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Youcef Amirat[♭], Gregory A. Chechkin[♮], Rustem R. Gadyl'shin[♯]

[♭] Laboratoire de Mathématiques
CNRS UMR 6620
Université Blaise Pascal
63177 Aubière cedex, France
amirat@math.univ-bpclermont.fr

[♮] Department of Differential Equations, Faculty of Mechanics and Mathematics,
Moscow Lomonosov State University, Moscow 119991, Russia,
&
Narvik University College, Postboks 385, 8505 Narvik, Norway,
chechkin@mech.math.msu.su

[♯] Department of Mathematical Analysis
Faculty of Physics and Mathematics
Bashkir State Pedagogical University
Ufa 450000, Russia
gadylshin@yandex.ru

Abstract

We study the asymptotic behavior of the eigenelements of the Dirichlet problem for the Laplacian in a bounded domain, a part of whose boundary, depending on a small parameter ε, is highly oscillating; the frequency of oscillations of the boundary is of order ε and the amplitude is fixed. We construct and analyze second-order asymptotic approximations, as $\varepsilon \to 0$, of the eigenelements in the case of simple eigenvalues of the limit problem.

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Introduction

Boundary-value problems involving rapidly oscillating boundaries or interfaces frequently arise when modeling problems of physics and engineering sciences, such as the scattering of acoustic and electromagnetic waves on small periodic obstacles, the free vibrations of strongly nonhomogeneous elastic bodies, electric current through rough interfaces, fluids over rough walls, and coupled fluid-solid periodic structures. Recent years many mathematical works were devoted to asymptotic analysis of these problems, see for instance [1]–[13], [16], [18], [20], [22]–[30]. The mathematical analysis of these problems consists in studying the large scale behavior of the solution. The goal is to construct accurate asymptotic approximations or to determine effective boundary conditions. The main difficulty comes from the presence of boundary layers near the rough region, which effects on correctors or error estimates have to be taken into account.

In this paper we continue to study the asymptotic behavior of eigenvalues and eigenfunctions to boundary-value problems in domains with oscillating boundaries, with the homogeneous Dirichlet condition on the rough part of the boundary. In [6] the authors considered problems with Dirichlet boundary conditions on the oscillating part of the boundary; they proved a convergence theorem for the eigenelements of a general $2m$-order elliptic operator for a special type of domains. In [4] and [5] we considered a spectral problem for the Laplace operator in a bounded domain, a part of whose boundary, depending on a small parameter ε, is rapidly oscillating; the frequency and the amplitude of oscillations of the boundary are of the same order ε. We constructed the leading terms of the asymptotic expansions for the eigenelements and verified the asymptotics in the cases of simple eigenvalues and multiple eigenvalues of the limit problem, respectively. The case of totally oscillating boundary was considered in [27].

In this paper we deal with the case where a part of the boundary is highly oscillating; the frequency of oscillations of the boundary is of order ε and the amplitude is fixed. Our aim is to construct accurate asymptotic approximations, as $\varepsilon \to 0$, of the eigenvalues and corresponding eigenfunctions.

The outline of the paper is as follows. In Section 1 we introduce notations, formulate the problem and present our main results. In Section 2 we prove some uniform estimates and results of convergence. In Section 3 we construct formal asymptotics for the eigenvalues, while in Section 4 we prove the main theorem. In Section 5 we construct and justify the asymptotics for the corresponding eigenfunctions.

1 Setting of the problem and main results

Let Ω^+ be a bounded domain in $\mathbb{R}^2$, located in the upper half space. We assume the boundary $\partial\Omega^+$ to be piecewise smooth, consisting of the parts: $\partial\Omega^+ = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the segment $\left(-\frac{1}{2}, \frac{1}{2}\right)$ on the abscissa axis, Γ_1 coincides with the straight lines $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$ at neighborhood of the abscissa axis. Let $\varepsilon = \frac{1}{2\mathcal{N}+1}$ be a small parameter, where $\mathcal{N}$ is a large positive number. Assume that $0 < a < \frac{1}{2}$, $h > 0$ and define (see Figure 1)

$$\Omega^-_{j,\varepsilon} = \{x \in \mathbb{R}^2 \; : \; -\varepsilon a < x_1 - \varepsilon j < \varepsilon a, \; -h < x_2 \leq 0\},$$

$$\Omega^-_\varepsilon = \bigcup_{j=-\mathcal{N}}^{\mathcal{N}} \Omega^-_{j,\varepsilon},$$

$$\Omega^\varepsilon = \Omega^+ \cup \Omega^-_\varepsilon,$$

$$\Gamma^\varepsilon = \partial\Omega^\varepsilon \backslash \overline{\Gamma_1}.$$

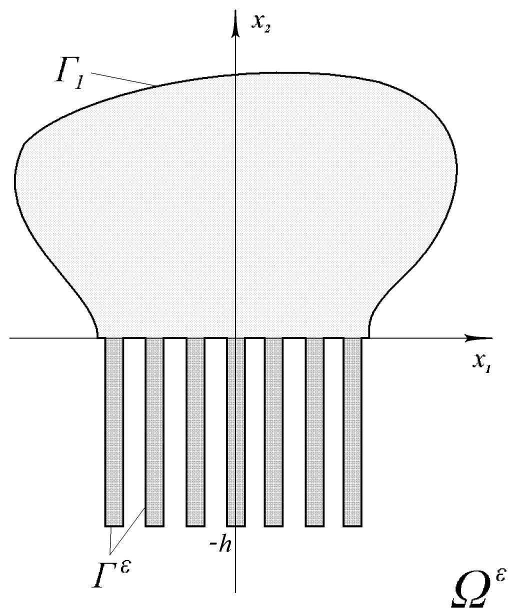

Figure 1: Membrane with oscillating boundary.

We consider the spectral problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon \;\text{ in } \Omega^\varepsilon, \\ u_\varepsilon = 0 \;\text{ on } \partial\Omega^\varepsilon, \end{cases} \tag{1}$$

and study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalue λ_ε and the corresponding eigenfunction u_ε. We first state some results of uniform estimates, convergence of solutions to nonhomogeneous boundary-value problems associated with (1), and convergence of eigenelements. The following statements will be proven in the next section.

Theorem 1.1. *Let $F_\varepsilon \in L_2(\Omega^\varepsilon)$, let Q be an arbitrary compact in the complex plane which does not contain eigenvalues of the boundary-value problem*

$$\begin{cases} -\Delta u_0^+ = \lambda_0 u_0^+ \quad \text{in } \Omega^+, \\ u_0^+ = 0 \quad \text{on } \partial\Omega^+, \end{cases} \tag{2}$$

and let $\lambda \in Q$. Then:

(i) The boundary-value problem

$$\begin{cases} -\Delta U_\varepsilon = \lambda U_\varepsilon + F_\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \\ U_\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \end{cases} \tag{3}$$

has, for ε small enough, a unique solution satisfying the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}, \tag{4}$$

uniformly with respect to ε and λ.

(ii) Assume that there is $F_0 \in L_2(\Omega)$ such that, as $\varepsilon \to 0$,

$$\|F_\varepsilon - F_0\|_{L_2(\Omega^+)} + \|F_\varepsilon\|_{L_2(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0, \tag{5}$$

and let U_0 be the solution of the boundary–value problem

$$\begin{cases} -\Delta U_0 = \lambda U_0 + F_0 \quad \text{in } \Omega^+, \\ U_0 = 0 \quad \text{on } \partial\Omega^+. \end{cases} \tag{6}$$

Then

$$\|U_\varepsilon - U_0\|_{H^1(\Omega^+)} + \|U_\varepsilon\|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0,$$

uniformly with respect to λ.

Theorem 1.2. *Assume that the multiplicity of the eigenvalue λ_0 of problem* (2) *is equal to p. Then:*

(i) there are p eigenvalues of problem (1) (with multiplicities taken into account) converging to λ_0, as $\varepsilon \to 0$;

(ii) if $\lambda_\varepsilon^1, \dots, \lambda_\varepsilon^p$ are the eigenvalues of problem (1)*, which converge to λ_0 and $u_\varepsilon^1, \dots, u_\varepsilon^p$ are the corresponding eigenfunctions, orthonormal in the space $L_2(\Omega^\varepsilon)$, then for any sequence $\varepsilon_k \underset{k\to\infty}{\to} 0$ there exists a subsequence $\varepsilon_{k'} \to 0$ such that*

$$\|u_\varepsilon^j - u_0^{+,j}\|_{H^1(\Omega^+)} + \|u_\varepsilon^j\|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0,$$

as $\varepsilon = \varepsilon_{k'} \to 0$. Here, $u_0^{+,1}, \dots, u_0^{+,p}$ denote the eigenfunctions of problem (2)*, corresponding to λ_0 and orthonormal in $L_2(\Omega^\varepsilon)$.*

Theorem 1.2 implies:

Corollary 1. *If the eigenvalue λ_0 of problem* (2) *is simple, then there is unique sequence of simple eigenvalues λ_ε of problem* (1) *converging to λ_0, as $\varepsilon \to 0$, and the sign of the corresponding eigenfunctions u_ε, normalized in $L_2(\Omega^\varepsilon)$, can be chosen so that*

$$\|u_\varepsilon - u_0^+\|_{H^1(\Omega^+)} + \|u_\varepsilon\|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0.$$

Here u_0^+ is the eigenfunction of problem (2)*, corresponding to λ_0 and normalized in $L_2(\Omega^\varepsilon)$.*

For later reference we state the following result.

