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K-THEORY FOR THE MAXIMAL ROE ALGEBRA OF CERTAIN EXPANDERS

HERVÉ OYONO-OYONO AND GUOLIANG YU

Abstract. We study in this paper the maximal version of the coarse Baum-Connes assembly map for families of expanding graphs arising from residually finite groups. Unlike for the usual Roe algebra, we show that this assembly map is closely related to the (maximal) Baum-Connes assembly map for the group and is an isomorphism for a class of expanders. We also introduce a quantitative Baum-Connes assembly map and discuss its connections to K-theory of (maximal) Roe algebras.

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Contents

1. Introduction

In this paper, we study K-theory of (maximal) Roe algebras for a class of expanders. The Roe algebra was introduced by John Roe in his study of higher index theory of elliptic operators on noncompact spaces [12]. The K-theory of Roe algebra is the receptacle for the higher indices of elliptic operators. If a space is coarsely embeddable into Hilbert space, then K-theory of Roe algebra and higher indices of elliptic operators are computatble [17]. Gromov discovered that expanders do not admit coarse embedding into Hilbert space [5]. The purpose of this paper is to completely or partially compute K-theory of the (maximal) Roe algebras associated to certain expanders. In particular, we prove the maximal version of the coarse Baum-Connes conjecture for a special class of expanders. The coarse Baum-Connes conjecture is a geometric analogue of the Baum-Connes conjecture [1] and provides an algorithm of computing K-theory of Roe algebras and higher indices of elliptic operators. We also prove the (maximal) coarse Novikov conjecture for a class of expanders. The coarse Novikov conjecture gives a partial computation of K-theory of Roe algebras and an algorithm to determine non-vanishing of higher indices for elliptic operators. Our results on the coarse Novikov conjecture are more general than results obtained in [3, 4, 6]. The question of computing K-theory of (maximal) Roe algebras associated to general expanders remains open. We show that this question is closely related to certain quantitative Novikov conjecture and the quantitative Baum-Connes conjecture for the K-theory of (maximal) Roe algebras. We explore this connection to prove the quantitative Novikov conjecture and the quantitative Baum-Connes conjecture in some cases.

The class of expanders under examination in this paper is those associated to a finitely generated and residually finite group Γ with respect to a family

$$\Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots$$

of finite index normal subgroups. The behaviour of the Baum-Connes assembly map for Γ and of the coarse Baum-Connes assembly map for the metric space $X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$ can differ quite substantially: if Γ has the property τ with respect to the family $(\Gamma_i)_{i\in\mathbb{N}}$, then $(\Gamma/\Gamma_i)_{i\in\mathbb{N}}$ is a family of expanders and the coarse assembly map for $X(\Gamma)$ fails to be an isomorphism, althougth for example for $\Gamma = SL_2(\mathbb{Z})$, the assemply maps is an isomorphism. In [4] was introduced the maximal Roe algebra of a coarse space and a maximal coarse assembly map with value in this algebra was defined. As we shall see, the behaviour of this maximal coarse Baum-Connes assembly map for $X(\Gamma)$, and of the maximal Baum-Connes assemply map for the group Γ with coefficients in $\ell^\infty(X(\Gamma), \mathcal{K}(H))/C_0(X(\Gamma), \mathcal{K}(H))$ turn out to be equivalent. In particular, as a consequence of [14], if Γ sastifies the strong Baum-Connes conjecture, then the maximal coarse assembly map for $X(\Gamma)$ is an isomorphism. As a as a spin-off we also obtain the injectivity of the coarse assembly map when Γ coarsely embeds in a Hilbert Space.

This suggests that the properties of the maximal coarse assembly map for $X(\Gamma)$ is closely related to some asymptotic properties of the maximal Baum-Connes assembly maps for Γ with coefficients in the family $\{C(\Gamma/\Gamma_i\}_{i\in\mathbb{N}}$. For this purpose, we define quantitative assembly maps that take into account the propagation in the crossed product $\{C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma\}_{i\in\mathbb{N}}$. Notice that althought $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$ and $C^*_{\max}(\Gamma_i)$ are Morita equivalent, the imprimitivy bimodule between these two algebras introduces some distortion in the propagation and the relevant propagation is the one coming from Γ. In this setting, we show that injectivity and bijectivity of the maximal coarse assembly map are equivalent to some asymptotic statements for these quantative assembly maps. For surjectivity, and up to a slight modification in the sequence of normal subgroups, we also obtain similar results.

