Packing of R^3 by Crosses

Catarina M. N. Cruz
University of Aveiro
Campus Universitário de Santiago
3810-183 Aveiro, Portugal
(+351) 234 370 200
cmcruz@ua.pt

Ana M. D'Azevedo Breda
University of Aveiro
Campus Universitário de Santiago
3810-183 Aveiro, Portugal
(+351) 234 370 200
ambreda@ua.pt

ABSTRACT
The existence of tilings of R^n by crosses, a cluster of unit cubes comprising a central one and $2n$ arms, has been studied by several authors. We have completely solved the problem for R^2 characterizing the crosses which lattice tile R^2 as well as determining the maximum packing density for the crosses which do not lattice tile the plane. In this paper we motivate a similar approach to study lattice packings of R^3 by crosses.

CCS Concepts
•Mathematics of computing → Enumeration

Keywords
Packing; Tiling; Lattice; Cross; Homomorphism; Abelian group

1. INTRODUCTION
Problems involving tilings and packings of spaces have been studied for thousands of years. Many of these problems are motivated by real-life applications, namely constituting different types of error correcting codes [1], as well as by other applications that can be found in [2] and [3].

The Minkowski's conjecture triggered research in the area of tilings by clusters of cubes. This conjecture was formulated, originally, in 1896, in algebraic terms [4], later, in 1907, Minkowski used a geometric setting [5].

Conjecture (Minkowski). *Each lattice tiling of R^n by unit cubes contains a pair of cubes that share a complete (n-1)-dimensional face.*

The Minkowski's conjecture was answered in the affirmative in 1941 by Hajós [6]. Since then, tilings of R^n by different clusters of cubes have been considered, namely (k,n)-*crosses* and (k,n)-*semicrosses*, with k and n positive integers. We note that, the (k,n)-cross consists of $2kn+1$ unit cubes, a central cube and $2n$ arms of length k, composed by k unit cubes, attached to all its facets. The (k,n)-semicross consists of $kn+1$ unit cubes, arms of length k are attached at non-opposite facets of the center of the semicross. Kárteszi [7], in 1966, was the first one to consider this type of a tiling, asking whether the $(1,3)$-cross tiles R^3, being this tiling found by Freller. However, a much stronger result was proved by Korchmáros and Golomb-Welch [8] showing that there exists a tiling of R^n by $(1,n)$-crosses for all $n \geq 2$. It turned out that all mentioned tilings were lattice tilings. Molnár [9] has enumerated such tiling:

Theorem (Molnár). *The number of non-congruent lattice tilings of R^n by $(1,n)$-crosses equals the number of non-isomorphic Abelian groups of order $2n+1$.*

It seems that tilings by (k,n)-crosses, for $k \geq 2$, were first considered by Stein in [10]. In the monograph [11], is presented the following problem:

For which positive integers k and n does the (k,n)-cross/semicross tile R^n? Lattice tile R^n?

This problem is far from being solved. It is not surprising that there is no tiling of the n-space if the arms of the crosses are too long. Next theorem was proved in [10].

Theorem (Stein). *If $n \geq 2$ and $k \geq 2n-1$, the (k,n)-cross does not tile R^n.*

In [12] was given a necessary condition for the existence of a tiling by (k,n)-crosses.

Theorem (Szabó). *If $n \geq 2$ and the (k,n)-cross lattice tiles R^n, then $k \leq n-1$.*

Schwartz in [2] have considered a more general type of crosses in connection to a disturb and a retention error in flash memories. A (k_+,k_-,n) *-quasi-cross* in R^n consists of $n(k_+ + k_-)+1$ unit cubes, so that, at each of two opposite facets of the central cube, we attach one arm of length k_+ and another arm of length k_-. In [2] it is proved:

Theorem (Schwartz).

