RIEMANN-HILBERT PROBLEM AND MATRIX BIORTHOGONAL POLYNOMIALS

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ABSTRACT. Recently the Riemann-Hilbert problem, with jumps supported on appropriate curves in the complex plane, has been presented for matrix biorthogonal polynomials, in particular non-Abelian Hermite matrix biorthogonal polynomials in the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with coefficients first order matrix polynomials. We will explore this discussion, present some achievements and consider some new examples of weights for matrix biorthogonal polynomials.

1. INTRODUCTION

Matrix extensions of real orthogonal polynomials where first discussed back in 1949 by Krein [47, 48] and thereafter were studied sporadically until the last decade of the XX century, being some relevant papers [13], [42] and [57]. Then, in 1984, Aptekarev and Nikishin, for a kind of discrete Sturm–Liouville operators, solved the corresponding scattering problem in [57], and found that the polynomials that satisfy a relation of the form

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + A^*_{k-1} P_{k-1}(x), \qquad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite measure; i.e., they derived a matrix version of Favard's theorem.

In a period of 20 years, from 1990 to 2010, it was found that matrix orthogonal polynomials (MOP) satisfy, in some cases, properties as do the classical orthogonal polynomials.

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Let us mention, for example, that for matrix versions of Laguerre, Hermite and Jacobi polynomials, i.e., the scalar-type Rodrigues' formula [36, 35] and a second order differential equation [33, 34, 14] has been discussed. It also has been proven [38] that operators of the form $D{=}\partial^2 F_2(t){+}\partial^1 F_1(t){+}\partial^0 F_0$ have as eigenfunctions different infinite families of MOP's. A new family of MOP's satisfying second order differential equations whose coefficients do not behave asymptotically as the identity matrix was found in [14]; see also [17]. We have studied [3, 5] matrix extensions of the generalized polynomials studied in [1, 2]. Recently, in [6], the Christoffel transformation to m,atrix orthogonal polynomials in the real line (MOPRL) have extended to obtaining a new matrix Christoffel formula, and in [7, 8] more general transformations –of Geronimus and Uvarov type– where also considered.

It was 26 years ago, on 1992, when Fokas, Its and Kitaev, in the context of 2D quantum gravity, discovered that certain Riemann-Hilbert problem was solved in terms of orthogonal polynomials in the real line (OPRL), [39]. Namely, it was found that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials in the real line and its Cauchy transforms. Later, Deift and Zhou combined these ideas with a non-linear steepest descent analysis in a series of papers [28, 29, 31, 32] which was the seed for a large activity in the field. To mention just a few relevant results let us cite the study of strong asymptotic with applications in random matrix theory, [28, 30], the analysis of determinantal point processes [25, 26, 49, 50], orthogonal Laurent polynomials [53, 54] and Painlevé equations [45, 27].

In this work we obtain Sylvester systems of differential equations for the orthogonal polynomials and its second kind functions, directly from a Riemann–Hilbert problem, with jumps supported on appropriate curves in the complex plane. The differential properties for the weight function are fundamental. In this case we consider a Sylvester type differential Pearson equation for the matrix of weights. We also study whenever the orthogonal polynomials and its second kind functions are solutions of a second order linear differential operators with matrix eigenvalues. This is done by stating an appropriate boundary value problem for the matrix of weights. In particular, special attention is paid to non-Abelian Hermite biorthogonal polynomials in the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with given first order matrix polynomials coefficients.

2. Riemann–Hilbert problem for Matrix Biorthogonal Polynomials

2.1. **Matrix biorthogonal polynomials**. Let

$$W = \begin{bmatrix} W^{(1,1)} & \dots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \dots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N\times N}$$

be a $N \times N$ matrix of weights with support on a smooth oriented non self-intersecting curve γ in the complex plane $\mathbb{C}$, i.e. $W^{(j,k)}$ is, for each $j, k \in \{1, \dots, N\}$, a complex weight with support on γ. We define the *moment of order n* associated with W as

$$W_n = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^n W(z)\,\mathrm{d}\,z, \qquad\qquad n \in \mathbb{N} := \{0, 1, 2, \dots\}.$$

We say that W is *regular* if $\det\left[W_{j+k}\right]_{j,k=0,\dots n} \neq 0$, $n \in \mathbb{N}$. In this way, we define a *sequence of matrix monic polynomials*, $\{P^{\mathsf{L}}_n(z)\}_{n\in\mathbb{N}}$, *left orthogonal* and *right orthogonal*, $\{P^{\mathsf{R}}_n(z)\}_{n\in\mathbb{N}}$ with respect to a regular matrix measure W, by the conditions,

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P^{\mathsf{L}}_n(z)W(z)z^k\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \tag{1}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^k W(z)P^{\mathsf{R}}_n(z)\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \tag{2}$$

for $k = 0, 1, \dots, n$ and $n \in \mathbb{N}$, where C_n is an nonsingular matrix.