Lemma 1.1. *Assume that the multiplicity of the eigenvalue λ_0 of problem* (2) *is equal to p and $\lambda_\varepsilon^1, \ldots, \lambda_\varepsilon^p$ are the eigenvalues of problem* (1)*, which converge to λ_0, as $\varepsilon \to 0$. Then:*

(i) for any λ close to λ_0, the solution U_ε to problem (3) *satisfies the estimate*

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)} \left(1 + \sum_{j=1}^{p} \frac{1}{|\lambda_\varepsilon^j - \lambda|}\right); \tag{7}$$

(ii) if a solution U_ε to problem (3) *is orthogonal in $L_2(\Omega^\varepsilon)$ to the eigenfunction u_ε^i of problem* (1) *corresponding to λ_ε^i, then it satisfies the estimate*

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)} \left(1 + \sum_{j=1; j\neq i}^{p} \frac{1}{|\lambda_\varepsilon^j - \lambda|}\right). \tag{8}$$

Our main aim is to construct accurate asymptotic approximations, as $\varepsilon \to 0$, of the eigenvalues of problem (1), which converges to the simple eigenvalue of problem (2). The following statement will be proven by using the method of matching asymptotic expansions [17].

Theorem 1.3. *Let λ_0 be a simple eigenvalue of problem* (2) *and u_ε be the corresponding eigenfunction, normalized in $L_2(\Omega^\varepsilon)$. Then,*

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2) \tag{9}$$

where

$$\lambda_1 = -\, q(a) \int\limits_{\Gamma_0} \left(\frac{\partial u_0^+}{\partial \nu}\right)^2 \, ds, \tag{10}$$

$$q(a) = \frac{a}{\pi}\Big(4\ln 2 - \big[(1-2a)\ln(1-2a) + (1+2a)\ln(1+2a)\big]\Big), \tag{11}$$

and ν is the outward unit normal to Ω^+.

Note that it is easy to deduce from (11) that $q(a) > 0$ as $0 < a < \frac{1}{2}$. In the paper we also construct and analyze an asymptotic approximation of the corresponding eigenfunction u_ε at order $O(\varepsilon^2)$ in the norms of $L_2(\Omega^\varepsilon)$ and $H^1(\Omega^\varepsilon)$ (see Theorem 5.1).

2 Uniform bounds and convergence results: proofs

In this section we prove Theorem 1.1, Theorem 1.2 and Lemma 1.1. We introduce the notations

$$\begin{cases} \Omega^- = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (-h, 0)\,, \\ \\ \Omega = \Omega^+ \cup \Omega^- \cup \Gamma_0. \end{cases}$$

Observe that the Hausdorff limit of the sequence $(\overline{\Omega^\varepsilon})_{\varepsilon>0}$ is the closed set $\overline{\Omega}$.

We define weak solutions of problem (3) in a classical way (see, for instance, [31]). A function $U_\varepsilon \in H_0^1(\Omega^\varepsilon)$ is a solution of problem (3) if it satisfies the integral identity

$$(\nabla U_\varepsilon, \nabla V)_{L_2(\Omega^\varepsilon)} = \lambda(U_\varepsilon, V)_{L_2(\Omega^\varepsilon)} + (F_\varepsilon, V)_{L_2(\Omega^\varepsilon)} \tag{12}$$

for any $V \in H_0^1(\Omega^\varepsilon)$. In analogues way we define weak solutions to the boundary-value problems (1), (2) and (6), respectively. Note that, from (12) follows directly the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_1 \left(\|U_\varepsilon\|_{L_2(\Omega^\varepsilon)} + \|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}\right), \tag{13}$$

uniformly with respect to ε small enough and $\lambda \in Q$.

2.1 Proof of Theorem 1.1

To prove the item (i) it suffices to prove estimate (4). Assume that this estimate is false. Then there exist sequences $\varepsilon_k \to 0$, $\lambda = \lambda_k \in Q$ and $F_\varepsilon = F_{\varepsilon_k} \in L_2(\Omega^{\varepsilon_k})$ such that, for the corresponding weak solution U_{ε_k}, the inequality

$$\|U_{\varepsilon_k}\|_{H^1(\Omega^{\varepsilon_k})} \ge k\|F_{\varepsilon_k}\|_{L_2(\Omega^{\varepsilon_k})} \tag{14}$$

holds true. Without loss of generality, let us assume that the functions U_ε satisfies

$$\|U_\varepsilon\|_{L_2(\Omega^\varepsilon)} = 1. \tag{15}$$

Substituting (15) and (14) in (13), we obtain

$$\|U_{\varepsilon_k}\|_{H^1(\Omega^{\varepsilon_k})} \le C_2, \tag{16}$$

uniformly with respect to ε and λ. On the other hand, it follows from (14) and (16) that

$$\|F_{\varepsilon_k}\|_{L_2(\Omega^{\varepsilon_k})} \underset{k\to\infty}{\to} 0. \tag{17}$$

It is clear that the functions of $H^1_0(\Omega^+)$ and of $H^1_0(\Omega^\varepsilon)$, extended by zero in Ω^- and in $\Omega\backslash\overline{\Omega^\varepsilon}$, respectively, belong to $H^1(\Omega)$. In the sequel, we keep the same notation for the extended functions. Under this notation, $U_\varepsilon \in H^1_0(\Omega)$ and (16) takes the form

$$\|U_{\varepsilon_k}\|_{H^1(\Omega)} \le C_3. \tag{18}$$

Hence, using the compactness of Q and the compact embedding of $H^1(\Omega)$ in $L_2(\Omega)$, we conclude that there exists a subsequence $\varepsilon_{k'}$ such that

$$U_{\varepsilon_{k'}} \underset{k'\to\infty}{\longrightarrow} U_* \in H^1_0(\Omega) \;\text{ weakly in } H^1(\Omega) \text{ and strongly in } L_2(\Omega), \tag{19}$$

$$\lim_{k'\to\infty} \lambda_{k'} = \lambda_* \in Q. \tag{20}$$

Clearly, it follows from (15) and (19) that

$$U_* \ne 0. \tag{21}$$

Let us show that

$$U_* = 0 \;\text{ in } \Omega^-. \tag{22}$$

Denote $\widetilde{\Omega}^-_{j,\varepsilon} = \Omega^-_{j,\varepsilon}\backslash\Gamma_0$, $\widetilde{\Omega}^-_\varepsilon = \Omega^-_\varepsilon\backslash\Gamma_0$, $\gamma_{j,\varepsilon} = \partial\Omega^-_{j,\varepsilon}\backslash\left(\Gamma^-_0 \cup \Gamma_0\right)$, where $\Gamma^-_0 = \left\{x:\; x_1 \in \left(-\frac{1}{2}, \frac{1}{2}\right),\; x_2 = -h\right\}$. Since $U_\varepsilon = 0$ on $\gamma_{j,\varepsilon}$, the Poincaré–Friedrichs inequality (see for instance [30, Ch XVII, Lemma 1.1]) reads

$$\int\limits_{\widetilde{\Omega}^-_{j,\varepsilon}} U^2_\varepsilon \mathrm{d}x \le C\varepsilon^2 \int\limits_{\widetilde{\Omega}^-_{j,\varepsilon}} |\nabla U_\varepsilon|^2 \mathrm{d}x. \tag{23}$$

By summation on j from $-\mathcal{N}$ to $\mathcal{N}$, we deduce the inequality

$$\int\limits_{\Omega^-} U^2_\varepsilon \mathrm{d}x \le C\varepsilon^2 \int\limits_{\Omega^-} |\nabla U_\varepsilon|^2 \mathrm{d}x$$

which together with (18) gives

$$\int\limits_{\Omega^-} U^2_{\varepsilon_k} \mathrm{d}x \underset{k\to\infty}{\to} 0,$$

and (22) follows from this convergence.

Consider now the restriction of U_* to Ω^+ which we still denote U_*. Let V be an arbitrary function in $H^1_0(\Omega^+)$ which we extend by zero outside Ω^+.

Obviously, this function belongs to $H_0^1(\Omega^\varepsilon)$. Passing to the limit in (12) as $\varepsilon = \varepsilon_{k'} \to 0$ and $\lambda_{\varepsilon_{k'}}$, keeping in mind (17), (19) and (20), we obtain

$$(\nabla U_*, \nabla V)_{L_2(\Omega^+)} = \lambda_*(U_*, V)_{L_2(\Omega^+)}.$$

Then, according to (21), (22) and the integral identity associated with problem (6) we conclude that U_* is an eigenfunction and λ_* is a corresponding eigenvalue of the limit problem, but this contradicts (20), then estimate (4) is proved.