(1) *Let $0 < k_- < k_+$ and l a positive integer. There is a lattice tiling of R^n by a $\left(k_+,k_-,n=\frac{p^l-1}{p-1}\right)$-quasi-cross, where $k_+ + k_- = p-1$ and p prime;*
(2) *For any $n \geq 2$, if a lattice tiling of R^n by (k_+,k_-,n)-quasi-crosses exists, where $0 < k_- < k_+$, then $k_- \leq n-1$;*
(3) *For any $n \geq 2$, if*
$$\frac{2k_+(k_-+1)-k_-^2}{k_+ + k_-} > n,$$
where $0 < k_- < k_+$, then there is not lattice tiling of R^n by (k_+,k_-,n)-quasi-crosses.
(4) *In particular, there is not lattice tiling of R^2 by $(k_+,k_-,2)$ quasi-crosses, where $0 < k_- < k_+$.*

In [13] we have considered in R^2 a more general type of crosses. We have dealt with (a, b, c, d)-*crosses*, with a,b,c,d nonnegative integers, comprising a central cube with four arms of length a,b,c

ICoMS'19, July 8–10, 2019, Prague, Czech Republic

ACM ISBN 978-1-4503-7168-1/19/07…$15.00
DOI: https://doi.org/10.1145/3343485.3343502

and d, respectively, see an example in Figure 1. In [13] we have determined the maximum density of a lattice packing of $\mathrm{R}^2$by (*a*, *b*, *c*, *d*)-crosses. Consequently, we have characterized the crosses for which there exists a lattice tiling of R^2.

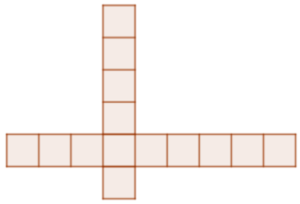

Figure 1. (5, 4, 3, 1)-cross in $\mathbf{R}^2$

The next two results are proved in [13].

Theorem 1.1.*Let $a \geq c, b \geq d$ be nonnegative integers. The maximum density of a lattice packing of$R^2$by (a, b, c, d)-crosses equals:*

(1) $\frac{a+b+c+1}{a+b(c+1)+1}$ *for $d = 0$ and a, b, c positive integers;*
(2) $\frac{2a+b+d+1}{(a+1)(b+1)+1}$ *for positive integers a, b, c, d such that $a = c$;*
(3) $\frac{a+b+c+d+1}{a+b(c+1)+(a-c)d+1}$ *for $a > c > 0$ and $b > d > 0$.*

Next theorem is an immediate consequence of the Theorem 1.1.

Theorem 1.2.*Let a, b, c, d be nonnegative integers. Then there is a lattice tiling of $R^2$by (a, b, c, d)-crosses if and only if either: two or more arms of the cross have length equal to 0; or $d = 0, b = 1$ and $a, c > 0$; or $a, b, c, d > 0$, such that, $a = c$ and $b = d = 1$.*

In this paper we motivate the study of lattice packings of R^3 by crosses applying a similar approach to that one used in [13].

The paper is organized as follows. In Section 2 are presented definitions, notation and important results for the study of lattice packings. In the last section, we present results which are immediately derived from the study done in [13] and we analyze lattice packings of R^3 by certain types of crosses.

2. HOMOMORPHISMS AND PACKINGS

Let S be a space and $F = \{F_i : i \in I\}$ a family of subsets of S. Then F is a *packing* of S if $int(F_i) \cap int(F_j) = \emptyset$, for all $i \neq j, i, j \in I$. If, in addition, $\bigcup_{i \in I} F_i = S$, F is a *tiling* of S.

In this paper we deal with tilings and packings of the Euclidean *n*-space, R^n, by clusters of unit cubes whose edges are parallels to its orthogonal axes. The unit cube centered at $X = (x_1, \dots, x_n)$ in R^n is the set:

$$\left\{Y = (y_1, \dots, y_n) \in \mathrm{R}^n : y_i = x_i + \alpha_i, \alpha_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], 1 \leq i \leq n\right\}.$$

By a cluster of cubes we will understand a set of cubes that are "rookwise" connected.

