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Quadratic Lyapunov Functions for Stability of the Generalized Proportional Fractional Differential Equations with Applications to Neural Networks

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Abstract: A fractional model of the Hopfield neural network is considered in the case of the application of the generalized proportional Caputo fractional derivative. The stability analysis of this model is used to show the reliability of the processed information. An equilibrium is defined, which is generally not a constant (different than the case of ordinary derivatives and Caputo-type fractional derivatives). We define the exponential stability and the Mittag–Leffler stability of the equilibrium. For this, we extend the second method of Lyapunov in the fractional-order case and establish a useful inequality for the generalized proportional Caputo fractional derivative of the quadratic Lyapunov function. Several sufficient conditions are presented to guarantee these types of stability. Finally, two numerical examples are presented to illustrate the effectiveness of our theoretical results.

Keywords: generalized Caputo proportional fractional derivative; stability; exponential stability; Mittag–Leffler stability; quadratic Lyapunov functions; Hopfield neural networks

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1. Introduction

In [1], Jarad, Abdeljawad, and Alzabut introduced a new type of fractional derivative, the so-called generalized proportional fractional derivative. This type of derivative preserves the semigroup property, possesses a nonlocal character, and converges to the original function and its derivative upon limiting cases [2]. Some stability properties of the Ulam type for generalized proportional fractional differential equations were studied in [3] and in [4]. We emphasize that the regular stability has not been investigated yet. In this paper, we develop some necessary tools for the generalized Caputo proportional fractional derivatives, starting with an important inequality concerning an estimate of that derivative of quadratic functions. We derive some inequalities for quadratic Lyapunov functions and some connections between the solutions and the Lyapunov functions. These results are applied to study the stability properties of the Hopfield neural network with time-variable coefficients and Lipschitz activation functions. Due to its long-term memory, nonlocality, and weak singularity characteristics, fractional calculus has been successfully applied to various models of neural networks. For instance, Boroomand constructed the Hopfield neural networks based on fractional calculus [5], Kaslik analyzed the stability of Hopfield neural networks [6], Wang applied the fractional steepest descent algorithm to train BP neural networks and proved the monotonicity and convergence of a three-layer example [7]. The three features for the generalized proportional fractional derivative—the kernel of the fractional operator, the semi-group property of the generated fractional integrals, and obtaining the Riemann–Liouville and Caputo fractional derivatives as a special case—offers a possibility for more adequate modeling of some properties of the neural network.

The equilibrium of the studied model as well as its exponential stability and the Mittag–Leffler stability are defined and investigated.

The paper is organized as follows. In Section 2, some basic definitions and results are given. In Section 3, we present several auxiliary results for the generalized Caputo proportional fractional derivatives of the quadratic Lyapunov function. Section 4 contains the main results. The Hopfield neural model with time-variable coefficients and the generalized proportional fractional derivatives of the Caputo type are set up. The equilibrium is defined in an appropriate way. Exponential stability and Mittag–Leffler stability are defined, and several sufficient conditions are obtained. The paper concludes with Section 5, in which some detailed examples of neural networks are presented and simulated.

2. Preliminary Results

We recall that the generalized proportional fractional operators of a function $u \in C^1([a,b],\mathbb{R})$, ($a < b \leq \infty$ are real numbers, and in the case of $b = \infty$, the interval is open) are defined respectively by (see [2]):

- the generalized proportional fractional integral

$$({}_a\mathscr{I}^{\alpha,\rho}u)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}u(s)\,ds, \text{ for } t \in (a,b],\ \alpha > 0;$$

- the generalized proportional Caputo fractional derivative

$$\begin{aligned}({}_a^C\mathscr{D}^{\alpha,\rho}u)(t) &= ({}_a\mathscr{I}^{1-\alpha,\rho}(\mathscr{D}^{1,\rho}u))(t)\\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}(\mathscr{D}^{1,\rho}u)(s)\,ds, \text{ for } t \in (a,b],\ \alpha \in (0,1),\ \rho \in (0,1],\end{aligned} \tag{1}$$

where $(\mathscr{D}^{1,\rho}u)(t) = (\mathscr{D}^{\rho}u)(t) = (1-\rho)u(t) + \rho u'(t)$ and $\rho \in (0,1]$ are fixed parameters.

Remark 1. *The generalized proportional Caputo fractional derivative defined by* (1) *is a generalization of the Caputo fractional derivative (with $\rho = 1$).*

Remark 2. *Note that, in some works (for example, see [8–10]), the so-called tempered fractional integral and tempered fractional derivative are applied and defined by the following:*

$${}_aI_t^{\alpha,\lambda}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}x(s)ds, \ \textit{for } \alpha > 0$$

and

$${}_a^C D_t^{\alpha,\lambda}x(t) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \int_a^t \frac{e^{\lambda s}}{(t-s)^\alpha}(\lambda x(s) + x'(s))ds, \ \textit{for } \alpha \in (0,1),$$

where $\lambda \geq 0$ is a fixed parameter. Tempered fractional integrals and tempered fractional derivatives are similar to the generalized proportional fractional integrals and derivatives (if $\lambda = (1-\rho)/\rho$, $\rho \in (0,1]$, then ${}_aI_t^{\alpha,\frac{1-\rho}{\rho}}u(t) = \rho^\alpha({}_a\mathscr{I}^{\alpha,\rho}u)(t)$ and $\rho^\alpha {}_a^C D_t^{\alpha,\frac{1-\rho}{\rho}}u(t) = ({}_a^C\mathscr{D}^{\alpha,\rho}u)(t)$).

Proposition 1 (Proposition 5.2, [1])**.** *Let $\alpha \in (0,1)$ and $\rho \in (0,1]$. Then,*

$$({}_a^C\mathscr{D}^{\alpha,\rho}u)(t) = 0,\ t > a \quad \textit{with } u(s) = e^{\frac{\rho-1}{\rho}s},\ s > a.$$

There is an explicit formula for the solution in the scalar linear case provided in Example 5.7 [1], which is (with an appropriate correction):

Proposition 2. *The solution of the linear Caputo proportional fractional initial value problem*

$$ {}_{a}^{C}\mathscr{D}^{\alpha,\rho}x(t) = \rho^{\alpha}\lambda x(t) + f(t), \quad x(a) = x_0, \tag{2}$$

is given by

$$x(t) = x_0 e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha}(\lambda(t-a)^{\alpha}) + \rho^{-\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(s-a)}(s-a)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-a)^{\alpha})f(s)ds, \tag{3}$$

where

$$E_{\alpha}(Az) = \sum_{k=0}^{\infty}\frac{(Az)^k}{\Gamma(1+k\alpha)} \quad \text{and} \quad E_{\alpha,\beta}(Az) = \sum_{k=0}^{\infty}\frac{(Az)^k}{\Gamma(\beta+k\alpha)}$$

are Mittag–Leffler functions with one parameter and two parameters, respectively.