Notice that neither the matrix of weights is requested to be Hermitian nor the curve γ to be the real line, i.e., we are dealing, in principle with nonstandard orthogonality and, consequently, with biorthogonal matrix polynomials instead of orthogonal matrix polynomials.

The matrix of weights induce a sesquilinear form in the set of matrix polynomials $\mathbb{C}^{N\times N}[z]$ given by

$$\langle P, Q\rangle_W := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma P(z)W(z)Q(z)\,\mathrm{d}\,z. \tag{3}$$

Moreover, we say that $\{P^{\mathsf{L}}_n(z)\}_{n\in\mathbb{N}}$ and $\{P^{\mathsf{R}}_n(z)\}_{n\in\mathbb{N}}$ are biorthogonal with respect to a matrix weight functions W if

$$\langle P^{\mathsf{L}}_n, P^{\mathsf{R}}_m\rangle_W = \delta_{n,m}C_n^{-1}, \qquad\qquad n, m \in \mathbb{N}. \tag{4}$$

As the polynomials are chosen to be monic, we can write

$$\begin{aligned} P^{\mathsf{L}}_n(z) &= I_N z^n + p^1_{\mathsf{L},n}z^{n-1} + p^2_{\mathsf{L},n}z^{n-2} + \dots + p^n_{\mathsf{L},n}, \\ P^{\mathsf{R}}_n(z) &= I_N z^n + p^1_{\mathsf{R},n}z^{n-1} + p^2_{\mathsf{R},n}z^{n-2} + \dots + p^n_{\mathsf{R},n}, \end{aligned}$$

with matrix coefficients $p^k_{\mathsf{L},n}, p^k_{\mathsf{R},n} \in \mathbb{C}^{N\times N}$, $k = 0, \dots, n$ and $n \in \mathbb{N}$ (imposing that $p^0_{\mathsf{L},n} = p^0_{\mathsf{R},n} = I$, $n \in \mathbb{N}$). Here $I \in \mathbb{C}^{N\times N}$ denotes the identity matrix.

We define the *sequence of second kind matrix functions* by

$$Q^{\mathsf{L}}_n(z) := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{P^{\mathsf{L}}_n(z')}{z'-z}W(z')\,\mathrm{d}\,z', \tag{5}$$

$$Q^{\mathsf{R}}_n(z) := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma W(z')\frac{P^{\mathsf{R}}_n(z')}{z'-z}\,\mathrm{d}\,z', \tag{6}$$

for $n \in \mathbb{N}$. From the orthogonality conditions (1) and (2) we have, for all $n \in \mathbb{N}$, the following asymptotic expansion near infinity for the sequence of functions of the second kind

$$Q^{\mathsf{L}}_n(z) = -C_n^{-1}(I_N z^{-n-1} + q^1_{\mathsf{L},n} z^{-n-2} + \cdots), \tag{7}$$

$$Q^{\mathsf{R}}_n(z) = -(I_N z^{-n-1} + q^1_{\mathsf{R},n} z^{-n-2} + \cdots)C_n^{-1}. \tag{8}$$

Assuming that the measures $W^{(j,k)}$, $j,k \in \{1,\dots,N\}$ are Hölder continuous we obtain, by the Plemelj's formula applied to (5) and (6), the following fundamental jump identities

$$(Q^{\mathsf{L}}_n(z))_+ - (Q_n(z)^{\mathsf{L}})_- = P^{\mathsf{L}}_n(z)W(z), \tag{9}$$

$$(Q^{\mathsf{R}}_n(z))_+ - (Q^{\mathsf{R}}_n(z))_- = W(z)P^{\mathsf{R}}_n(z), \tag{10}$$

$z \in \gamma$, where, $(f(z))_\pm = \lim_{\epsilon\to 0^\pm} f(z+i\epsilon)$; here $\pm$ indicates the the positive/negative region according to the orientation of the curve γ.