We prove the item (ii) by means of the same scheme. Let $\varepsilon_k \to 0$ be an arbitrary sequence and λ be fixed in Q. Using inequality (4), the compact embedding of $H^1(\Omega)$ in $L_2(\Omega)$ and inequality (23), we deduce that there exists a subsequence $\varepsilon_{k'}$ such that (19) and (22) hold. Let us remind that, due to (22), the restriction U_* to Ω^+ is an element of $H_0^1(\Omega^+)$. Passing to the limit in (12), for $V \in H_0^1(\Omega)$, as $\varepsilon_{k'} \to 0$, by means of (19) and (5), we get that $U_* = U_0$ is a solution of problem (6). Since this problem has a unique solution and the sequence $\varepsilon_k \to 0$ is arbitrary, we deduce that

$$U_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} U_0 \ \text{ weakly in } H^1(\Omega) \text{ and strongly in } L_2(\Omega). \tag{24}$$

Rewriting (12) in the form

$$(\nabla U_\varepsilon, \nabla V)_{L_2(\Omega)} = \lambda(U_\varepsilon, V)_{L_2(\Omega)} + (F_\varepsilon, V)_{L_2(\Omega^\varepsilon)}, \tag{25}$$

taking $V = U_\varepsilon$ then passing to the limit in (25), as $\varepsilon \to 0$, and keeping in mind (5) and (24), we conclude that

$$\|\nabla U_\varepsilon\|^2_{L_2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} \|\nabla U_0\|^2_{L_2(\Omega)}. \tag{26}$$

It follows from (24) and (26) that

$$\|U_\varepsilon - U_0\|^2_{H^1(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

so we proved the item (ii) of the theorem, for any fixed $\lambda \in Q$. Since, for any fixed ε, the solution U_ε is holomorphic with respect to $\lambda \in Q$, the convergence (5) is uniform with respect to $\lambda \in Q$. Theorem 1.1 is proved.

2.2 Proof of Theorem 1.2

Denote by Σ^ε and Σ_0^+ the set of eigenvalues of problems (1) and (2), respectively, and by $S(z,t)$ an open disk in the complex plane, centered in z and with radius t. Let λ_0 be an arbitrary eigenvalue of the limit problem (2). Since the set of eigenvalues of problem (2) does not have any accumulation point, there exists $t > 0$ such that $\overline{S(\lambda_0,t)}$ can contain no eigenvalue of the limit problem other than λ_0. Hence, due to the statement (i) of Theorem 1.1, for ε small enough, we have $\partial S(\lambda_0, t) \cap \Sigma^\varepsilon = \emptyset$, and due to the statement (ii) of the same theorem we have

$$\int\limits_{\partial S(\lambda_0,t)} U_\varepsilon(x,\lambda)\, d\lambda \underset{\varepsilon\to 0}{\to} \int\limits_{\partial S(\lambda_0,t)} U_0(x,\lambda)\, d\lambda \tag{27}$$

in $H^1(\Omega)$. It is well-known (see, for instance [19]) that the resolvents of problems (3) and (6) have only simple poles which are the eigenvalues of these problems, respectively, and the residues at these poles are the projection operators from $L_2(\Omega^\varepsilon)$ and $L_2(\Omega^+)$, respectively, onto the corresponding eigenspaces. Conversely, any eigenvalue is a simple pole of the resolvent. Let F_0 be a function such that problem (6) is insoluble as $\lambda = \lambda_0$. Then the right-hand side of (27) is not equal to zero and hence, for ε small enough, the left-hand side of (27) does not equal to zero. Consequently, $S(\lambda_0, t) \cap \Sigma^\varepsilon \neq \emptyset$ and since t is arbitrary, there exists an eigenvalue of problem (1) converging to λ_0, as $\varepsilon \to 0$.

Suppose that $\varepsilon_k \underset{k\to\infty}{\to} 0$ is an arbitrary sequence, $\lambda^j_{\varepsilon_k}$ are the eigenvalues of problem (1) converging to λ_0, and $u^j_{\varepsilon_k}$ are the corresponding eigenfunctions, orthonormalized. Obviously, we have

$$\|u^j_{\varepsilon_k}\|_{H^1(\Omega)} \le C.$$

Using this estimate and following the proof of statement (ii) of Theorem 1.1, it is easy to show that there exists a subsequence $\varepsilon_{k'} \to 0$ such that

$$\|u^j_{\varepsilon_{k'}} - u_0^{+,j}\|_{H^1(\Omega)} \to 0 \tag{28}$$

where $u_0^{+,j}$ are the orthonormalized eigenfunctions of problem (2) corresponding to the eigenvalue λ_0. Hence the total multiplicity of the eigenvalues of problem (1) converging to λ_0 is less then or equal to p.

Now, assume that there exists a subsequence of the sequence $\varepsilon_{k'} \to 0$ (not relabeled for convenience) such that the total multiplicity of the eigenvalues $\lambda^j_{\varepsilon_{k'}}$ of problem (1), converging to λ_0, equals P and $P < p$. Then, first, for λ close to λ_0, for the solutions $U_{\varepsilon_{k'}}$ and U_0 of problems (3) and (6), respectively, we have the representations

$$U_{\varepsilon_{k'}} = \sum_{j=1}^{P} \frac{u^j_{\varepsilon_{k'}}}{\lambda^j_{\varepsilon_{k'}} - \lambda} \int\limits_{\Omega^{\varepsilon_{k'}}} u^j_{\varepsilon_{k'}} F_{\varepsilon_{k'}}\, dx + \widetilde{U}_{\varepsilon_{k'}}, \tag{29}$$

$$U_0 = \sum_{j=1}^{p} \frac{u_0^{+,j}}{\lambda_0 - \lambda} \int\limits_{\Omega^+} u_0^{+,j} F_0\, dx + \widetilde{U}_0, \tag{30}$$

where $\widetilde{U}_{\varepsilon_{k'}}$ and $\widetilde{U}_0$ are holomorphic functions with respect to λ in the neighborhood of λ_0. Second, there exists an eigenfunction $u_0^{+,P+1}$ of problem (2), orthogonal to $u_0^{+,j}$, as $j \le P$. Suppose that

$$F_{\varepsilon_{k'}} = F_0 = u_0^{+,P+1} \;\text{ in } \Omega^+, \;\; F_{\varepsilon_{k'}} = 0 \;\text{ in } \Omega^-.$$

Then, on one hand

$$\int\limits_{\Omega^{\varepsilon_{k'}}} u^j_{\varepsilon_{k'}} F_{\varepsilon_{k'}}\, dx \underset{k'\to\infty}{\rightarrow} 0, \;\; j \le P,$$

and hence, due to (29), the left-hand side of (27) converges to zero. On the other hand

$$\int\limits_{\Omega^+} u_0^{+,P+1} F_0\, dx = 1$$

and consequently, due to (30), the right-hand side of (27) does not converge to zero. From this contradiction it follows that $P = p$ and the statement (i) of the theorem follows.

The statement (ii) follows from the convergence (28). This completes the proof of Theorem 1.2.

2.3 Proof of Lemma 1.1

It follows from Theorem 1.2 (i) that, for λ close to λ_0, the solution U_ε of problem (3) admits the representation

$$U_\varepsilon = \sum_{j=1}^{p} \frac{u^j_\varepsilon}{\lambda^j_\varepsilon - \lambda} \int\limits_{\Omega^\varepsilon} u^j_\varepsilon F_\varepsilon\, dx + \widetilde{U}_\varepsilon \tag{31}$$

where the eigenfunctions $u^1_\varepsilon, \dots, u^p_\varepsilon$ are chosen normalized in $L_2(\Omega^\varepsilon)$ and $\widetilde{U}_\varepsilon$ is a holomorphic function of $\lambda \in \overline{S(\lambda_0, t)}$, for sufficiently small t. It is easy to see that $\widetilde{U}_\varepsilon$ is a solution of the boundary-value problem

$$\begin{cases} -\Delta \widetilde{U}_\varepsilon = \lambda \widetilde{U}_\varepsilon + \widetilde{F}_\varepsilon \;\text{ in }\; \Omega^\varepsilon, \\ \widetilde{U}_\varepsilon = 0 \;\text{ on }\; \partial\Omega^\varepsilon, \end{cases}$$

with

$$\widetilde{F}_\varepsilon = F_\varepsilon - \sum_{j=1}^{j=p} u^j_\varepsilon \int_{\Omega^\varepsilon} u^j_\varepsilon F_\varepsilon\, dx.$$

Then, employing Theorem 1.1 (i) we have

$$\|\widetilde{U}_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_1 \|\widetilde{F}_\varepsilon\|_{L_2(\Omega^\varepsilon)} \le C_2 \|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}, \tag{32}$$

uniformly with respect to ε and $\lambda \in \underline{\partial S(\lambda_0, t)}$; actually we show by analytic continuation that (32) holds for $\lambda \in \overline{S(\lambda_0, t)}$.

Moreover, multiplying equation (1) by u_ε^j, integrating over Ω^ε and integrating by parts yields

$$\int_{\Omega^\varepsilon} |\nabla u_\varepsilon^j|^2\,dx = \lambda_\varepsilon^j \quad (1 \le j \le p).$$

Then, using the Poincaré–Friedrichs inequality, we obtain

$$\begin{aligned}\left\| \sum_{j=1}^{p} \frac{u_\varepsilon^j}{\lambda_\varepsilon^j - \lambda} \int\limits_{\Omega^\varepsilon} u_\varepsilon^j F_\varepsilon\,dx \right\|_{H^1(\Omega^\varepsilon)} &\le C_3 \sum_{j=1}^{p} \frac{\sqrt{\lambda_\varepsilon^j}}{|\lambda_\varepsilon^j - \lambda|}\, \|F_\varepsilon\|_{L^2(\Omega^\varepsilon)} \\ &\le C_4 \|F_\varepsilon\|_{L_2(\Omega^\varepsilon)} \sum_{j=1}^{p} \frac{1}{|\lambda_\varepsilon^j - \lambda|},\end{aligned}$$

and (7) follows from this estimate, (31) and (32).