3. Quadratic Lyapunov Functions and Their Generalized Proportional Derivatives

Initially, we will prove the following results for scalar functions:

Lemma 1. *Let the function $u \in C^1([a,b],\mathbb{R})$ with $a, b \in \mathbb{R}$, $b \le \infty$ (if $b = \infty$, then the interval is half open) and $\alpha \in (0,1)$, $\rho \in (0,1]$ be two reals. Then,*

$$({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u^2)(t) \le 2u(t)({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u)(t), \quad t \in (a,b]. \tag{4}$$

Proof. From definition (1), we have that for any $t \in (a,b]$,

$$\begin{aligned}
&({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u^2)(t) - 2u(t)({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u)(t)\\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}\Big\{(1-\rho)[(u(s))^2 - 2u(t)u(s) + u^2(t) - u^2(t)]\\
&\quad + 2\rho\Big[(u(s))'(u(s)) - (u(s))'u(t)\Big]\Big\}\,ds\\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}\Big\{(1-\rho)e^{-\frac{1-\rho}{\rho}t}e^{\frac{1-\rho}{\rho}s}[(u(s)-u(t))^2 - u^2(t)]\\
&\quad + 2\rho e^{-\frac{1-\rho}{\rho}t}e^{\frac{1-\rho}{\rho}s}u(s)'[u(s)-u(t)]\Big\}\,ds\\
&\le \frac{e^{-\frac{1-\rho}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}\Big\{(1-\rho)e^{\frac{1-\rho}{\rho}s}[u(s)-u(t)]^2 + 2\rho e^{\frac{1-\rho}{\rho}s}u(s)'[u(s)-u(t)]\Big\}\,ds.
\end{aligned} \tag{5}$$

Use integration by parts and obtain the following:

$$\begin{aligned}
&({}_{a}^{C}\mathscr{D}^{\alpha,\rho}(u)^2)(t) - 2u(t)({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u)(t)\\
&\le \frac{e^{-\frac{1-\rho}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\Big\{\rho\int_a^t (t-s)^{-\alpha}[u(s)-u(t)]^2 de^{\frac{1-\rho}{\rho}s} + 2\rho\int_a^t (t-s)^{-\alpha}e^{\frac{1-\rho}{\rho}s}u(s)'[u(s)-u(t)]ds\Big\}\\
&= \frac{e^{-\frac{1-\rho}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\Big\{\rho\left(\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^{\alpha}}\right)\Big|_{s=a}^{s=t} - \rho\int_a^t e^{\frac{1-\rho}{\rho}s}d\Big((t-s)^{-\alpha}[u(s)-u(t)]^2\Big)\\
&\quad + 2\rho\int_a^t (t-s)^{-\alpha}e^{\frac{1-\rho}{\rho}s}u(s)'[u(s)-u(t)]ds\Big\}\\
&= \frac{e^{-\frac{1-\rho}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\Big\{\rho\left(\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^{\alpha}}\right)\Big|_{s=a}^{s=t} - \rho\alpha\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{1-\alpha}[u(s)-u(t)]^2 ds\\
&\quad - 2\rho\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{-\alpha}u'(s)[u(s)-u(t)]ds + 2\rho\int_a^t (t-s)^{-\alpha}e^{\frac{1-\rho}{\rho}s}u(s)'[u(s)-u(t)]ds\Big\}\\
&= \frac{e^{-\frac{1-\rho}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\Big\{\rho\left(\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^{\alpha}}\right)\Big|_{s=a}^{s=t} - \rho\alpha\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{-1-\alpha}[u(s)-u(t)]^2 ds.
\end{aligned} \tag{6}$$

The integral

$$\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{-1-\alpha}[u(s)-u(t)]^2 ds$$

has a singularity at the upper limit t, but it is a removable singularity because by the L'Hopital rule, we obtain the following:

$$\lim_{s\to t^-}\frac{[u(s)-u(t)]^2}{(t-s)^{1+\alpha}} = \lim_{s\to t^-}\frac{2u'(s)(u(s)-u(t))}{-(1+\alpha)(t-s)^{\alpha}} = \lim_{s\to t^-}\frac{2(u'(s)(u(s)-u(t)))'}{(1+\alpha)\alpha}(t-s)^{1-\alpha} = 0.$$

Thus,

$$\begin{aligned}({}_a^C\mathscr{D}^{\alpha,\rho}(u)^2)(t) - 2u(t)({}_a^C\mathscr{D}^{\alpha,\rho}u)(t) &\le \frac{\rho^\alpha e^{-\frac{1-\rho}{\rho}t}}{\Gamma(1-\alpha)}\left(\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^\alpha}\right)\Big|_{s=a}^{s=t} \\ &= \frac{\rho^\alpha e^{-\frac{1-\rho}{\rho}t}}{\Gamma(1-\alpha)}\left(\lim_{s\to t}\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^\alpha} - \frac{[u(a)-u(t)]^2 e^{\frac{1-\rho}{\rho}a}}{(t-a)^\alpha}\right).\end{aligned} \tag{7}$$

By the L'Hopital rule we get the following:

$$\begin{aligned}({}_a^C\mathscr{D}^{\alpha,\rho}(u)^2)(t) - 2u(t)({}_a^C\mathscr{D}^{\alpha,\rho}u)(t) &\le \frac{\rho^\alpha e^{-\frac{1-\rho}{\rho}t}}{\Gamma(1-\alpha)}\lim_{s\to t}\frac{[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{(t-s)^\alpha} \\ &= -\frac{\rho^\alpha e^{-\frac{1-\rho}{\rho}t}}{\Gamma(1-\alpha)}\lim_{s\to t}\frac{2u'(s)[u(s)-u(t)]e^{\frac{1-\rho}{\rho}s} + \frac{1-\rho}{\rho}[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}}{\alpha(t-s)^{\alpha-1}} \\ &= -\frac{\rho^\alpha e^{-\frac{1-\rho}{\rho}t}}{\Gamma(1-\alpha)\alpha}\lim_{s\to t}\left(2u'(s)[u(s)-u(t)]e^{\frac{1-\rho}{\rho}s} + \frac{1-\rho}{\rho}[u(s)-u(t)]^2 e^{\frac{1-\rho}{\rho}s}\right)(t-s)^{1-\alpha} = 0.\end{aligned} \tag{8}$$

Inequality (8) proves the claim of Lemma 1. □

Inequality (4) is true in the vector case:

Corollary 1. *Let the function $u \in C^1([a,b],\mathbb{R}^n)$ with $a,b \in \mathbb{R}$, $b \le \infty$ (if $b = \infty$, then the interval is half open) and $\alpha \in (0,1)$, $\rho \in (0,1]$. Then,*

$$({}_a^C\mathscr{D}^{\alpha,\rho}u^T(t)u(t)) \le 2u^T(t)({}_a^C\mathscr{D}^{\alpha,\rho}u)(t), \quad t \in (a,b]. \tag{9}$$

The proof follows from the decomposition of the scalar product $u^T(t)u(t)$ into a sum of products and the application of Lemma 1.

