A new approach to transport equations associated to a regular field: trace results and well-posedness.

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A NEW APPROACH TO TRANSPORT EQUATIONS ASSOCIATED TO A REGULAR FIELD: TRACE RESULTS AND WELL-POSEDNESS.

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ABSTRACT. We generalize known results on transport equations associated to a Lipschitz field $\mathscr{F}$ on some subspace of $\mathbb{R}^N$ endowed with some general space measure μ. We provide a new definition of both the transport operator and the trace measures over the incoming and outgoing parts of $\partial\mathbf{\Omega}$ generalizing known results from [9, 16]. We also prove the well-posedness of some suitable boundary-value transport problems and describe in full generality the generator of the transport semigroup with no-incoming boundary conditions.

1. INTRODUCTION

In this paper we present new methodological tools to investigate the well-posedness of the general transport equation

$$\partial_t f(\mathbf{x},t) + \mathscr{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x},t) = 0 \qquad (\mathbf{x} \in \mathbf{\Omega},\ t > 0), \tag{1.1a}$$

supplemented by boundary condition

$$f_{|\Gamma_-}(\mathbf{y},t) = \psi_-(\mathbf{y},t), \qquad (\mathbf{y} \in \Gamma_-, t > 0), \tag{1.1b}$$

and the initial condition

$$f(\mathbf{x},0) = f_0(\mathbf{x}), \qquad (\mathbf{x} \in \mathbf{\Omega}). \tag{1.1c}$$

Here $\mathbf{\Omega}$ is a sufficiently smooth open subset of $\mathbb{R}^N$, $\Gamma_\pm$ are suitable boundaries of the phase space and ψ_- is a given function of the trace space $L^1(\Gamma_-, \mathrm{d}\mu_-)$ corresponding to the boundary Γ_- (see Section 2 for details).

The present paper is part of a series of papers on transport equations with general vector fields [5, 6] and introduce all the methodological tools that allow us not only to solve the initial-boundary problem (1.1) but also to treat in [6] the case of abstract boundary conditions relying the incoming and outgoing fluxes, generalizing the results of [9].

The main novelty of our approach is that we assume $\mathbb{R}^N$ to be endowed with a general positive Radon measure μ. Here by a Radon measure we understand a Borel measure (or its completions, see [15, p. 332]) which is finite on compact sets. As we shall see it further on, taking into account such general Radon measure μ leads to a large amount of technical difficulties, in particular in the definition of trace spaces and in the derivation of Green's formula. Moreover, for such a measure μ, it is far from being trivial to identify the vector field $\mathscr{F} \cdot \nabla_x$ to the time derivative along the characteristic curves (as done in [9, Formulae (5.4) & (5.5), p.392]): the main difficulty stemming from the impossibility of applying classical convolution arguments (and the so-called Friedrich's lemma). We overcome this difficulty by introducing new mollification techniques along the characteristic curves. Let us explain in more details our general assumptions:

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1.1. **General assumption and motivations.** The transport coefficient $\mathscr{F}$ is a *time independent* vector field $\mathscr{F} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ which is (globally) *Lipschitz-continuous* with Lipschitz constant $\kappa > 0$, i.e.

$$|\mathscr{F}(\mathbf{x}_1) - \mathscr{F}(\mathbf{x}_2)| \leqslant \kappa|\mathbf{x}_1 - \mathbf{x}_2| \qquad \text{for any } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N. \tag{1.2}$$

Clearly, one can associate a flow $(T_t)_{t\in\mathbb{R}}$ to this field $\mathscr{F}$ (with the notations of Section 2.1, $T_t = \mathbf{\Theta}(\cdot, t, 0)$) and we make the following fundamental assumption (known as ***Liouville's Theorem*** whenever μ is the Lebesgue measure) on $\mathscr{F}$:

Assumption 1. *The measure μ is invariant under the flow $(T_t)_{t\in\mathbb{R}}$, i.e. $\mu(T_tA) = \mu(A)$ for any measurable subset $A \subset \mathbb{R}^N$ and any $t \in \mathbb{R}$.*

Remark 1.1. *Notice that, whenever μ is the Lebesgue measure over $\mathbb{R}^N$, it is well-known that Assumption 1 is equivalent to $\mathrm{div}(\mathscr{F}(\mathbf{x})) = 0$ for any $\mathbf{x} \in \mathbb{R}^N$. More generally, by virtue of* [2, Remark 3 & Proposition 4], *Assumption 1 holds for a general Borel measure μ provided the field $\mathscr{F}$ is locally integrable with respect to μ and **divergence-free** with respect to μ in the sense that*

$$\int_{\mathbb{R}^N} \mathscr{F}(T_t(\mathbf{x})) \cdot \nabla_{\mathbf{x}} f(T_t(\mathbf{x}))\mathrm{d}\mu(\mathbf{x}) = 0, \qquad \forall t \in \mathbb{R}$$

for any infinitely differentiable function f with compact support.

A typical example of such a transport equation is the so-called Vlasov equation for which:

i) The phase space $\mathbf{\Omega}$ is given by the cylindrical domain $\mathbf{\Omega} = \mathcal{D} \times \mathbb{R}^3 \subset \mathbb{R}^6$ where $\mathcal{D}$ is a sufficiently smooth open subset of $\mathbb{R}^3$, referred to as the *position space*, while the so–called *velocity space* is here given by $\mathbb{R}^3$. The measure μ is given by $\mathrm{d}\mu(\mathbf{x}) = \mathrm{d}x\mathrm{d}\beta(v)$ where β is a suitable Radon measure on $\mathbb{R}^3$, e.g. Lebesgue measure over $\mathbb{R}^3$ for continuous models or combination of Lebesgue measures over suitable spheres for the multigroup model.

ii) For any $\mathbf{x} = (x, v) \in \mathcal{D} \times \mathbb{R}^3$,

$$\mathscr{F}(\mathbf{x}) = (v, \mathbf{F}(x, v)) \in \mathbb{R}^6 \tag{1.3}$$

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is a time independent force field over $\mathcal{D}\times\mathbb{R}^3$ satisfying Assumption 1 and (1.2). The free transport case, investigated in [16, 4], corresponds to $\mathbf{F} = 0$.

The existence of solution to the transport equation (1.1a) is a classical matter when considering the whole space $\mathbf{\Omega} = \mathbb{R}^N$. In particular, the concept of renormalized solutions allows to consider irregular transport coefficient $\mathscr{F}(\cdot)$ (see [10] and the recent contributions [2, 13]) which is of particular relevance in fluid mechanics.

On the other hand, there are few results addressing the initial-boundary value problem (1.1), possibly due to difficulties created by the boundary conditions (1.1b). We mention here the seminal works by C. Bardos [8], and by R. Beals and V. Protopopescu [9] (see also [11, 14]). Let us however mention that the results of [9, 11] introduce restrictive assumptions on the characteristics of the equation. For instance, fields with 'too many' periodic trajectories create serious difficulties. They are however covered in a natural way by the theory presented here, see Examples 2.5 & 2.6.

1.2. **Presentation of the results.** In this paper, we revisit and generalize the afore-mentioned results to the general case $\mathbf{F} \neq 0$ and for a general Radon measure μ. The latter, in particular, leads to numerous technical problems such as e.g. determination of suitable measures $\mu_\pm$ over the 'incoming' and 'outgoing' parts $\Gamma_\pm$ of $\partial\mathbf{\Omega}$. We provide here a general construction of these 'trace measures' generalizing, and making more precise, the results of [9, 11]. This construction allows us to establish Proposition 2.12 which allows to compute integrals over $\mathbf{\Omega}$ via integration along the integral curves of $\mathscr{F}(\cdot)$ coming from the boundary $\partial\mathbf{\Omega}$, and which is free from some restrictive assumptions of *op. cit.* In particular, we present a new proof of the Green formula clarifying and removing gaps of the proofs in [9, 11]. Of course, the boundary condition (1.1b) we treat here is less general than the abstract ones investigated in [9, 11] but, as we already mentioned it, the tools we introduce here will allow us to generalize, in a subsequent paper [6], the results of the *op. cited* by dealing with abstract boundary conditions.

Another major difficulty, when dealing with a general Radon measure μ, is to provide a precise definition of the transport operator $\mathcal{T}_{\max}$ associated to (1.1). It appears quite natural to define the transport operator $\mathcal{T}_{\max}$ (with its maximal domain on $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$) as a *weak directional derivative along the characteristic curves* in the L^1-sense. However, it is not clear *a priori* that any function f for which the weak directional derivative exists in $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ (with appropriate and minimal class of test-functions) admits a trace over $\Gamma_{\pm}$. With the aim of proving such a trace result, we provide here a new characterization of the transport operator related to a *mild representation* of the solution to (1.1). Namely, we prove (Theorem 3.6) that the domain $\mathscr{D}(\mathcal{T}_{\max})$ (as defined in Section 3), is precisely the set of functions $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ that admits a representative which is *absolutely continuous along almost any characteristic curve*.

Note that in the classical case when μ is the Lebesgue measure, such a representation is known to be true [10, Appendix]. Actually, in this case, one defines the domain $\mathscr{D}(\mathcal{T}_{\max})$ as the set of all $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ for which the directional derivative $-\mathscr{F} \cdot \nabla f$ exists in the distributional sense and belongs to $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Then, by convolution arguments, it is well-known that the set $\mathscr{C}_0^1(\mathbf{\Omega}) \cap \mathscr{D}(\mathcal{T}_{\max})$ is dense in $\mathscr{D}(\mathcal{T}_{\max})$ for the graph norm $\|f\| = \|f\| + \|\mathscr{F} \cdot \nabla f\|$.

The question is much more delicate for a general Radon measure μ. Indeed, in such a case, the convolution argument used in the case of the Lebesgue measure does not apply anymore. Our strategy to prove the characterization of $\mathcal{T}_{\max}$ is also based on a convolution argument but it uses *mollification technique along the characteristic curves* as developed in Section 3. Such a result shall allow us to obtain a rigorous derivation of Green's formula, clarifying some results of [9].

1.3. **Plan of the paper.** The organization of the paper is as follows. In Section 2 we introduce main tools used throughout the paper and present the aforementioned new results concerning *integration over the characteristic curves* of $\mathscr{F}$ as well as *a new construction of the boundary measures* over the 'incoming' and 'outgoing' parts $\Gamma_{\pm}$ of $\partial\mathbf{\Omega}$ which generalizes and clarifies that of [9, 11]. In Section 3 we provide a construction of the maximal transport operator $\mathcal{T}_{\max}$. It is defined in a weak sense, through its action on suitably defined test functions. The fundamental result of this section shows that any function in the domain $\mathscr{D}(\mathcal{T}_{\max})$ admits a representation which is absolutely continuous along almost any characteristic which, in turn, allows for existence of its traces on the outgoing and incoming parts of the boundary. In Section 4 we apply the results of Section 3 to prove well-posedness of the time–dependent transport problem with no reentry boundary conditions associated with $\mathcal{T}_{\max}$. Moreover, we consider the corresponding stationary problem and, as a by-product, we recover a new proof of the *Green formula*.

2. Integration along the characteristics

2.1. **Characteristic curves.** A crucial role in our study is played by the characteristic curves associated to the field $\mathscr{F}$. Precisely, for any $\mathbf{x} \in \mathbb{R}^N$ and $t \in \mathbb{R}$, consider the initial-value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathbf{X}(s) = \mathscr{F}(\mathbf{X}(s)), \qquad (s \in \mathbb{R}); \\ \mathbf{X}(t) = \mathbf{x}. \end{cases} \tag{2.1}$$

Since $\mathscr{F}$ is Lipschitz continuous on $\mathbb{R}^N$, Eq. (2.1) has a unique *global in time* solution and this allows to define the flow–mapping $\mathbf{\Theta} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$, such that, for $(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R}$, the mapping:

$$\mathbf{X}(\cdot)\ :\ s \in \mathbb{R} \longmapsto \mathbf{\Theta}(\mathbf{x}, t, s)$$

is the only solution of Eq. (2.1). Being concerned with solutions to the transport equation (1.1) in the region $\mathbf{\Omega}$, we have to introduce the definition of stay times of the characteristic curves in $\mathbf{\Omega}$:

Definition 2.1. *For any* $\mathbf{x} \in \mathbf{\Omega}$*, define* $\tau_{\pm}(\mathbf{x}) = \inf\{s > 0\,;\, \mathbf{\Theta}(\mathbf{x}, 0, \pm s) \notin \mathbf{\Omega}\}$*, with the convention that* $\inf \varnothing = \infty$*, and set* $\tau(\mathbf{x}) = \tau_+(\mathbf{x}) + \tau_-(\mathbf{x})$.

In other words, given $\mathbf{x} \in \mathbf{\Omega}$, $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is the maximal interval for which $\mathbf{\Theta}(\mathbf{x}, 0, s)$ lies in $\mathbf{\Omega}$ for any $s \in I_{\mathbf{x}}$ and $\tau(\mathbf{x})$ is the length of the interval $I_{\mathbf{x}}$. Notice that $0 \leqslant \tau_{\pm}(\mathbf{x}) \leqslant \infty$. Thus, the function $\mathbf{\Theta}$ restricted to the set

$$\mathbf{\Lambda} := \Big\{ (\mathbf{x}, t, s)\,;\, \mathbf{x} \in \mathbf{\Omega},\, t \in \mathbb{R}\,,\, s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))\ \Big\}$$

is such that $\mathbf{\Theta}(\mathbf{\Lambda}) = \mathbf{\Omega}$. Note that here we ***do not*** assume that the length of the interval $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}),\ \tau_+(\mathbf{x}))$ is ***finite***. In particular, $I_{\mathbf{x}} = \mathbb{R}$ for any stationary point $\mathbf{x}$ of $\mathscr{F}$, i.e. $\mathscr{F}(\mathbf{x}) = 0$. If $\tau(\mathbf{x})$ is finite, then the function $\mathbf{X}\ :\ s \in I_{\mathbf{x}} \longmapsto \mathbf{\Theta}(\mathbf{x}, 0, s)$ is bounded since $\mathscr{F}$ is Lipschitz continuous. Moreover, still by virtue of the Lipschitz continuity of $\mathscr{F}$, the only case when $\tau_{\pm}(\mathbf{x})$ is finite is when $\mathbf{\Theta}(\mathbf{x}, 0, \pm s)$ reaches the boundary $\partial\mathbf{\Omega}$ so that $\mathbf{\Theta}(\mathbf{x}, 0, \pm\tau_{\pm}(\mathbf{x})) \in \partial\mathbf{\Omega}$. We note that, since $\mathscr{F}$ is Lipschitz around each point of $\partial\mathbf{\Omega}$, the points of the set $\{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \mathscr{F}(\mathbf{y}) = 0\}$ (introduced in [9, 11]) are equilibrium points of the $\mathscr{F}$ and cannot be reached in finite time.

Remark 2.2. *We emphasize that periodic trajectories which do not meet the boundaries have* $\tau_{\pm} = \infty$ *and thus are treated as infinite though geometrically they are bounded.*

Finally we mention that it is not difficult to prove that the mappings $\tau_{\pm} : \mathbf{\Omega} \to \mathbb{R}^+$ are lower semicontinuous and therefore measurable, see e.g., [7, p. 301]

The flow $\mathbf{\Theta}(\mathbf{x}, t, s)$ defines, at each instant t, a mapping of the phase space $\mathbf{\Omega}$ into $\mathbb{R}^N$. Through this mapping, to each point $\mathbf{x}$ there corresponds the point $\mathbf{x}_{s,t} = \mathbf{\Theta}(\mathbf{x}, t, s)$ reached at time s by the point which was at $\mathbf{x}$ at the 'initial' time t. The flow $\mathbf{\Theta}$, restricted to $\mathbf{\Lambda}$, has the properties:

Proposition 2.3. *Let* $\mathbf{x} \in \mathbf{\Omega}$ *and* $t \in \mathbb{R}$ *be fixed. Then,*

(i) $\mathbf{\Theta}(\mathbf{x}, t, t) = \mathbf{x}$.
(ii) $\mathbf{\Theta}(\mathbf{\Theta}(\mathbf{x}, t, s_1), s_1, s_2) = \mathbf{\Theta}(\mathbf{x}, t, s_2), \quad \forall s_1, s_2 \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
(iii) $\mathbf{\Theta}(\mathbf{x}, t, s) = \mathbf{\Theta}(\mathbf{x}, t - s, 0) = \mathbf{\Theta}(\mathbf{x}, 0, s - t), \qquad \forall s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
(iv) $|\mathbf{\Theta}(\mathbf{x}_1, t, s) - \mathbf{\Theta}(\mathbf{x}_2, t, s)| \leqslant \exp(\kappa|t - s|)|\mathbf{x}_1 - \mathbf{x}_2|$ *for any* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{\Omega}$*,* $s - t \in I_{\mathbf{x}_1} \cap I_{\mathbf{x}_2}$.