If U_ε is orthogonal in $L_2(\Omega^\varepsilon)$ to u_ε^i, then multiplying equations (1) and (3) by U_ε and u_ε^i, respectively, integrating over Ω^ε and integrating by parts, we find

$$\int_{\Omega^\varepsilon} \nabla U_\varepsilon \nabla u_\varepsilon^i\,dx = 0, \qquad \int_{\Omega^\varepsilon} \nabla U_\varepsilon \nabla u_\varepsilon^i\,dx = \int_{\Omega^\varepsilon} u_\varepsilon^i F_\varepsilon\,dx,$$

hence

$$\int_{\Omega^\varepsilon} u_\varepsilon^i F_\varepsilon\,dx = 0$$

and repeating the arguments used for (7) we obtain (8). Lemma 1.1 is proved.

3 Formal asymptotic construction

Let us define a real number λ_1 and a function u_1^+ in Ω^+ satisfying the boundary-value problem

$$\begin{cases} -\Delta u_1^+ = \lambda_0 u_1^+ + \lambda_1 u_0^+ \ \text{ in } \Omega^+, \\ u_1^+ = 0 \ \text{ on } \Gamma_1, \\ u_1^+ = q(a) \dfrac{\partial u_0^+}{\partial x_2} \ \text{ on } \Gamma_0, \end{cases} \tag{33}$$

where $q(a)$ is now an arbitrary constant. The constant λ_1 can be defined from the solvability condition of problem (33). Multiplying the equation in (33) by u_0^+ and keeping in mind its normalization in $L_2(\Omega^+)$ yields

$$-\int\limits_{\Omega^+} \Delta u_1^+ u_0^+\,dx = \lambda_0 \int\limits_{\Omega^+} u_1^+ u_0^+\,dx + \lambda_1. \tag{34}$$

Applying two times the Green's formula to the left-hand side of this equation, we obtain

$$-\int\limits_{\Omega^+} \Delta u_1^+ u_0^+ \, dx = \int\limits_{\Omega^+} \nabla u_1^+ \cdot \nabla u_0^+ \, dx - \int\limits_{\partial\Omega^+} \frac{\partial u_1^+}{\partial \nu} u_0^+ \, ds =$$

$$= -\int\limits_{\Omega^+} u_1^+ \Delta u_0^+ \, dx - \int\limits_{\partial\Omega^+} \frac{\partial u_1^+}{\partial \nu} u_0^+ \, ds + \int\limits_{\partial\Omega^+} \frac{\partial u_0^+}{\partial \nu} u_1^+ \, ds =$$

$$= \lambda_0 \int\limits_{\Omega^+} u_1^+ u_0^+ \, dx - q(a) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial u_0^+}{\partial x_2} \right)^2 (x_1) \, dx_1. \tag{35}$$

It follows from (34) and (35) that

$$\lambda_1 = -q(a) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial u_0^+}{\partial x_2} \right)^2 (x_1) \, dx_1 = -q(a) \int\limits_{\Gamma_0} \left(\frac{\partial u_0^+}{\partial \nu} \right)^2 \, ds.$$

To determine uniquely the solution of problem (33), we assume in addition that

$$\int\limits_{\Omega} u_1^+(x) \, u_0^+(x) \, dx = 0.$$

Remark 3.1. *Due to the geometry of* Ω^+ *and the boundary-value problem* (2), *the function* u_0^+ *belongs to* $C^\infty(\overline{\Omega^+})$ *and we have*

$$u_0^+(x) = \alpha_0(x_1) x_2 + \frac{1}{6} \alpha_2(x_1) x_2^3 + O(x_2^5), \quad \textit{as } x_2 \to 0, \tag{36}$$

$$\alpha_0(x_1) = \frac{\partial u_0^+}{\partial x_2}(x_1, 0), \quad \alpha_2(x_1) = -\left(\alpha_0''(x_1) + \lambda_0 \alpha_0(x_1) \right), \tag{37}$$

$$\alpha_j^{(2n)} \left(\pm \frac{1}{2} \right) = 0 \quad \textit{for any } n \geq 0. \tag{38}$$

It follows from (33) *that* $u_1^+ \in C^\infty(\overline{\Omega^+})$ *and we have*

$$u_1^+(x) = q(a) \alpha_0(x_1) + \alpha_1(x_1) x_2 + \frac{1}{2} q(a) \alpha_2(x_1) x_2^2 + O(x_2^4), \quad \textit{as } x_2 \to 0, \tag{39}$$

$$\alpha_1(x_1) = \frac{\partial u_1^+}{\partial x_2}(x_1, 0),$$

and the formula (38) *holds for* $j = 1$.

Denote

$$\widetilde{u}_\varepsilon^+(x) = u_0^+(x) + \varepsilon u_1^+(x), \tag{40}$$
$$\widetilde{\lambda}_\varepsilon = \lambda_0 + \varepsilon\lambda_1. \tag{41}$$

It follows from (2) and (33) that $\widetilde{u}_\varepsilon^+$ belongs to $C^\infty(\overline{\Omega^+})$ and satisfies the boundary-value problem

$$\begin{cases} -\Delta\widetilde{u}_\varepsilon^+ = \widetilde{\lambda}_\varepsilon\widetilde{u}_\varepsilon^+ + \widetilde{f}_\varepsilon^+ \quad \text{in } \Omega^+, \\ \widetilde{u}_\varepsilon^+ = 0 \quad \text{on } \Gamma_1, \end{cases} \tag{42}$$

where

$$\widetilde{f}_\varepsilon^+ = -\varepsilon^2\lambda_1 u_1^+.$$

Obviously,

$$\|\widetilde{f}_\varepsilon^+\|_{L_2(\Omega^+)} = O(\varepsilon^2), \tag{43}$$
$$\|\widetilde{u}_\varepsilon^+\|_{L_2(\Omega^+)} = 1 + o(1). \tag{44}$$

Remark 3.2. *In accordance with* (42) and (43), *the pair* $(\widetilde{u}_\varepsilon^+, \widetilde{\lambda}_\varepsilon)$ *given by* (40) and (41) *is defined to be an asymptotic approximation of the solution of problem* (1) *in* Ω^+.

Let us now consider the domain Ω_ε^-. Introduce the notations (see Figure 2):

$$\begin{aligned}
&\Pi^+ = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, +\infty), \\
&\Pi_a^- = (-a, a) \times (-\infty, 0), \\
&\gamma(a) = (-a, a) \times \{0\}, \\
&\Pi_a = \Pi^+ \cup \Pi_a^- \cup \gamma(a), \\
&\Gamma^+ = \left(\left\{-\frac{1}{2}\right\} \times (0, +\infty)\right) \cup \left(\left\{\frac{1}{2}\right\} \times (0, +\infty)\right), \\
&\Gamma_a^- = \partial\Pi_a \backslash \overline{\Gamma^+}, \\
&\widetilde{\Pi}_a = \overline{\Pi_a} \backslash \left(\{(-a, 0)\} \cup \{(a, 0)\}\right), \\
&\Pi_a(R) = \{\xi \in \Pi_a : \xi_2 < R\}, \\
&\widetilde{\Pi}_a(R) = \left\{\xi \in \widetilde{\Pi}_a : \xi_2 < R\right\}.
\end{aligned}$$

Consider the boundary-value problem

$$\begin{cases} \Delta_\xi X = 0 \quad \text{in } \Pi_a, \\ X = 0 \quad \text{on } \Gamma_a^-, \\ \dfrac{\partial X}{\partial \xi_1} = 0 \quad \text{on} \quad \Gamma^+. \end{cases} \tag{45}$$

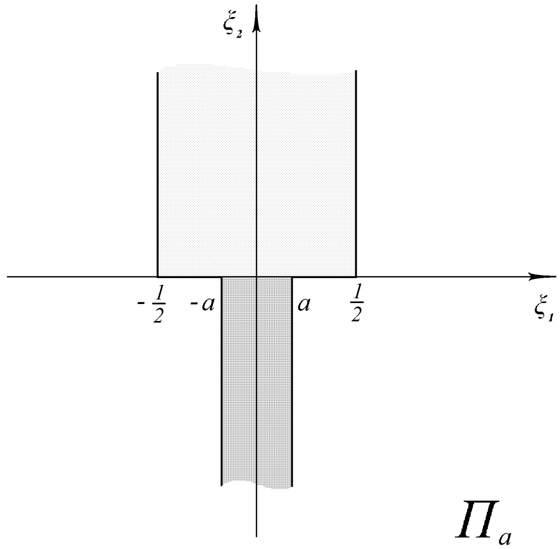

Figure 2: Cell of periodicity.