Remark 3. *In the case of the Caputo fractional derivative, i.e., $\rho = 1$, the results of Lemma 1 and Corollary 1 are reduced to Lemma 1 [11] and Remark 1 [11].*

Consider the following system of nonlinear fractional differential equations with the generalized proportional Caputo fractional derivative:

$$\begin{aligned}&({}_{t_0}^C\mathscr{D}^{\alpha,\rho}u)(t) = F(t,u(t)), \ \text{ for } t > t_0, \\ &u(t_0) = u_0,\end{aligned} \tag{10}$$

where $t_0 \ge 0$, $({}_{t_0}^C\mathscr{D}^{\alpha,\rho}u)(t)$ is the generalized proportional Caputo fractional derivative of the function $u \in C^1([t_0,\infty),\mathbb{R}^n)$, $\rho \in (0,1], \alpha \in (0,1)$ are two reals, and $F : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a function.

Remark 4. *We will assume that for any initial value u_0, the initial value problem (10) has a solution $u(t;t_0,u_0)$ defined for $t \ge t_0$.*

Next, we will obtain two types of bounds for the solutions of (10).

Lemma 2. *Assume that:*

1. *The function* $u(\cdot) = u(\cdot; t_0, u_0) \in C^1([t_0,\infty),\mathbb{R}^n)$ *is a solution of the IVP for the nonlinear system of generalized proportional Caputo fractional differential equations* (10)*;*
2. *For any point* $t \geq t_0$*, the inequality*

$$ {}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}(\|u(t)\|^2) \leq 0 \tag{11}$$

holds.

Then,

$$\|u(t)\| \leq \|u_0\| e^{\frac{\rho-1}{2\rho}(t-t_0)}, \ \textit{for } t \geq t_0. \tag{12}$$

Proof. Define the function $m(t) = u^T(t)u(t) = \|u(t)\|^2 : [t_0,\infty) \to \mathbb{R}_+$. Let $\varepsilon > 0$ be an arbitrary number. We will prove that

$$m(t) < (\|u_0\|^2 + \varepsilon)e^{\frac{\rho-1}{\rho}(t-t_0)}, \ \ t \geq t_0. \tag{13}$$

For $t = t_0$, we get

$$m(t_0) = \|u_0\|^2 < (\|u_0\|^2 + \varepsilon) = (\|u_0\|^2 + \varepsilon)e^{\frac{\rho-1}{\rho}(t_0-t_0)},$$

i.e., inequality (13) is true for $t = t_0$.

Now, assume that (13) is not true. Then there exist $t^* \in (t_0,\infty)$, such that

$$m(t) < (\|u_0\|^2 + \varepsilon)e^{\frac{\rho-1}{\rho}(t-t_0)}, \ t \in [t_0, t^*), \qquad m(t^*) = (\|u_0\|^2 + \varepsilon)e^{\frac{\rho-1}{\rho}(t^*-t_0)}. \tag{14}$$

Denote $\eta(t) = \|u(t)\|^2 - (\|u_0\|^2 + \varepsilon)e^{\frac{\rho-1}{\rho}(t-t_0)} : [t_0, t^*] \to (-\infty, 0]$. From (14), it follows that $\eta(t^*) = 0$, $\eta(t) < 0$ for $t \in [t_0, t^*)$. Therefore,

$$\begin{aligned}
({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}\eta)(t)\Big|_{t=t^*} &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)}(t^*-s)^{-\alpha}\Big((1-\rho)\eta(s) + \rho\eta'(s)\Big)\, ds \\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \Big[\rho \int_{t_0}^{t^*} (t^*-s)^{-\alpha}\eta(s) de^{\frac{\rho-1}{\rho}(t-s)} + \rho \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)}(t^*-s)^{-\alpha}\eta'(s)ds, \\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \Big[\rho(t^*-s)^{-\alpha}\eta(s)e^{\frac{\rho-1}{\rho}(t^*-s)}|_{s=t_0}^{s=t^*} - \rho \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} d((t^*-s)^{-\alpha}\eta(s)) \\
&+ \rho \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)}(t^*-s)^{-\alpha}\eta'(s)ds.
\end{aligned} \tag{15}$$

Thus, by the L'Hopital rule, we get the following:

$$(t^*-s)^{-\alpha}\eta(s)e^{\frac{\rho-1}{\rho}(t^*-s)}\Big|_{s=t^*} = \lim_{s\to t^*-0} \frac{\eta(s)e^{\frac{\rho-1}{\rho}(t^*-s)}}{(t^*-s)^{\alpha}} = \lim_{s\to t^*-0} \frac{\eta'(s)e^{\frac{\rho-1}{\rho}(t^*-s)} - \frac{\rho-1}{\rho}\eta(s)e^{\frac{\rho-1}{\rho}(t^*-s)}}{\alpha}(t^*-s)^{1-\alpha} = 0. \tag{16}$$

From (15) and (16), we obtain the following:

$$\begin{aligned}
({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}\eta)(t)\Big|_{t=t^*} &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \Bigg[-\rho(t^*-t_0)^{-\alpha}\eta(t_0)e^{\frac{\rho-1}{\rho}(t-t_0)} - \rho\alpha \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{\eta(s)}{(t^*-s)^{1+\alpha}} ds \\
&- \rho \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)}(t^*-s)^{-\alpha}\eta'(s)ds + \rho \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t-s)}(t^*-s)^{-\alpha}\eta'(s)ds\Bigg] \\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \Bigg[-\rho\eta(t_0)\frac{e^{\frac{\rho-1}{\rho}(t^*-t_0)}}{(t^*-t_0)^{\alpha}} - \rho\alpha \int_{t_0}^{t^*} \frac{e^{\frac{\rho-1}{\rho}(t^*-s)}}{(t^*-s)^{1+\alpha}}\eta(s)ds\Bigg] > 0.
\end{aligned} \tag{17}$$

From Proposition 1, inequality (17), and condition 2 for $t = t^*$, we obtain the equation below:

$$0 < ({}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}\eta)(t)\Big|_{t=t^*} = {}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}(\|u(t)\|^2 - (\|u_0\|^2+\varepsilon)e^{\frac{\rho-1}{\rho}(t-t_0)})\Big|_{t=t^*}$$
$$= {}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}\|u(t)\|^2\Big|_{t=t^*} - (\|u_0\|^2+\varepsilon)e^{\frac{1-\rho}{\rho}t_0}{}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}\Big|_{t=t^*} = {}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}\|u(t)\|^2\Big|_{t=t^*} \leq 0. \quad (18)$$

The obtained contradiction proves the validity of (13) for any $\varepsilon > 0$. Therefore,

$$\|u(t)\|^2 < \|u_0\|^2 e^{\frac{\rho-1}{\rho}(t-t_0)},$$

i.e., the claim of Lemma 2 is true. □

Corollary 2. *Assume that the conditions of Lemma 2 are satisfied. Then,* $\|x(t)\| \leq \|u_0\|$ *for all* $t \geq t_0$.

