parametric space $[0, T]$ we need to solve the integral equation:

$$\lambda\varphi(t) = \int_0^T \left(1 - \frac{|t-s|}{T}\right) \varphi(s)\mathrm{d}s,$$

After differentiating twice we have:

$$\ddot{\varphi}(t) + \frac{2}{\lambda T}\varphi(t) = 0 \quad \forall t \in [0, T].$$

The general solution of this equation together outline conditions take us to conclude that the eigenvalues associated to the process verify:

Table 2 Eigenvalues, accumulated variances and eigenfunctions associated to the first eight random variables of our process, for $T = 2$

n	Eigen values λ_n	Accumulated variance	Eigen functions φ_n
ξ_1	1.3510	33.7758	$0.7919\cos\left[0.8603\,(t-1)\right]$
ξ_2	0.0852	67.5516	$0.9629\cos\left[3.4256\,(t-1)\right]$
ξ_3	0.0241	69.6820	$0.9884\cos\left[6.4373\,(t-1)\right]$
ξ_4	0.0110	70.2853	$0.9946\cos\left[9.5293\,(t-1)\right]$
ξ_5	0.0063	70.5606	$0.9969\cos\left[12.6453\,(t-1)\right]$
ξ_6	0.0040	70.7170	$0.9980\cos\left[15.7713\,(t-1)\right]$
ξ_7	0.0028	70.8175	$0.9986\cos\left[18.9024\,(t-1)\right]$
ξ_8	0.0021	70.8875	$0.9990\cos\left[22.0365\,(t-1)\right]$

$$\cot\left(\frac{T}{2}\sqrt{\frac{2}{\lambda_n T}}\right) = \frac{T}{2}\sqrt{\frac{2}{\lambda_n T}} \quad \forall n \in N.$$

and it can also be written as:

$$\cot\left(x_n\right) = x_n \quad \forall n \in N,$$

where $x_n = \frac{T}{2}\sqrt{\frac{2}{\lambda_n T}}$. In addition, this equation can be solved by different approximative methods, we have used the Mathematica program.

To obtain the eigenfunction associated to the process we take into account the ortonormality property, $\int_0^T \varphi^2(t)\mathrm{d}t = 1$, and then:

$$\varphi_n(t) = \sqrt{\frac{2\sqrt{\frac{2}{\lambda_n T}}}{T\sqrt{\frac{2}{\lambda_n T}} + \sin\left(\sqrt{\frac{2}{\lambda_n T}}T\right)}}\cos\left[\sqrt{\frac{2}{\lambda_n T}}\left(t - \frac{T}{2}\right)\right].$$

Table 2 shows, for $T = 2$, information about eigenvalues, λ_i, eigenfunctions, φ_i, and accumulated variance, $\frac{\sum_{j=1}^{n}\lambda_j}{V} \times 100\%$, for the first eight random variables associated to the process. The total variance associated to the process is given by:

$$V = \sum_{i=1}^{\infty}\lambda_i = \int_{-T}^{T} C(t,t)\mathrm{d}t = \int_{-T}^{T} 1 - \frac{|t-t|}{T}\mathrm{d}t = 2T.$$

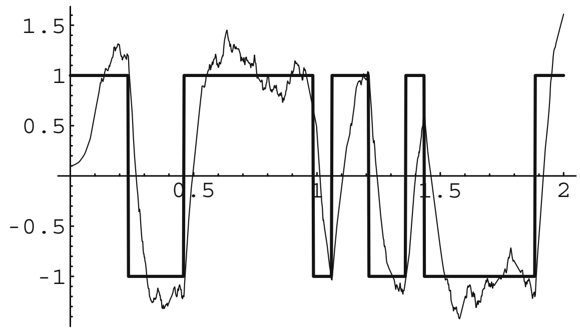

Fig. 5 Real *thick line* and filter *thin line* of the random binary signal with deterministic input

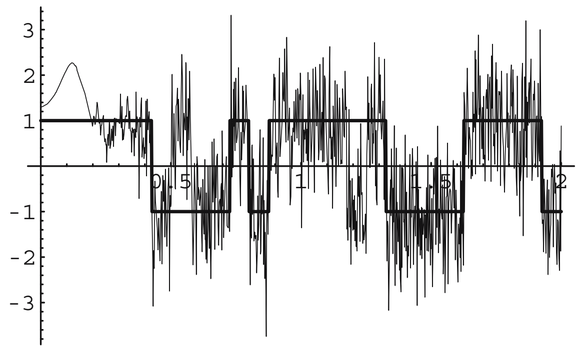

Fig. 6 Real *thick line* and filter *thin line* of the random binary signal with random input