Let us show problem (45) has a solution X belonging to $C^\infty(\widetilde{\Pi}_a(R)) \cap H^1(\Pi_a(R))$ for any $R > 0$, even with respect to ξ_1 and which has the differentiable asymptotics

$$\begin{aligned} \partial_\xi^\beta \left(X(\xi) - \xi_2 - q(a)\right) =& O\left(e^{-2\pi\xi_2}\right), \quad \xi_2 \to \infty, \\ \partial_\xi^\beta X(\xi) =& O\left(e^{\frac{\pi}{a}\xi_2}\right), \quad \xi_2 \to -\infty, \end{aligned} \tag{46}$$

where the constant $q(a)$ is defined by formula (11). For this purpose we introduce the function

$$\begin{aligned} \mathcal{F}(z) =& \ln\left[8a^2z - 1 + \sqrt{(8a^2z-1)^2 - 1}\right] \\ =& \ln\left[8a^2z - 1 + 4a\sqrt{z\left(4a^2z-1\right)}\right], \end{aligned} \tag{47}$$

where $z = \eta_1 + \mathrm{i}\eta_2$ is a complex variable. It follows from (47) that the function

$$Y(\eta) = \frac{a}{\pi}\operatorname{Re}\mathcal{F}(z) \tag{48}$$

satisfies the boundary-value problem (see, for instance, [14])

$$\begin{cases} \Delta_\eta Y = 0, \quad \eta_2 > 0, \\ Y(\eta_1, 0) = 0, \quad 0 < \eta_1 < \dfrac{1}{2a^2}, \\ \dfrac{\partial Y}{\partial \eta_2}(\eta_1, 0) = 0, \quad \eta_1 \in (-\infty, 0) \cup (\dfrac{1}{2a^2}, \infty). \end{cases} \tag{49}$$

Let $w = \xi_1 + \mathrm{i}\xi_2$ and

$$\mathcal{W}(z) = \frac{\mathrm{i}a}{\pi} \int\limits_1^z \sqrt{\frac{s-1}{4a^2 s - 1}} \frac{ds}{s} + a. \tag{50}$$

It is known (see, for instance, [21, Ch. II, Section 3]) that $w = \mathcal{W}(z)$ is a conformal mapping of the half-plane $\eta_2 > 0$ to the right half of the domain Π_a (see Figure 3). In addition, the point B_j on the plane z goes to the points b_i on the plane w, respectively. Then, according to (48) and (49), the

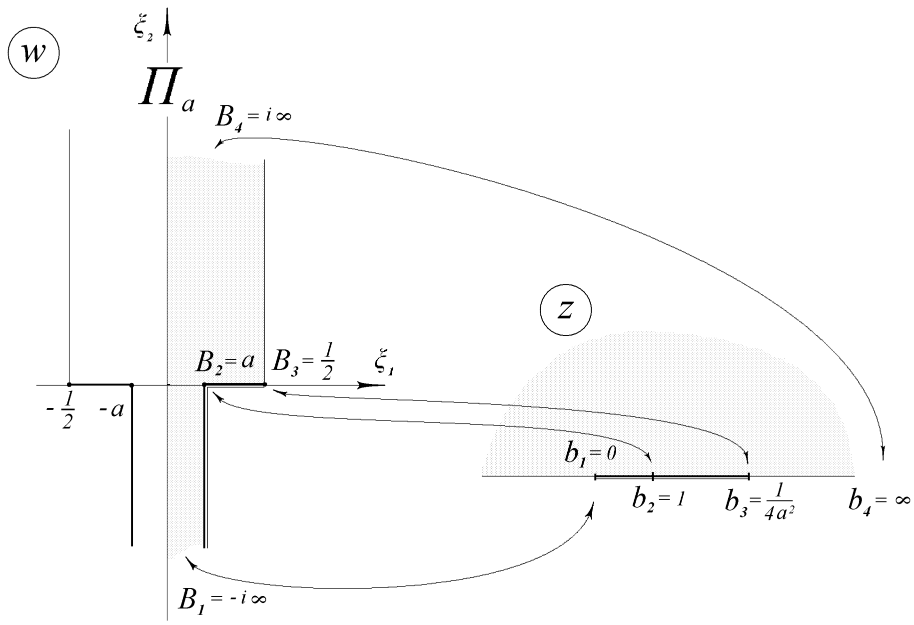

Figure 3: Conformal mapping.

even extension in $\xi_1 < 0$ of the function

$$\frac{a}{\pi} \operatorname{Re} \mathcal{F}\left(\mathcal{W}^{-1}(w)\right)$$

is a solution of problem (45). We denote this function by $X(\xi)$, and it follows from (49) and (50) that $X \in C^\infty(\widetilde{\Pi}_a(R)) \cap H^1(\Pi_a(R))$ for any $R > 0$ and $X(\xi)$ has differentiable asymptotics at infinity.

Due to well-known results for harmonic functions defined in a semi-infinite strip, to prove (46) it suffices to show that

$$X(\xi) = \xi_2 + q(a) + o(1), \quad \text{as } \xi_2 \to \infty, \tag{51}$$

$$X(\xi) = o(1), \quad \text{as } \xi_2 \to -\infty. \tag{52}$$

Since, according to the mapping under consideration, the passage $w \to \infty$ in the lower semi-infinite strip Π_a^- corresponds to the convergence $z \to 0$, then (52) follows from (47) and (48). Further, it follows from (47) and (50) that in the half-plane $\eta_2 > 0$, the expansions

$$\begin{aligned} \mathcal{F}(z) &= \ln z + 2(2\ln 2 + \ln a) + O(z^{-1})s \quad \text{as } z \to \infty, \\ -\frac{\mathrm{i}\pi}{a}\mathcal{W}(z) &= \ln z + C(a) - \mathrm{i}\pi + O(z^{-1}) \quad \text{as } z \to \infty, \end{aligned} \tag{53}$$

hold, where

$$C(a) = \int\limits_1^\infty \left(\sqrt{\frac{s-1}{4a^2 s - 1}} - 1\right)\frac{ds}{s}.$$

We infer from (53) that

$$\begin{aligned} \mathcal{F}(\mathcal{W}^{-1}(w)) = &-\frac{\mathrm{i}\pi}{a}w + 2(2\ln 2 + \ln a) - C(a) + \mathrm{i}\pi \\ &+ o(1), \quad \text{as } w \to \infty,\ \xi_2 > 0. \end{aligned} \tag{54}$$

Taking the imaginary part of (54), we obtain (51) with

$$q(a) = \frac{a}{\pi}\Big(2(2\ln 2 + \ln a) - c(a)\Big) \tag{55}$$

and

$$c(a) = \int\limits_{\frac{1}{4a^2}}^\infty \left(\sqrt{\frac{s-1}{4a^2 s - 1}} - 1\right)\frac{ds}{s}.$$

Calculating the integral, we get

$$c(a) = 2\ln a + [(1-2a)\ln(1-2a) + (1+2a)\ln(1+2a)] \tag{56}$$

and formula (11) follows from (55) and (56). Hereinafter we assume that $q(a)$ is defined by formula (11). Note that the solution X is even in ξ_1 and can be extended to a 1-periodic function in ξ_1. Later on we use the same notation X for the extension.

Then, let us consider the boundary-value problems:

$$\begin{cases} \Delta_\xi \widetilde{X} = \frac{\partial X}{\partial \xi_1} \ \text{ in } \ \Pi_a, \\ \widetilde{X} = 0 \ \text{ on } \ \partial\Pi_a, \end{cases} \tag{57}$$

$$\begin{cases} \Delta_\xi X_1 = \frac{\partial \widetilde{X}}{\partial \xi_1} \ \text{ in } \ \Pi_a, \\ X_1 = 0 \quad \text{on } \Gamma_a^-, \\ \dfrac{\partial X_1}{\partial \xi_1} = 0 \ \text{ on } \quad \Gamma^+, \end{cases} \tag{58}$$

$$\begin{cases} \Delta_\xi X_2 = X \quad \text{in } \Pi_a, \\ X_2 = 0 \ \text{ on } \Gamma_a^-, \\ \dfrac{\partial X_2}{\partial \xi_1} = 0 \ \text{ on } \Gamma^+. \end{cases} \tag{59}$$

Arguing as in [4] and [15], we can prove that there exists a constant $0 < c < \frac{\pi}{a}$ such that problems (57)–(59) have solutions in $C^\infty(\widetilde{\Pi}_a(R)) \cap H^1(\Pi_a(R))$ for any $R > 0$, with the differentiable asymptotics

$$
\begin{aligned}
&\partial_\xi^\beta \widetilde{X}(\xi) = O\left(e^{\mp c\xi_2}\right) \quad \text{as } \xi_2 \to \pm\infty, \\
&\partial_\xi^\beta \left(X_1(\xi) - q_1(a)\right) = O\left(e^{-c\xi_2}\right), \quad \xi_2 \to \infty, \\
&\partial_\xi^\beta \left(X_2(\xi) - \frac{1}{6}\xi_2^3 - \frac{1}{2}q(a)\xi_2 - q_2(a)\right) = O\left(e^{-c\xi_2}\right), \quad \text{as } \xi_2 \to \infty, \\
&\partial_\xi^\beta X_j(\xi) = O\left(e^{c\xi_2}\right), \quad \text{as } \xi_2 \to -\infty,
\end{aligned}
\tag{60}
$$

where $q_j(a)$ denote some constants. Due to the evenness of the function X, $\widetilde{X}$ is odd in ξ_1, X_j is even in ξ_1 and thus $\widetilde{X}$ and X_j have 1-periodic extensions in ξ_1 for which we keep the same notations $\widetilde{X}$, X_j.