The proof follows from inequality (12), $\rho \in (0,1]$, and the inequality $e^{\frac{\rho-1}{2\rho}(t-t_0)} \leq 1$.

Lemma 3. *Assume that:*

1. *The function* $u(\cdot) = u(\cdot; t_0, u_0) \in C^1([t_0,\infty),\mathbb{R}^n)$ *is a solution of the IVP for the nonlinear system of generalized proportional Caputo fractional differential equations (10);*
2. *There exists a positive constant* $K > 0$*, such that at any point* $t \geq t_0$*, the inequality*

$$ {}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}\left(\|u(t)\|^2\right) \leq -K\|u(t)\|^2 \quad (19)$$

holds.

Then,

$$\|u(t)\| \leq \|u_0\| e^{\frac{\rho-1}{2\rho}(t-t_0)}\sqrt{E_\alpha\left(\frac{-K}{\rho^\alpha}t^\alpha\right)} \text{ for } t \geq t_0. \quad (20)$$

Proof. Define the function $m(t) = u^T(t)u(t) = \|u(t)\|^2 : [t_0,\infty) \to \mathbb{R}_+$. From inequality (19), it follows that there exists a function $\xi : [t_0,\infty) \to [0,\infty)$, such that

$$({}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}m)(t) \leq -Km(t) - \xi(t),\ t \geq t_0. \quad (21)$$

According to Proposition 2, with $a = 0$, $\lambda = -K/\rho^\alpha$, $f(t) = -\xi(t)$, and $x_0 = m(0)$, the solution of the linear Caputo proportional fractional initial value problem (21) is given by

$$m(t) = m(0)e^{\frac{\rho-1}{\rho}t}E_\alpha\left(\frac{-K}{\rho^\alpha}t^\alpha\right) - \rho^{-\alpha}\int_0^t e^{\frac{\rho-1}{\rho}s}s^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-K}{\rho^\alpha}s^\alpha\right)\xi(s)ds \leq m(0)e^{\frac{\rho-1}{\rho}t}E_\alpha\left(\frac{-K}{\rho^\alpha}t^\alpha\right). \quad (22)$$

□

4. Stability of Neural Networks with a Generalized Proportional Caputo Fractional Derivative

The fractional-order Hopfield neural networks with the generalized proportional Caputo fractional derivative is described by the following equation:

$$({}_{0}^{C}\mathscr{D}^{\alpha,\rho}x_i)(t) = -a_i(t)x_i(t) + \sum_{k=1}^{n} b_{i,k}(t)f_k(x_k(t)) + I_i(t),\ \ t > 0,\ i = 1,2,\dots,n, \quad (23)$$

where n is the number of units in a neural network, ${}_{0}^{C}\mathscr{D}^{\alpha,\rho}$ denotes the generalized proportional Caputo fractional derivative of order $\alpha \in (0,1)$, $\rho \in (0,1]$, $x_i(t)$ is the state of the i-th unit at time t, $f_k(u)$ denotes the activation function of the k-th neuron, $b_{i,k}(t) : [0, \infty) \to \mathbb{R}$ denotes the connection weight of the k-th neuron on the i-th neuron at time t, $a_i(t) : [0, \infty) \to (0, \infty)$ represents the rate at which the i-th neuron resets its potential to the resting state when disconnected from the network at time t, and $I_i(t)$ denotes the external inputs at time t.

We will now define the equilibrium of the neural network (23). Different than the classical case of ordinary derivatives and the Caputo fractional derivatives, in the general case, the equilibrium of (23) could not be a constant because the generalized proportional derivative of a nonzero constant is not 0. Applying Proposition 1, we define the equilibrium of (23):

Definition 1. *The function* $x^*(t) = Ce^{\frac{\rho-1}{\rho}t} : C = (C_1, C_2, \dots, C_n) \in \mathbb{R}^n$, $c_i = const$, $i = 1, 2, \dots, n$, *is called an equilibrium of* (23) *if*

$$a_i(t)C_i e^{\frac{\rho-1}{\rho}t} = \sum_{k=1}^{n} b_{i,k}(t) f_k(C_k e^{\frac{\rho-1}{\rho}t}) + I_i(t),\ t \geq 0,\ \ i = 1, 2 \dots, n.$$

Remark 5. *The constant vector* $C \in \mathbb{R}^n$ *in Definition 1 could be a zero vector (zero equilibrium) or a nonzero vector (nonzero equilibrium).*

Remark 6. *The zero vector is an equilibrium of* (23) *if* $f_k(0) = 0$ *and* $I_k(t) \equiv 0$ *for all* $k = 1, 2, \dots, n$.

Let $x^*(t) = Ce^{\frac{\rho-1}{\rho}t}$ be an equilibrium of (23). Consider the change in the variables $u(t) = x(t) - x^*(t),\ t \geq 0$, in system (23), use Proposition 1 and obtain the following:

$$\begin{aligned}
&({}_0^C\mathscr{D}^{\alpha,\rho}u_i)(t) = ({}_0^C\mathscr{D}^{\alpha,\rho}x_i)(t) - ({}_0^C\mathscr{D}^{\alpha,\rho}x_i^*)(t) = ({}_0^C\mathscr{D}^{\alpha,\rho}x_i)(t) \\
&= -a_i(t)(u_i(t) + x_i^*(t)) + \sum_{k=1}^{n} b_{i,k}(t) f_k(u_k(t) + x_k^*(t)) + I_i(t) \\
&= -a_i(t)u_i(t) + \sum_{k=1}^{n} b_{i,k}(t)[f_k(u_k(t) + x_k^*(t)) - f_k(x_k^*(t))] - a_i(t)x_i^*(t) + \sum_{k=1}^{n} b_{i,k}(t) f_k(x_k^*(t)) + I_i(t) \\
&= -a_i(t)u_i(t) + \sum_{k=1}^{n} b_{i,k}(t) F_k(t, u_k(t)),\quad t > 0,\ i = 1, 2, \dots, n,
\end{aligned} \tag{24}$$

where $F_k(t, v) = f_k(v + x_k^*(t)) - f_k(x_k^*(t))$, i.e., if $x^*(t)$ is an equilibrium of (23), then the system

$$({}_0^C\mathscr{D}^{\alpha,\rho}u_i)(t) = -a_i(t)u_i(t) + \sum_{k=1}^{n} b_{i,k}(t) F_k(t, u_k(t)),\ \ t > 0,\ i = 1, 2, \dots, n, \tag{25}$$

has a zero solution, and vice versa.