A set L of points of R^n is a *lattice* if it is an Abelian group under vector addition with no accumulation points. Further, if $K \subset \mathrm{R}^n$, then by a translation of K we mean the set:

$$K + v = \{X + v : X \in K\}, \text{ where } v \in \mathrm{R}^n.$$

Definition 2.1. Let $F = \{K + v : v \in L\}$ be a tiling/packing of R^n by translations of $K \subset \mathrm{R}^n$. If L is a lattice, then F is called *a lattice tiling/packing* of R^n.

We note that, if F is a lattice packing of R^n by K, then there is a superset K' of K such that $\{K' + v : v \in L\}$ is a lattice tiling of R^n.

We will deal with lattice tilings/packings where all coordinates of each vector of L are integers. These tilings/packings are sometimes also called *lattice*Z*-tiling/packing* of R^n.

Let $L \subset \mathrm{R}^n$ be a lattice with a generating matrix $M(L)$, that is, a matrix whose columns $\{l_1, \dots, l_n\}$ form a basis of L. A *fundamental region* of L is the set $\{\sum_{i=1}^{n} \alpha_i l_i : \alpha_i \in [0, 1[\}$. The *volume* of the fundamental region of L does not depend on the choice of a basis of L and it is equal to $|\det M(L)|$. The *density* of L is defined by $\frac{1}{|\det M(L)|}$.

In this paper we study lattice*Z-tilings/packings* of R^3 by crosses. The crosses which we will consider in $\mathrm{R}^3$are similar to the ones considered in [13]. In R^3 we consider *(a, b, c, d, e, f)-crosses*, with *a, b, c, d, e* and *f* nonnegative integers. An *(a, b, c, d, e, f)-cross* is a set of $a + b + c + d + e + f + 1$ unit cubes, a central cube and six arms of length *a, b, c, d, e* and *f*, respectively. In Figure 2 is represented a (2, 3, 6, 7, 4, 5)-cross. As we are considering packings where the center of each unit cube in the cross has only integer coordinates, the (*a, b, c, d, e, f*)-cross is fully determined by a set $K \subset \mathrm{Z}^3$ of centers of its cubes. Therefore, this problem can be seen as a tiling/packing problem of Z^3 by translations of the set of centers of the unit cubes of the (*a, b, c, d, e, f*)-cross.

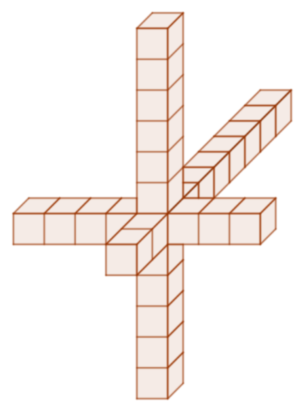

Figure 2. (2, 3, 6, 7, 4, 5)-cross in $\mathbf{R}^3$

If L induces a packing of $\mathrm{R}^3$by an (*a, b, c, d, e, f*)-cross, with K the set of centers of the unit cubes of the cross, then the *packing density* of L is defined by

$$\rho(L) = \frac{|K|}{|detM(L)|} = \frac{a+b+c+d+e+f+1}{|detM(L)|}.$$

Intuitively, the packing density measures the ratio of the volume of the fundamental region covered by (*a, b, c, d, e, f*)-cross to the total volume of the fundamental region. Thus, L induces a lattice tiling of R^3 by (*a, b, c, d, e, f*)-crosses if and only if $\rho(L)$=1, that is, $|detM(L)| = a + b + c + d + e + f + 1$.

Next theorem is stated in [14] and will be a crucial tool in the presented study.

Theorem 2.2. *Let V be a set in Z^n. Then there is a lattice tiling/packing of Z^n by translations of V if and only if there exists a homomorphism $\varphi: \mathrm{Z}^n \rightarrow G$, where G is an Abelian group, such that the restriction of φ to V is a bijection / injection.*

The following corollary is an immediate consequence of the previous result.

Corollary 2.3.*Let an (a, b, c, d, e, f)-cross and K the set of centers of their unit cubes. Let G be an Abelian group. If*

$\varphi: Z^3 \rightarrow G$ is a homomorphism such that the restriction of φ to K is an injection, then φ induces a lattice packing of $Z^3$by K with the density $\frac{|K|}{|G|}$.