The paper is organised as follows. In section 2, we review results from [4] and [8] concerning maximal Roe algebras and coarse assembly maps. We also show the existence of a short exact sequence (see section 2.2)

$$0\longrightarrow\mathcal{K}(\ell^2(X(\Gamma))\otimes H)\longrightarrow C^*_{max}(X(\Gamma))\longrightarrow A_\Gamma\rtimes_{\max}\Gamma\longrightarrow 0. \tag{1.1}$$

In section 3, we collect results about Baum-Connes assembly maps that we use later on. In section 4, we state for the left hand side of the Baum-Connes assembly map an analogue of the exact sequence of equation 1.1 . We show that assembly maps intertwines this exact sequence with the one induced in K-theory by the exact sequence 1.1, and obtain injectivity and bijectivity results for the maximal coarse assembly map for $X(\Gamma)$. In section 5.3 we set asymptotic statements concerning some quantitative assembly maps and we discuss examples of groups that satisfy these statements.

2. K-theory for maximal Roe algebras

2.1. **Maximal Roe algebra of a locally compact metric space.**

In this section, we collect from [4] results concerning the maximal Roe algebra of a locally compact metric space that we will need in this paper.

2.1.1. *The case of a discrete space.* Let Σ be a discrete space equipped with a proper distance d. Let us denote by $C[\Sigma]$ the algebra of locally compact operators with finite propagation of $\ell^2(\Sigma)\otimes H$, where H is a separable Hilbert space, i.e (bounded) operators T of $\ell^2(\Sigma)\otimes H$ such that when written as a family $(T_{x,y})_{(x,y)\in\Sigma^2}$ of operator on H, then

- $T_{x,y}$ is compact for all x and y in Σ;
- there exists a real r such that $d(x,y)>r$ implies that $T_{x,y}=0$ (T is said to have propagation less than r).

For any real r, we define $C_r[\Sigma]$ as the set of elements of $C[\Sigma]$ with propagation less than r. It is straightforward to check that $C[\Sigma]$ is a $*$-algebra. The (usual) Roe algebra $C^*(\Sigma)$ is the closure of $C[\Sigma]$ viewed as a subalgebra of operator of $\mathcal{L}(\ell^2(\Sigma)\otimes H)$. The next lemma, proved in [4], shows that if Σ has bounded geometry, then $C[\Sigma]$ admits an envelopping algebra.

Lemma 2.1. *Let Σ be a discrete metric space with bounded geometry. For any positive number r, there exists a real c_r such that for any $*$-representation ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ and any T in $C_r[\Sigma]$, then $\|\phi(T)\|_{\mathcal{L}(H_\phi)}\leq c_r\|T\|_{\ell^2(\Sigma)\otimes H}$.*

This envelopping algebra is then

Definition 2.2. [4] *The maximal Roe algebra of a discrete metric space Σ with bounded geometry, denoted by $C^*_{max}(\Sigma)$, is the completion of $C[\Sigma]$ with respect to the $*$-norm*

$$\|\phi(T)\|=\sup_{(\phi,H_\phi)}\|\phi(T)\|_{\mathcal{L}(H_\phi)},$$

when (ϕ,H_ϕ) runs through representations ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ.

2.1.2. *The general case.* Let X be a locally compact space, equipped with a metric d. A X-module is a Hilbert space H_X together with a $*$-representation ρ_X of $C_0(X)$ in H_X. We shall often write f instead of $\rho_X(f)$ for the action of f on H_X. If the representation is non-degenerate, the X-module is said to be non-degenerate. A X-module is called standard if no non-zero function of $C_0(X)$ acts as a compact operator on H_X. In the litterature, the terminology $C_0(X)$-ample is also used for such a representation [8, 15].