Definition 2. *Let* $\alpha \in (0, 1)$ *and* $\rho \in (0, 1)$. *The equilibrium* $x^*(\cdot)$ *of* (23) *is called exponentially stable if, for any solution* $x(t)$ *of* (23), *the inequality*

$$\|x(t) - x^*(t)\| \leq m(\|x(0) - x^*(0)\|)e^{\lambda\frac{\rho-1}{\rho}t},\ t \geq 0,$$

holds, where $\lambda > 0$ *is a constant, and* $m(s) \geq 0,\ m(0) = 0$, *is a given locally Lipschitz function.*

Remark 7. *Note that the exponential stability is defined only for* $\rho \in (0, 1)$.

Remark 8. *The exponential stability of the equilibrium* $x^*(\cdot)$ *implies that every solution* $x(\cdot)$ *of* (23) *satisfies* $\lim_{t\to\infty} \|x(t) - x^*(t)\| = 0$.

Definition 3. *Let $\alpha \in (0,1)$ and $\rho \in (0,1]$. The equilibrium $x^*(\cdot)$ of (23) is called generalized Mittag–Leffler stable if there exist the positive constants λ, μ, and γ, such that for any solution $x(\cdot)$ of (23), the inequality*

$$\|x(t) - x^*(t)\| \le m(\|x(0) - x^*(0)\|)e^{\lambda \frac{\rho-1}{\rho} t}\Big(E_\alpha(-\mu t^\alpha)\Big)^\gamma, \ t \ge 0,$$

holds, where $E_\alpha(z)$ is the Mittag–Leffler function with one parameter, $m(s) \ge 0$, $m(0) = 0$, is a given locally Lipschitz function.

Remark 9. *Note that the generalized Mittag–Leffler stability is defined for $\rho \in (0,1]$ and for $\rho = 1$, and it generalizes the corresponding results for the Caputo fractional differential equations [6,12–15].*

Remark 10. *Note that the Mittag–Leffler stability for the Hopfield neural network with tempered fractional derivatives is studied in [10,16], but only for zero equilibrium, zero internal perturbations, and constant coefficients.*

Remark 11. *The generalized Mittag–Leffler stability of the equilibrium $x^*(\cdot)$ implies that every solution $x(\cdot)$ of (23) satisfies $\lim_{t\to\infty} \|x(t) - x^*(t)\| = 0$.*

Theorem 1 (Exponential stability)**.** *Let the following assumptions hold:*

1. *$\alpha \in (0,1)$ and $\rho \in (0,1)$;*
2. *The functions $a_i \in C(\mathbb{R}_+, (0,\infty)), b_{i,k}, I_i \in C(\mathbb{R}_+, \mathbb{R})$, $i, k = 1,2,\dots,n$;*
3. *There exist positive constants M_i, $i = 1,2,\dots,n$, such that the activation functions $f_i \in C(\mathbb{R}, \mathbb{R})$ satisfy $|f_i(v) - f_k(w)| \le M_i|v - w|$ for $v, w \in \mathbb{R}$;*
4. *Equation (23) has an equilibrium $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), \dots, x_n^*(\cdot))$;*
5. *The inequality*

$$2a_i(t) \ge \sum_{k=1}^n \Big(|b_{i,k}(t)| + M_i^2|b_{k,i}(t)|\Big), \ t \ge 0, \ i = 1,2,\dots,n$$

holds.

Then, the equilibrium $x^*(\cdot)$ of (23) is exponentially stable.

Proof. Let $x(\cdot)$ be a solution of (23), and consider the system (25) with $u(t) = x(t) - x^*(t)$, $t \ge 0$. From condition 3, we have the following equation:

$$|F_k(t,v)| = |f_k(v + x_k^*(t)) - f_k(0 + x_k^*(t))| \le M_k|v|,$$

for $v \in \mathbb{R}$, $t \ge 0$. Then,

$$\begin{aligned} u_i(t)({}_0^C\mathscr{D}^{\alpha,\rho}u_i)(t) &= -a_i(t)u_i^2(t) + \sum_{k=1}^n b_{i,k}(t)(u_i(t)F_k(t,u_k(t))) \\ &\le -a_i(t)u_i^2(t) + \sum_{k=1}^n |b_{i,k}(t)|0.5(u_i^2(t) + F_k^2(t,u_k(t))) \le 0.5\Big(-2a_i(t) + \sum_{k=1}^n |b_{i,k}(t)|\Big)u_i^2(t) + 0.5\sum_{k=1}^n |b_{i,k}(t)M_k^2u_k^2(t), \end{aligned} \tag{26}$$

and by applying Condition 5, we get the following:

$$\begin{aligned}
({}_0^C\mathscr{D}^{\alpha,\rho}(u^T(t)u(t)) &\leq 2u^T(t)({}_0^C\mathscr{D}^{\alpha,\rho}u)(t) = 2\sum_{i=1}^n u_i(t)({}_0^C\mathscr{D}^{\alpha,\rho}u_i)(t) \\
&\leq 0.5\sum_{i=1}^n \Big(-2a_i(t) + \sum_{k=1}^n |b_{i,k}(t)|\Big)u_i^2(t) + \sum_{i=1}^n\sum_{k=1}^n |b_{i,k}(t)M_k^2 u_k^2(t) \\
&= 0.5\sum_{i=1}^n \Big(-2a_i(t) + \sum_{k=1}^n |b_{i,k}(t)|\Big)u_i^2(t) + 0.5\sum_{i=1}^n M_i^2 u_i^2(t)\sum_{k=1}^n |b_{k,i}(t)| \\
&= 0.5\sum_{i=1}^n \Big[-2a_i(t) + \sum_{k=1}^n \Big(|b_{i,k}(t)| + M_i^2|b_{k,i}(t)|\Big)\Big]u_i^2(t) \leq 0.
\end{aligned} \tag{27}$$

According to Lemma 2 applied to the system in (25), with $t_0 = 0$, the inequality

$$\|u(t)\| \leq \|u(0)\|e^{\frac{\rho-1}{2\rho}t}, \quad t \geq 0 \tag{28}$$

holds. This proves the claim of the Theorem, with $\lambda = 0.5$ and $m(s) = s$. □

From Corollary 2 we obtain the following (applied to (23) with $t_0 = 0$):

Corollary 3 (Boundedness)**.** *Let $\alpha \in (0,1)$, $\rho \in (0,1]$, and conditions 2–5 of Theorem 1 are satisfied. Then, any solution $x(\cdot)$ of (23) satisfies $\|x(t) - x^*(t)\| \leq \|x(0) - x^*(0)\|$ for all $t \geq 0$.*

Theorem 2 (Generalized Mittag–Leffler stability)**.** *Let the following assumptions hold:*

1. *Conditions 1–4 of Theorem 1 are satisfied;*
2. *There exists a positive constant L, such that inequality*

$$2a_i(t) - \sum_{k=1}^n \Big(|b_{i,k}(t)| + M_i|b_{k,i}(t)|\Big) \geq L, \ t \geq 0, \ i = 1,2,\dots,n$$

holds.