An important consequence of (iii) above is that $\mathbf{\Theta}(\mathbf{x}, 0, s) = \mathbf{\Theta}(\mathbf{x}, -s, 0)$ for any $\mathbf{x} \in \mathbf{\Omega}$, $0 \leqslant s \leqslant \tau_+(\mathbf{x})$. Therefore, from now on, to shorten notations we shall denote

$$\mathbf{\Phi}(\mathbf{x}, t) = \mathbf{\Theta}(\mathbf{x}, 0, t), \qquad \forall t \in \mathbb{R},$$

so that $\mathbf{\Phi}(\mathbf{x}, -t) = \mathbf{\Theta}(\mathbf{x}, t, 0)$, $t \in \mathbb{R}$. We define the incoming and outgoing part of the boundary $\partial\mathbf{\Omega}$ through the flow $\mathbf{\Phi}$:

Definition 2.4. *The incoming* Γ_- *and the outgoing* Γ_+ *parts of the boundary* $\partial\mathbf{\Omega}$ *are defined by:*

$$\Gamma_{\pm} := \{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \exists \mathbf{x} \in \mathbf{\Omega},\ \tau_{\pm}(\mathbf{x}) < \infty \text{ and } \mathbf{y} = \mathbf{\Phi}(\mathbf{x}, \pm\tau_{\pm}(\mathbf{x}))\,\}\,. \tag{2.2}$$

Properties of $\mathbf{\Phi}$ and of $\tau_\pm$ imply that $\Gamma_\pm$ are Borel sets. It is possible to extend the definition of $\tau_\pm$ to $\Gamma_\pm$ as follows. If $\mathbf{x}\in\Gamma_-$ then we put $\tau_-(\mathbf{x})=0$ and denote $\tau_+(\mathbf{x})$ the length of the integral curve having $\mathbf{x}$ as its left end–point; similarly if $\mathbf{x}\in\Gamma_+$ then we put $\tau_+(\mathbf{x})=0$ and denote $\tau_-(\mathbf{x})$ the length of the integral curve having $\mathbf{x}$ as its right endpoint. Note that this definition implies that $\tau_\pm$ are measurable over $\mathbf{\Omega}\cup\Gamma_-\cup\Gamma_+$.

Let us illustrate the above definition of $\Gamma_\pm$ by two simple $2D$ examples:

Example 2.5 (***Harmonic oscillator in a rectangle***)**.** Let $\mathbf{\Omega}=(-a,a)\times(-\xi,\xi)$ with $a,\xi>0$ and let us consider the harmonic oscillator force field

$$\mathscr{F}(\mathbf{x})=(v,-\omega^2 x),\qquad \text{for any } \mathbf{x}=(x,v)\in\mathbf{\Omega} \tag{2.3}$$

where $\omega>0$. We take as μ the Lebesgue measure over $\mathbb{R}^2$ and, since $\mathscr{F}$ is divergence-free, Assumption 1 is fulfilled. In this case, for any $\mathbf{x}_0=(x_0,v_0)\in\mathbf{\Omega}$, the solution $(x(t),v(t))=\mathbf{\Phi}(\mathbf{x}_0,t)$ to the characteristic equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t)=\mathscr{F}(\mathbf{X}(t))\,,\mathbf{X}(0)=\mathbf{x}_0$, given by

$$\mathbf{\Phi}(\mathbf{x}_0,t)=\Big(x_0\cos(\omega t)+\frac{v_0}{\omega}\sin(\omega t)\,;\,-x_0\omega\sin(\omega t)+v_0\cos(\omega t)\Big)\,,$$

is such that

$$\omega^2x^2(t)+v^2(t)=\omega^2x_0^2+v_0^2,\qquad t\in(-\tau_-(\mathbf{x}_0),\tau_+(\mathbf{x}_0))$$

which means that the integral curves associated to $\mathscr{F}$ are *ellipses* centered at $(0,0)$ and oriented in the counterclockwise direction. Now,

$$\partial\mathbf{\Omega}=\Big(\{-a\}\times[-\xi,\xi]\Big)\bigcup\Big(\{a\}\times[-\xi,\xi]\Big)\bigcup\Big([-a,a]\times\{-\xi\}\Big)\bigcup\Big([-a,a]\times\{\xi\}\Big)$$

and it is easy to check that

$$\Gamma_\pm=\Big(\{\pm a\}\times(-\xi,0]\Big)\bigcup\Big(\{\mp a\}\times[0,\xi)\Big)\bigcup\Big([0,a)\times\{\pm\xi\}\Big)\bigcup\Big((-a,0]\times\{\mp\xi\}\Big).$$

Notice that $\Gamma_+\cap\Gamma_-=\{(a,0),\ (0,\xi),\ (-a,0),\ (0,-\xi)\}$ and

$$\partial\mathbf{\Omega}\setminus(\Gamma_+\cup\Gamma_-)=\{(a,\xi),(a,-\xi),(-a,\xi),(-a,\xi)\}$$

is a discrete set (of linear Lebesgue measure zero).

Example 2.6 (***Hamonic oscillator in a stadium***)**.** Consider now the two-dimensional phase space (where $\mathbb{R}^2$ is still endowed with the Lebesgue measure μ):

$$\mathbf{\Omega}=\{\mathbf{x}=(x,v)\in\mathbb{R}^2\,;\,x^2+v^2<2\ \text{ and }\ -1<v<1\}$$

and consider the harmonic oscillator force field $\mathscr{F}$ given by (2.3) with $\omega=1$ for simplicity. Then, the integral curves associated to $\mathscr{F}$ are *circles* centered at $(0,0)$ and oriented in the counterclockwise direction. In this case, one can see that

$$\Gamma_\pm=\{(x,-1)\,;\,-1<\pm x\leqslant 0\}\cup\{(x,1)\,;\,0\leqslant\pm x<1\}.$$

In particular, one sees that $\partial\mathbf{\Omega}\setminus\big(\Gamma_+\cup\Gamma_-\big)=\big\{(x,v)\in\mathbb{R}^2\,;\,x^2+v^2=2\,;\,-1\leqslant v\leqslant 1\big\}$ is a 'big' part of the boundary $\partial\mathbf{\Omega}$ (with positive linear Lebesgue measure). Notice also that $\tau_+(\mathbf{x})=+\infty$ for any $\mathbf{x}=(x,v)$ with $x^2+v^2<1$.

The main aim of the present discussion is to represent $\mathbf{\Omega}$ as a collection of characteristics running between points of Γ_- and Γ_+ so that the integral over $\mathbf{\Omega}$ can be split into integrals over Γ_- (or Γ_+) and along the characteristics. However, at present we cannot do this in a precise way since, in general, the sets Γ_+ and Γ_- do not provide a partition of $\partial\mathbf{\Omega}$ as there may be 'too many' characteristics which extend to infinity on either side. Since we have not assumed $\mathbf{\Omega}$ to be bounded, Γ_- or Γ_+ may be empty and also we may have characteristics running from $-\infty$ to $+\infty$ such as periodic ones. Thus, in general, characteristics starting from Γ_- or ending at Γ_+ would not fill the whole $\mathbf{\Omega}$ and, to proceed, we have to construct an auxiliary set by extending $\mathbf{\Omega}$ into the time domain and use the approach of [9] which is explained below.

2.2. **Integration along characteristics.** For any $0 < T < \infty$, we define the domain

$$\mathbf{\Omega}_T = \mathbf{\Omega} \times (0, T)$$

and the measure $\mathrm{d}\mu_T = \mathrm{d}\mu \otimes \mathrm{d}t$ on $\mathbf{\Omega}_T$. Consider the vector field over $\mathbf{\Omega}_T$:

$$Y = \partial_t + \mathscr{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} = \mathscr{A}(\xi) \cdot \nabla_\xi$$

where $\mathscr{A}(\xi) = (\mathscr{F}(\mathbf{x}), 1)$ for any $\xi = (\mathbf{x}, t)$. We can define the characteristic curves of $\mathscr{A}$ as the solution $\xi(s) = (\mathbf{X}(s), \theta(s))$ to the system $\dfrac{\mathrm{d}}{\mathrm{d}s}\xi(s) = \mathscr{A}(\xi(s))$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{X}(s) = \mathscr{F}(\mathbf{X}(s)), \quad \frac{\mathrm{d}}{\mathrm{d}s}\theta(s) = 1, \qquad (s \in \mathbb{R}),$$

with

$$\mathbf{X}(0) = \mathbf{x}, \quad \theta(0) = t.$$

It is clear that the solution $\xi(s)$ to the above system is given by

$$\mathbf{X}(s) = \mathbf{\Phi}(\mathbf{x}, s), \qquad \theta(s) = s + t,$$

and we can define the flow of solution $\mathbf{\Psi}(\xi, s) = (\mathbf{\Phi}(\mathbf{x}, s), s+t)$ associated to $\mathscr{A}$ and the existence times of the characteristic curves of Y are defined, for any $\xi = (\mathbf{x}, t) \in \mathbf{\Omega}_T$, as

$$\ell_\pm(\xi) = \inf\{s > 0, (\mathbf{\Phi}(\mathbf{x}, \pm s), \pm s + t) \notin \mathbf{\Omega}_T\}.$$

The flow $\mathbf{\Psi}(\cdot, \cdot)$ enjoys, *mutatis mutandis*, the properties listed in Proposition 2.3 and μ_T is invariant under $\mathbf{\Psi}$. Moreover, since $\mathscr{A}$ is clearly Lipschitz continuous on $\overline{\mathbf{\Omega}_T}$, no characteristic of Y can escape to infinity in finite time. In other words, all characteristic curves of Y now have finite lengths. Indeed, if $\mathbf{\Phi}(\mathbf{x}, \pm s)$ does not reach $\partial\mathbf{\Omega}$, then the characteristic curve $\mathbf{\Psi}(\xi, \pm s)$ enters or leaves $\mathbf{\Omega}_T$ through the bottom $\mathbf{\Omega} \times \{0\}$, or through the top $\mathbf{\Omega} \times \{T\}$ of it. Precisely, it is easy to verify that for $\xi = (\mathbf{x}, t) \in \mathbf{\Omega}_T$ we have

$$\ell_+(\xi) = \tau_+(\mathbf{x}) \wedge (T - t) \quad \text{and} \quad \ell_-(\xi) = \tau_-(\mathbf{x}) \wedge t,$$

where $\wedge$ denotes minimum. This clearly implies $\sup\{\ell_\pm(\xi)\,;\, \xi \in \mathbf{\Omega}_T\} \leqslant T$. Define now

$$\Sigma_{\pm,T} = \{\zeta \in \partial\mathbf{\Omega}_T\,;\, \exists \xi \in \mathbf{\Omega}_T \text{ such that } \zeta = \mathbf{\Psi}(\xi, \pm\ell_\pm(\xi))\}.$$

The definition of $\Sigma_{\pm,T}$ is analogous to $\Gamma_\pm$ with the understanding that now the charateristic curves correspond to the vector field $\mathscr{A}$. In other words, $\Sigma_{-,T}$ (resp. $\Sigma_{+,T}$) is the subset of $\partial\mathbf{\Omega}_T$ consisting of all left (resp. right) limits of characteristic curves of $\mathscr{A}$ in $\mathbf{\Omega}_T$ whereas Γ_- (resp. Γ_+) is the subset of $\partial\mathbf{\Omega}$ consisting of all left (resp. right) limits of characteristic curves of $\mathscr{F}$ in $\mathbf{\Omega}$. The main difference (and the interest of such a lifting to $\mathbf{\Omega}_T$) is the fact that *each characteristic curve of $\mathscr{A}$ does reach the boundaries $\Sigma_{\pm,T}$ in finite time*. The above formulae allow us to extend functions $\ell_\pm$ to $\Sigma_{\pm,T}$ in the same way as we extended the functions $\tau_\pm$ to $\Gamma_\pm$. With these considerations, we can represent, up to a set of zero measure, the phase space $\mathbf{\Omega}_T$ as

$$\begin{aligned}\mathbf{\Omega}_T &= \{\mathbf{\Psi}(\xi, s)\,;\, \xi \in \Sigma_{-,T}\,,\, 0 < s < \ell_+(\xi)\} \\ &= \{\mathbf{\Psi}(\xi, -s)\,;\, \xi \in \Sigma_{+,T}\,,\, 0 < s < \ell_-(\xi)\}.\end{aligned} \tag{2.4}$$

With this realization we can prove the following:

Proposition 2.7. *Let $T > 0$ be fixed. There are unique positive Borel measures $\mathrm{d}\nu_{\pm}$ on $\Sigma_{\pm,T}$ such that $\mathrm{d}\mu_T = \mathrm{d}\nu_+ \otimes \mathrm{d}s = \mathrm{d}\nu_- \otimes \mathrm{d}s$.*

Proof. For any $\delta > 0$, define $\mathscr{E}_\delta$ as the set of all bounded Borel subsets E of $\Sigma_{-,T}$ such that $\ell_+(\xi) > \delta$ for any $\xi \in E$. Let us now fix $E \in \mathscr{E}_\delta$. For all $0 < \sigma \leqslant \delta$ put

$$E_\sigma = \{\mathbf{\Psi}(\xi, s)\,;\, \xi \in E, 0 < s \leqslant \sigma\}.$$

Clearly E_σ is a measurable subset of $\mathbf{\Omega}_T$. Define the mapping $h : \sigma \in (0, \delta] \mapsto h(\sigma) = \mu_T(E_\sigma)$ with $h(0) = 0$. If σ_1 and σ_2 are two positive numbers such that $\sigma_1 + \sigma_2 \leqslant \delta$, then

$$E_{\sigma_1+\sigma_2} \setminus E_{\sigma_1} = \{\mathbf{\Psi}(\xi, s)\,;\, \xi \in E, \sigma_1 < s \leqslant \sigma_1 + \sigma_2\} = \{\mathbf{\Psi}(\eta, \sigma_1)\,;\, \eta \in E_{\sigma_2}\}.$$

The properties of the flow $\mathbf{\Psi}$ (see Proposition 2.3) ensure that the mapping $\eta \mapsto \mathbf{\Psi}(\eta, \sigma_1)$ is one-to-one and measure preserving, so that

$$\mu_T(E_{\sigma_1+\sigma_2} \setminus E_{\sigma_1}) = \mu_T(E_{\sigma_2}) = h(\sigma_2).$$

Since $E_{\sigma_1+\sigma_2} = E_{\sigma_1} \cup (E_{\sigma_1+\sigma_2} \setminus E_{\sigma_1})$, we immediately obtain

$$h(\sigma_1 + \sigma_2) = h(\sigma_1) + h(\sigma_2) \qquad \text{for any} \qquad \sigma_1,\, \sigma_2 > 0 \ \text{with}\ \sigma_1 + \sigma_2 \leqslant \delta. \tag{2.5}$$

This is the well-known Cauchy equation, though defined only on an interval of the real line. It can be solved in a standard way using non-negativity instead of continuity, yielding:

$$h(\sigma) = c_E \sigma \qquad \text{for any} \quad 0 < \sigma \leqslant \delta$$

where $c_E = h(\delta)/\delta$. We define $\nu_-(E) = c_E$. It is not difficult to see that, with the above procedure, the mapping $\nu_-(\cdot)$ defines a positive measure on the ring $\mathscr{E} = \bigcup_{\delta>0} \mathscr{E}_\delta$ of all the Borel subsets of $\Sigma_{-,T}$ on which the function $\ell_+(\xi)$ is bounded away from 0. Such a measure ν_- can be uniquely extended to the σ-algebra of the Borel subsets of $\Sigma_{-,T}$ (see e.g. [12, Theorem A, p. 54]). Consider now a Borel subset E of $\Sigma_{-,T}$ and a Borel subset I of $\mathbb{R}^+$, such that for all $\xi \in E$ and $s \in I$ we have $0 < s < \ell_+(\xi)$. Then

$$E \times I = \{\mathbf{\Psi}(\xi, s)\,;\, \xi \in E, s \in I\} \subset \mathbf{\Omega}_T.$$

Thanks to the definition of $\nu_-(\cdot)$, we can state that $\mu_T(E \times I) = \nu_-(E)\mathrm{meas}(I)$ where $\mathrm{meas}(I)$ denotes the linear Lebesgue measure of $I \subset \mathbb{R}$. This shows that $\mathrm{d}\mu_T = \mathrm{d}\nu_- \otimes \mathrm{d}s$. Similarly we can define a measure ν_+ on $\Sigma_{+,T}$ and prove that $\mathrm{d}\mu_T = \mathrm{d}\nu_+ \otimes \mathrm{d}s$. The uniqueness of the measures $\mathrm{d}\nu_{\pm}$ is then obvious. □

Remark 2.8. *Note that the above construction of the Borel measures $\mathrm{d}\nu_{\pm}$ differs from that of* [11, Lemmas XI.3.1 & 3.2], [9, Propositions 7 & 8] *which , moreover, only apply when μ is absolutely continuous with respect to the Lebesgue measure. Our construction is much more general and can also be generalized to the case of a non–divergence force field $\mathscr{F}$,* [5].