Definition 2.3. *Let H_X be a standard non-degenerate X-module and let T be a bounded operator on H_X.*

(i) *The support of T is the complement of the open subset of $X \times X$*

$$\{(x, y) \in X \times X \text{ s.t. there exist } f \text{ and } g \text{ in } C_0(X) \text{ satisfying} \\ f(x) \neq 0,\ g(y) \neq 0 \text{ and } f \cdot T \cdot g = 0\}.$$

(ii) *If there exists a real r such that for any x and y in X such that $d(x, y) > r$, then (x, y) is not in the support of T, then the operator T is said to have finite propagation (in this case propagation less than r).*

(iii) *The operator T is said to be locally compact if $f \cdot T$ and $T \cdot f$ are compact for any f in $C_0(X)$. We then define $C[X]$ as the set of locally compact and finite propagation bounded operators of H_X.*

(iv) *The operator T is said to pseudo-local if $[f, T]$ is compact for all f in $C_0(X)$.*

It is straightforward to check that $C[X]$ is a $*$-algebra and that for a discrete space, this definition coincides with the previous one. Moreover, up to (non-canonical) isomorphism, $C[X]$ does not depend on the choice of H_X. The Roe algebra $C^*(X)$ is then the norm closure of $C[X]$ in the algebra $\mathcal{L}(H_X)$ of bounded operators on H_X. Although $C^*(X)$ is not canonically defined, we shall see later on that up to canonical isomorphism, its K-theory does not depend on the choice a non-degenerated standard X-module.

Definition 2.4. *A net in a locally compact space X is a countable subset Σ such that there exists numbers ε and r satisfying*

- *$d(y, y') \geq \varepsilon$ for any distinct elements y and y' of Σ;*
- *For any x in X, there exists y in Σ such that $d(x, y) \leq r$.*

The following result was proved in [4]

Lemma 2.5. *If a locally compact space X contains a net with bounded geometry, then with notation of definition 2.4, there exists a unitary map $\Psi : H_\Sigma \to H_X$ that fullfills the following conditions:*

(i) *The homomorphism $\mathcal{L}(H_\Sigma) \to \mathcal{L}(H_X); T \mapsto \Psi^* \cdot T \cdot \Psi$ restricts to an algebra $*$-isomorphism $C[\Sigma] \to C[X]$;*

(ii) *There exists a number r such that for every x in X and y in Σ with $d(x, y) < r$, then there exists f in $C_0(X)$ and g in $C_0(\Sigma)$ which satisfy $f(x) \neq 0$, $g(y) \neq 0$ and $f \cdot \Psi \cdot g = 0$ (i.e Ψ has propagation less than r).*

Then, if $\Psi : H_\Sigma \to H_X$ is a unitary map as in lemma 2.5, the $*$-isomorphism $C[\Sigma] \to C[X]; T \mapsto \Psi^* \cdot T \cdot \Psi$ extends to an isomorphism $Ad_\Psi : C^*(\Sigma) \to C^*(X)$.

As a consequence of lemma 2.5, lemma 2.1 admits the following generalisation to spaces that contain a net with bounded geometry.

Lemma 2.6. *Let X be a locally compact metric that contains a net with bounded geometry and let H_X be a standard non-degenerate X-module. Then for any positive number r, there exists a real c_r such that for any $*$-representation ϕ of $C[X]$ on a Hilbert space H_ϕ and any T in $C[X]$ with propagation less than r, then $\|\phi(T)\|_{\mathcal{L}(H_\phi)} \leq c_r \|T\|_{H_X}$.*

This allowed to define for X the maximal Roe algebra as in the discrete case.

Definition 2.7. [4] *Let X be a locally compact metric space that contains a net with bounded geometry . The maximal Roe algebra of X, denoted by $C^*_{max}(X)$, is the completion of $C[X]$ with respect to the $*$-norm*

$$\|\phi(T)\| = \sup_{(\phi, H_\phi)} \|\phi(T)\|_{\mathcal{L}(H_\phi)},$$

when (ϕ, H_ϕ) runs through representation ϕ on of $C[X]$ a Hilbert space H_ϕ.