In this paper we study lattice tilings/packings of R^3 by (a, b, c, d, e, f)-crosses. For certain parameters *a*, *b*, *c*, *d*, *e* and *f* the results occur immediately from the study done in [13], as we will see in the next section. The main aim of this paper is to initiate the study of lattice packings of R^3 by these crosses from the analysis of a nontrivial case.

Our approach consists of two phases:

i) We begin by finding a lower bound *U* on the order of an Abelian group G such that there exists a homomorphism $\varphi: Z^3 \rightarrow G$ so that the restriction of φ to K, the set of centers of the unit cubes in (*a, b, c, d, e, f*)-cross, is an injection. We note that, by Corollary 2.3, the density of the lattice packing of Z^3 induced by φ is given by $\frac{|K|}{|G|}$, since *U* is the smallest possible order of G, $\frac{|K|}{|U|}$ it is the maximum packing density of Z^3 by K.

ii) We find a homomorphism $\varphi: Z^3 \rightarrow G$, with G an Abelian group of order *U*, such that the restriction of φ to K is an injection, that is, we show that there exists a lattice packing of Z^3 by K where the packing density is given by $\frac{|K|}{|U|}$.

3. LATTICE PACKINGS OF R^3 BY CROSSES

3.1 First Results

Let *P* be a lattice packing of R^3 by (*a, b, c, d, e, f*)-crosses. We may assume, without loss of generality, that the (*a, b, c, d, e, f*)-cross whose central cube is centered at $O = (0, 0, 0)$ belongs to *P*. Thus, whenever we refer to an (*a, b, c, d, e, f*)-cross, we will assume that the central cube of the cross is centered at *O*.

Let K be the set of centers of the unit cubes in an (*a, b, c, d, e, f*)-cross, with a, b, c, d, e, f nonnegative integers. Then, K is given by:

$$\{(x, 0, 0): x \in [-d, a]\} \cup \{(0, y, 0): y \in [-e, b]\} \cup \{(0,0, z): z \in [-f, c]\}.$$

As we are dealing with unit cubes whose coordinates of the centers are integers, by $[\alpha, \beta]$ we assume the set

$\{\gamma \in Z : \alpha \leq \gamma \leq \beta\}$, with Z the set of integer numbers.

We will denote by A, B, C, D, E and F, respectively, the following subsets of K:

$A = \{(x, 0,0): x \in [1, a]\}; D = \{(x, 0,0): x \in [-d, -1]\};$

$B = \{(0, y, 0): y \in [1, b]\}; E = \{(0, y, 0): x \in [-e, -1]\};$

$C = \{(0,0, z): z \in [1, c]\}; F = \{(0,0, z): x \in [-f, -1]\}.$

Furthermore, $H_0 = H \cup \{(0,0,0)\}$, for $H \in \{A, B, C, D, E, F\}$.

For some parameters *a*, *b*, *c*, *d*, *e* and *f* the maximum packing density of R^3 by (*a, b, c, d, e, f*)-crosses is immediately deduced from the study already done in [13]. As example, let us consider an (*0, b, c, 0, e, f*)-cross, where b, c, e, f are positive integers, $b = e$ and $c \geq f$. We are considering a cross of the type represented below.

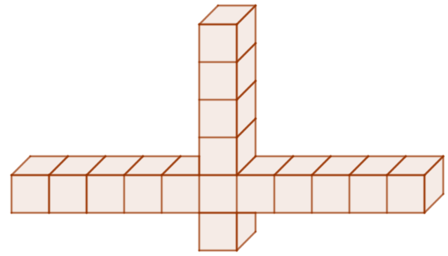

Figure 3. (0, 5, 4, 0, 5, 1)-cross in R^3

The set of centers of the unit cubes of the (*0, b, c, 0, e, f*)-cross, $K = \{(0, y, 0): y \in [-e, b]\} \cup \{(0, 0, z): z \in [-f, c]\}$, is contained in the plane yOz. In [13] we have proved the following lemma.