2.2. **Reductions: from biorthogonality to orthogonality**. We consider two possible reductions for the matrix of weights, the symmetric reduction and the Hermitian reduction.

i) A matrix of weights $W(z)$ with support on γ is said to be symmetric if

$$(W(z))^\top = W(z), \qquad z \in \gamma.$$

ii) A matrix of weights $W(x)$ with support on $\mathbb{R}$ is said to be Hermitian if

$$(W(x))^\dagger = W(x), \qquad x \in \mathbb{R}.$$

These two reductions leads to orthogonal polynomials, as the two biorthogonal families are identified; i.e., for the symmetric case

$$P^{\mathsf{R}}_n(z) = (P^{\mathsf{L}}_n(z))^\top, \qquad Q^{\mathsf{R}}_n(z) = (Q^{\mathsf{L}}_n(z))^\top, \qquad z \in \mathbb{C},$$

and for the Hermitian case, with $\gamma = \mathbb{R}$,

$$P^{\mathsf{R}}_n(z) = (P^{\mathsf{L}}_n(\bar z))^\dagger, \qquad Q^{\mathsf{R}}_n(z) = (Q^{\mathsf{L}}_n(\bar z))^\dagger, \qquad z \in \mathbb{C}.$$

In both cases biorthogonality collapses into orthogonality, that for the symmetric case reads as

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P_n(z)W(z)(P_m(z))^\top \,\mathrm{d}\,z = \delta_{n,m}C_n^{-1}, \qquad n,m \in \mathbb{N},$$

while for the Hermitian case can be written as follows

$$\frac{1}{2\pi\,\mathrm{i}}\int_{\mathbb{R}} P_n(x)W(x)(P_m(x))^\dagger \,\mathrm{d}\,x = \delta_{n,m}C_n^{-1}, \qquad n,m \in \mathbb{N},$$

where $P_n = P^{\mathsf{L}}_n$.

2.3. **The Riemann-Hilbert problem.** Let us consider the particular case when the $N\times N$ matrix of weights with support on a smooth oriented non self-intersecting curve γ has entrywise power logarithmic type singularities at the end points of the support of the measure, that is the entries $W^{(j,k)}$ of the matrix measure W can be described as

$$W^{(j,k)}(z) = (z-c)^{\alpha_{j,k}} \log^{p_{j,k}}(z) \tilde{W}^{(j,k)}(z)$$

where $\alpha_{j,k} > -1$, $p_{j,k} \in \mathbb{N}$ and $\tilde{W}_{j,k}(x)$ is Hölder continuous, bounded and non-vanishing on γ.

The biorthogonality can be characterized in terms of a left and right Riemann-Hilbert formulation,

Theorem 1.

i) *The matrix function*

$$Y_n^{\mathsf{L}}(z) := \begin{bmatrix} P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z) \\ -C_{n-1}P_{n-1}^{\mathsf{L}}(z) & -C_{n-1}Q_{n-1}^{\mathsf{L}}(z) \end{bmatrix}$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RHL1): $Y_n^{\mathsf{L}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$*;*

(RHL2): *has the following asymptotic behaviour near infinity,*

$$Y_n^{\mathsf{L}}(z) = (I_N + \sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{L}}) \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix};$$

(RHL3): *satisfies the jump condition*

$$(Y_n^{\mathsf{L}}(z))_+ = (Y_n^{\mathsf{L}}(z))_- \begin{bmatrix} I_N & W(z) \\ 0_N & I_N \end{bmatrix}, \qquad z \in \gamma.$$

(RHL4): $Y_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(s_1^{\mathsf{L}}(z)) \\ \mathrm{O}(1) & \mathrm{O}(s_2^{\mathsf{L}}(z)) \end{bmatrix}$, *as* $z \to c$,

where c denotes any of the end points of the curve γ if they exists,

$$\lim_{z\to c}(z-c)s_j^{\mathsf{L}}(z) = 0_N, j = 1,2$$

and the O *conditions are understood entrywise.*

ii) *The matrix function*

$$Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix}$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RHR1): $Y_n^{\mathsf{R}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$*;*

(RHR2): *has the following asymptotic behaviour near infinity,*

$$Y_n^{\mathsf{R}}(z) = \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix} (I_N + \sum_{j=1}^{\infty} (z^{-j}) Y_n^{j,\mathsf{R}});$$

(RHR3): *satisfies the jump condition*

$$(Y_n^{\mathsf{R}}(z))_+ = \begin{bmatrix} I_N & 0_N \\ W(z) & I_N \end{bmatrix} (Y_n^{\mathsf{R}}(z))_-, \qquad z \in \gamma.$$