2.2. **Maximal Roe algebra associated to a residually finite group.** Let Γ be a residually finite group, finitely generated. Let $\Gamma_0 \supset \Gamma_1 \supset \dots \Gamma_n \supset \dots$ be a decreasing sequence of finite index subgroups of Γ such that $\bigcap_{i\in\mathbb{N}} \Gamma_i = \{e\}$. Let d be a left invariant metric associated to a given finite set of generators. Let us endow Γ/Γ_i with the metric $d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2), \gamma_1$ and γ_2 in $\Gamma_i\}$. We set $X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$ and we equip $X(\Gamma)$ with a metric d such that,

- on Γ/Γ_i, then d is the metric defined above;
- $d(\Gamma/\Gamma_i, \Gamma/\Gamma_i) \geq i + j$ if $i \neq j$.
- the group Γ acts on $X(\Gamma)$ by isometries.

Let us define by $\mathcal{K}(H)$ the algebra of compact operators of H. Then the C^*-algebra $\ell^\infty(X(\Gamma), \mathcal{K}(H))$ acts on $\ell^2(X(\Gamma))\otimes H$ by pointwise action of $\mathcal{K}(H)$. This action is clearly by propagation zero locally compact operators. The group Γ acts diagonally on the Hilbert space $\ell^2(X(\Gamma))\otimes H$ by finite propagation operators, the action being on $\ell^2(X(\Gamma))$ induced by the action on $X(\Gamma)$ and trivial on H. From this, we get a covariant representation of $(\ell^\infty(X(\Gamma), \mathcal{K}(H)), \Gamma)$ on $\ell^2(X(\Gamma))$, where the action of Γ on $\ell^\infty(X(\Gamma), \mathcal{K}(H))$ is induced by the action of Γ on $X(\Gamma)$ by translations. This yields to a $*$-homomorphism $C_c(\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H))) \longrightarrow C[X(\Gamma)]$ (where $C_c(\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H)))$ is equipped with the convolution product) and thus, setting $B_\Gamma = \ell^\infty(X(\Gamma), \mathcal{K}(H))$, to a $*$-homomorphism

$$B_\Gamma \rtimes_{\max} \Gamma \longrightarrow C^*_{max}(X(\Gamma)).$$

Under this map, the image of $B_{\Gamma,0} \overset{\text{def}}{=\!=} C_0(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma$ is contained in $\mathcal{K}(\ell^2(X(\Gamma))\otimes H)$. Thus if we set $A_\Gamma = B_\Gamma / B_{\Gamma,0}$, then we finally get a $*$-homomorphism

$$\Phi_\Gamma : A_\Gamma \rtimes_{\max} \Gamma \longrightarrow C^*_{max}(X(\Gamma))/\mathcal{K}(\ell^2(X(\Gamma))\otimes H).$$

Proposition 2.8. Φ_Γ *is a $*$-isomorphism, i.e we have an short exact sequence*

$$0 \longrightarrow \mathcal{K}(\ell^2(X(\Gamma))\otimes H) \longrightarrow C^*_{max}(X(\Gamma)) \longrightarrow A_\Gamma \rtimes_{max} \Gamma \longrightarrow 0.$$

Proof. Let us construct an inverse for Φ_Γ. Let T be an element in $C[X(\Gamma)]$ with propagation less than r. Let n be any integer such that $n \geq r$ and $B_\Gamma(e, r) \cap \Gamma_n = \{e\}$. Then there is a decomposition $T = T' + T''$ with T' in $\mathcal{K}(\ell^2(\coprod_{i=0}^{n-1} \Gamma/\Gamma_i)\otimes H)$ and $T'' = (T''_i)_{i\geq n} \in \prod_{i\geq n} \mathcal{K}(\ell^2(\Gamma/\Gamma_i)\otimes H)$. Let us denote for γ in Γ by $L_{\gamma\Gamma_i}$ the diagonal operator on $\ell^2(\Gamma/\Gamma_i)\otimes H$ given by left translation by $\gamma\Gamma_i$ on $\ell^2(\Gamma/\Gamma_i)$