Next, by the cylindrical structure of $\mathbf{\Omega}_T$, and the representation of $\Sigma_{\pm,T}$ as

$$\Sigma_{-,T} = (\Gamma_- \times (0, T)) \cup \mathbf{\Omega} \times \{0\} \quad \text{and} \quad \Sigma_{+,T} = (\Gamma_+ \times (0, T)) \cup \mathbf{\Omega} \times \{T\},$$

the measures $\mathrm{d}\nu_{\pm}$ over $\Gamma_{\pm} \times (0, T)$ can be written as $\mathrm{d}\nu_{\pm} = \mathrm{d}\mu_{\pm} \otimes \mathrm{d}t$, where $\mathrm{d}\mu_{\pm}$ are Borel measures on $\Gamma_{\pm}$. This leads to the following

Lemma 2.9. *There are unique positive Borel measures* $\mathrm{d}\mu_\pm$ *on* $\Gamma_\pm$ *such that, for any* $f \in L^1(\mathbf{\Omega}_T, \mathrm{d}\mu_T)$

$$\begin{aligned}\int_{\mathbf{\Omega}_T} f(\mathbf{x},t)\mathrm{d}\mu_T(\mathbf{x},t) &= \int_0^T \mathrm{d}t \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})\wedge t} f(\mathbf{\Phi}(\mathbf{y},-s), t-s)\mathrm{d}s \\ &+ \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})\wedge T} f(\mathbf{\Phi}(\mathbf{x},-s), T-s)\mathrm{d}s,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\int_{\mathbf{\Omega}_T} f(\mathbf{x},t)\mathrm{d}\mu_T(\mathbf{x},t) &= \int_0^T \mathrm{d}t \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})\wedge(T-t)} f(\mathbf{\Phi}(\mathbf{y},s), t+s)\mathrm{d}s \\ &+ \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x})\wedge T} f(\mathbf{\Phi}(\mathbf{x},s), s)\mathrm{d}s.\end{aligned} \tag{2.7}$$

The above fundamental result allows to compute integrals over the cylindrical phase-space $\mathbf{\Omega}_T$ through integration along the characteristic curves. Let us now generalize it to the phase space $\mathbf{\Omega}$. Here the main difficulty stems from the fact that the characteristic curves of the vector field $\mathscr{F}$ are no longer assumed to be of finite length. In order to extend Lemma 2.9 to possibly infinite existence times, first we prove the following:

Lemma 2.10. *Let* $T > 0$ *be fixed. Then,* $\tau_+(\mathbf{x}) < T$ *for any* $\mathbf{x} \in \mathbf{\Omega}$ *if and only if* $\tau_-(\mathbf{x}) < T$ *for any* $\mathbf{x} \in \mathbf{\Omega}$.

Proof. It is easy to see that $\tau_+(\mathbf{x}) < T$ for any $\mathbf{x} \in \mathbf{\Omega}$ is equivalent to $\tau(\mathbf{x}) < T$ for any $\mathbf{x} \in \mathbf{\Omega}$ and this is also equivalent to $\tau_-(\mathbf{x}) < T$ for any $\mathbf{x} \in \mathbf{\Omega}$. □

Hereafter, the support of a measurable function f defined on $\mathbf{\Omega}$ is defined as $\mathrm{Supp} f = \mathbf{\Omega} \setminus \omega$ where ω is the maximal open subset of $\mathbf{\Omega}$ on which f vanishes $\mathrm{d}\mu$–almost everywhere.

Proposition 2.11. *Let* $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. *Assume that there exists* $\tau_0 > 0$ *such that* $\tau_\pm(\mathbf{x}) < \tau_0$ *for any* $\mathbf{x} \in \mathrm{Supp}(f)$. *Then,*

$$\begin{aligned}\int_{\mathbf{\Omega}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y},-s)\right)\mathrm{d}s \\ &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s.\end{aligned} \tag{2.8}$$

Proof. For any $T > \tau_0$, define the domain $\mathbf{\Omega}_T = \mathbf{\Omega} \times (0,T)$. Since $T < \infty$, it is clear that $f \in L^1(\mathbf{\Omega}_T, \mathrm{d}\mu\mathrm{d}t)$ and, by (2.6), we get

$$T \int_{\mathbf{\Omega}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_0^T \mathrm{d}t \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{t\wedge\tau_-(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s + \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})} f(\mathbf{\Phi}(\mathbf{x},-s))\mathrm{d}s.$$

Since the formula is valid for any $T > \tau_0$, differentiating with respect to T leads to the first assertion. The second assertion is proved in the same way by using formula (2.7). □

To drop the finiteness assumption on $\tau_\pm(\mathbf{x})$, first we introduce the sets

$$\mathbf{\Omega}_\pm = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \tau_\pm(\mathbf{x}) < \infty\}, \qquad \mathbf{\Omega}_{\pm\infty} = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \tau_\pm(\mathbf{x}) = \infty\},$$

and

$$\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_\pm\,;\, \tau_\mp(\mathbf{y}) = \infty\}.$$

Then

Proposition 2.12. *Given $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, one has*

$$\int_{\mathbf{\Omega}_\pm} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_\pm} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\tau_\mp(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y}, \mp s)\right)\mathrm{d}s, \tag{2.9}$$

and

$$\int_{\mathbf{\Omega}_\pm \cap \mathbf{\Omega}_{\mp\infty}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\infty} f\left(\mathbf{\Phi}(\mathbf{y}, \mp s)\right)\mathrm{d}s. \tag{2.10}$$

Proof. Assume first $f \geqslant 0$. Let us fix $T > 0$. It is clear that $\mathbf{x} \in \mathbf{\Omega}$ satisfies $\tau_+(\mathbf{x}) < T$ if and only if $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, -s)$, with $\mathbf{y} \in \Gamma_+$ and $0 < s < T \wedge \tau_-(\mathbf{y})$. Then, by Proposition 2.11,

$$\int_{\{\tau_+(\mathbf{x})<T\}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{T\wedge\tau_-(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, -s))\mathrm{d}s.$$

Since $f \geqslant 0$, the inner integral is increasing with T and, using the monotone convergence theorem, we let $T \to \infty$ to get

$$\int_{\mathbf{\Omega}_+} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y}, -s)\right)\mathrm{d}s$$

which coincides with (2.9). We proceed in the same way with integration on Γ_- and get the second part of (2.9). Next we consider the set

$$\Delta = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \mathbf{x} = \mathbf{\Phi}(\mathbf{y}, -s),\ \mathbf{y} \in \mathbf{\Omega}_{+\infty},\ 0 < s < T\}.$$

Proposition 2.11 asserts that

$$\int_\Delta f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{+\infty}} \mathrm{d}\mu_+(\mathbf{y}) \int_0^T f(\mathbf{\Phi}(\mathbf{y}, -s))\mathrm{d}s.$$

Letting again $T \to \infty$, we get (2.10). We extend the results to arbitrary f by linearity. □

Finally, with the following, we show that it is possible to transfer integrals over Γ_- to Γ_+:

Proposition 2.13. *For any $\psi \in L^1(\Gamma_-, \mathrm{d}\mu_-)$,*

$$\int_{\Gamma_-\setminus\Gamma_{-\infty}} \psi(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}) = \int_{\Gamma_+\setminus\Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{z}, -\tau_-(\mathbf{z})))\mathrm{d}\mu_+(\mathbf{z}). \tag{2.11}$$

Proof. For any $\epsilon > 0$, let f_ϵ be the function defined on $\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-$ by

$$\psi_\epsilon(\mathbf{x}) = \begin{cases} \dfrac{\psi(\mathbf{\Phi}(\mathbf{x}, -\tau_-(\mathbf{x})))}{\tau_+(\mathbf{x}) + \tau_-(\mathbf{x})} & \text{if} \quad \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) > \epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $\psi_\epsilon \in L^1(\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-, \mathrm{d}\mu)$, Eqs. (2.9) and (2.10) give

$$\begin{aligned}\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} \psi_\epsilon(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\{\tau_+(\mathbf{y})>\epsilon\}\setminus\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{y})\frac{\mathrm{d}s}{\tau_+(\mathbf{y})}\\ &= \int_{\{\tau_+(\mathbf{y})>\epsilon\}\setminus\Gamma_{-\infty}} \psi(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}).\end{aligned}$$

In the same way,

$$\begin{aligned}\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} \psi_\epsilon(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\{\tau_-(\mathbf{y})>\epsilon\}\setminus\Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y},-\tau_-(\mathbf{y})))\frac{\mathrm{d}s}{\tau_-(\mathbf{y})}\\ &= \int_{\{\tau_-(\mathbf{y})>\epsilon\}\setminus\Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{y},-\tau_-(\mathbf{y})))\mathrm{d}\mu_-(\mathbf{y}),\end{aligned}$$

which leads to

$$\int_{\{\tau_-(\mathbf{y})>\epsilon\}\setminus\Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{y},-\tau_-(\mathbf{y})))\mathrm{d}\mu_+(\mathbf{y}) = \int_{\{\tau_+(\mathbf{y})>\epsilon\}\setminus\Gamma_{-\infty}} \psi(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y})$$

for any $\epsilon > 0$. Passing to the limit as $\epsilon \to 0$ we get the conclusion. □

We end this section with a technical result we shall need in the sequel (see Lemma 3.3):

Proposition 2.14. *Let K be a compact subset of $\mathbf{\Omega}$. Denote*

$$K_\pm := \{\mathbf{y} \in \Gamma_\pm\ ;\ \exists t_0 \in \mathbb{R}\ \text{ such that } \mathbf{\Phi}(\mathbf{y}, \pm t) \in K\ \text{ for any } t \geqslant t_0\}\,.$$

Then $\mu_\pm(K_\pm) = 0$.

Proof. Let K be a fixed compact subset of $\mathbf{\Omega}$. Applying Eq. (2.9) or (2.10) to the function $f(\mathbf{x}) = \chi_K(\mathbf{x})$, one has

$$\infty > \mu(K) \geqslant \int_{K_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^\infty \chi_K(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t. \tag{2.12}$$

By definition, if $\mathbf{y} \in K_-$, then for some $t_0 \in \mathbb{R}$, $\chi_K(\mathbf{\Phi}(\mathbf{y}, t)) = 1$ for any $t \geqslant t_0$. Therefore,

$$\int_0^\infty \chi_K(\mathbf{\Phi}(\mathbf{y}, t)) = \infty, \qquad \forall \mathbf{y} \in K_-.$$

Inequality (2.12) implies that $\mu_-(K_-) = 0$. One proves the result for K_+ in the same way. □

3. The maximal transport operator and trace results

The results of the previous section allow us to define the (maximal) transport operator $\mathcal{T}_{\max}$ as the weak derivative along the characteristic curves. To be precise, let us define the space of *test functions* $\mathfrak{Y}$ as follows:

Definition 3.1 (*Test–functions*). *Let $\mathfrak{Y}$ be the set of all measurable and bounded functions $\psi : \mathbf{\Omega} \to \mathbb{R}$ with compact support in $\mathbf{\Omega}$ and such that, for any $\mathbf{x} \in \mathbf{\Omega}$, the mapping*

$$s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \longmapsto \psi(\mathbf{\Phi}(\mathbf{x}, s))$$

is continuously differentiable with

$$\mathbf{x} \in \mathbf{\Omega} \longmapsto \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x}, s))\Big|_{s=0} \quad \textit{measurable and bounded}. \tag{3.1}$$

Remark 3.2. *Notice that the class of test-functions $\mathfrak{Y}$ is not defined as a subset of $L^\infty(\mathbf{\Omega}, \mathrm{d}\mu)$; that is, we do not identify functions equal μ-almost everywhere. It is however a natural question to know whether two test-functions coinciding μ-almost everywhere are such that there derivatives (defined by (3.1)) do coincide μ-almost everywhere. We provide a positive answer to this question at the end of the paper (see Appendix).*

An important property of test-functions is the following consequence of Proposition 2.14:

Lemma 3.3. *Let $\psi \in \mathfrak{Y}$ be given. For $\mu_{\mp}$-almost any $\mathbf{y} \in \Gamma_{\mp}$ there exists a sequence $(t_n^{\pm})_n$ (depending on $\mathbf{y}$) such that*

$$\lim_{n\to\infty} t_n^{\pm} = \tau_{\pm}(\mathbf{y}) \qquad \textit{and} \qquad \psi(\mathbf{\Phi}(\mathbf{y}, \pm t_n^{\pm})) = 0 \quad \forall n \in \mathbb{N}.$$

Proof. Let $\psi \in \mathfrak{Y}$ be given and let $K = \mathrm{Supp}(\psi)$. For any $\mathbf{y} \in \Gamma_-$ with $\tau_+(\mathbf{y}) < \infty$ one has $\mathbf{\Phi}(\mathbf{y}, \tau_+(\mathbf{y})) \in \Gamma_+$ and, since K is compact in $\mathbf{\Omega}$, $\psi(\mathbf{\Phi}(\mathbf{y}, \tau_+(\mathbf{y}))) = 0$ and the existence of a sequence $(t_n^+)_n$ converging to $\tau_+(\mathbf{y})$ with the above property is clear. Now, Proposition 2.14 applied to K shows that there exists a set $\Gamma'_- \subset \Gamma_-$ with $\mu_-(\Gamma \setminus \Gamma'_-) = 0$ and such that, for any $\mathbf{y} \in \Gamma'_-$, there is a sequence $(t_n^+)_n$ converging to ∞ such that $\mathbf{\Phi}(\mathbf{y}, t_n) \notin K$ for any $n \in \mathbb{N}$. This proves the result. The statement for Γ_+ is proved in the same way. □

In the next step we define the transport operator $(\mathcal{T}_{\max}, \mathscr{D}(\mathcal{T}_{\max}))$.