Consider now the functions defined by

$$
\begin{aligned}
v_1(\xi; x_1) =& \alpha_0(x_1)X(\xi), \\
v_2(\xi; x_1) =& \alpha_1(x_1)X(\xi) - 2\alpha_0'(x_1)\widetilde{X}(\xi), \\
v_3(\xi; x_1) =& \alpha_2(x_1)X_2(\xi) + 4\alpha_0''(x_1)X_1(\xi) - 2\alpha_1'(x_1)\widetilde{X}(\xi), \\
\widetilde{v}_\varepsilon(\xi; x_1) =& \varepsilon v_1(\xi; x_1) + \varepsilon^2 v_2(\xi; x_1) + \varepsilon^3 v_3(\xi; x_1).
\end{aligned}
\tag{61}
$$

In view of (38) and the boundary conditions in (57) we have

$$
\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right) = 0, \quad \text{as } x_1 = \pm\frac{1}{2}.
$$

Denote

$$
\widetilde{\Omega}_\varepsilon = \overline{\Omega_\varepsilon} \setminus \left(\bigcup_{j=-\mathcal{N}}^{\mathcal{N}} \left\{-\varepsilon a + \varepsilon\frac{j}{2}\right\} \times \{0\} \bigcup_{j=-\mathcal{N}}^{\mathcal{N}} \left\{\varepsilon a + \varepsilon\frac{j}{2}\right\} \times \{0\} \right).
$$

We easily verify that the function $\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right)$ belongs to $C^\infty(\widetilde{\Omega}^\varepsilon) \cap H^1(\Omega^\varepsilon)$. Moreover, since in (ξ, x_1) variables the Laplacian has the form

$$
\Delta = \varepsilon^{-2}\Delta_\xi + 2\varepsilon^{-1}\frac{\partial^2}{\partial x_1 \partial \xi_1} + \frac{\partial^2}{\partial x_1^2},
\tag{62}
$$

using (45), (57)–(59), (61) and (62), we easily verify that, for fixed $r > 0$ and sufficiently small such that Γ_1 coincides with the straight lines $x_1 = \pm\frac{1}{2}$ as $0 < x_2 < r$, the function $\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right)$ satisfies the boundary-value-problem

$$
\left\{
\begin{array}{l}
-\Delta\widetilde{v}_\varepsilon = \widetilde{\lambda}_\varepsilon\widetilde{v}_\varepsilon + \widetilde{f}_\varepsilon^- \quad \text{in } \Omega^\varepsilon, \\
\widetilde{v}_\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \cap \left((-\infty, \infty) \times (-h, r)\right),
\end{array}
\right.
\tag{63}
$$

where

$$\widetilde{f}_\varepsilon^-(x) = -\varepsilon^2\Bigg(\left(\frac{\partial^2}{\partial x_1^2} + \widetilde{\lambda}_\varepsilon \right) (v_2(\xi; x_1) + \varepsilon v_3(\xi; x_1)) \\ + 2\frac{\partial^2}{\partial x_1 \partial \xi_1} v_3(\xi; x_1) \Bigg)\Bigg|_{\xi = \frac{x}{\varepsilon}}. \tag{64}$$

Moreover, it follows from (60) and (61) that

$$\|\widetilde{f}_\varepsilon^-\|_{L_2(\Omega_\varepsilon^-)} = O\left(\varepsilon^{\frac{5}{2}}\right). \tag{65}$$

Remark 3.3. *According to (63) and (65), the pair $(\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right), \widetilde{\lambda}_\varepsilon)$ given by (61) and (41) is defined to be an asymptotic approximation of the solution of problem* (1) *in* Ω_ε^-.

4 Proof of Theorem 1.3

Consider the pairs of asymptotic approximations $(\widetilde{u}_\varepsilon^+, \widetilde{\lambda}_\varepsilon)$ and $(\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right), \widetilde{\lambda}_\varepsilon)$, where $\widetilde{u}_\varepsilon^+$ is defined by (40), $\widetilde{\lambda}_\varepsilon$ is defined by (41) and $\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right)$ is defined by (61), see Remark 3.2 and Remark 3.3. Using (36) and (39) we can write

$$\widetilde{u}_\varepsilon^+(x) = V_\varepsilon^+(x) + O(\varepsilon x_2^3 + x_2^5), \quad \text{as } x_2 \to 0, \tag{66}$$

where

$$V_\varepsilon^+(x) = \alpha_0(x_1)x_2 + \frac{1}{6}\alpha_2(x_1)x_2^3 \\ + \varepsilon\left(q(a)\alpha_0(x_1) + \alpha_1(x_1)x_2 + \frac{1}{2}\alpha_2(x_1)x_2^2\right). \tag{67}$$

Note that, due to (38),

$$V_\varepsilon^+\left(\pm\frac{1}{2}, x_2\right) = 0. \tag{68}$$

Then, denote

$$\widetilde{u}_\varepsilon(x) = \begin{cases} \widetilde{u}_\varepsilon^+(x) - V_\varepsilon^+(x) & \text{in } \Omega^+, \\ 0 \ \text{ in } \Omega_\varepsilon^-. \end{cases} \tag{69}$$

Due to (66), the function $\widetilde{u}_\varepsilon$ belongs to $H^2(\Omega^\varepsilon)$ and then the function defined in Ω^ε by

$$\widetilde{U}_\varepsilon(x) = \widetilde{u}_\varepsilon(x) + \widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right) \tag{70}$$

belongs to $H^1(\Omega^\varepsilon)$ and $H^2(Q)$ for any subdomain Q of Ω^ε, separated from the angels equal to $\frac{3}{2}\pi$ in Ω^ε. According to (42), the function $\widetilde{u}_\varepsilon$ satisfies the boundary-value problem

$$\begin{cases} -\Delta\widetilde{u}_\varepsilon = \widetilde{\lambda}_\varepsilon\widetilde{u}_\varepsilon + \widetilde{f}_\varepsilon & \text{in } \Omega^\varepsilon, \\ \widetilde{u}_\varepsilon = -V_\varepsilon^+ & \text{on } \Gamma_1, \\ \widetilde{u}_\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \end{cases}$$

where

$$\widetilde{f}_\varepsilon(x) = \begin{cases} \widetilde{f}_\varepsilon^{+}(x) + \left(\Delta + \widetilde{\lambda}_\varepsilon\right) V_\varepsilon^{+}(x) & \text{in } \Omega^{+}, \\ 0 \quad \text{in } \Omega_\varepsilon^{-}. \end{cases} \tag{71}$$

Let us now formulate a boundary-value problem in Ω^ε satisfied by $\widetilde{U}_\varepsilon$. For this we introduce, for $\xi_2 > 0$, the functions

$$\begin{aligned} X^{+}(\xi) &= X(\xi) - \xi_2 - q(a), \\ X_1^{+}(\xi) &= X_1(\xi) - q_1(a), \\ X_2^{+} &= X_2(\xi) - \frac{1}{6}\xi_2^3 - \frac{1}{2}q(a)\xi_2 - q_2(a), \end{aligned} \tag{72}$$

and

$$\begin{aligned} v_1^{+}(\xi; x_1) =& \alpha_0(x_1) X^{+}(\xi), \\ v_2^{+}(\xi; x_1) =& \alpha_1(x_1) X^{+}(\xi) - 2\alpha_0'(x_1)\widetilde{X}(\xi), \\ v_3^{+}(\xi; x_1) =& \alpha_2(x_1) X_2^{+}(\xi) + 4\alpha_0''(x_1) X_1^{+}(\xi) - 2\alpha_1'(x_1)\widetilde{X}(\xi), \\ \widetilde{v}_\varepsilon^{+}(\xi; x_1) =& \varepsilon v_1^{+}(\xi; x_1) + \varepsilon^2 v_2^{+}(\xi; x_1) + \varepsilon^3 v_3^{+}(\xi; x_1). \end{aligned} \tag{73}$$

Due to (46), (60), (72) and (73), the function $\widetilde{v}_\varepsilon^{+}(\xi; x_1)$ has the differentiable asymptotics

$$\partial_{\xi x_1}^{\beta} \widetilde{v}_\varepsilon^{+}(\xi; x_1) = O\left(e^{-c\xi_2}\right), \quad \text{as } \xi_2 \to \infty, \tag{74}$$

and due to (67), we have in Ω^{+}

$$\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}; x_1\right) = \widetilde{v}_\varepsilon^{+}\left(\frac{x}{\varepsilon}; x_1\right) + V_\varepsilon^{+}(x) + \widetilde{V}_\varepsilon^{+}(x), \tag{75}$$

where V_ε^{+} is defined by (67) and

$$\widetilde{V}_\varepsilon^{+}(x) = \varepsilon^2 q(a)\alpha_1(x_1) + \varepsilon^3\big(\alpha_2(x_1) q_2(a) + 4\alpha_0''(x_1) q_1(a)\big). \tag{76}$$

Denote

$$\begin{aligned} \widehat{f}_{\varepsilon,1}^{+} &= -(\Delta + \widetilde{\lambda}^\varepsilon)\widetilde{v}_\varepsilon^{+}, \\ \widehat{f}_{\varepsilon,2}^{+} &= -\varepsilon^2\left(\frac{d^2}{dx_1^2} + \widetilde{\lambda}^\varepsilon\right)\widetilde{V}_\varepsilon^{+}(x). \end{aligned} \tag{77}$$