Lemma 3.1.1. *Let $K \subset Z^2$be the set of centers of the unit cubes of an (a, b, c, d)-cross, where a, b, c, d are positive integers such that $a = c$ and $b \geq d$. If there exists a homomorphism $\varphi: Z^2 \rightarrow G$,G an Abelian group, so that the restriction of φ to K is an injection, then $|G| \geq (a + 1)(b + 1) + 1$.*

The following corollary is an immediate consequence of the previous result.

Corollary 3.1.2. *Let $K \subset Z^3$be the set of centers of the unit cubes of an $(0, b, c, 0, e, f)$-cross, where b, c, e, f are positive integers such that $b = e$ and $c \geq f$. If there exists a homomorphism $\varphi: Z^3 \rightarrow G$, G an Abelian group, so that the restriction of φ to K is an injection, then $|G| \geq (b + 1)(c + 1) + 1$.*

Proving the existence of such homomorphism satisfying the conditions of Corollary 3.1.2, we get the maximum packing density of R^3 by (*0, b, c, 0, e, f*)-crosses, as shown in the following theorem.

Theorem 3.1.3. *Let b, c, e, f positive integers such that $b = e$ and $c \geq f$. The maximum packing density of $R^3$by (0, b, c, 0, e, f)-crosses is $\frac{2b+c+f+1}{(b+1)(c+1)+1}$.*

Proof. Let $\varphi: Z^3 \rightarrow Z_{(b+1)(c+1)+1}$ be a homomorphism defined by $\varphi(0, 1, 0) = 1$, $\varphi(0, 0, 1) = b + 1$ and $\varphi(1, 0, 0) = e$, with e the neutral element. Then,

$\varphi(B_0) = \{y: y \in [0, b]\};$

$\varphi(E) = \{(b + 1)(c + 1) + 1 + y: y \in [-b, -1]\};$

$\varphi(C) = \{z(b + 1): z \in [1, c]\};$

$\varphi(F) = \{(b + 1)(c + 1 + z) + 1: z \in [-f, -1]\}.$

The largest element of $\varphi(B_0)$ is smaller than the smallest element of $\varphi(C) \cup \varphi(E) \cup \varphi(F)$. The largest element of $\varphi(C) \cup \varphi(F)$ is smaller than the smallest element of $\varphi(E)$. By the other side, $\varphi(C) \cap \varphi(F) = \emptyset$. In fact, supposing that there are $z_1 \in [1, c]$ and $z_2 \in [-f, -1]$ such that $z_1(b + 1) = (b + 1)(c + 1 + z_2) + 1$, we get $(b + 1)(z_1 - c - 1 - z_2) = 1$, which is not possible since $b + 1 > 1$. Therefore, φ is injective on the set of centers of the unit cubes of the (*0, b, c, 0, e, f*)-cross and, by Corollaries 2.3 and 3.1.2, the maximum packing density of of R^3 by (*0, b, c, 0, e, f*)-crosses is $\frac{2b+c+f+1}{(b+1)(c+1)+1}$.■

Considering an (a, b, c, d, e, f)-cross, if $a = d = 0$ or $b = e = 0$ or $c = f = 0$, the set of centers of the unit cubes of the cross is contained in one of the planes xOy, xOz or yOz. Taking into account the study of lattice packings of R^2 by crosses done in

[13], applying a similar reasoning to the used in the example considered before, we get analogous results to the ones presented in Theorem 1.1, unless of the choice of the parameters equals to 0, as we can see in the following theorem.

Theorem 3.1.4. *Let $a \geq d$, $b \geq e$ be nonnegative integers. The maximum density of a lattice packing of* R^3 *by* $(a, b, 0, d, e, 0)$*-crosses equals:*

(1) $\frac{a+b+d+1}{a+b(d+1)+1}$ for $e = 0$ *and* a, b, d *positive integers;*

(2) $\frac{2a+b+e+1}{(a+1)(b+1)+1}$ *for positive integers* a, b, d, e *so that* $a = d$;

(3) $\frac{a+b+d+e+1}{a+b(d+1)+(a-d)e+1}$ *for* $a > d > 0$ *and* $b > e > 0$.