Definition 3.4 (***Transport operator*** $\mathcal{T}_{\max}$). *The domain of the maximal transport operator $\mathcal{T}_{\max}$ is the set $\mathscr{D}(\mathcal{T}_{\max})$ of all $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ for which there exists $g \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ such that*

$$\int_{\mathbf{\Omega}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} f(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x}, s))\Big|_{s=0} \mathrm{d}\mu(\mathbf{x}), \qquad \forall \psi \in \mathfrak{Y}.$$

In this case, $g =: \mathcal{T}_{\max} f$.

Remark 3.5. *Of course, in some weak sense, $\mathcal{T}_{\max} f = -\mathscr{F} \cdot \nabla f$. Precisely, for any $\varphi \in \mathscr{C}_0^1(\mathbf{\Omega})$, the following formula holds:*

$$\int_{\mathbf{\Omega}} (\mathscr{F}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x}))\, f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \mathcal{T}_{\max} f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

3.1. **Fundamental representation formula: mild formulation.** Recall that, if f_1 and f_2 are two functions defined over $\mathbf{\Omega}$, we say that f_2 is a *representative of* f_1 if $\mu\{\mathbf{x} \in \mathbf{\Omega}\,;\, f_1(\mathbf{x}) \neq f_2(\mathbf{x})\} = 0$, i.e. when $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for μ-almost every $\mathbf{x} \in \mathbf{\Omega}$. The following fundamental result provides a characterization of the domain of $\mathscr{D}(\mathcal{T}_{\max})$:

Theorem 3.6. *Let $f \in L^1(\mathbf{\Omega}, \mu)$. The following are equivalent:*

(1) *There exists $g \in L^1(\mathbf{\Omega}, \mu)$ and a representative $f^\sharp$ of f such that, for μ-almost every $\mathbf{x} \in \mathbf{\Omega}$ and any $-\tau_-(\mathbf{x}) < t_1 \leqslant t_2 < \tau_+(\mathbf{x})$:*

$$f^\sharp(\mathbf{\Phi}(\mathbf{x}, t_1)) - f^\sharp(\mathbf{\Phi}(\mathbf{x}, t_2)) = \int_{t_1}^{t_2} g(\mathbf{\Phi}(\mathbf{x}, s))\mathrm{d}s. \tag{3.2}$$

(2) *$f \in \mathscr{D}(\mathcal{T}_{\max})$. In this case, $g = \mathcal{T}_{\max} f$.*

The proof of the theorem is made of several steps. The difficult part of the proof is the implication $(2) \implies (1)$. It is carried out through several technical lemmas based upon ***mollification along the characteristic curves*** (recall that, whenever μ is not absolutely continuous with respect to the Lebesgue measure, no global convolution argument is available). Let us make precise what this is all about. Consider a sequence $(\varrho_n)_n$ of one dimensional mollifiers supported in $[0,1]$, i.e. for any $n \in \mathbb{N}$, $\varrho_n \in \mathscr{C}_0^\infty(\mathbb{R})$, $\varrho_n(s) = 0$ if $s \notin [0, 1/n]$, $\varrho_n(s) \geqslant 0$ and $\int_0^{1/n} \varrho_n(s)\mathrm{d}s = 1$. Then, for any $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, define the (extended) mollification:

$$\varrho_n \diamond f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\mathbf{\Phi}(\mathbf{x}, -s))\mathrm{d}s.$$

As we shall see later, such a definition corresponds precisely to a time convolution over any characteristic curves (see e.g. (3.4)). Note that, with such a definition, it is not clear *a priori* that $\varrho_n \diamond f$ defines a measurable function, finite almost everywhere. It is proved in the following that actually such a function is integrable.

Lemma 3.7. *Given $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, $\varrho_n \diamond f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ for any $n \in \mathbb{N}$. Moreover,*

$$\|\varrho_n \diamond f\| \leqslant \|f\|, \qquad \forall f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu), n \in \mathbb{N}. \tag{3.3}$$

Proof. One considers, for a given $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, the extension of f by zero outside $\mathbf{\Omega}$:

$$\overline{f}(\mathbf{x}) = f(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{\Omega}, \qquad \overline{f}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^N \setminus \mathbf{\Omega}.$$

Then $\overline{f} \in L^1(\mathbb{R}^N, \mathrm{d}\mu)$. Let us consider the transformation:

$$\Upsilon \, : \, (\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \Upsilon(\mathbf{x}, s) = (\mathbf{\Phi}(\mathbf{x}, -s), -s) \in \mathbb{R}^N \times \mathbb{R}.$$

As a homeomorphism, Υ is measure preserving for pure Borel measures. It is also measure preserving for completions of Borel measures (such as a Lebesgue measure) since it is measure-preserving on Borel sets and the completion of a measure is obtained by adding to the Borel σ-algebra all sets contained in a measure-zero Borel sets, see [12, Theorem 13.B, p. 55]. Then, according to [12, Theorem 39.B, p. 162], the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \overline{f}(\mathbf{\Phi}(\mathbf{x}, -s))$$

is measurable as the composition of Υ with the measurable function $(\mathbf{x}, s) \mapsto \overline{f}(\mathbf{x})$. Define now $\Lambda = \{(\mathbf{x}, s)\,;\, \mathbf{x} \in \mathbf{\Omega},\, 0 < s < \tau_-(\mathbf{x})\}$, Λ is a measurable subset of $\mathbb{R}^N \times \mathbb{R}$. Therefore, the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \longmapsto \overline{f}(\mathbf{\Phi}(\mathbf{x}, -s))\chi_\Lambda(\mathbf{x}, s)\varrho_n(s)$$

is measurable. Since ϱ_n is compactly supported, it is also integrable over $\mathbb{R}^N \times \mathbb{R}$ and, according to Fubini's Theorem

$$[\varrho_n \diamond f](\mathbf{x}) := \int_{\mathbb{R}} \overline{f}(\mathbf{\Phi}(\mathbf{x}, -s))\chi_\Lambda(\mathbf{x}, s)\varrho_n(s)\mathrm{d}s = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\mathbf{\Phi}(\mathbf{x}, -s))\mathrm{d}s$$

is finite for almost every $\mathbf{x} \in \mathbf{\Omega}$ the and the associated application $\varrho_n \diamond f$ is integrable.

Let us prove now (3.3). Since $|\varrho_n \diamond f| \leqslant \varrho_n \diamond |f|$, to show that $\varrho_n \diamond f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, it suffices to deal with a *nonnegative function* $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. One sees easily that, for any $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$,

$$(\varrho_n \diamond f)(\mathbf{\Phi}(\mathbf{y}, t)) = \int_0^t \varrho_n(s) f(\mathbf{\Phi}(\mathbf{y}, t - s))\mathrm{d}s = \int_0^t \varrho_n(t - s) f(\mathbf{\Phi}(\mathbf{y}, s))\mathrm{d}s. \tag{3.4}$$

Thus,

$$\begin{aligned}\int_0^{\tau_+(\mathbf{y})}[\varrho_n\diamond f](\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t &= \int_0^{\tau_+(\mathbf{y})}\mathrm{d}s\int_s^{\tau_+(\mathbf{y})}\varrho_n(s)f(\mathbf{\Phi}(\mathbf{y},t-s))\mathrm{d}t\\ &= \int_0^{\tau_+(\mathbf{y})\wedge 1/n}\varrho_n(s)\mathrm{d}s\int_0^{\tau_+(\mathbf{y})-s}f(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r.\end{aligned}$$

Therefore,

$$\begin{aligned}0\leqslant\int_0^{\tau_+(\mathbf{y})}[\varrho_n\diamond f](\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t &\leqslant \int_0^{1/n}\varrho_n(s)\mathrm{d}s\int_0^{\tau_+(\mathbf{y})}f(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r\\ &= \int_0^{\tau_+(\mathbf{y})}f(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r,\qquad \forall \mathbf{y}\in\Gamma_-,\ n\in\mathbb{N}\end{aligned}$$

so that

$$\int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})}[\varrho_n\diamond f](\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t\leqslant\int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})}f(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r.$$

This proves, thanks to Proposition 2.12, that

$$\int_{\mathbf{\Omega}_-}[\varrho_n\diamond f]\mathrm{d}\mu\leqslant\int_{\mathbf{\Omega}_-}f\mathrm{d}\mu. \tag{3.5}$$

Now, in the same way:

$$\begin{aligned}\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}}[\varrho_n\diamond f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_{+\infty}}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\infty}[\varrho_n\diamond f](\mathbf{\Phi}(\mathbf{y},-t))\mathrm{d}t\\ &= \int_{\Gamma_{+\infty}}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\infty}\mathrm{d}t\int_0^{\infty}\varrho_n(s)f(\mathbf{\Phi}(\mathbf{y},-s-t))\mathrm{d}s\\ &= \int_{\Gamma_{+\infty}}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\infty}\mathrm{d}t\int_t^{\infty}\varrho_n(r-t)f(\mathbf{\Phi}(\mathbf{y},-r))\mathrm{d}r.\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}}[\varrho_n\diamond f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_{+\infty}}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\infty}f(\mathbf{\Phi}(\mathbf{y},-r))\mathrm{d}r\int_0^{r}\varrho_n(r-t)\mathrm{d}t\\ &\leqslant \int_{\Gamma_{+\infty}}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\infty}f(\mathbf{\Phi}(\mathbf{y},-r))\mathrm{d}r\end{aligned}$$

i.e.

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}}\varrho_n\diamond f(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\leqslant\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}}f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{3.6}$$

Finally

$$\begin{aligned}\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}[\varrho_n\diamond f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}\mathrm{d}\mu(\mathbf{x})\int_0^{\infty}\varrho_n(s)f(\mathbf{\Phi}(\mathbf{x},-s))\mathrm{d}s\\ &= \int_0^{\infty}\varrho_n(s)\mathrm{d}s\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}f(\mathbf{\Phi}(\mathbf{x},-s))\mathrm{d}\mu(\mathbf{x}).\end{aligned}$$

Now, from Assumption 1, for any $s\geqslant 0$,

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}f(\mathbf{\Phi}(\mathbf{x},-s))\mathrm{d}\mu(\mathbf{x})=\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}),$$

so that

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}}[\varrho_n \diamond f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), one finally gets $\|\varrho_n \diamond f\| \leqslant \|f\|$. □

As it is the case for classical convolution, the family $(\varrho_n \diamond f)_n$ approximates f in L^1-norm:

Proposition 3.8. *Given* $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$,

$$\lim_{n\to\infty}\int_{\mathbf{\Omega}}\Big|(\varrho_n \diamond f)(\mathbf{x}) - f(\mathbf{x})\Big|\mathrm{d}\mu(\mathbf{x}) = 0. \tag{3.8}$$

Proof. According to (3.3) and from the density of $\mathscr{C}_0(\mathbf{\Omega})$ in $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, it suffices to prove the result for any f continuous over $\mathbf{\Omega}$ and compactly supported. Splitting f into positive and negative parts, $f = f^+ - f^-$, one can also assume f to be nonnegative. From the continuity of both f and $\mathbf{\Phi}(\cdot,\cdot)$, one has

$$\mathscr{K}_n := \mathrm{Supp}(\varrho_n \diamond f) = \overline{\Big\{\mathbf{x} \in \mathbf{\Omega}\,,\; \exists s_0 \in \mathrm{Supp}(\varrho_n) \text{ such that } \mathbf{\Phi}(\mathbf{x}, -s_0) \in \mathrm{Supp}(f)\Big\}}.$$

Moreover, it is easily seen that $\mathscr{K}_{n+1} \subset \mathscr{K}_n$ for any $n \geqslant 1$. Finally, it is clear that

$$\mathscr{K}_1 \subset \{\mathbf{x} \in \overline{\mathbf{\Omega}}\,;\exists \mathbf{y} \in \mathrm{Supp}(f) \;\text{ with } |\mathbf{x}-\mathbf{y}| \leqslant d\}$$

where $d = \sup\{|\mathbf{\Phi}(\mathbf{x}, s) - \mathbf{x}|\,;\, 0 \leqslant s \leqslant 1\,; \mathbf{x} \in \mathrm{Supp}(f)\} < \infty$. Therefore, $\mathscr{K}_1$ is compact. Set now

$$\mathscr{O}_n := \mathscr{K}_n \cup \mathrm{Supp}(f) \qquad \text{and} \qquad \mathscr{O}_n^- = \{\mathbf{x} \in \mathscr{O}_n\,;\, \tau_-(\mathbf{x}) < 1/n\}.$$

Noticing that $\mu(\mathscr{O}_1)$ is finite, one can see easily that $\lim_n \mu(\mathscr{O}_n^-) = 0$. Since $\sup_{\mathbf{x}\in\mathbf{\Omega}} |\varrho_n \diamond f(\mathbf{x})| \leqslant \sup_{\mathbf{x}\in\mathbf{\Omega}} |f(\mathbf{x})|$, for any $\varepsilon > 0$, there exists $n_0 \geqslant 1$ such that

$$\int_{\mathscr{O}_n^-}|f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon, \quad \text{and} \quad \int_{\mathscr{O}_n^-}|\varrho_n \diamond f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon \qquad \forall n \geqslant n_0.$$

Now, noticing that $\mathrm{Supp}(\varrho_n \diamond f - f) \subset \mathscr{O}_n$, one has for any $n \geqslant n_0$,

$$\int_{\mathbf{\Omega}}|\varrho_n \diamond f - f|\mathrm{d}\mu = \int_{\mathscr{O}_n}|\varrho_n \diamond f - f| \leqslant 2\varepsilon + \int_{\mathscr{O}_n\setminus\mathscr{O}_n^-}|\varrho_n \diamond f - f|\mathrm{d}\mu.$$

For any $\mathbf{x} \in \mathscr{O}_n \setminus \mathscr{O}_n^-$, since ϱ is supported in $[0, 1/n]$, one has

$$\begin{aligned}[\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x}) &= \int_0^{1/n}\varrho_n(s) f(\mathbf{\Phi}(\mathbf{x}, -s))\mathrm{d}s - f(\mathbf{x}) \\ &= \int_0^{1/n}\varrho_n(s)\left(f(\mathbf{\Phi}(\mathbf{x}, -s)) - f(\mathbf{x})\right)\mathrm{d}s.\end{aligned}$$

Note that, thanks to Gronwall's lemma,

$$|\mathbf{\Phi}(\mathbf{x}, -s) - \mathbf{x}| \leqslant \frac{L}{\kappa}(\exp(ks) - 1) \leqslant \frac{L}{\kappa}(\exp(\kappa/n) - 1), \qquad \forall \mathbf{x} \in \mathscr{O}_1, s \in (0, 1/n)$$

where $L = \sup\{|\mathscr{F}(\mathbf{x})|, \mathbf{x} \in \mathscr{O}_1\}$. Since f is uniformly continuous on $\mathscr{O}_1$, it follows that

$$\lim_{n\to\infty}\sup\Big\{|f(\mathbf{\Phi}(\mathbf{x}, -s) - f(\mathbf{x})|\,;\, \mathbf{x} \in \mathscr{O}_1,\;\, s \in (0, 1/n)\Big\} = 0$$

from which we deduce that there exists some $n_1 \geqslant 0$, such that $|\varrho_n \diamond f(\mathbf{x}) - f(\mathbf{x})| \leqslant \varepsilon$ for any $\mathbf{x} \in \mathscr{O}_n \setminus \mathscr{O}_n^-$ and any $n \geqslant n_1$. One obtains then, for any $n \geqslant n_1$,

$$\int_{\mathbf{\Omega}} |\varrho_n \diamond f - f| \mathrm{d}\mu \leqslant 2\varepsilon + \varepsilon\mu(\mathscr{O}_n \setminus \mathscr{O}_n^-) \leqslant 2\varepsilon + \varepsilon\mu(\mathscr{O}_1)$$

which proves the result. □

We saw that, for a given $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, $\varrho_n \diamond f$ is also integrable ($n \in \mathbb{N}$). Actually, we shall see that $\varrho_n \diamond f$ is even more regular than f:

Lemma 3.9. *Given $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, set $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then, $f_n \in \mathscr{D}(\mathcal{T}_{\max})$ with*

$$[\mathcal{T}_{\max} f_n](\mathbf{x}) = -\int_0^{\tau_-(\mathbf{x})} \varrho_n'(s) f(\mathbf{\Phi}(\mathbf{x}, -s)) \mathrm{d}s, \qquad \mathbf{x} \in \mathbf{\Omega}.$$

Proof. Set $g_n(\mathbf{x}) = -\int_0^{\tau_-(\mathbf{x})} \varrho_n'(s) f(\mathbf{\Phi}(\mathbf{x}, -s)) \mathrm{d}s$, $\mathbf{x} \in \mathbf{\Omega}$. It is easy to see that $g_n \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Now, given $\psi \in \mathfrak{Y}$, let us consider the quantity

$$I = \int_{\mathbf{\Omega}} f_n(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{\Phi}(\mathbf{x}, s)) \Big|_{s=0} \mathrm{d}\mu(\mathbf{x}).$$

One has to prove that $I = \int_{\mathbf{\Omega}} g_n(\mathbf{x}) \psi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x})$. We split the above integral over $\mathbf{\Omega}$ into three integrals I_-, I_+ and I_∞ over $\mathbf{\Omega}_-$, $\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_+$ and $\mathbf{\Omega}_{+\infty} \cap \mathbf{\Omega}_{-\infty}$ respectively. Recall that, for any $\mathbf{x} \in \mathbf{\Omega}_-$, there is some $\mathbf{y} \in \Gamma_-$ and some $t \in (0, \tau_+(\mathbf{y}))$ such that $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, t)$. In such a case

$$\frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{\Phi}(\mathbf{x}, s)) \Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)). \tag{3.9}$$

Then, according to Prop. 2.12 and Eq. (3.4):

$$\begin{aligned} I_- &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f_n(\mathbf{\Phi}(\mathbf{y}, t)) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \mathrm{d}t \\ &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \mathrm{d}t \int_0^t \varrho_n(t - s) f(\mathbf{\Phi}(\mathbf{y}, s)) \mathrm{d}s \\ &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, s)) \mathrm{d}s \int_s^{\tau_+(\mathbf{y})} \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \varrho_n(t - s) \mathrm{d}t. \end{aligned} \tag{3.10}$$

Let us now investigate more carefully this last integral. Let $\mathbf{y} \in \Gamma_-$ be fixed. If $\tau_+(\mathbf{y}) < \infty$ then, since ψ is compactly supported, we have $\psi(\mathbf{\Phi}(\mathbf{y}, \tau_+(\mathbf{y}))) = 0$ and integration by part (together with $\varrho_n(0) = 0$) leads to

$$\int_s^{\tau_+(\mathbf{y})} \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \varrho_n(t - s) \mathrm{d}t = -\int_s^{\tau_+(\mathbf{y})} \varrho_n'(t - s) \psi(\mathbf{\Phi}(\mathbf{y}, t)) \mathrm{d}t.$$

If now $\tau_+(\mathbf{y}) > \infty$, then, since ϱ_n is supported in $[0, 1/n]$, one has

$$\begin{aligned} \int_s^{\tau_+(\mathbf{y})} \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \varrho_n(t - s) \mathrm{d}t &= \int_s^{s+1/n} \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t)) \varrho_n(t - s) \mathrm{d}t \\ &= -\int_s^{\tau_+(y)} \varrho_n'(t - s) \psi(\mathbf{\Phi}(\mathbf{y}, t)) \mathrm{d}t \end{aligned}$$

Finally, we obtain,

$$\begin{aligned} I_- &= -\int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s \int_s^{\tau_+(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y},t))\varrho_n'(t-s)\mathrm{d}t \\ &= -\int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t \int_0^t \varrho_n'(s) f(\mathbf{\Phi}(\mathbf{y},t-s))\mathrm{d}s. \end{aligned}$$

Using again Prop. 2.12, we finally get

$$I_- = \int_{\mathbf{\Omega}_-} g_n(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

One proves in the same way that

$$I_+ = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} f_n(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} g_n(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

It remains to consider $I_\infty = \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f_n(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x})$. One has

$$\begin{aligned} I_\infty &= \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x}) \int_0^\infty \varrho_n(t) f(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}t \\ &= \int_0^\infty \varrho_n(t)\mathrm{d}t \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0} f(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}\mu(\mathbf{x}). \end{aligned}$$

For any $\mathbf{x} \in \mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}$ and any $t \geqslant 0$, setting $\mathbf{y} = \mathbf{\Phi}(\mathbf{x},-t)$, one has $\mathbf{y} \in \mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))$ from which Liouville's Theorem (Assumption 1) yields

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0} f(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t)) f(\mathbf{y})\mathrm{d}\mu(\mathbf{y}).$$

Therefore,

$$\begin{aligned} I_\infty &= \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{y})\mathrm{d}\mu(\mathbf{y}) \int_0^\infty \varrho_n(t)\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t \\ &= -\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{y})\mathrm{d}\mu(\mathbf{y}) \int_0^\infty \varrho_n'(t)\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t \\ &= -\int_0^\infty \varrho_n'(t)\mathrm{d}t \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{y})\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}\mu(\mathbf{y}). \end{aligned}$$

Arguing as above, one can "turn back" to the $\mathbf{x}$ variable to get

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{y})\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}\mu(\mathbf{y}) = \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{\Phi}(\mathbf{x},-t))\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}),$$

i.e.

$$I_\infty = -\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \int_0^\infty \varrho_n'(t) f(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}t = \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} \psi(\mathbf{x})g_n(\mathbf{x})\mathrm{d}\mu(\mathbf{x})$$

and the Lemma is proven. □

Remark 3.10. *Notice that Proposition 3.8 together with Lemma 3.9 prove that $\mathscr{D}(\mathcal{T}_{\max})$ is a dense subset of $L^1(\mathbf{\Omega},\mathrm{d}\mu)$.*

Now, whenever $f \in \mathscr{D}(\mathcal{T}_{\max})$, one has the following more precise result:

Proposition 3.11. *If $f \in \mathscr{D}(\mathcal{T}_{\max})$, then*

$$[\mathcal{T}_{\max}(\varrho_n \diamond f)](\mathbf{x}) = [\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{x}), \qquad (\mathbf{x} \in \mathbf{\Omega}\,,\, n \in \mathbb{N}). \tag{3.11}$$

Before proving this result, we need the following very simple lemma:

Lemma 3.12. *For any $\psi \in \mathfrak{Y}$ and any $n \in \mathbb{N}$, define*

$$\chi_n(\mathbf{x}) = \int_0^{\tau_+(\mathbf{x})} \varrho_n(s)\psi(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s, \qquad \mathbf{x} \in \mathbf{\Omega}.$$

Then, χ_n belongs to $\mathfrak{Y}$.

Proof. Since τ_+ is measurable and ϱ_n is compactly supported, it is easy to see that χ_n is measurable and bounded over $\mathbf{\Omega}$. Now, for any $\mathbf{x} \in \mathbf{\Omega}$, and any $t \in (\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$, one has

$$\chi_n(\mathbf{\Phi}(\mathbf{x},t)) = \int_t^{\tau_+(\mathbf{x})} \varrho_n(s-t)\psi(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s.$$

It is clear then from the properties of ϱ_n that the mapping $t \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto \chi_n(\mathbf{\Phi}(\mathbf{x},t))$ is continuously differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_n(\mathbf{\Phi}(\mathbf{x},t)) = -\int_t^{\tau_+(\mathbf{x})} \varrho_n'(s-t)\psi(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s = \int_t^{\tau_+(\mathbf{x})} \varrho_n(s-t)\frac{\mathrm{d}}{\mathrm{d}s}\left[\psi(\mathbf{\Phi}(\mathbf{x},s))\right]\mathrm{d}s. \tag{3.12}$$

In particular, for $t = 0$, one gets

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\chi_n(\mathbf{\Phi}(\mathbf{x},t))\right|_{t=0} = -\int_0^{\tau_+(\mathbf{x})} \varrho_n'(s)\psi(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s.$$

Since ϱ_n' is compactly supported and $\psi \in \mathfrak{Y}$, the application $\mathbf{x} \in \mathbf{\Omega} \longmapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\chi_n(\mathbf{\Phi}(\mathbf{x},t))\right|_{t=0}$ is measurable and bounded. □

PROOF OF PROPOSITION 3.11. We use the notations of Lemma 3.9. Since $\varrho_n \diamond \mathcal{T}_{\max} f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, it suffices to show that

$$\int_{\mathbf{\Omega}} f_n(\mathbf{x}) \left.\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\right|_{s=0} \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \psi(\mathbf{x})[\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}), \qquad \forall \psi \in \mathfrak{Y}.$$

Here again, we shall deal separately with the integrals over $\mathbf{\Omega}_-$, $\mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}$ and $\mathbf{\Omega}_{+\infty} \cap \mathbf{\Omega}_{-\infty}$. Let χ_n be defined as in Lemma 3.12, as we already saw it (see (3.12)), for any $\mathbf{y} \in \Gamma_-$, and any $0 < s < \tau_+(\mathbf{y})$, $\frac{\mathrm{d}}{\mathrm{d}s}\chi_n(\mathbf{\Phi}(\mathbf{y},s)) = \int_s^{\tau_+(\mathbf{y})} \varrho_n(t-s)\frac{\mathrm{d}}{\mathrm{d}t}[\psi(\mathbf{\Phi}(\mathbf{y},t))]\mathrm{d}t$. Consequently, according to (3.10),

$$\begin{aligned}\int_{\mathbf{\Omega}_-} f_n(\mathbf{x}) \left.\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\right|_{s=0} \mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_-} \mathrm{d}\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y},r))\frac{\mathrm{d}}{\mathrm{d}r}\chi_n(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r \\ &= \int_{\mathbf{\Omega}_-} f(\mathbf{x}) \left.\frac{\mathrm{d}}{\mathrm{d}s}\chi_n(\mathbf{\Phi}(\mathbf{x},s))\right|_{s=0} \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_-} \chi_n(\mathbf{x})[\mathcal{T}_{\max} f](\mathbf{x})\mathrm{d}\mu(\mathbf{x})\end{aligned}$$

where, for the two last identities, we used (3.9) and the fact that $\chi_n \in \mathfrak{Y}$. Now, using Prop. 2.12

$$\int_{\mathbf{\Omega}_-} \chi_n(\mathbf{x})[\mathcal{T}_{\max} f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_-} [\mathcal{T}_{\max} f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x})} \varrho_n(r)\psi(\mathbf{\Phi}(\mathbf{x},r))\mathrm{d}r$$
$$= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s \int_0^s \varrho_n(s-t)[\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t.$$

Therefore, Eq. (3.4) leads to

$$\int_{\mathbf{\Omega}_-} \chi_n(\mathbf{x})[\mathcal{T}_{\max} f](\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y},s))[\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s$$
$$= \int_{\mathbf{\Omega}_-} \psi(\mathbf{x}) \left[\varrho_n \diamond \mathcal{T}_{\max} f\right](\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

The integrals over $\mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}$ and $\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}$ are evaluated in the same way. □

We are in position to prove the following

Proposition 3.13. *Let $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ and $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then, for μ_-– a. e. $\mathbf{y} \in \Gamma_-$,*

$$f_n(\mathbf{\Phi}(\mathbf{y},s)) - f_n(\mathbf{\Phi}(\mathbf{y},t)) = \int_s^t [\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r \qquad \forall 0 < s < t < \tau_+(\mathbf{y}). \tag{3.13}$$

In the same way, for almost every $\mathbf{z} \in \Gamma_+$,

$$f_n(\mathbf{\Phi}(\mathbf{z},-s)) - f_n(\mathbf{\Phi}(\mathbf{z},-t)) = \int_s^t \mathcal{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{z},-r))\mathrm{d}r, \qquad \forall 0 < s < t < \tau_-(\mathbf{z}).$$

Proof. We focus only on (3.13), the second assertion following the same lines. Since $f \in L^1(\mathbf{\Omega}_-, \mathrm{d}\mu)$, Proposition 2.12 implies that the integral $\int_0^{\tau_+(\mathbf{y})} |f(\mathbf{\Phi}(\mathbf{y},r))|\mathrm{d}r$ exists and is finite for μ_--almost every $\mathbf{y} \in \Gamma_-$. Therefore, for μ_--almost every $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$, the quantities $\int_0^t \varrho_n(t-s) f(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s$ and $\int_0^t \varrho_n'(t-s) f(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s$ are well-defined and finite. Moreover, thanks to Eq. (3.4) Lemma 3.9, they are respectively equal to $f_n(\mathbf{\Phi}(\mathbf{y},t))$ and $[\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},t))$. In particular, the mapping $t \in (0, \tau_+(\mathbf{y})) \mapsto [\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},t)) \in \mathbb{R}$ is continuous. It is then easy to see that, for any $0 < s < t < \tau_+(\mathbf{y})$

$$\int_s^t [\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r = -\int_s^t \mathrm{d}r \int_0^r \varrho_n'(r-u) f(\mathbf{\Phi}(\mathbf{y},u))\mathrm{d}u$$
$$= -\int_0^s f(\mathbf{\Phi}(\mathbf{y},u))\mathrm{d}u \int_s^t \varrho_n'(r-u)\mathrm{d}r - \int_s^t f(\mathbf{\Phi}(\mathbf{y},u))\mathrm{d}u \int_u^t \varrho_n'(r-u)\mathrm{d}r$$
$$= -\int_0^t f(\mathbf{\Phi}(\mathbf{y},u))\varrho_n(t-u)\mathrm{d}u + \int_0^s f(\mathbf{\Phi}(\mathbf{y},u))\varrho_n(s-u)\mathrm{d}u,$$

which is nothing but (3.13). □

As a consequence, one gets the following result :

Proposition 3.14. *For any $f \in \mathscr{D}(\mathcal{T}_{\max})$, there exists some functions $\widetilde{f}_\pm \in L^1(\mathbf{\Omega}_\pm, \mathrm{d}\mu)$ such that $\widetilde{f}_\pm(\mathbf{x}) = f(\mathbf{x})$ for μ- almost every $\mathbf{x} \in \mathbf{\Omega}_\pm$ and, for μ_-–almost every $\mathbf{y} \in \Gamma_-$:*

$$\widetilde{f}_-(\mathbf{\Phi}(\mathbf{y},s)) - \widetilde{f}_-(\mathbf{\Phi}(\mathbf{y},t)) = \int_s^t [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r \qquad \forall 0 < s < t < \tau_+(\mathbf{y}), \tag{3.14}$$

while, for μ_+–almost every $\mathbf{z} \in \Gamma_+$:

$$\widetilde{f}_+(\mathbf{\Phi}(\mathbf{z},-s)) - \widetilde{f}_+(\mathbf{\Phi}(\mathbf{z},-t)) = \int_s^t [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{z},-r))\mathrm{d}r \qquad \forall 0 < s < t < \tau_-(\mathbf{z}).$$

Proof. Define, for any $n \geqslant 1$, $f_n = \varrho_n \diamond f$, so that, from Propositions 3.11 and 3.8, $\lim_{n\to\infty} \|f_n - f\| + \|\mathcal{T}_{\max} f_n - \mathcal{T}_{\max} f\| = 0$. In particular,

$$\lim_{n\to\infty} \int_{\mathbf{\Omega}_-} |f_n(\mathbf{x}) - f(\mathbf{x})| + |[\mathcal{T}_{\max} f_n](\mathbf{x}) - [\mathcal{T}_{\max} f](\mathbf{x})| \,\mathrm{d}\mu(\mathbf{x}) = 0.$$