and the identity on H. For any integer i, we have a unique decomposition $T''_i = \sum_{k=1}^p h_k L_{\gamma_k\Gamma_i}$, where $\gamma_k\Gamma_i$ belongs to $B_{\Gamma/\Gamma_i}(\Gamma_i, r)$ and h_k belongs to $C(\Gamma/\Gamma_i, \mathcal{K}(H))$ and is viewed as an operator acting on $\ell^2(\Gamma/\Gamma_i)\otimes H$ by pointwise action of operators of $\mathcal{K}(H)$. Since $B_\Gamma(e,r)\cap\Gamma_i = \{e\}$ for $i \geq n$, the element $\gamma_k\Gamma_i$ lifts to a unique element of $B_\Gamma(e,r)$. If $B_\Gamma(e,r) = \{g_1,\dots,g_m\}$, then there is a unique decomposition $T''_i = \sum_{k=1}^m f_k^i L_{g_k\Gamma_i}$, with f_k^i in $C(\Gamma/\Gamma_i, \mathcal{K}(H)$. Let us denote for $k = 1,\dots,m$ by $\phi_k(T)$ the image of $(f_k^i)_{i\geq n}$ under the projection

$$\prod_{i\geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H))\longrightarrow \prod_{i\geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H)/\bigoplus_{i\geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H) \cong A_\Gamma.$$

It is then straightforward to check that we obtain in this way a well defined map

$$\Lambda_r : C_r[X(\Gamma)]\longrightarrow C_c(\Gamma, A_\Gamma); T \mapsto \sum_{k=1}^m \phi_k(T)\delta_{g_k},$$

where δ_g is the Dirac function at an element g of Γ. Moreover, if $r' \geq r$,then $\Lambda_{r'}$ restricts to Λ_r on $C_r[X(\Gamma)]$ and the maps Λ_r extends to a $*$-homomorphism $C[X(\Gamma)]\longrightarrow C_c(\Gamma, A_\Gamma)$ and thus to a $*$-homomorphism $C^*_{max}(X(\Gamma))\longrightarrow A_\Gamma\rtimes_{\max}\Gamma$. This homomorphism clearly factorizes through a $*$-homomorphism

$$C^*_{max}(X(\Gamma))/\mathcal{K}(\ell^2(X(\Gamma))\otimes H)\longrightarrow A_\Gamma\rtimes_{\max}\Gamma$$

which provides an inverse for Φ_Γ.

□

We shall denote by $\Psi_{A_\Gamma,\Gamma,\max} : C^*_{max}(X(\Gamma))\longrightarrow A_\Gamma\rtimes_{\max}\Gamma$ the projection map corresponding to the exact sequence of the previous proposition. Let $\lambda_{\Gamma,A_\Gamma} : A_\Gamma\rtimes_{\max}\Gamma \to A_\Gamma \rtimes_{\mathrm{red}} \Gamma$ be the homomorphism given by the regular representation of the covariant system (A_Γ, Γ). The next lemma shows that $\lambda_{\Gamma,A_\Gamma}\circ\Psi_{A_\Gamma,\Gamma,\max}$ factorizes through $C^*(X(\Gamma))$ (see [7]).

Lemma 2.9. *There exists a unique homomorphism*

$$\Psi_{A_\Gamma,\Gamma,red} : C^*(X(\Gamma)) \to A_\Gamma \rtimes_{red} \Gamma$$

such that $\lambda_{X(\Gamma)}\circ\Psi_{A_\Gamma,\Gamma,red} = \Psi_{A_\Gamma,\Gamma,max}\circ\lambda_{\Gamma,A_\Gamma}$.

Proof. If such an homomorphism exists, it is clearly unique. Let us prove the existence. Let T be an element of $C_r(X(\Gamma))$ such that $\|T\|_{\mathcal{L}(\ell^2(X(\Gamma))\otimes H))} = 1$ and let us set $x_T = \lambda_{\Gamma,A_\Gamma}\circ\Psi_{A_\Gamma,\Gamma,\max}(T)$. Let us view $A_\Gamma \rtimes_{\mathrm{red}} \Gamma$ as an algebra of adjointable operator on the right A_Γ-Hilbert module $A_\Gamma\otimes\ell^2(\Gamma)$. For any positive real ε, let ξ be an element of $A_\Gamma\otimes\ell^2(\Gamma)$ such that $\|\xi\|_{A_\Gamma\otimes\ell^2(\Gamma)} = 1$ and

$$\|x_T.\xi\|_{A_\Gamma\otimes\ell^2(\Gamma)} \geq \|x_T\|_{A_\Gamma\rtimes_{\mathrm{red}}\Gamma} - \varepsilon.$$