Then, the equilibrium $x^*(\cdot)$ of (23) is Mittag–Leffler stable.

Proof. Let $x(\cdot)$ be a solution of (23) and consider the system in (25) with $u(t) = x(t) - x^*(t)$. Similar to the proof of Theorem 1, we prove the following inequality:

$$({}_0^C\mathscr{D}^{\alpha,\rho}(u^T(t)u(t)) \leq 0.5\sum_{i=1}^n \Big[-2a_i(t) + \sum_{k=1}^n \Big(|b_{i,k}(t)| + M_i^2|b_{k,i}(t)|\Big)\Big]u_i^2(t) \leq -0.5L\|u(t)\|^2. \tag{29}$$

Denote $m(t) = \|u(t)\|^2$, and from (29) and Condition 2 of Theorem 2, it follows that there exists a function $g(t) : [0,\infty) \to (-\infty, 0]$, such that

$$({}_0^C\mathscr{D}^{\alpha,\rho}m)(t) = -0.5Lm(t) + g(t), \quad t > 0. \tag{30}$$

According to Proposition 2, the solution of the linear Caputo proportional fractional initial value problem (30) is given by the following equation:

$$m(t) = m(0)e^{\frac{\rho-1}{\rho}t}E_\alpha\Big(-\frac{0.5L}{\rho^\alpha}t^\alpha\Big) + \rho^{-\alpha}\int_0^t e^{\frac{\rho-1}{\rho}(t-a)}s^{\alpha-1}E_{\alpha,\alpha}\Big(-\frac{0.5L}{\rho^\alpha}s^\alpha\Big)g(s)ds \leq m(0)e^{\frac{\rho-1}{\rho}t}E_\alpha\Big(-\frac{0.5L}{\rho^\alpha}t^\alpha\Big). \tag{31}$$

From inequality (31), it follows that

$$\|x(t) - x^*(t)\| \leq \|x(0) - x^*(0)\|e^{\frac{\rho-1}{2\rho}t}\Big(E_\alpha\Big(-\frac{0.5L}{\rho^\alpha}t^\alpha\Big)\Big)^{0.5}. \tag{32}$$

□

5. Applications

Example 1. *Consider the following neural networks of $n = 3$ neurons with a ring structure [6] with the following generalized proportional fractional derivatives:*

$$\begin{aligned}
({}_0^C\mathscr{D}^{\alpha,\rho}x_1)(t) &= -6x_1(t) + 2\sin(x_1(t)) + 2\sin(x_2(t)) + \sin(x_3(t)) + I_1(t),\\
({}_0^C\mathscr{D}^{\alpha,\rho}x_2)(t) &= -5x_2(t) - 2\sin(x_1(t)) - 0.4\sin(x_2(t)) + \sin(x_3(t)) + I_2(t),\\
({}_0^C\mathscr{D}^{\alpha,\rho}x_3)(t) &= -8x_3(t) + \sin(x_1(t)) - 2.5\sin(x_2(t)) + 3.5\sin(x_3(t)) + I_3(t), \quad t > 0,
\end{aligned} \tag{33}$$

where the activation functions are $f_k(x) = \sin(x)$, $k = 1,2,3$, i.e., condition 3 of Theorem 1 is satisfied by $M_i = 1$, $i = 1,2,3$.

Case 1. Let $I_i(t) \equiv K_i \neq 0$, $i = 1,2,3$ be constants. Then, for $\rho \in (0,1)$, the system in (33) has no equilibrium because, for example, the following equality:

$$-6C_1 e^{\frac{\rho-1}{\rho}t} = 2\sin\left(C_1 e^{\frac{\rho-1}{\rho}t}\right) + \sin\left(C_2 e^{\frac{\rho-1}{\rho}t}\right) - 3\sin\left(C_3 e^{\frac{\rho-1}{\rho}t}\right) + K_1,\ t \geq 0$$

is not satisfied by any constant C_i, $i = 1,2,3$ (compare with the case of the Caputo fractional derivative $\rho = 1$, [15]).

Case 2. Let $I_i(t) \equiv 0$, $i = 1,2,3$, $t \geq 0$. Then, for any $a_i(t) > 0$, the system in (33) has zero equilibrium because $\sin(0) = 0$ (see Remark 6).

Case 3. Consider the following neural network:

$$\begin{aligned}
({}_0^C\mathscr{D}^{\alpha,\rho}x_1)(t) &= -\sin\left(e^{\frac{\rho-1}{\rho}t}\right)x_1(t) + \frac{6}{\sin\left(e^{\frac{\rho-1}{\rho}t}\right)}\sin(x_1(t)) + e^{\frac{\rho-1}{\rho}t}\sin(x_3(t)) - 6,\\
({}_0^C\mathscr{D}^{\alpha,\rho}x_2)(t) &= -\sin\left(e^{\frac{\rho-1}{\rho}t}\right)x_2(t) + 0.5e^{\frac{\rho-1}{\rho}t}\sin(x_1(t)) + 0.5e^{\frac{\rho-1}{\rho}t}\sin(x_2(t)),\\
({}_0^C\mathscr{D}^{\alpha,\rho}x_3)(t) &= -\sin^2\left(e^{\frac{\rho-1}{\rho}t}\right)x_3(t) + \sin\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t}\sin(x_1(t)) - 2\sin(x_3(t)) + 2\sin\left(e^{\frac{\rho-1}{\rho}t}\right).
\end{aligned} \tag{34}$$

Thus, the coefficients are as follows:

$$a_1(t) = \sin\left(e^{\frac{\rho-1}{\rho}t}\right) > 0,\ a_2(t) = \sin\left(e^{\frac{\rho-1}{\rho}t}\right) > 0,\ a_3(t) = \sin^2\left(e^{\frac{\rho-1}{\rho}t}\right) > 0,$$

$$B = \{b_{i,k}(t)\} = \begin{bmatrix} \frac{6}{\sin\left(e^{\frac{\rho-1}{\rho}t}\right)} & 0 & e^{\frac{\rho-1}{\rho}t} \\ 0.5e^{\frac{\rho-1}{\rho}t} & 0.5e^{\frac{\rho-1}{\rho}t} & 0 \\ \sin\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t} & 0 & -2 \end{bmatrix},$$

and

$$I_1(t) = -6,\ I_2(t) = 0,\ I_3(t) = 2\sin\left(e^{\frac{\rho-1}{\rho}t}\right).$$

Then, for $\rho \in (0,1)$, the system in (33) has the equilibrium

$$\left(e^{\frac{\rho-1}{\rho}t}, e^{\frac{\rho-1}{\rho}t}, e^{\frac{\rho-1}{\rho}t}\right),$$

because

$$a_1 x_1^*(t) = \sin\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t} = \frac{6}{\sin\left(e^{\frac{\rho-1}{\rho}t}\right)}\sin\left(e^{\frac{\rho-1}{\rho}t}\right) + 0 + e^{\frac{\rho-1}{\rho}t}\sin\left(e^{\frac{\rho-1}{\rho}t}\right) - 6$$

$$a_2(t)x_2^*(t) = \sin\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t} = 0.5e^{\frac{\rho-1}{\rho}t}\sin\left(e^{\frac{\rho-1}{\rho}t}\right) + 0.5e^{\frac{\rho-1}{\rho}t}\sin\left(e^{\frac{\rho-1}{\rho}t}\right) + 0 + 0$$

$$a_3(t)x_3^*(t) = \sin^2\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t} = \sin\left(e^{\frac{\rho-1}{\rho}t}\right)e^{\frac{\rho-1}{\rho}t}\sin\left(e^{\frac{\rho-1}{\rho}t}\right) + 0 - 2\sin\left(e^{\frac{\rho-1}{\rho}t}\right) + 2\sin\left(e^{\frac{\rho-1}{\rho}t}\right)$$

hold.