Then Eq. (2.9) yields

$$\int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |f_n(\mathbf{\Phi}(\mathbf{y},s)) - f(\mathbf{\Phi}(\mathbf{y},s))| \,\mathrm{d}s \\ + \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |[\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},s)) - [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},s))| \,\mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0$$

since $\mathcal{T}_{\max} f$ and $\mathcal{T}_{\max} f_n$ both belong to $L^1(\mathbf{\Omega},\mathrm{d}\mu)$. Consequently, for almost every $\mathbf{y} \in \Gamma_-$ (up to a subsequence, still denoted by f_n) we get

$$\begin{cases} f_n(\mathbf{\Phi}(\mathbf{y},\cdot)) \longrightarrow f(\mathbf{\Phi}(\mathbf{y},\cdot)) \\ \mathcal{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{y},\cdot)) \longrightarrow [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},\cdot)) \quad \text{in} \quad L^1((0,\tau_+(\mathbf{y})),\mathrm{d}s) \end{cases}$$

as $n \to \infty$. Let us fix $\mathbf{y} \in \Gamma_-$ for which this holds. Passing again to a subsequence, we may assume that $f_n(\mathbf{\Phi}(\mathbf{y},s))$ converges (pointwise) to $f(\mathbf{\Phi}(\mathbf{y},s))$ for almost every $s \in (0,\tau_+(\mathbf{y}))$. Let us fix such a s_0. Then,

$$f_n(\mathbf{\Phi}(\mathbf{y},s_0)) - f_n(\mathbf{\Phi}(\mathbf{y},s)) = \int_{s_0}^s [\mathcal{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r \qquad \forall s \in (0,\tau_+(\mathbf{y})).$$

Now, the right-hand-side has a limit as $n \to \infty$ so that the first term on the left-hand side also must converge as $n \to \infty$. Thus, for any $s \in (0,\tau_+(\mathbf{y}))$, the limit

$$\lim_{n\to\infty} f_n(\mathbf{\Phi}(\mathbf{y},s)) = \widetilde{f}_-(\mathbf{\Phi}(\mathbf{y},s))$$

exists and, for any $0 < s < \tau_+(\mathbf{y})$

$$\widetilde{f}_-(\mathbf{\Phi}(\mathbf{y},s)) = \widetilde{f}_-(\mathbf{\Phi}(\mathbf{y},s_0)) - \int_{s_0}^s [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r.$$

It is easy to check then that $\widetilde{f}_-(\mathbf{x}) = f(\mathbf{x})$ for almost every $\mathbf{x} \in \mathbf{\Omega}_-$. The same arguments lead to the existence of $\widetilde{f}_+$. □

The above result shows that the mild formulation of Theorem 3.6 is fulfilled for any $\mathbf{x} \in \mathbf{\Omega}_- \cup \mathbf{\Omega}_+$. It remains to deal with $\mathbf{\Omega}_\infty := \mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}$.

Proposition 3.15. *Let $f \in \mathscr{D}(\mathcal{T}_{\max})$. Then, there exists a set $\mathcal{O} \subset \mathbf{\Omega}_\infty$ with $\mu(\mathcal{O}) = 0$ and a function $\widetilde{f}$ defined on $\{\mathbf{z} = \mathbf{\Phi}(\mathbf{x},t),\ \mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{O},\ t \in \mathbb{R}\}$ such that $f(\mathbf{x}) = \widetilde{f}(\mathbf{x})$ μ-almost every $\mathbf{x} \in \mathbf{\Omega}_\infty$ and*

$$\widetilde{f}(\mathbf{\Phi}(\mathbf{x},s)) - \widetilde{f}(\mathbf{\Phi}(\mathbf{x},t)) = \int_s^t [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{x},r))\mathrm{d}r, \qquad \forall\, \mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{O},\ s < t.$$

Proof. Since $(\mathbf{x},t) \mapsto (\mathbf{z},t) = (\mathbf{\Phi}(\mathbf{x},t),t)$ is a measurable and measure preserving mapping from $\mathbf{\Omega}_\infty \times \mathbb{R}$ onto itself, Propositions 3.8 and 3.11 give

$$\lim_{n\to\infty} \int_{\mathbf{\Omega}_\infty} \mathrm{d}\mu(\mathbf{x}) \int_{I_k} |f_n(\mathbf{\Phi}(\mathbf{x},t)) - f(\mathbf{\Phi}(\mathbf{x},t))|\, dt = 0 \tag{3.15}$$

$$\lim_{n\to\infty} \int_{\mathbf{\Omega}_\infty} \mathrm{d}\mu(\mathbf{x}) \int_{I_k} |\mathcal{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{x},t)) - \mathcal{T}_{\max} f(\mathbf{\Phi}(\mathbf{x},t))|\, \mathrm{d}t = 0, \tag{3.16}$$

for any $I_k = [-k,k]$, $k \in \mathbb{N}$. This shows, in particular, that there is (a maximal) $\mathcal{E} \subset \mathbf{\Omega}_\infty$ with $\mu(\mathcal{E}) = 0$ such that, for almost every $\mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{E}$ and any bounded interval $I \subset \mathbb{R}$:

$$\int_I |f(\mathbf{\Phi}(\mathbf{x},t))|\mathrm{d}t + \int_I |[\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{x},t))|\mathrm{d}t < \infty$$

and we can argue as in Proposition 3.13 that

$$f_n(\mathbf{\Phi}(\mathbf{x},s)) - f_n(\mathbf{x}) = -\int_0^s \mathcal{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{x},r))\mathrm{d}r, \qquad \forall s \in \mathbb{R}.$$

Proposition 3.8 yields the existence of a subsequence $(f_{n_p})_p$ and a μ-null set A_0 with $\mathcal{E} \subset A_0 \subset \mathbf{\Omega}_\infty$ such that

$$\lim_{p\to\infty} f_{n_p}(\mathbf{x}) = f(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{A}_0.$$

Now, for any $k \in \mathbb{N}$,

$$\lim_{p\to\infty} \int_{\mathbf{\Omega}_\infty} \mathrm{d}\mu(\mathbf{x}) \int_{I_k} \left|\mathcal{T}_{\max} f_{n_p}(\mathbf{\Phi}(\mathbf{x},t)) - \mathcal{T}_{\max} f(\mathbf{\Phi}(\mathbf{x},t))\right| \mathrm{d}t = 0$$

so that, there is a subsequence (depending on k) and a μ-null set A_k with $A_0 \subset A_k \subset \mathbf{\Omega}_\infty$ such that

$$\lim_{p(k)\to\infty} \int_{I_k} \left|\mathcal{T}_{\max} f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},t)) - \mathcal{T}_{\max} f(\mathbf{\Phi}(\mathbf{x},t))\right| \mathrm{d}t = 0, \qquad \forall \mathbf{x} \in \mathbf{\Omega}_\infty \setminus A_k.$$

Let $\mathbf{x} \in \mathbf{\Omega}_\infty \setminus A_k$ and $|s| < k$ be fixed. From

$$f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},s)) - f_{n_{p(k)}}(\mathbf{x}) = -\int_0^s \mathcal{T}_{\max} f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},r))\mathrm{d}r$$

we deduce that the limit $\lim_{p(k)\to\infty} f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},s))$ exists and is equal to

$$\lim_{p(k)\to\infty} f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},s)) = f(\mathbf{x}) - \int_0^s \mathcal{T}_{\max} f(\mathbf{\Phi}(\mathbf{x},r))\mathrm{d}r.$$

We define then $\widetilde{f}$ by

$$\widetilde{f}(\mathbf{\Phi}(\mathbf{x},s)) = \lim_{p(k)\to\infty} f_{n_{p(k)}}(\mathbf{\Phi}(\mathbf{x},s)), \qquad \mathbf{x} \in \mathbf{\Omega}_\infty \setminus A_k,\ |s| < k$$

and defining $\mathcal{O} = \bigcup_{k\geqslant 1} A_k$, we get the result. □

Before the proof of Theorem 3.6, we have to establish existence of the trace on Γ_-.

Proposition 3.16. *Let f satisfies condition (1) of Theorem 3.6. Then*

$$\lim_{t\to 0+} f^\sharp(\mathbf{\Phi}(\mathbf{y},t))$$

exists for almost every $\mathbf{y} \in \Gamma_-$. Similarly, $\lim_{t\to 0+} f^\sharp(\mathbf{\Phi}(\mathbf{y},-t))$ exists for almost every $\mathbf{y} \in \Gamma_+$.

Proof. First we note that there is $\widetilde{\Omega}_- \subset \Omega_-$ with $\mu(\Omega_- \setminus \widetilde{\Omega}_-) = 0$ such that (3.2) is valid any $\mathbf{x} \in \widetilde{\Omega}_-$. Let $\widetilde{\Gamma}_- = \{\mathbf{y} \in \Gamma_-;\ \mathbf{y} = \mathbf{\Phi}(\mathbf{x}, \tau_-(\mathbf{x})), \mathbf{x} \in \widetilde{\Omega}_-\}$. It is easy to see that $\mu_-(\Gamma_- \setminus \widetilde{\Gamma}_-) = 0$. Indeed, otherwise, by (2.9), there would be a subset of Ω_- of positive μ-measure, not intersecting $\widetilde{\Omega}_-$, which would contradict (3.2). Consequently, any $\mathbf{x} \in \widetilde{\Omega}_-$ can be written as $\mathbf{x} = \Phi(\mathbf{y}, \tau_-(\mathbf{y}))$, $\mathbf{y} \in \widetilde{\Gamma}_-$ and (3.2) can be recast as

$$f^\sharp(\mathbf{\Phi}(\mathbf{y}, t)) - f^\sharp(\mathbf{\Phi}(\mathbf{y}, t_0)) = \int_t^{t_0} g(\mathbf{\Phi}(\mathbf{y}, s)\mathrm{d}s. \tag{3.17}$$

for almost any $\mathbf{y} \in \Gamma_-$, where $0 < t \leqslant t_0 < \tau_+(\mathbf{y})$. Using again (2.9), $s \mapsto g(\mathbf{\Phi}(\mathbf{y}, s)$ is integrable on $(0, \tau_+(\mathbf{y})$ for almost any $\mathbf{y} \in \Gamma_-$. Consequently, for almost every $\mathbf{y} \in \Gamma_-$ we can pass to the limit in (3.17) with $t \to 0$; it is easy to check that this limit does not depend on t_0. The existence of $\lim_{t \to 0+} f^\sharp(\mathbf{\Phi}(\mathbf{y}, -t))$ for a. e. $\mathbf{y} \in \Gamma_+$ follows by the same argument. □

The above proposition allows to define the trace operators.

Definition 3.17. *For any $f \in \mathscr{D}(\mathcal{T}_{\max})$, define the traces $\mathsf{B}^\pm f$ by*

$$\mathsf{B}^+ f(\mathbf{y}) := \lim_{t \to 0+} f^\sharp(\mathbf{\Phi}(\mathbf{y}, -t)) \qquad \text{and} \qquad \mathsf{B}^- f(\mathbf{y}) := \lim_{t \to 0+} f^\sharp(\mathbf{\Phi}(\mathbf{y}, t))$$

for any $\mathbf{y} \in \Gamma_\pm$ for which the limits exist, where $f^\sharp$ is a suitable representative of f.

PROOF OF THEOREM 3.6. To prove that $(2) \implies (1)$, given $f \in \mathscr{D}(\mathcal{T}_{\max})$, set

$$f^\natural(\mathbf{x}) = \begin{cases} \widetilde{f}_-(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbf{\Omega}_-, \\ \widetilde{f}_+(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}, \\ \widetilde{f}(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}, \end{cases}$$

where $\widetilde{f}_\pm$ are given by Proposition 3.14 while $\widetilde{f}$ is provided by Prop. 3.15. Then, it is clear that for any $\mathbf{x} \in \mathbf{\Omega}$ and any $-\tau_-(\mathbf{x}) < t_1 \leqslant t_2 < \tau_+(\mathbf{x})$

$$f^\sharp(\mathbf{\Phi}(\mathbf{x}, t_1)) - f^\sharp(\mathbf{\Phi}(\mathbf{x}, t_2)) = \int_{t_1}^{t_2} [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{x}, s))\mathrm{d}s$$

and (3.2) is proven.

Let us now prove that $(1) \implies (2)$. Let us fix $\psi \in \mathfrak{Y}$, one has

$$\int_{\mathbf{\Omega}_-} f(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{\Phi}(\mathbf{x}, s))\big|_{s=0} \mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, t)) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t$$
$$= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f^\sharp(\mathbf{\Phi}(\mathbf{y}, t)) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t.$$

Notice that since both $\int_{\mathbf{\Omega}_-} f(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{\Phi}(\mathbf{x}, s))\big|_{s=0} \mathrm{d}\mu(\mathbf{x})$ and $\int_{\mathbf{\Omega}_-} \psi(\mathbf{x}) g(\mathbf{x}) \mathrm{d}\mu(\mathbf{x})$ exist, Proposition 2.12 and Fubini's Theorem, the integrals

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\mathbf{\Phi}(\mathbf{y}, t)) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t \quad \text{and} \quad \int_0^{\tau_+(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y}, t)) \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t$$

are well-defined for μ_--almost every $\mathbf{y} \in \Gamma_-$. Let us prove that these two integrals coincide for almost-every $\mathbf{y} \in \Gamma_-$. According to Lemma 3.3, for almost every $\mathbf{y} \in \Gamma_-$, there is a sequence

$(t_n)_n$ (depending on $\mathbf{y}$) such that $\psi(\mathbf{\Phi}(\mathbf{y},t_n))=0$ and $t_n\to\tau_+(\mathbf{y})$. Thus,

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\mathbf{\Phi}(\mathbf{y},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t=\lim_{n\to\infty}\int_0^{t_n} f^\sharp(\mathbf{\Phi}(\mathbf{y},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t$$

and

$$\int_0^{\tau_+(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y},t))\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t=\lim_{n\to\infty}\int_0^{t_n}\psi(\mathbf{\Phi}(\mathbf{y},t))g(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t.$$

Further, for almost every $\mathbf{y}\in\Gamma_-$, according to (3.2),

$$f^\sharp(\mathbf{\Phi}(\mathbf{y},t))=\mathsf{B}^-f(\mathbf{y})-\int_0^t g(\mathbf{\Phi}(\mathbf{y},r))\mathrm{d}r,\qquad\forall t\in(0,\tau_+(\mathbf{y})).$$

Integration by parts, using the fact that $\psi(\mathbf{\Phi}(\mathbf{y},0))=\psi(\mathbf{\Phi}(\mathbf{y},t_n))=0$ for any n, leads to

$$\int_0^{t_n} f^\sharp(\mathbf{\Phi}(\mathbf{y},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t=\int_0^{t_n} g(\mathbf{\Phi}(\mathbf{y},t))\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t.$$

Consequently, for μ_- almost every $\mathbf{y}\in\Gamma_-$:

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\mathbf{\Phi}(\mathbf{y},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t=\int_0^{\tau_+(\mathbf{y})}\psi(\mathbf{\Phi}(\mathbf{y},s))g(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t. \tag{3.18}$$

Finally, we get

$$\begin{aligned}\int_{\mathbf{\Omega}_-} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x})&=\int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})}\psi(\mathbf{\Phi}(\mathbf{y},t))g(\mathbf{\Phi}(\mathbf{y},t))\mathrm{d}t\\&=\int_{\mathbf{\Omega}_-} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).\end{aligned} \tag{3.19}$$

Using now parametrization over Γ_+, we prove in the same way that

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x})=\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{3.20}$$

It remains now to evaluate $A:=\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x})$. According to Assumption 1

$$A=\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f^\sharp(\mathbf{\Phi}(\mathbf{x},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}\mu(\mathbf{x}),\qquad\forall t\in\mathbb{R}.$$

Let us integrate the above identity over $(0,1)$, so that

$$A=\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}}\mathrm{d}\mu(\mathbf{x})\int_0^1 f^\sharp(\mathbf{\Phi}(\mathbf{x},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t.$$