(RHR4): $Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(1) \\ \mathrm{O}(s_1^{\mathsf{R}}(z)) & \mathrm{O}(s_2^{\mathsf{R}}(z)) \end{bmatrix}$, *as $z \to c$,*
where c denotes any of the end points of the curve γ if they exists,

$$\lim_{z \to c} (z-c) s_j^{\mathsf{R}}(z) = 0_N, \; j = 1, 2$$

and the O *conditions are understood entrywise.*

ii) *The determinant of $Y_n^{\mathsf{L}}(z)$ and $Y_n^{\mathsf{L}}(z)$ are both equal to* 1, *for every $z \in \mathbb{C}$.*

Proof. Using the standard calculations from the scalar case it follows that the matrices Y_n^{L} and Y_n^{R} satisfy (RHL1) – (RHL3) and (RHR1) – (RHR3) respectively.
The entries $W^{(j,k)}$ of the matrix measure W can be described as

$$W^{(j,k)}(z) = (z-c)^{\alpha_{j,k}} \log^{p_{j,k}}(z) \tilde{W}^{(j,k)}(z)$$

where $\alpha_{j,k} > -1$, $p_{j,k} \in \mathbb{N}$ and $\tilde{W}_{j,k}(x)$ is Hölder continuous, bounded and non-vanishing on γ. At the boundary values of the curve γ if they exists and are denote by c, it holds [41] that in a neighbourhood of the point c, the Cauchy transform of the function

$$\frac{1}{2\pi i} \int_{\gamma} \frac{p(z')(z'-c)^{\alpha_{j,k}} \log^{p_{j,k}}(z') \tilde{W}^{(j,k)}(z')}{z'-z} dz'$$

where $p(z')$ denotes any polynomial in z', behaves like $\mathrm{O}((z-c)^{\alpha_{j,k}} \log^{p_{j,k}}(z)) + \mathrm{O}(z-c)^{\alpha'_{j,k}}$, where $-1 < \alpha'_{j,k} < \alpha_{j,k}$. It follows that

$$\lim_{z \to c} (z-c)(z-c)^{\alpha_{j,k}} \log^{p_{j,k}}(z) = 0$$

and the condition (RHL4), is fulfilled for the matrix Y_n^{L} and respectively the condition (RHR4), is fulfilled for the matrix Y_n^{R}. Now let us consider

$$G(z) = Y_n^{\mathsf{L}}(z) \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix}$$

It can easily be proved that G has no jump on the curve γ. In a neighbourhood of the point c

$$G(z)=\begin{bmatrix}\mathrm{O}(s_1^{\mathsf{L}}(z))+\mathrm{O}(s_2^{\mathsf{R}}(z)) & \mathrm{O}(s_1^{\mathsf{L}}(z))+\mathrm{O}(s_1^{\mathsf{R}}(z))\\ \\ \mathrm{O}(s_2^{\mathsf{L}}(z))+\mathrm{O}(s_2^{\mathsf{R}}(z)) & \mathrm{O}(s_2^{\mathsf{L}}(z))+\mathrm{O}(s_1^{\mathsf{R}}(z))\end{bmatrix}$$

so

$$\lim_{z\to c}(z-c)G(z)=0$$

and at the point c the singularity is removable. Now using the behaviour for $z\to\infty$,

$$G(z)=\begin{bmatrix}I_N z^n & 0_N\\ 0_N & I_N z^{-n}\end{bmatrix}\begin{bmatrix}0_N & I_N\\ -I_N & 0_N\end{bmatrix}\begin{bmatrix}I_N z^n & 0_N\\ 0_N & I_N z^{-n}\end{bmatrix}\begin{bmatrix}0_N & -I_N\\ I_N & 0_N\end{bmatrix}=\begin{bmatrix}I_N & 0_N\\ 0_N & I_N\end{bmatrix}$$

and using Liouville's Theorem it holds that $G(z)=I$, the identity matrix. From this follows the unicity of the solution of each of the Riemann-Hilbert problems stated in this theorem.