By means of the equation in (63) and formulae (64) and (74), we have

$$\begin{aligned} \widehat{f}_{\varepsilon,1}^{+}(x) = -\varepsilon^2\Bigg(&\left(\frac{\partial^2}{\partial x_1^2} + \widetilde{\lambda}_\varepsilon\right)\left(v_2^{+}(\xi; x_1) + \varepsilon v_3^{+}(\xi; x_1)\right) \\ &+ 2\frac{\partial^2}{\partial x_1 \partial \xi_1} v_3^{+}(\xi; x_1)\Bigg)\Bigg|_{\xi = \frac{x}{\varepsilon}}. \end{aligned} \tag{78}$$

Due to (75)–(78), the formula (64) in Ω^{+} could be rewritten in the form

$$\widetilde{f}_\varepsilon^{-}(x) = -(\Delta + \widetilde{\lambda}^\varepsilon) V_\varepsilon(x) + \widehat{f}_\varepsilon^{+} \tag{79}$$

where

$$\widehat{f}_\varepsilon^+ = \widehat{f}_{\varepsilon,1}^+ + \widehat{f}_{\varepsilon,2}^+. \tag{80}$$

Now, we conclude with (63), (68), (70), (71) and (79) that the function $\widetilde{U}_\varepsilon$ satisfies the boundary-value problem

$$\begin{cases} -\Delta\widetilde{U}_\varepsilon = \widetilde{\lambda}_\varepsilon\widetilde{U}_\varepsilon + \widetilde{F}_\varepsilon \ \text{ in } \ \Omega^\varepsilon, \\ \widetilde{U}_\varepsilon = 0 \ \text{ on } \ \Gamma^\varepsilon\backslash\left((-\infty,\infty)\times\{-h\}\right), \\ \widetilde{U}_\varepsilon = \widetilde{v}_\varepsilon \ \text{ on } \ \Gamma^\varepsilon\cap\left((-\infty,\infty)\times\{-h\}\right), \\ \widetilde{U}_\varepsilon = \widetilde{V}_\varepsilon^+ + \widetilde{v}_\varepsilon^+ \ \text{ on } \ \Gamma_1\backslash\left((-\infty,\infty)\times(0,r)\right), \\ \widetilde{U}_\varepsilon = 0 \ \text{ on } \ \Gamma_1\cap\left((-\infty,\infty)\times(0,r)\right), \end{cases} \tag{81}$$

where

$$\widetilde{F}_\varepsilon = \begin{cases} \widetilde{f}_\varepsilon^+ + \widehat{f}_\varepsilon^+ \quad \text{in } \Omega^+, \\ \widetilde{f}_\varepsilon^- \quad \text{in } \Omega_\varepsilon^-, \end{cases} \tag{82}$$

and $r > 0$ is an arbitrary number, small enough so that Γ_1 coincides with the straight lines $x_1 = \pm\frac{1}{2}$, as $0 < x_2 < r$.

It follows from (76) and (77) that

$$\|\widehat{f}_{\varepsilon,2}^+\|_{L_2(\Omega^+)} = O\left(\varepsilon^2\right),$$

and from (74) and (78) that

$$\|\widehat{f}_{\varepsilon,1}^+\|_{L_2(\Omega^+)} = O\left(\varepsilon^{\frac{5}{2}}\right).$$

Then we infer from (80), (82), (43) and (65) that

$$\|\widetilde{F}_\varepsilon\|_{L_2(\Omega^\varepsilon)} = O\left(\varepsilon^2\right).$$

Note that, according to (69), (70) and (75), we have

$$\widetilde{U}_\varepsilon = \begin{cases} \widetilde{u}_\varepsilon^+ + \widetilde{v}_\varepsilon^+ + \widetilde{V}_\varepsilon^+ \ \text{ in } \Omega^+, \\ \widetilde{v}_\varepsilon \ \text{ in } \Omega_\varepsilon^-. \end{cases} \tag{83}$$

We deduce from the definition (76) that

$$\|\widetilde{V}_\varepsilon^+\|_{L_2(\Omega^+)} = O\left(\varepsilon^2\right), \tag{84}$$

and (74) and (75) imply that

$$\|\widetilde{v}_\varepsilon^+\|_{L_2(\Omega^+)} = O\left(\varepsilon^{\frac{3}{2}}\right).$$

Then we derive from (83) and (44) that

$$\|\widetilde{U}_\varepsilon\|_{L_2(\Omega^+)} = 1 + o(1). \tag{85}$$

Now, let $\chi(s)$ be a smooth cut-off function, equals to zero as $s<1$ and equals to one as $s>2$. Denote

$$\widehat{U}_\varepsilon(x)=\widetilde{U}_\varepsilon(x)-\widetilde{U}_{\varepsilon,1}(x)-\widetilde{U}_{\varepsilon,2}(x) \tag{86}$$

where

$$\begin{aligned}\widetilde{U}_{\varepsilon,1}(x)=&\left(\widetilde{v}_\varepsilon^{+}\left(\frac{x}{\varepsilon};x_1\right)+\widetilde{V}_\varepsilon^{+}(x)\right)\chi\left(\frac{2x_2}{r}\right),\\ \widetilde{U}_{\varepsilon,2}(x)=&\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon};x_1\right)\left(1-\chi\left(\frac{4(x_2+h)}{h}\right)\right).\end{aligned} \tag{87}$$

Hence, due to (81) the function $\widehat{U}_\varepsilon$ is a solution of the boundary-value problem

$$\begin{cases} -\Delta\widehat{U}_\varepsilon=\widetilde{\lambda}_\varepsilon\widehat{U}_\varepsilon+\widehat{F}_\varepsilon \quad \text{in } \Omega^\varepsilon,\\ \widetilde{U}_\varepsilon=0 \quad \text{on } \partial\Omega^\varepsilon,\end{cases} \tag{88}$$

where

$$\widehat{F}_\varepsilon=\widetilde{F}_\varepsilon+\widetilde{F}_{\varepsilon,1}+\widetilde{F}_{\varepsilon,2}, \tag{89}$$

$$\widetilde{F}_{\varepsilon,j}(x)=\left(\Delta+\widetilde{\lambda}_\varepsilon\right)\widetilde{U}_{\varepsilon,j}. \tag{90}$$

Using (46), (60), (61), (74), (76), (87) and (90), we deduce that

$$\begin{aligned}\widetilde{U}_{\varepsilon,1}=\widetilde{F}_{\varepsilon,1}\equiv 0 \ \text{ in } \Omega_\varepsilon^{-}, \quad \widetilde{U}_{\varepsilon,2}=\widetilde{F}_{\varepsilon,2}\equiv 0 \ \text{ in } \Omega^{+},\\ \|\widetilde{U}_{\varepsilon,1}\|_{H^1(\Omega^\varepsilon)}+\|\widetilde{F}_{\varepsilon,1}\|_{L_2(\Omega^\varepsilon)}=&O(\varepsilon^2),\\ \|\widetilde{U}_{\varepsilon,2}\|_{H^1(\Omega^\varepsilon)}+\|\widetilde{F}_{\varepsilon,2}\|_{L_2(\Omega^\varepsilon)}=&O(\varepsilon^N), \ \ \forall\, N.\end{aligned}$$

Finally, we deduce from (85)–(87) and (89) that

$$\|\widehat{F}_\varepsilon\|_{L_2(\Omega^\varepsilon)}=O\left(\varepsilon^2\right), \tag{91}$$

and

$$\|\widehat{U}_\varepsilon-\widetilde{U}_\varepsilon\|_{H^1(\Omega^\varepsilon)}=O\left(\varepsilon^2\right), \tag{92}$$

$$\|\widehat{U}_\varepsilon\|_{L_2(\Omega^+)}=1+o(1). \tag{93}$$

Applying Lemma 1.1 (i) with $p=1$, $\lambda=\widetilde{\lambda}_\varepsilon$, $F_\varepsilon=\widehat{F}_\varepsilon$ and $U_\varepsilon=\widehat{U}_\varepsilon$, we obtain

$$|\lambda_\varepsilon-\widetilde{\lambda}_\varepsilon|\le C\,\frac{\|\widehat{F}_\varepsilon\|_{L_2(\Omega^\varepsilon)}}{\|\widehat{U}_\varepsilon\|_{L_2(\Omega^\varepsilon)}}.$$

From this inequality, (91) and (93) it then follows that

$$\lambda_\varepsilon=\widetilde{\lambda}_\varepsilon+O\left(\varepsilon^2\right)$$

which with (41) gives the estimate (9). Theorem 1.3 is completely proved.