Let us fix $\mathbf{x}\in\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}$. For any $t\in(0,1)$, one has $f^\sharp(\mathbf{\Phi}(\mathbf{x},t))=f^\sharp(\mathbf{x})-\int_0^t g(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s$ and integration by parts yields

$$\begin{aligned}\int_0^1 f^\sharp(\mathbf{\Phi}(\mathbf{x},t))\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t&=\int_0^1\psi(\mathbf{\Phi}(\mathbf{x},t))g(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t-\psi(\mathbf{x})f^\sharp(\mathbf{x})\\&\quad+\psi(\mathbf{\Phi}(\mathbf{x},1))\left(f^\sharp(\mathbf{x})-\int_0^1 g(\mathbf{\Phi}(\mathbf{x},s))\mathrm{d}s\right)\\&=\int_0^1\psi(\mathbf{\Phi}(\mathbf{x},t))g(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t+\psi(\mathbf{\Phi}(\mathbf{x},1))f^\sharp(\mathbf{\Phi}(\mathbf{x},1))-\psi(\mathbf{x})f^\sharp(\mathbf{x})\end{aligned}$$

where we used again (3.2). Integrating over $\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}$ we see from Liouville's Theorem (Assumption 1) that

$$\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{x},1))f^{\sharp}(\mathbf{\Phi}(\mathbf{x},1))\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} \psi(\mathbf{x})f^{\sharp}(\mathbf{x})\mathrm{d}\mu(\mathbf{x}),$$

i.e.

$$A = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} \mathrm{d}\mu(\mathbf{x}) \int_0^1 \psi(\mathbf{\Phi}(\mathbf{x},t))g(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t$$

which, thanks to Liouville's Theorem, is nothing but

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g(\mathbf{x})\,\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we obtain

$$\int_{\mathbf{\Omega}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}), \qquad \forall \psi \in \mathfrak{Y}$$

which exactly means that $f \in \mathscr{D}(\mathcal{T}_{\max})$ with $g = \mathcal{T}_{\max}$ and the proof is complete. □

Corollary 3.18. *Traces $B^{\pm}f$ on $\Gamma_{\pm}$ can be defined for any $f \in \mathscr{D}(\mathcal{T}_{\max})$. For μ_{-}- almost any $\mathbf{y} \in \Gamma_{-}$ we have*

$$\mathsf{B}^{-}f(\mathbf{y}) = f^{\sharp}(\mathbf{\Phi}(\mathbf{y},t)) + \int_0^t [\mathcal{T}_{\max}f](\mathbf{\Phi}(y,s))\mathrm{d}s, \qquad \forall t \in (0,\tau_{+}(y)),$$

where $f^{\sharp}$ is a suitable representative of f. An analogous formula holds for $\mathsf{B}^{+}f$.

Lemma 2.9 provides the existence of Borel measures $\mathrm{d}\mu_{\pm}$ on $\Gamma_{\pm}$, which allow us to define the natural trace spaces associated to Problem (1.1), namely,

$$L^1_{\pm} := L^1(\Gamma_{\pm}, \mathrm{d}\mu_{\pm}).$$

However, the traces $B^{\pm}f$, $f \in \mathscr{D}(\mathcal{T}_{\max})$, not necessarily belong to $L^1_{\pm}$.

4. Well-posedness for initial and boundary- value problems

4.1. **Absorption semigroup.**

From now on, we will denote $X = L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ endowed with its natural norm $\|\cdot\|_X$. Let $\mathcal{T}_0$ be the free streaming operator with *no re–entry boundary conditions*:

$$\mathcal{T}_0\psi = \mathcal{T}_{\max}\psi, \qquad \text{for any } \psi \in \mathscr{D}(\mathcal{T}_0),$$

where the domain $\mathscr{D}(\mathcal{T}_0)$ is defined by

$$\mathscr{D}(\mathcal{T}_0) = \{\psi \in \mathscr{D}(\mathcal{T}_{\max})\,;\, \mathsf{B}^{-}\psi = 0\}.$$

We state the following generation result:

Theorem 4.1. *The operator $(\mathcal{T}_0, \mathscr{D}(\mathcal{T}_0))$ is the generator of a nonnegative C_0-semigroup of contractions $(U_0(t))_{t\geqslant 0}$ in $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ given by*

$$U_0(t)f(\mathbf{x}) = f(\mathbf{\Phi}(\mathbf{x},-t))\chi_{\{t<\tau_{-}(\mathbf{x})\}}(\mathbf{x}), \qquad (\mathbf{x} \in \mathbf{\Omega},\ f \in X),$$

where χ_A denotes the characteristic function of a set A.

Proof. The proof is divided into three steps:

• *Step 1.* Let us first check that the family of operators $(U_0(t))_{t\geqslant 0}$ is a nonnegative contractive C_0-semigroup in X. Thanks to Proposition 2.3, we can prove that, for any $f \in X$ and any $t \geqslant 0$, the mapping $U_0(t)f : \mathbf{\Omega} \to \mathbb{R}$ is measurable and the semigroup properties $U_0(0)f = f$ and $U_0(t)U_0(s)f = U_0(t+s)f$ $(t,s \geqslant 0)$ hold. Let us now show that $\|U_0(t)f\|_X \leqslant \|f\|_X$. We have

$$\|U_0(t)f\|_X = \int_{\mathbf{\Omega}_+} |U_0(t)f|\mathrm{d}\mu + \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu + \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu.$$

Propositions 2.12 and 2.3 yield

$$\begin{aligned}\int_{\mathbf{\Omega}_+} |U_0(t)f|\mathrm{d}\mu &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |U_0(t)f(\mathbf{\Phi}(\mathbf{y},-s))|\mathrm{d}s\\ &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\max(0,\tau_-(\mathbf{y})-t)} |f(\mathbf{\Phi}(\mathbf{y},-s-t))|\mathrm{d}s\\ &\leqslant \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_t^{\max(t,\tau_-(\mathbf{y}))} |f(\mathbf{\Phi}(\mathbf{y},-r))|\mathrm{d}r \leqslant \int_{\mathbf{\Omega}_+} |f|\mathrm{d}\mu.\end{aligned}$$

In the same way we obtain

$$\int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu = \int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\infty} |U_0(t)f(\mathbf{\Phi}(\mathbf{y},s))|\mathrm{d}s = \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} |f|\mathrm{d}\mu,$$

and

$$\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} |f|\mathrm{d}\mu.$$

This proves contractivity of $U_0(t)$. Let us now show that $U_0(t)f$ is continuous, i.e.

$$\lim_{t\to 0} \|U_0(t)f - f\|_X = 0.$$

It is enough to show that this property holds for any $f \in \mathscr{C}_0(\mathbf{\Omega})$. In this case, $\lim_{t\to 0} U_0(t)f(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{\Omega}$. Moreover, $\sup_{\mathbf{x}\in\mathbf{\Omega}} |U_0(t)f(\mathbf{x})| \leqslant \sup_{\mathbf{x}\in\mathbf{\Omega}} |f(\mathbf{x})|$ and the support of $U_0(t)f$ is bounded, so that the Lebesgue dominated convergence theorem leads to the result. This proves that $(U_0(t))_{t\geqslant 0}$ is a C_0-semigroup of contractions in X. Let $\mathcal{A}_0$ denote its generator.

• *Step 2.* To show that $\mathscr{D}(\mathcal{A}_0) \subset \mathscr{D}(\mathcal{T}_0)$, fix $f \in \mathscr{D}(\mathcal{A}_0)$, $\lambda > 0$ and $g = (\lambda - \mathcal{A}_0)f$. Then,

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}t, \qquad (\mathbf{x} \in \mathbf{\Omega}). \tag{4.1}$$

To prove that $f \in \mathscr{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} f = \mathcal{A}_0 f$, it suffices to prove that

$$\int_{\mathbf{\Omega}} (\lambda f(\mathbf{x}) - g(\mathbf{x}))\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} f(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0} \mathrm{d}\mu(\mathbf{x}), \qquad \forall \psi \in \mathfrak{Y}.$$

Let us fix $\psi \in \mathfrak{Y}$, set $\varphi(\mathbf{x}) := \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\big|_{s=0}$ and write

$$\begin{aligned}\int_{\mathbf{\Omega}} f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\mathbf{\Omega}_+} f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) + \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_-} f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\\ &\quad + \int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = I_1 + I_2 + I_3.\end{aligned}$$

We first deal with I_1. For any $\mathbf{y}\in\Gamma_+$ and $t\in(0,\tau_-(\mathbf{y}))$ we have $\varphi(\mathbf{\Phi}(\mathbf{y},-t))=-\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},-t))$ and $f(\mathbf{\Phi}(\mathbf{y},-t))=\int_t^{\tau_-(\mathbf{y})}\exp(-\lambda(s-t))g(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s$. Then, by Proposition 2.12,

$$\begin{aligned}I_1&=-\int_{\Gamma_+}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{y},-t))\mathrm{d}t\int_t^{\tau_-(\mathbf{y})}\exp(-\lambda(s-t))g(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s\\
&=-\int_{\Gamma_+}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}g(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s\int_0^s\exp(-\lambda(s-t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi(\mathbf{\Phi}(\mathbf{y},-t))\right)\mathrm{d}t\\
&=\int_{\Gamma_+}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}g(\mathbf{\Phi}(\mathbf{y},-s))\times\\
&\qquad\times\left\{\lambda\int_0^s\exp(-\lambda(s-t))\psi(\mathbf{\Phi}(\mathbf{y},-t))\mathrm{d}t-\psi(\mathbf{\Phi}(\mathbf{y},-s))\right\}\mathrm{d}s\end{aligned}$$

where we used the fact that $\psi(\mathbf{\Phi}(\mathbf{y},0))=0$ for any $\mathbf{y}\in\Gamma_+$ since ψ is compactly supported. Thus

$$\begin{aligned}I_1&=\lambda\int_{\Gamma_+}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}\psi(\mathbf{\Phi}(\mathbf{y},-t))\mathrm{d}t\int_t^{\tau_-(\mathbf{y})}\exp(-\lambda(s-t))g(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s\\
&\qquad-\int_{\Gamma_-}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}g(\mathbf{\Phi}(\mathbf{y},-s))\psi(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s\\
&=\int_{\Gamma_+}\mathrm{d}\mu_+(\mathbf{y})\int_0^{\tau_-(\mathbf{y})}\psi(\mathbf{\Phi}(\mathbf{y},-t))\big(\lambda f(\mathbf{\Phi}(\mathbf{y},-t))-g(\mathbf{\Phi}(\mathbf{y},-t))\big)\mathrm{d}t.\end{aligned}$$

Using again Proposition 2.12, we obtain

$$I_1=\int_{\mathbf{\Omega}_+}\left(\lambda f(\mathbf{x})-g(\mathbf{x})\right)\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).\tag{4.2}$$

Arguing in a similar way, we prove that

$$I_2=-\int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}}\left(\lambda f(\mathbf{x})-g(\mathbf{x})\right)\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).\tag{4.3}$$

Finally, since

$$f(\mathbf{x})=\int_0^\infty\exp(-\lambda t)g\left(\mathbf{\Phi}(\mathbf{x},-t)\right)\mathrm{d}t\qquad\text{for any}\quad\mathbf{x}\in\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty},$$

one has

$$\begin{aligned}I_3&=\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}}\varphi(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\int_0^\infty\exp(-\lambda t)g(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}t\\
&=\int_0^\infty\exp(-\lambda t)\mathrm{d}t\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}}\varphi(\mathbf{x})g(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}\mu(\mathbf{x}).\end{aligned}$$

Now, Assumption 1 asserts that

$$\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}}\varphi(\mathbf{x})g(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}\mu(\mathbf{x})=\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}}g(\mathbf{x})\varphi(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}\mu(\mathbf{x}),\qquad\forall t\geqslant0,$$

and, since $\varphi(\mathbf{\Phi}(\mathbf{x},t)) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{\Phi}(\mathbf{x},t))$, finally

$$\begin{aligned}I_3 &= \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \int_0^\infty \exp(-\lambda t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi(\mathbf{\Phi}(\mathbf{x},t))\right)\mathrm{d}t \\ &= -\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) + \lambda\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\int_0^\infty \exp(-\lambda t)\psi(\mathbf{\Phi}(\mathbf{x},t))\mathrm{d}t.\end{aligned}$$

Using again Assumption 1, this finally gives

$$I_3 = -\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} \left(g(\mathbf{x}) - \lambda f(\mathbf{x})\right)\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{4.4}$$

Combining (4.2)–(4.4) leads to

$$\int_{\mathbf{\Omega}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = -\int_{\mathbf{\Omega}}\left(g(\mathbf{x}) - \lambda f(\mathbf{x})\right)\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x})$$

which proves that $f \in \mathscr{D}(\mathcal{T}_{\max})$ and $(\lambda - \mathcal{T}_{\max})f = g$. Next, for $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$ we write $t = \tau_-\left(\mathbf{\Phi}(\mathbf{y},t)\right)$ and, by Proposition 2.3 and (4.1), we obtain

$$f(\mathbf{\Phi}(\mathbf{y},t)) = \int_0^t \exp(-\lambda(t-s))\, g(\mathbf{\Phi}(\mathbf{y},s))\mathrm{d}s. \tag{4.5}$$

Consequently, $\lim_{t\to 0^+} f(\mathbf{\Phi}(\mathbf{y},t)) = 0$ a.e. $\mathbf{y} \in \Gamma_-$, i.e. $\mathsf{B}^- f = 0$ so that $f \in \mathscr{D}(\mathcal{T}_0)$ and $\mathcal{A}_0 f = \mathcal{T}_0 f = \lambda f - g$.