Again using the standard arguments as in the scalar case we can conclude that $\det Y_n^{\mathsf{L}}(z)$ and $\det Y_n^{\mathsf{R}}(z)$ are both equal to 1. □

We recover a representation for the inverse matrix $(Y_n^{\mathsf{L}})^{-1}$ given by the following result

Corollary 1. *It holds that*

$$(Y_n^{\mathsf{L}})^{-1}(z)=\begin{bmatrix}0_N & I_N\\ -I_N & 0_N\end{bmatrix}Y_n^{\mathsf{R}}(z)\begin{bmatrix}0_N & -I_N\\ I_N & 0_N\end{bmatrix} \tag{11}$$

Corollary 2. *In the conditions of theorem 1 we have that for all $n\in\mathbb{N}$,*

$$Q_n^{\mathsf{L}}(z)P_{n-1}^{\mathsf{R}}(z)-P_n^{\mathsf{L}}(z)Q_{n-1}^{\mathsf{R}}(z)=C_{n-1}^{-1}, \tag{12}$$

$$P_{n-1}^{\mathsf{L}}(z)Q_n^{\mathsf{R}}(z)-Q_{n-1}^{\mathsf{L}}(z)P_n^{\mathsf{R}}(z)=C_{n-1}^{-1}, \tag{13}$$

$$Q_n^{\mathsf{L}}(z)P_n^{\mathsf{R}}(z)-P_n^{\mathsf{L}}(z)Q_n^{\mathsf{R}}(z)=0. \tag{14}$$

Proof. As we have already proven the matrix

$$\begin{bmatrix}-Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z)\\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z)\end{bmatrix},$$

is the inverse of $Y_n^{\mathsf{L}}(z)$, i.e.

$$Y_n^{\mathsf{L}}(z)\begin{bmatrix}-Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z)\\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z)\end{bmatrix}=I;$$

and multiplying the two matrices we get the result. □

2.4. **Three term recurrence relation**. Following the standard arguments from the Riemann-Hilbert formulation we can prove

$$Y^{\mathsf{L}}_{n+1}(z) = T^{\mathsf{L}}_n(z)Y^{\mathsf{L}}_n(z), \quad n \in \mathbb{N}.$$

where

$$T^{\mathsf{L}}_n(z) = \begin{bmatrix} zI_N - \beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N \end{bmatrix},$$

denotes the left *transfer matrix*

For the right orthogonality, we similarly obtain from (16) that

$$Y^{\mathsf{R}}_{n+1}(z) = Y^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z), \quad n \in \mathbb{N}.$$

where

$$T^{\mathsf{R}}_n(z) = \begin{bmatrix} zI_N - \beta^{\mathsf{R}}_n & -C_n \\ C_n^{-1} & 0_N \end{bmatrix},$$

Hence, we conclude that the sequence of monic polynomials $\{P^{\mathsf{L}}_n(z)\}_{n\in\mathbb{N}}$ satisfies the three term recurrence relations

$$zP^{\mathsf{L}}_n(z) = P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n P^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{N}, \tag{15}$$

with recursion coefficients

$$\beta^{\mathsf{L}}_n := p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1}, \qquad \gamma^{\mathsf{L}}_n := C_n^{-1}C_{n-1},$$

with initial conditions, $P^{\mathsf{L}}_{-1} = 0_N$ and $P^{\mathsf{L}}_0 = I_N$. We can also assert that

$$zP^{\mathsf{R}}_n(z) = P^{\mathsf{R}}_{n+1}(z) + P^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + P^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n \in \mathbb{N}, \tag{16}$$

where

$$\beta^{\mathsf{R}}_n := C_n\beta^{\mathsf{L}}_n C_n^{-1}, \qquad \gamma^{\mathsf{R}}_n := C_n\gamma^{\mathsf{L}}_n C_n^{-1} = C_{n-1}C_n^{-1},$$

3. Matrix weights supported on a curve γ on the complex plane that connects the point 0 to the point ∞: Laguerre weights

Motivated by different attempts that appear in the literature we try to consider some classes of weights with the aim to use the Riemann-Hilbert formulation. In this matrix case it is not so obvious which are the conditions we should to impose in order to guarantee the integrability of the matrix measure that we want to consider. If we consider

3.1. **Matrix weights supported on a curve γ without end points**: $W(z) = z^A H(z)$.

We begin considering the weight $W(z) = z^A H(z)$ supported on a curve γ on the complex plane that connects the point 0 to the point ∞, where

i) The function z^A is defined as $z^A = e^{A \log z}$, where γ is the branch cut of the logarithmic function, and we define por $t \in \gamma$, the $t^A := (z^A)_+$, where $(z^A)_+$ is the non-tangential limit as $z \to t$, from the left side of the oriented curve γ.

ii) A is a constant matrix such that the minimum of the real part of the eigenvalues of the matrix A is greater than -1.

iii) The factor $H(t)$ is the restriction to the curve γ of $H(z)$, a matrix of entire functions, $z \in \mathbb{C}$ such that $H(z)$ is invertible for all $z \in \mathbb{C}$.

iv) The left logarithmic derivative $h(z) := (H(z))^{-1}(H(z))'$ is an entire function.