The following result is an immediate consequence of the Theorem 3.1.4.

Corollary3.1.5. *Let* a, b, d, e *be nonnegative integers. Then, there is a lattice tiling of* R^3 *by* $(a, b, 0, d, e, 0)$*-crosses if and only if either: two or more parameters in* $\{a, b, d, e\}$ *are equal to 0; or* $e = 0$, $b = 1$ *and* $a, d > 0$*; or* $a, b, d, e > 0$*, such that,* $a = d$ *and* $b = e = 1$.

3.2 Lattice Packing of R^3 by (*a*, *b*, *c*, 0, 0, 0)-Crosses

In this subsection we analyze the density of lattice packings of R^3 by $(a, b, c, 0,0,0)$-crosses, where a, b, c are positive integers and, without loss of generality, $a \geq b$. Our study is supported in Corollary 2.3, however we have felt difficulties into find a lower bound on the order of an Abelian group assuming only that φ is injective on the cross. Therefore, we have considered an additional condition. Since we have in view the maximum packing density, considering the $(a, b, c, 0, 0, 0)$-cross with the central cube centered at $O = (0, 0, 0)$, we begin by analyzing a possible "good position" for another $(a, b, c, 0, 0, 0)$-cross. Thus, having in mind the minimal space between the crosses, we have considered as a good position that ones whose distance between the centers of the crosses is minimal. To avoid superposition between the crosses, the minimal Euclidean distance between the centers is equal to $\sqrt{2}$. There exist some possibilities for the center W of another cross, let us consider, without loss of generality, $W = (1, 1, 0)$, as shown in Figure 4. From now, we are considering this additional condition in the following results. We begin by determining the maximum packing density of R^3 by $(a, b, c, 0,0,0)$-crosses.

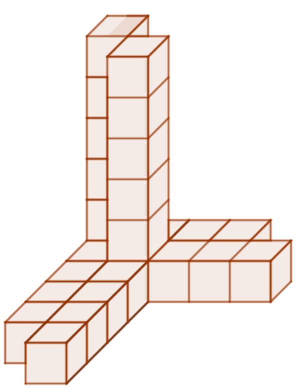

Figure 4. (4, 3, 5, 0, 0, 0)-crosses in R^3

Theorem 3.2.1. *Let an* $(a, b, c, 0, 0, 0)$*-cross, where* a, b, c *are positive integers and* $a \geq b$. *If there exists a homomorphism* $\varphi: Z^3 \to G$, G *an Abelian group, such that the restriction of* φ *to* K, *the set of centers of the unit cubes of the cross, is an injection and* $\varphi(1,1,0) = e$, e *the neutral element of* G, *then* $|G| \geq a + c(b + 1) + 1$. *There exists a homomorphism* φ *in these conditions with* $|G| = a + c(b + 1) + 1$. *Consequently, the maximum density of a lattice packing of* R^3 *by* $(a, b, c, 0, 0, 0)$*-crosses equals:* $\frac{a+b+c+1}{a+c(b+1)+1}$.

Proof. Let $\varphi: Z^3 \to G$ be a homomorphism, where G is an Abelian group, such that φ is injective on $K = A_0 \cup B \cup C$. Note that, A_0, B and C denote the sets defined in Subsection 3.1, the arms of the $(a, b, c, 0, 0, 0)$-cross, with a, b, c positive integers and $a \geq b$. Suppose that $\varphi(1,1,0) = e$, the neutral element of G.

Let us consider the set

$$X = \{(x, 0, z): 1 \leq x \leq b \wedge 1 \leq z \leq c\}.$$

We will prove the theorem showing that:

i) $\varphi(X) \cap \varphi(A_0) = \emptyset$;

ii) $\varphi(X) \cap \varphi(C) = \emptyset$;

iii) φ is injective in *X*.

Proof of i).