Neither the conditions of Theorem 1 nor the conditions of Theorem 2 are satisfied. For example, the following inequality:

$$2\sin\left(e^{\frac{\rho-1}{\rho}t}\right) \geq \gamma(\rho,t) := 2\frac{6}{\sin\left(e^{\frac{\rho-1}{\rho}t}\right)} + 0.5e^{\frac{\rho-1}{\rho}t} + e^{\frac{\rho-1}{\rho}t}\left(1+\sin\left(e^{\frac{\rho-1}{\rho}t}\right)\right),\ t \geq 0 \quad (35)$$

is not satisfied (see Figure 1, top left). Therefore, we are not able to conclude the stability properties of the equilibrium (see Figure 2).

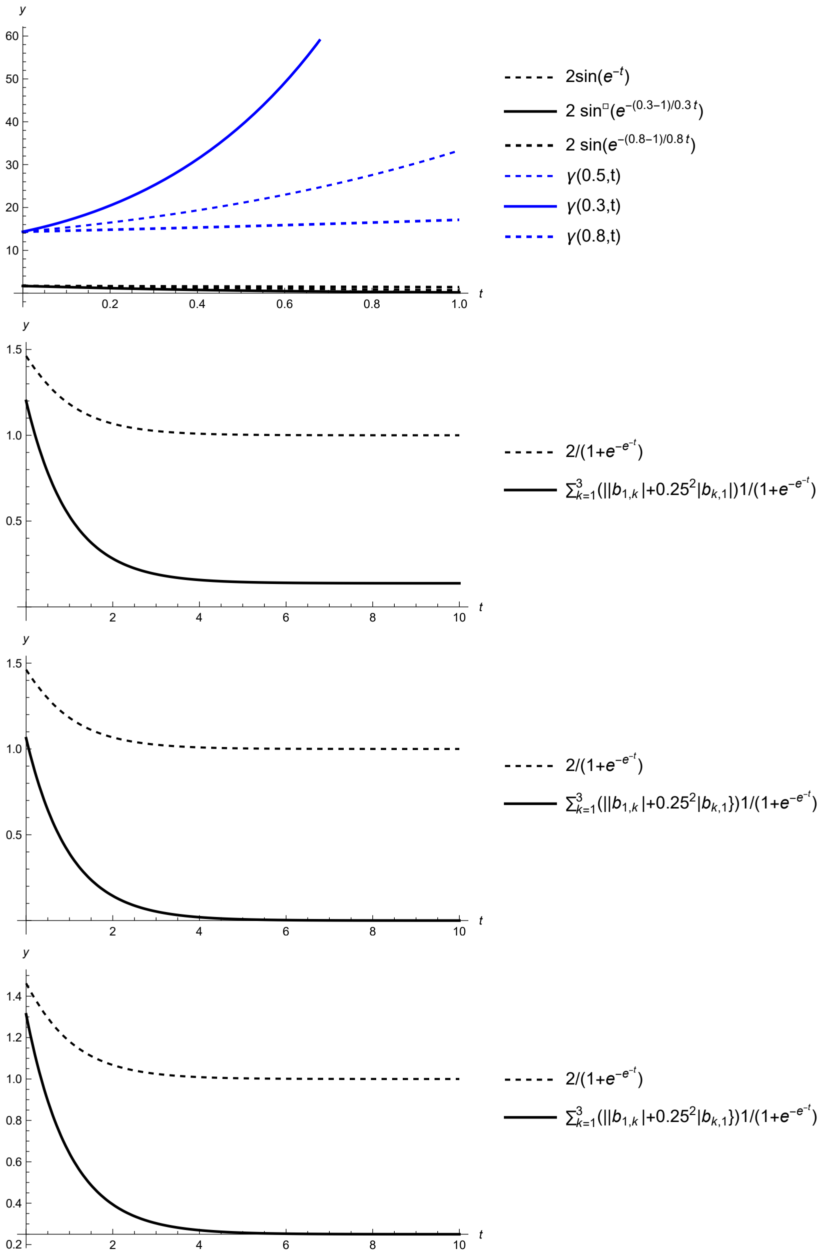

Figure 1. Graph of inequality (35) for various ρ (**1st plot**). Graph of inequality (37) for $\rho = 0.5$ (**2nd plot**), (38) for $\rho = 0.5$ (**3rd plot**), and (39) for $\rho = 0.5$ (**4th plot**).

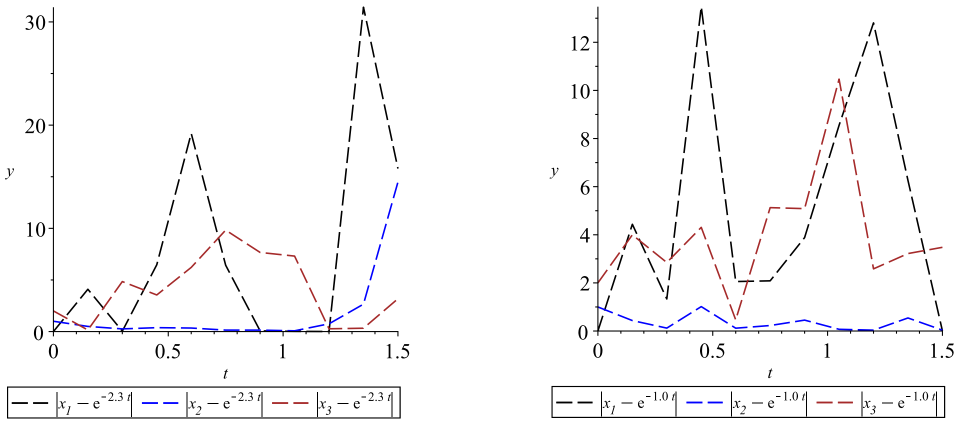

Figure 2. Graphs of functions $|x_i(t) - e^{\frac{\rho-1}{\rho}t}|$, with $i = 1,2,3$ and $\alpha = 0.6$. On the (**left**), $\rho = 0.3$, and on the (**right**), $\rho = 0.5$.