5 Asymptotic approximation of the eigenfunction

Denote

$$\widetilde{u}_{\varepsilon,0}(x) = \begin{cases} u_0^+(x) + \varepsilon u_1^+ + \varepsilon v_1^+\left(\frac{x}{\varepsilon}; x_1\right) & \text{in } \Omega^+, \\ \varepsilon v_1\left(\frac{x}{\varepsilon}; x_1\right) & \text{in } \Omega_\varepsilon^-, \end{cases} \tag{94}$$

and

$$\widetilde{u}_{\varepsilon,1}(x) = \begin{cases} u_0^+(x) + \varepsilon u_1^+ + \varepsilon v_1^+\left(\frac{x}{\varepsilon}; x_1\right) + \varepsilon^2 v_2^+\left(\frac{x}{\varepsilon}; x_1\right) & \text{in } \Omega^+, \\ \varepsilon v_1\left(\frac{x}{\varepsilon}; x_1\right) + \varepsilon^2 v_2\left(\frac{x}{\varepsilon}; x_1\right) & \text{in } \Omega_\varepsilon^-. \end{cases} \tag{95}$$

We observe that $\widetilde{u}_{\varepsilon,0} \in H^1(\Omega^\varepsilon)$ and $\widetilde{u}_{\varepsilon,1} \notin H^1(\Omega^\varepsilon)$ since it has a jump as $x_2 = 0$. We also verify that

$$\|\widetilde{u}_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)} \to 1, \quad \text{as } \varepsilon \to 1.$$

Set

$$u_{\varepsilon,j} = \frac{\widetilde{u}_{\varepsilon,j}}{\|\widetilde{u}_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)}}, \quad j = 0, 1. \tag{96}$$

We have the following result.

Theorem 5.1. *Let u_ε be an eigenfunction, normalized in $L_2(\Omega^\varepsilon)$ and corresponding to the eigenvalue λ_ε and let $u_{\varepsilon,j}$ $(j = 0, 1)$ the normalized functions defined by (94)–(96). We have*

$$\|u_\varepsilon - u_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)} = O(\varepsilon^2),$$
$$\|u_\varepsilon - u_{\varepsilon,1}\|_{H^1(\Omega^+)} + \|u_\varepsilon - u_{\varepsilon,1}\|_{H^1(\Omega_\varepsilon^-)} = O(\varepsilon^2).$$

Proof. It consists in two steps.

(i) For given $g_\varepsilon \in L^2(\Omega^\varepsilon)$, let us denote

$$g_\varepsilon^\perp = g_\varepsilon - (g_\varepsilon, u_\varepsilon)u_\varepsilon$$

where $(\cdot,\cdot)$ is the inner product in $L_2(\Omega^\varepsilon)$. Clearly, $(g_\varepsilon^\perp, u_\varepsilon) = 0$. Then, consider the functions $\widehat{U}_\varepsilon$ and $\widehat{F}_\varepsilon$ defined by (86), (87) and (89), (90), respectively. It follows from Corollary 1 and (93) that

$$(\widehat{U}_\varepsilon, u_\varepsilon) \underset{\varepsilon \to 0}{\to} 1 \tag{97}$$

and due to (88) the function

$$\widehat{U}_\varepsilon^\perp = \widehat{U}_\varepsilon - (\widehat{U}_\varepsilon, u_\varepsilon)u_\varepsilon$$

satisfies problem (3) with $\lambda = \widetilde{\lambda}_\varepsilon$ and

$$F_\varepsilon = \widehat{F}_\varepsilon + (\widehat{U}_\varepsilon, u_\varepsilon)(\lambda_\varepsilon - \widetilde{\lambda}_\varepsilon)u_\varepsilon.$$

It follows from (4), (91) and (97) that

$$\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)} = O\left(\varepsilon^2\right),$$

hence, using estimate (8) we have

$$\|\widehat{U}_\varepsilon^\perp\|_{H^1(\Omega^\varepsilon)} = O\left(\varepsilon^2\right). \tag{98}$$

Due to (83), (84) and (92) we have

$$\begin{aligned}
&\|\widetilde{u}_\varepsilon^+ + \widetilde{v}_\varepsilon^+ - \widehat{U}_\varepsilon\|_{H^1(\Omega^+)} = O(\varepsilon^2),\\
&\|\widetilde{v}_\varepsilon - \widehat{U}_\varepsilon\|_{H^1(\Omega_\varepsilon^-)} = O(\varepsilon^2).
\end{aligned} \tag{99}$$

From the definitions (61), (72) and (73) of v_j and v_j^+ and also from formula (60), we deduce that

$$\begin{aligned}
&\|\varepsilon^3 v_3^+\|_{H^1(\Omega^+)} + \|\varepsilon^3 v_3\|_{H^1(\Omega_\varepsilon^-)} = O\left(\varepsilon^{\frac{5}{2}}\right),\\
&\|\varepsilon^2 v_2^+\|_{L_2(\Omega^+)} + \|\varepsilon^2 v_2\|_{L_2(\Omega_\varepsilon^-)} = O\left(\varepsilon^{\frac{5}{2}}\right).
\end{aligned} \tag{100}$$

Using again the definitions (61) and (73) of $\widetilde{v}_\varepsilon$ and $\widetilde{v}_\varepsilon^+$ and using formulae (99) and (100), we deduce that

$$\begin{aligned}
&\|\widetilde{u}_\varepsilon^+ + \varepsilon v_1^+ - \widehat{U}_\varepsilon\|_{L_2(\Omega^+)} = O(\varepsilon^2),\\
&\|\varepsilon v_1 - \widehat{U}_\varepsilon\|_{L_2(\Omega_\varepsilon^-)} = O(\varepsilon^2),\\
&\|\widetilde{u}_\varepsilon^+ + \varepsilon v_1^+ + \varepsilon^2 v_2^+ - \widehat{U}_\varepsilon\|_{H^1(\Omega^+)} = O(\varepsilon^2),\\
&\|\varepsilon v_1 + \varepsilon^2 v_2 - \widehat{U}_\varepsilon\|_{H^1(\Omega_\varepsilon^-)} = O(\varepsilon^2),
\end{aligned} \tag{101}$$

and finally, we deduce from (98) and (101) that

$$\begin{aligned}
&\|\widetilde{u}_{\varepsilon,0}^\perp\|_{L_2(\Omega^\varepsilon)} = O(\varepsilon^2),\\
&\|\widetilde{u}_{\varepsilon,1}^\perp\|_{H^1(\Omega^+)} + \|\widetilde{u}_{\varepsilon,1}^\perp\|_{H^1(\Omega_\varepsilon^-)} = O(\varepsilon^2).
\end{aligned} \tag{102}$$

(ii) It is clear, since u_ε and $u_{\varepsilon,j}$ $(j = 0, 1)$ are normalized in $L_2(\Omega^\varepsilon)$, that

$$|(u_{\varepsilon,j}, u_\varepsilon)| \leq 1. \tag{103}$$

Multiplying both sides of the equality

$$u_{\varepsilon,j}^\perp = u_{\varepsilon,j} - (u_{\varepsilon,j}, u_\varepsilon)u_\varepsilon$$

by $u_{\varepsilon,j}$ and integrating over Ω^ε, we find that

$$\left(u_{\varepsilon,j}^\perp, u_{\varepsilon,j}\right) = 1 - (u_{\varepsilon,j}, u_\varepsilon)^2 \tag{104}$$

then we deduce from (103) and (104) that

$$0 \le 1 - (u_{\varepsilon,j}, u_\varepsilon) \le \|u^\perp_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)} \, \|u_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)} = \|u^\perp_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)},$$

hence, with (102),

$$0 \le 1 - (u_{\varepsilon,j}, u_\varepsilon) = O(\varepsilon^2). \tag{105}$$

Then writing

$$\|u_\varepsilon - u_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)} \le \|u_\varepsilon - u_{\varepsilon,0} + u^\perp_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)} + \|u^\perp_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)},$$

we obtain, according to (102) and (105),

$$\|u_\varepsilon - u_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)} \le 1 - (u_{\varepsilon,0}, u_\varepsilon) + O(\varepsilon^2) = O(\varepsilon^2).$$

As for the estimate of $(u_\varepsilon - u_{\varepsilon,1})$ in the H^1-norm, we write

$$\begin{aligned}
& \|u_\varepsilon - u_{\varepsilon,1}\|_{H_1(\Omega^+)} + \|u_\varepsilon - u_{\varepsilon,1}\|_{H_1(\Omega^-_\varepsilon)} \\
\le \; & \|u_\varepsilon - u_{\varepsilon,1} + u^\perp_{\varepsilon,1}\|_{H_1(\Omega^+)} + \|u^\perp_{\varepsilon,1}\|_{H_1(\Omega^+)} \\
& + \|u_\varepsilon - u_{\varepsilon,1} + u^\perp_{\varepsilon,1}\|_{H_1(\Omega^-_\varepsilon)} + \|u^\perp_{\varepsilon,1}\|_{H_1(\Omega^-_\varepsilon)} \\
\le \; & 2\|u_\varepsilon\|_{H_1(\Omega^\varepsilon)} \left(1 - (u_{\varepsilon,1}, u_\varepsilon)\right) + \|u^\perp_{\varepsilon,1}\|_{H_1(\Omega^+)} + \|u^\perp_{\varepsilon,1}\|_{H_1(\Omega^-_\varepsilon)},
\end{aligned}$$

then, since $\|u_\varepsilon\|_{H_1(\Omega^\varepsilon)}$ is uniformly bounded, using (102) and (105) we obtain

$$\|u_\varepsilon - u_{\varepsilon,1}\|_{H_1(\Omega^+)} + \|u_\varepsilon - u_{\varepsilon,1}\|_{H_1(\Omega^-_\varepsilon)} = O(\varepsilon^2).$$

Theorem 5.1 is proved. □

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