Let $(x, 0, z) \in X$ and $(\alpha, 0, 0) \in A_0$. Suppose, by contradiction, $\varphi(x, 0, z) = \varphi(\alpha, 0, 0)$. Consequently,

$$\varphi(0, 0, z) = \varphi(\alpha - x, 0, 0), \text{ with } -b \leq \alpha - x \leq a - 1.$$

If $0 \leq \alpha - x \leq a - 1$, the hypothesis φ is injective on K, is contradicted. If $-b \leq \alpha - x \leq -1$, as we are assuming $\varphi(1, 1, 0) = e$, then $\varphi(\alpha - x, \alpha - x, 0) = e$ and, consequently, $\varphi(\alpha - x, 0, 0) = \varphi(0, x - \alpha, 0)$. Therefore,

$$\varphi(0, 0, z) = \varphi(0, x - \alpha, 0), \text{ with } 1 \leq x - \alpha \leq b,$$

contradicting again the hypothesis.

Proof of ii).

Assume that there exist $(x, 0, z) \in X$ and $(0, 0, \gamma) \in C$ such that $\varphi(x, 0, z) = \varphi(0,0, \gamma)$. Then,

$$\varphi(x, 0, 0) = \varphi(0, 0, \gamma - z), \text{ with } 1 - c \leq \gamma - z \leq c - 1.$$

If $0 \leq \gamma - z \leq c - 1$, we get immediately a contradiction. Otherwise, if $1 - c \leq \gamma - z \leq -1$, by the hypothesis $\varphi(1,1,0) = e$, $\varphi(0, -x, 0) = \varphi(0, 0, \gamma - z)$, that is

$$\varphi(0, x, 0) = \varphi(0, 0, z - \gamma), \text{ with } 1 \leq z - \gamma \leq c - 1,$$

which is a contradiction.

Proof of iii).

Let $(x_1, 0, z_1), (x_2, 0, z_2) \in X$, $(x_1, 0, z_1) \neq (x_2, 0, z_2)$. Suppose $\varphi(x_1, 0, z_1) = \varphi(x_2, 0, z_2)$, in these conditions we have $\varphi(x_1 - x_2, 0, 0) = \varphi(0, 0, z_2 - z_1)$, $1 - b \leq x_1 - x_2 \leq b - 1$ and $1 - c \leq z_2 - z_1 \leq c - 1$.

Consider $0 \leq x_1 - x_2 \leq b - 1$. If $0 \leq z_2 - z_1 \leq c - 1$, the injection of φ on the cross is contradicted. If $1 - c \leq z_2 - z_1 \leq -1$, since $\varphi(x_1 - x_2, 0, 0) = \varphi(0, x_2 - x_1, 0)$, $\varphi(0, x_1 - x_2, 0) = \varphi(0, 0, z_1 - z_2)$, with $0 \leq x_1 - x_2 \leq b - 1$ and $1 \leq z_1 - z_2 \leq c - 1$, which is a contradiction.

Considering $1 - b \leq x_1 - x_2 \leq -1$, following a similar reasoning we obtain always a contradiction.

Therefore, φ is injective on $A_0 \cup C \cup X$. By hypothesis, φ is injective on $A_0 \cup B \cup C$, then

$$|\varphi(A_0 \cup B \cup C \cup X)| \geq a + 1 + b + c + bc - b,$$

since $|\varphi(B \cap X)| \leq b$. That is, $|G| \geq a + c(b+1) + 1$.

To prove the second part of the theorem we will consider the homomorphism $\varphi: Z^3 \to Z_{a+c(b+1)+1}$ given by $\varphi(1,0,0) = 1$, $\varphi(0,1,0) = a + c(b+1)$ and $\varphi(0,0,1) = a + c(b+1) - b$. In these conditions,

$\varphi(A_0) = \{x : x \in [0,a]\}$;

$\varphi(B) = \{a + c(b+1) + 1 - y : y \in [1,b]\}$;

$\varphi(C) = \{a + (c+1-z)(b+1) - b : z \in [1,c]\}$.

Note that, $0 \leq \varphi(A_0) \leq a$, $a+1 \leq \varphi(C) \leq a + c(b+1) - b$ and $a + c(b+1) + 1 - b \leq \varphi(B) \leq a + c(b+1)$.