• *Step 3.* Now let us show the converse inclusion $\mathscr{D}(\mathcal{T}_0) \subset \mathscr{D}(\mathcal{A}_0)$. Let $f \in \mathscr{D}(\mathcal{T}_0)$. Changing possibly f on a set of zero measure, we may write $f = f^\sharp$, where $f^\sharp$ is the representative of f given by Theorem 3.6. Then, for any $\mathbf{x} \in \mathbf{\Omega}$ and any $0 \leqslant t < \tau_-(\mathbf{x})$

$$f(\mathbf{\Phi}(\mathbf{x},-t)) - f(\mathbf{x}) = \int_0^t [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{x},-r))\mathrm{d}r$$

which, according to the explicit expression of $U_0(t)$, means that

$$U_0(t)f(\mathbf{x}) - f(\mathbf{x}) = \int_0^t U_0(r)\mathcal{T}_{\max} f(\mathbf{x})\mathrm{d}r \tag{4.6}$$

for any $\mathbf{x} \in \mathbf{\Omega}$ and $t < \tau_-(\mathbf{x})$. Letting t converge towards $\tau_-(\mathbf{x})$ we obtain

$$f(\mathbf{x}) = -\int_0^{\tau_-(\mathbf{x})} [\mathcal{T}_{\max} f](\mathbf{\Phi}(\mathbf{x},-r))\mathrm{d}r.$$

In particular, Eq. (4.6) holds true for any $\mathbf{x} \in \mathbf{\Omega}$ and any $t \geqslant \tau_-(\mathbf{x})$. Arguing exactly as in [16, p. 38], the pointwise identity (4.6) represents the X–integral, i.e, $U_0(t)f - f = \int_0^t U_0(r)\mathcal{T}_{\max} f\mathrm{d}r$ in $L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Consequently, $f \in \mathscr{D}(\mathcal{A}_0)$ with $\mathcal{A}_0 f = \mathcal{T}_{\max} f$. □

4.2. **Green's formula.** The above result allows us to treat more general boundary-value problem:

Theorem 4.2. *Let $u \in L^1_-$ and $g \in X$ be given. Then the function*

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x},-t))\mathrm{d}t + \chi_{\{\tau_-(\mathbf{x})<\infty\}} \exp(-\lambda\tau_-(\mathbf{x}))u(\mathbf{\Phi}(\mathbf{x},-\tau_-(\mathbf{x})))$$

is a ***unique*** *solution $f \in \mathscr{D}(\mathcal{T}_{\max})$ of the boundary value problem:*

$$\begin{cases}(\lambda - \mathcal{T}_{\max})f = g, \\ \mathsf{B}^- f = u,\end{cases} \tag{4.7}$$

where $\lambda > 0$*. Moreover,* $\mathsf{B}^+ f \in L^1_+$ *and*

$$\|\mathsf{B}^+ f\|_{L^1_+} + \lambda \|f\|_X \leqslant \|u\|_{L^1_-} + \|g\|_X, \tag{4.8}$$

with equality sign if $g \geqslant 0$ *and* $u \geqslant 0$*.*

Proof. Let us write $f = f_1 + f_2$ with $f_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x}, -t))\mathrm{d}t$, and

$$f_2(\mathbf{x}) = \chi_{\{\tau_-(\mathbf{x})<\infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u\big(\mathbf{\Phi}\,(\mathbf{x}, -\tau_-(\mathbf{x}))\,\big), \qquad \mathbf{x} \in \mathbf{\Omega}.$$

According to Theorem 4.1, $f_1 = (\lambda - \mathcal{T}_0)^{-1} g$, i.e. $f_1 \in \mathscr{D}(\mathcal{T}_{\max})$ with $(\lambda - \mathcal{T}_{\max}) f_1 = g$ and $\mathsf{B}^- f_1 = 0$. Therefore, to prove that f is a solution of (4.7) it suffices to check that $f_2 \in \mathscr{D}(\mathcal{T}_{\max})$, $(\lambda - \mathcal{T}_{\max}) f_2 = 0$ and $\mathsf{B}^- f_2 = u$. It is easy to see that $f_2 \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ (see also (4.10)). To prove that $f_2 \in \mathscr{D}(\mathcal{T}_{\max})$ one argues as in the proof of Theorem 4.1. Precisely, let $\psi \in \mathfrak{Y}$, noticing that f_2 vanishes outside $\mathbf{\Omega}_-$, one has thanks to (4.9)

$$\begin{aligned}\int_{\mathbf{\Omega}} f_2(\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{\Phi}(\mathbf{x}, s))\big|_{s=0} \mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f_2(\mathbf{\Phi}(\mathbf{y}, t)) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t \\ &= \int_{\Gamma_-} u(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \exp(-\lambda t) \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t.\end{aligned}$$

For almost every $\mathbf{y} \in \Gamma_-$, we compute the integral over $(0, \tau_+(\mathbf{y}))$ by parts, which yields $f_2 \in \mathscr{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} f_2 = \lambda f_2$. Also,

$$f_2(\mathbf{\Phi}(\mathbf{y}, t)) = \exp(-\lambda t) u(\mathbf{y}), \qquad \mathbf{y} \in \Gamma_-, \quad 0 < t < \tau_+(\mathbf{y}) \tag{4.9}$$

from which we see that $\mathsf{B}^- f_2 = u$.

Consequently, f is a solution to (4.7). To prove that the solution is unique, it is sufficient to prove that the only solution $h \in \mathscr{D}(\mathcal{T}_{\max})$ to $(\lambda - \mathcal{T}_{\max}) h = 0, \mathsf{B}^- h = 0$, is $h = 0$. This follows from the fact that such a solution h actually belongs to $\mathscr{D}(\mathcal{T}_0)$ if $\lambda \in \varrho(\mathcal{T}_0)$. Finally, it remains to prove (4.8). For simplicity, we denote the representative of f_i, $i = 1, 2$, defined in Proposition 3.16, with the same letter. Using (4.9) and the fact that f_2 vanishes on $\mathbf{\Omega}_{-\infty}$, from (2.9) we get

$$\begin{aligned}\lambda \int_{\mathbf{\Omega}} |f_2| \mathrm{d}\mu = \lambda \int_{\mathbf{\Omega}_-} |f_2| \mathrm{d}\mu &= \lambda \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} e^{-\lambda t} |u(\mathbf{y})| \mathrm{d}t \\ &= \int_{\Gamma_-} |u(\mathbf{y})| \left(1 - e^{-\lambda \tau_+(\mathbf{y})}\right) \mathrm{d}\mu_-(\mathbf{y}).\end{aligned} \tag{4.10}$$

Define $h : \mathbf{y} \in \Gamma_- \longmapsto h(\mathbf{y}) = |u(\mathbf{y})| e^{-\lambda \tau_+(\mathbf{y})}$. It is clear that h vanishes on $\Gamma_{-\infty}$ and $h(\mathbf{y}) \leqslant |u(\mathbf{y})|$ for a.e. $\mathbf{y} \in \Gamma_-$. In particular, $h \in L^1_-$ and, according to (2.11),

$$\begin{aligned}\int_{\Gamma_-} h(\mathbf{y}) \mathrm{d}\mu_-(\mathbf{y}) &= \int_{\Gamma_- \setminus \Gamma_{-\infty}} h(\mathbf{y}) \mathrm{d}\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} h(\mathbf{\Phi}(z, -\tau_-(z))) \mathrm{d}\mu_+(z) \\ &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} e^{-\lambda \tau_-(z)} |u(\mathbf{\Phi}(z, -\tau_-(z)))| \mathrm{d}\mu_+(z) = \int_{\Gamma_+} |\mathsf{B}^+ f_2(z)| \mathrm{d}\mu_+(z) = \|\mathsf{B}^+ f_2\|_{L^1_+}.\end{aligned}$$

Combining this with (4.10) leads to

$$\lambda \|f_2\|_X + \|\mathsf{B}^+ f_2\|_{L^1_+} = \|u\|_{L^1_-}. \tag{4.11}$$

Now, let us show that $\mathsf{B}^+ f_1 \in L^1_+$ and $\|\mathsf{B}^+ f_1\|_{L^1_+} + \lambda \|f_1\|_X \leqslant \|g\|_X$. For any $\mathbf{y} \in \Gamma_+$ and $0 < t < \tau_-(\mathbf{y})$, we see, as above, that $f_1(\mathbf{\Phi}(\mathbf{y}, -t)) = \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s - t)) g(\mathbf{\Phi}(\mathbf{y}, -s)) \mathrm{d}s$.

This shows that

$$\mathsf{B}^+ f_1(\mathbf{y}) = \lim_{t\to 0^+} f_1(\mathbf{\Phi}(\mathbf{y},-t)) = \int_0^{\tau_-(\mathbf{y})} \exp(-\lambda s))g(\mathbf{\Phi}(\mathbf{y},-s))\mathrm{d}s.$$

According to Proposition 2.12,

$$\int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |g(\mathbf{\Phi}(\mathbf{y},-s))|\,\mathrm{d}s = \int_{\mathbf{\Omega}_+} |g|\,\mathrm{d}\mu$$

which, since $\exp(-\lambda(s-t))|g(\mathbf{\Phi}(\mathbf{y},-s))| \leqslant |g(\mathbf{\Phi}(\mathbf{y},-s))|$, implies $\mathsf{B}^+ f_1 \in L^1_+$. Let us now assume $g \geqslant 0$. Then $f_1 \geqslant 0$ and hence

$$\lambda\|f_1\| = \lambda \int_{\mathbf{\Omega}} f_1\,\mathrm{d}\mu = \lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu + \lambda \int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu + \lambda \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu.$$

Using similar arguments to those used in the study of f_2, we have

$$\lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y},-t))\,(1-\exp(-\lambda t))\,\mathrm{d}t,$$

which, by Proposition 2.12, implies $\lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_+} g\,\mathrm{d}\mu - \int_{\Gamma_+} \mathsf{B}^+ f_1\,\mathrm{d}\mu_+$. Similar argument shows that $\lambda \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} g\,\mathrm{d}\mu$, while the equality

$$\lambda \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g\,\mathrm{d}\mu,$$

is a direct consequence of the invariance of μ with respect to $\mathbf{\Phi}(\cdot,t)$. This shows that $\lambda\|f\|_X = \|g\|_X - \|\mathsf{B}^+ f\|_{L^1_+}$ for $g \geqslant 0$. In general, defining

$$F_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda s)\;|g(\mathbf{\Phi}(\mathbf{x},-s)|\,\mathrm{d}s, \qquad \mathbf{x}\in\mathbf{\Omega},$$

we obtain $\|\mathsf{B}^+ f_1\|_{L^1_+} + \lambda\|f_1\|_X \leqslant \|B^+F_1\|_{L^1_+} + \lambda\|F_1\|_X = \|g\|_X$ which, combined with (4.11), gives (4.8). □

Remark 4.3. *Notice that, in order to get the existence and uniqueness of the solution f to (4.7), it is not necessary for u to belong to $L^1(\Gamma_-,\mathrm{d}\mu_-)$. Indeed, we only have to make sure that $f_2 \in L^1(\mathbf{\Omega},\mathrm{d}\mu)$, i.e., from (4.10), $\int_{\Gamma_-} |u(\mathbf{y})|\left(1-e^{-\lambda\tau_+(\mathbf{y})}\right)\mathrm{d}\mu_-(\mathbf{y}) < \infty$. Of course, to get (4.8), the assumption $u \in L^1(\Gamma_-,\mathrm{d}\mu_-)$ is necessary.*

Let us note that, with the notation of Theorem 4.2, we have

$$\int_{\Gamma_+} \mathsf{B}^+ f\mathrm{d}\mu_+ + \lambda\int_{\mathbf{\Omega}} f\,\mathrm{d}\mu = \int_{\Gamma_-} u\,\mathrm{d}\mu_- + \int_{\mathbf{\Omega}} g\,\mathrm{d}\mu. \tag{4.12}$$

Indeed, for nonnegative u and g, (4.8) turns out to be precisely (4.12). Then, for arbitrary $u \in L^1_-$ and $g \in X$, we get (4.12) by splitting functions into positive and negative parts. This leads to the following generalization of Green's formula:

Proposition 4.4 (*Green's formula*). *Let $f \in \mathscr{D}(\mathcal{T}_{\max})$ satisfies $\mathsf{B}^- f \in L^1_-$. Then $\mathsf{B}^+ f \in L^1_+$ and*

$$\int_{\mathbf{\Omega}} \mathcal{T}_{\max} f\mathrm{d}\mu = \int_{\Gamma_-} \mathsf{B}^- f\mathrm{d}\mu_- - \int_{\Gamma_+} \mathsf{B}^+ f\,\mathrm{d}\mu_+$$

Proof. For given $f \in \mathscr{D}(\mathcal{T}_{\max})$, we obtain the result by setting $u = \mathsf{B}^- f \in L^1_-$ and $g = (\lambda - \mathcal{T}_{\max})f \in X$ in Eq. (4.12). □

Remark 4.5. *If $\mathrm{d}\mu$ is the Lebesgue measure on $\mathbb{R}^N$, the above formula leads to a better understanding of the measures $\mathrm{d}\mu_\pm$. Indeed, comparing it to the classical Green's formula (see e.g. [8]), we see that the restriction of $\mathrm{d}\mu_\pm$ on the set $\Sigma_\pm = \{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \pm\mathscr{F}(\mathbf{y}) \cdot n(\mathbf{y}) > 0\}$ equals*

$$|\mathscr{F}(\mathbf{y}) \cdot n(\mathbf{y})|\, \mathrm{d}\gamma(\mathbf{y}),$$

where $\mathrm{d}\gamma(\cdot)$ is the surface Lebesgue measure on $\partial\mathbf{\Omega}$.

Appendix: About the class of test-functions

We answer in this Appendix a natural question concerning the definition of the class of test-functions $\mathfrak{Y}$. Precisely, we prove that two test-functions equal μ–almost everywhere are such that their derivatives (in the sense of (3.1)) also coincide μ-almost everywhere. To prove our claim, it clearly suffices to prove that, given $\psi \in \mathfrak{Y}$ such that $\psi(\mathbf{x}) = 0$ for μ-a. e. $\mathbf{x} \in \mathbf{\Omega}$, one has $\varphi(\mathbf{x}) = 0$ for μ-a. e. $\mathbf{x} \in \mathbf{\Omega}$ where $\varphi(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{\Phi}(\mathbf{x}, s))\big|_{s=0}$. Let

$$E := \big\{\mathbf{x} \in \mathbf{\Omega}\,;\, \psi(\mathbf{x}) = 0 \text{ and } \varphi(\mathbf{x}) \neq 0\big\}.$$

It is clear that E is measurable and that one has to prove that $\mu(E) = 0$. It is no loss of generality to assume that E *is bounded*. We observe that for any $\mathbf{x} \in E$, there exists $\delta_{\mathbf{x}} > 0$ such that

$$\psi(\mathbf{\Phi}(\mathbf{x}, t)) \neq 0, \qquad \forall\, 0 < |t| < \delta_{\mathbf{x}}. \tag{A.1}$$

Let us split E as follows

$$E = \big(E \cap \mathbf{\Omega}_-\big) \cup \big(E \cap \mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}\big) \cup \big(E \cap \mathbf{\Omega}_{+\infty} \cap \mathbf{\Omega}_{-\infty}\big) := E_- \cup E_+ \cup E_\infty$$

and prove that $\mu(E_-) = \mu(E_+) = \mu(E_\infty) = 0$.

(1) First consider E_-. Since $\psi(\mathbf{x}) = 0$ for μ-a. e. $\mathbf{x} \in \mathbf{\Omega}_-$ and using the fact that any $\mathbf{x} \in \mathbf{\Omega}_-$ can be written as $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, t)$ for some $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$, we observe that, for μ_- a. e. $\mathbf{y} \in \Gamma_-$, $\psi(\mathbf{\Phi}(\mathbf{y}, t)) = 0$ for almost every (in the sense of the Lebesgue measure in $\mathbb{R}$) $0 < t < \tau_+(\mathbf{y})$. For such a $\mathbf{y} \in \Gamma_-$, continuous differentiability of $t \mapsto \psi(\mathbf{\Phi}(\mathbf{y}, t))$ implies $\psi(\mathbf{\Phi}(\mathbf{y}, t)) = 0$ for *any* $0 < t < \tau_+(\mathbf{y})$. This means, according to (A.1) that, for μ_--a. e. $\mathbf{y} \in \Gamma_-$, $\mathbf{\Phi}(\mathbf{y}, t) \notin E$ for any $0 < t < \tau_+(\mathbf{y})$. Since

$$\mu(E \cap \mathbf{\Omega}_-) = \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \chi_E(\mathbf{\Phi}(\mathbf{y}, t))\mathrm{d}t$$

we see that $\mu(E_-) = 0$.

(2) In the same way, using Γ_+ instead of Γ_-, we show that $\mu(E \cap \mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}) = 0$.

(3) It remains to prove that $\mu(E_\infty) = 0$. In accordance with (A.1), we define for, any $n \in \mathbb{N}$,

$$E_n := \left\{\mathbf{x} \in E_\infty\,;\, \delta_{\mathbf{x}} \geqslant 1/n\right\} = \left\{\mathbf{x} \in E_\infty\,;\, \psi(\mathbf{\Phi}(\mathbf{x}, t)) \neq 0,\ \forall\, 0 < |t| < 1/n\right\}.$$

According to Assumption 1, it is easy to see that $\mu(E_n) = 0$ for any $n \in \mathbb{N}$ since $\psi(\mathbf{x}) = 0$ for μ-a.e. $\mathbf{x} \in \mathbf{\Omega}$. Moreover, $E_1 \subset E_2 \subset \ldots \subset E_n \subset E_{n+1} \subset \ldots$, and

$$\bigcap_{n=1}^{\infty} \Big(E_\infty \setminus E_n\Big) = \varnothing.$$

Since we assumed $\mu(E) < \infty$, we have $\mu(E_\infty \setminus E_1) < \infty$ and $\lim_{n\to\infty} \mu\big(E_\infty \setminus E_n\big) = 0$. Writing $E_\infty = E_n \cup \big(E_\infty \setminus E_n\big)$, we see that $\mu(E_\infty) = 0$.

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