Example 2. *Consider the following neural networks of $n = 3$ neurons with the following generalized proportional fractional derivatives:*

$$\begin{aligned} ({}_0^C\mathscr{D}^{\alpha,\rho}x_1)(t) &= -\frac{1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}}x_1(t) + \frac{0.1}{1+e^{-x_1(t)}} + e^{\frac{\rho-1}{\rho}t}\frac{1}{1+e^{-x_3(t)}} + \frac{-0.1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}}, \\ ({}_0^C\mathscr{D}^{\alpha,\rho}x_2)(t) &= -\frac{1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}}x_2(t) + e^{\frac{\rho-1}{\rho}t}\frac{1}{1+e^{-x_1(t)}}, \\ ({}_0^C\mathscr{D}^{\alpha,\rho}x_3)(t) &= -\frac{1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}}x_3(t) + e^{\frac{\rho-1}{\rho}t}\frac{1}{1+e^{-x_1(t)}} + \frac{1}{1+e^{-x_3(t)}} + \frac{-1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}}, \end{aligned} \tag{36}$$

with the coefficients

$$a_k(t) = \frac{1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}} > 0, k = 1,2,3,$$

the activation functions $f_k(x) = 1/(1+e^{-x}) > 0,\ k = 1,2,3$ are equal to the sigmoid function, with $M_k = 0.25$, the perturbations are thus given by

$$I_1(t) = \frac{-0.1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}},\ I_2(t) = 0,\ I_3(t) = \frac{-1}{1+e^{-e^{\frac{\rho-1}{\rho}t}}},$$

and

$$B = \{b_{i,k}(t)\} = \begin{bmatrix} 0.1 & 0 & e^{\frac{\rho-1}{\rho}t} \\ e^{\frac{\rho-1}{\rho}t} & 0 & 0 \\ e^{\frac{\rho-1}{\rho}t} & 0 & 1 \end{bmatrix}.$$

Then, for $\rho \in (0,1)$, the system in (36) has the following equilibrium:

$$x^*(t) = \left(e^{\frac{\rho-1}{\rho}t}, e^{\frac{\rho-1}{\rho}t}, e^{\frac{\rho-1}{\rho}t}\right)$$

because

$$a_1 x_1^*(t) = \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} e^{\frac{\varrho-1}{\varrho}t} = 0.1\frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} + 0 + \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} e^{\frac{\varrho-1}{\varrho}t} + \frac{-0.1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}},$$

$$a_2(t) x_2^*(t) = \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} e^{\frac{\varrho-1}{\varrho}t} = \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} e^{\frac{\varrho-1}{\varrho}t} + 0 + 0 + 0,$$

$$a_3(t) x_3^*(t) = \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} e^{\frac{\varrho-1}{\varrho}t} = e^{\frac{\varrho-1}{\varrho}t}\frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} + 0 + \frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} + \frac{-1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}}.$$

Moreover, condition 5 of Theorem 1 is satisfied because of the following inequalities:

$$2\frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} \geq 1 \geq (1+0.25^2)0.1 + (0+0.5*0.25^2) + (1+0.25^2)e^{\frac{\varrho-1}{\varrho}t}, \tag{37}$$

$$2\frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} \geq (1+0*0.25^2) + (1+0.25^2)*0 + (1+0.25^2)*0, \tag{38}$$

$$2\frac{1}{1+e^{-e^{\frac{\varrho-1}{\varrho}t}}} \geq (1+0.25^2)e^{\frac{\varrho-1}{\varrho}t} + 0 + (1+0.25^2), \tag{39}$$

(see Figure 1, top right, bottom left, and bottom right, respectively).

From Theorem 1, the equilibrium is exponentially stable, i.e., (see Figure 3)

$$|x_i(t) - e^{\frac{\varrho-1}{\varrho}t}| \leq |x_i(0) - 1|e^{0.5\frac{\varrho-1}{\varrho}t},\ t \geq 0,\ i = 1,2,3.$$

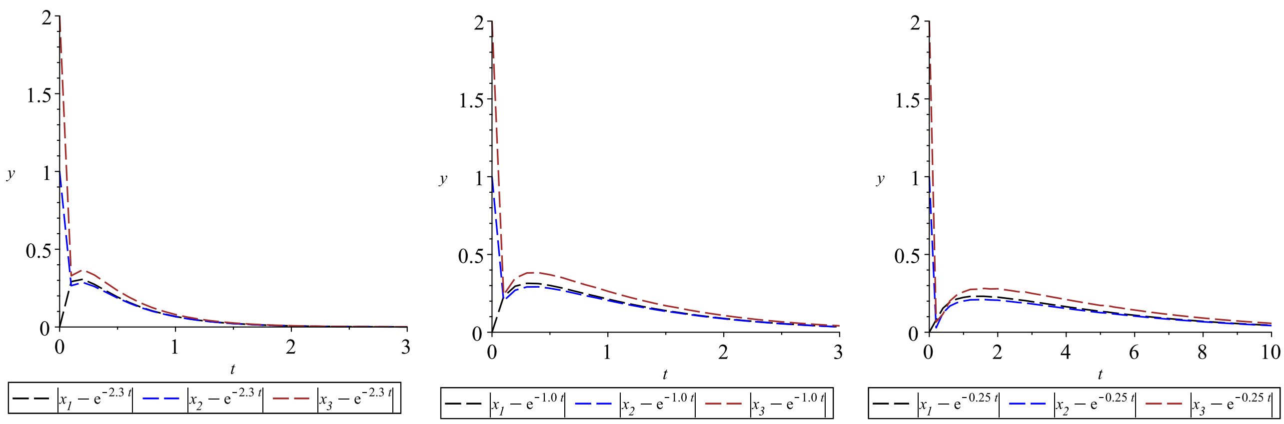

Figure 3. Graphs of the functions $|x_i(t) - e^{\frac{\varrho-1}{\varrho}t}|$, with $i = 1,2,3$, $\alpha = 0.6$, and $\rho = 0.3$ (**left**), $\rho = 0.5$ (**center**), and $\rho = 0.8$ (**right**).

6. Conclusions

Initially, we proved an important inequality concerning an estimate of the generalized proportional Caputo fractional derivative of quadratic functions. The result could be applied to the study of various types of stability for the solutions of various types of fractional differential equations with the generalized proportional Caputo fractional derivative. In our paper, we applied it to study the stability properties of the Hopfiel neural network with the generalized proportional Caputo type fractional derivative. An equilibrium of the studied model was then defined. This equilibrium is generally not a constant (different than the case of ordinary derivatives and the Caputo type fractional derivatives). We defined the exponential stability and the Mittag–Leffler stability of the equilibrium. Several sufficient conditions were presented to guarantee these types of stability. The theoretical results were illustrated, with two numerical examples.

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