Therefore, $\varphi(A_0)$, $\varphi(B)$ and $\varphi(C)$ are disjoint, consequently, φ is injective on the set of centers of unit cubes of the $(a,b,c,0,0,0) - cross$. By Corollary 2.3 and taking into account that $|G| \geq a + c(b+1) + 1$, in the considered conditions, that is, considering $\varphi(1,1,0) = e$, the maximum packing density of R^3 by the $(a,b,c,0,0,0) - cross$ equals $\frac{a+b+c+1}{a+c(b+1)+1}$. ■

From Theorem 3.2.1, we can characterize the crosses which tile R^3.

Theorem 3.2.2. *Let a, b, c be positive integers and* $a \geq b$. *There exists a lattice tiling of* R^3 *by* $(a,b,c,0,0,0)$*-crosses induced by an homomorphism* φ *with* $\{(0,0,0),(1,1,0)\} \subset Ker(\varphi)$ *if and only if c=1.*

Proof. By Theorem 3.2.1 the maximum packing density of R^3 by the $(a,b,c,0,0,0) - cross$ equals $\frac{a+b+c+1}{a+c(b+1)+1}$. Thus, $\frac{a+b+c+1}{a+c(b+1)+1} = 1$ if and only if $c = 1$. ■

4. CONCLUSION

In this paper we have motivated the study of lattice packings of R^3 by crosses following a similar idea to the one used in [13]. We have focused our attention in a particular case, imposing an additional condition, namely, the position of two crosses. In the future, we hope to study lattice packings of R^3 by all types of crosses as well as to analyze a possible influence which the additional condition considered in this paper have on the maximum packing density.

5. ACKNOWLEDGMENTS

The authors were supported by Fundação para a Ciência e a Tecnologia (FCT), within project UID/MAT/04106/2019 (CIDMA).

6. REFERENCES

[1] Golomb, S. W. and Welch, L. R. 1970. Perfect codes in the Lee metric and the packing of polyominoes. *SIAMJ. Appl. Math.* 18, 302-317.

[2] Schwartz, M. 2012. Quasi-cross lattice tilings with applications to flash memory. *IEEE Trans. Inform. Theory* 58, 2397-2405.

[3] Szabó, S. and Sands, A 2009. *Factoring groups into subsets*. Lect. Notes Pure Appl. Math. 257, CRC Press, Boca Raton, FL.

[4] Minkowski, H. 1896. *Geometrie der Zahlen*, Teubner, Leipzig.

[5] Minkowski, H. 1907. *DiophantischeApproximationen*, Teubner, Leipzig.

[6] Hajós, G. 1942. Über einfache und mehrfacheBedeckung des n-dimensionalen Raumes mit einem Würfelgitter. *Math. Z.* 47, 427-467.

[7] Kárteszi, F. 1966. *Szemléletesgeometria*, Gondolat, Budapest.

[8] Golomb, S. W. and Welch, L. R. 1968. Algebraic coding and the Lee metric. In *Error Correcting Codes*, Wiley, New York, 175-189.

[9] Molnár, E. 1971. Sui mosaic dellospazio di dimensione n. *Atti Accad. Naz. Lincei Rend. Sci. Fis. Mat. Natur.* 51, 177-185.

[10] Stein, S. 1967. Factoring by subsets. *Pacific J. Math.* 22, 523-541.

[11] Stein, S. and Szabó, S. 1994. *Algebra and Tiling: Homomorphism in the Service of Geometry*. Carus Math. Monogr. 25, Mathematical Assoc. America, Washington, DC.

[12] Szabó, S. 1984. A bound of *k* for tiling by (*k*, *n*)-crosses and semicrosses. *Acta Math. Hungar.* 44, 97-99.

[13] Cruz, C. N., Breda, A. M. D'Azevedo and Pinto, M. R. 2015. Packing of R^2 by crosses. Mathematica Slovaca 65, 5, 1-22.

[14] Horak, P. and Albdaiwi, B. F. 2012. Diameter perfect Lee codes. *IEEE Trans. Inform. Theory* 58, 5490-5499.