By considering the state vector: $Z(t) = \left[X(t)\ \dot{X}(t) \right]^{\mathrm{T}}$ we can built the following SSM:

$$\dot{Z}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Z(t) + \begin{bmatrix} 0 \\ -\frac{2}{T} \sum_{i=1}^{k} \left(\frac{1}{\lambda_i}\right) \varphi_i(t)\, \xi_i \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \end{bmatrix},$$
$$Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} Z(t) + V(t),$$

where $w(t) = \ddot{\omega}(t)$.
After simulating a process, we have built the SSM associated to the first four random variables and have applied the filter equations for both, deterministic and random inputs. The observed sample-path has been obtained by disturbing the simulated trajectory with a white noise with variance one.
Results of the filter for models with deterministic and random inputs, together with a simulated path are shown in Figs. 5 and 6. Mean square errors associated to these figures are 0.51085 and 0.38365, respectively.
Figures 7 and 8 contain the simple and partial autocorrelation functions of the innovations for the process with deterministic and random inputs, respectively.

3 Final comments

The mean square error associated to the models and the Figs. 1, 2, 5 and 6 show a better behavior of the model with random input. Nevertheless, from Figs. 3, 4, 7 and 8 both models are suitable for representing the time evolution this is due to the white noise character of the innovations.

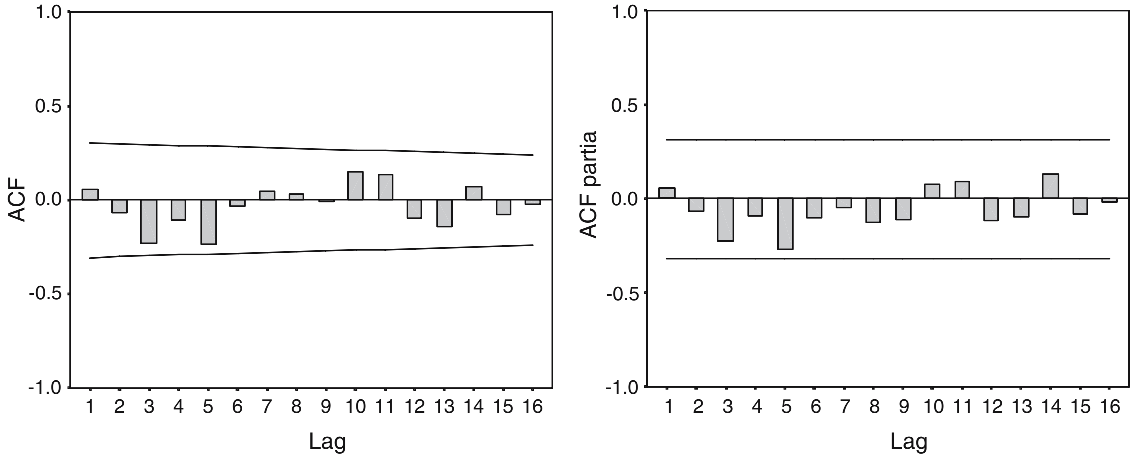

Fig. 7 Estimated simple and partial autocorrelation functions of the innovations for random binary signal with deterministic input

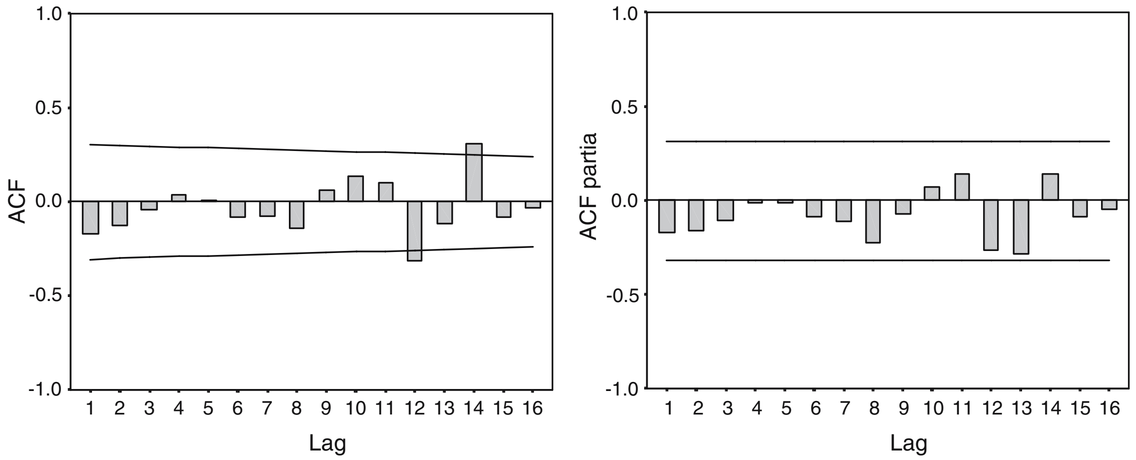

Fig. 8 Estimated simple and partial autocorrelation functions of the innovations for random binary signal with random input

We must take into account that although the random treatment of the input is more natural, the dimension of the system have increased from 2 to 5, and it will give rise to a higher operation complexity.

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