It is necessary in other to consider the Riemann-Hilbert problem related to the weight function $W(z)$ to clarify the behaviour of this weight function $W(z)$ on a neighbourhood of the point $z = 0$.

If we consider the Jordan decomposition of the matrix A, it holds that there exists an invertible matrix P such that

$$A = PJP^{-1}$$

where $J = D + N$, where D is the diagonal matrix formed whose entries are the eigenvalues of the matrix A and N is a nilpotent matrix that commutes with the matrix D. This commutation permits to obtain

$$z^A = z^{PJP^{-1}} = Pz^J P^{-1} = Pz^D z^N P^{-1}$$

where z^N is a polynomial in the variable $\log z$. The matrix z^D is a diagonal matrix whose entries are of the form $z^{\alpha_j + i\beta_j}$, where $\alpha_j + i\beta_j$ is a eigenvalue of the matrix A. Let us consider as an example the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case

$$z^A = \begin{bmatrix} z^{-\frac{1}{2}} & 0 & 0 \\ 0 & z^{-\frac{1}{2}} & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & \log z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So we observe that the matrix A is not diagonalizable, appears factors in the behaviour near 0, such as $z^{-\frac{1}{2}} \log z$, so it appears a power logarithmic type singularity. To assure the integrability of this kind of measure it is enough to ask that $\alpha > -1$, where is the minimum of the real part of the eigenvalues of the matrix A. So in this case for this weight we are in the conditions of the theorem (1).

It is also valuable to comment about the factor $H(z)$ of the measure $W(z)$. In other also to have integrability of this matrix weight function we should be careful. If for example we consider $H(z) = e^{Bz}$, then it is clear that, by reasoning similarly as before, that we should impose that the real part of the eigenvalues of the matrix B are negative. If we consider $h(z) := (H(z))^{-1}(H(z))'$ to be a matrix polynomial $h(z) = B_0 + B_1 z + \dots B_m z^m$, it should be enough in order to guarantee integrability of the measure, to impose that the real part of the eigenvalues of the matrix B_m to be negative.

3.2. **Matrix weights supported on a curve γ without end points**: $W(z) = z^\alpha H(z)G(z)z^B$.

In [34] appears different examples of Laguerre matrix weights for the Matrix Orthogonal polynomials on the real line. This motivates to consider the matrix weight We begin considering the weight $W(z) = z^\alpha H(z)G(z)z^B$ supported on a curve γ on the complex plane that connects the point 0 to the point ∞. with similar considerations as in the case treated before. Nevertheless when we try to apply the general methods from the Riemann-Hilbert formulation we find a lot of dificulties, derived of the non-commutativity of the matrix product and we should impose important restrictions.

This kind of matrix weights can be treating in a more general context. If we consider instead of a given matrix of weights we are provided with two matrices, say $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, of entire functions such that the following two matrix Pearson equations are satisfied

$$z\frac{\mathrm{d}\, W^{\mathsf{L}}}{\mathrm{d}\, z} = h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \tag{17}$$

$$z\frac{\mathrm{d}\, W^{\mathsf{R}}}{\mathrm{d}\, z} = W^{\mathsf{R}}(z)h^{\mathsf{R}}(z); \tag{18}$$

and given solutions to them we construct the corresponding matrix of weights $W = W^{\mathsf{L}}W^{\mathsf{R}}$. Moreover, this matrix of weights is also characterized by a Pearson equation.

Proposition 1 (Pearson Sylvester differential equation). *Given two matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, any solution of the Sylvester type matrix differential equation, which we call Pearson equation for the weight,*

$$z\frac{\mathrm{d}\, W}{\mathrm{d}\, z} = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z) \tag{19}$$

is of the form $W = W^{\mathsf{L}}W^{\mathsf{R}}$ where the factor matrices W^{L} and W^{R} are solutions of (17) *and* (18)*, respectively.*

Proof. Given solutions W^{L} and W^{R} of (17) and (18), respectively, it follows intermediately, just using the Leibniz law for derivatives, that $W = W^{\mathsf{L}}W^{\mathsf{R}}$ fulfills (19). Moreover, given a solution W of (19) we pick a solution W^{L} of (17), then it is easy to see that $(W^{\mathsf{L}})^{-1}W$ satisfies (18). □

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