We can assume without loss of generality that ξ lies indeed in $C_c(\Gamma, A_\Gamma)$ and is supported in some $B_\Gamma(e,s)$ for s positive real. Let us fix an integer k such that $B_\Gamma(e, 2r+2s)\cap\Gamma_k = \{e\}$. There is a decomposition $T = T' + T''$ with T' in $\mathcal{K}(\ell^2(\sqcup_{i=1}^{k-1}\Gamma/\Gamma_i)\otimes H)$ and T'' in $\prod_{i\geq k}\mathcal{K}(\ell^2(\Gamma/\Gamma_i)\otimes H) \cong \prod_{i\geq k} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes \Gamma/\Gamma_i$. Since this decomposition is diagonal, we get $\|T''\| \leq 1$. Actually T'' can be viewed as an adjointable operator on the right $\ell^\infty(\cup_{i\geq k}\Gamma/\Gamma_i, \mathcal{K}(H))$-Hilbert module $\prod_{i\geq k} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \otimes \ell^2(\Gamma/\Gamma_i)$. Let us chose a lift ξ' in $C_c(\Gamma, B_\Gamma)$ of ξ under the map induced by the canonical projection $B_\Gamma \to A_\Gamma$ such that $\|\xi'\|_{B_\Gamma\otimes\ell^2(\Gamma)} \leq 1+\varepsilon$ and ξ' is supported in $B_\Gamma(e,s)$. Under the identification

$$C_c(\Gamma, \prod_{i\geq k} C(\Gamma/\Gamma_i, \mathcal{K}(H))) \cong \prod_{i\geq k} C_c(\Gamma, C(\Gamma/\Gamma_i, \mathcal{K}(H))),$$

we can write $\xi' = (\xi'_i)_{i\geq k}$. Since $B_\Gamma(e,2r)\cap\Gamma_i = \{e\}$ for $i\geq k$, the map

$$B_\Gamma(e,r) \to B_{\Gamma/\Gamma_i}(\Gamma_i, r); \gamma \to \gamma\Gamma_i$$

is bijective. Hence, if we define ξ''_i in $C_c(\Gamma/\Gamma_i, \mathcal{K}(H))$ with support in $B_{\Gamma/\Gamma_i}(\Gamma_i, r)$ by $\xi''_i(\gamma\Gamma_i) = \xi'_i(\gamma)$ for any integer $i\geq k$ and γ in $B_\Gamma(e,r)$, then $\xi'' = (\xi''_i)_{i\leq k}$ is an element of $\prod_{i\geq k} C(\Gamma/\Gamma_i, \ell^2(\Gamma/\Gamma_i))$ such that $\|\xi''\| = \|\xi'\| \leq 1+\varepsilon$. Let us now define $\eta'' = (\eta''_i)_{i\geq k}$ in $\prod_{i\geq k} C(\Gamma/\Gamma_i, C(\Gamma/\Gamma_i, \mathcal{K}(H)))$ by $\eta'' = T''\cdot\xi''$. Then $\eta''_i \in C(\Gamma/\Gamma_i, C(\Gamma/\Gamma_i, \mathcal{K}(H)))$ has support in $B_{\Gamma/\Gamma_i}(\Gamma_i, r+s)$. Let us define $\eta'_i \in C(\Gamma, C(\Gamma/\Gamma_i, \mathcal{K}(H))$ with support in $B_\Gamma(e, r+s)$ by $\eta'_i(\gamma) = \eta''_i(\gamma\Gamma_i)$ for $i\geq k$ and γ in $B_\Gamma(e,r+s)$. Since $B_\Gamma(e,2r+2s)\cap\Gamma_i = \{e\}$ for $i\geq k$, the map

$$B_\Gamma(e,r+s) \to B_{\Gamma/\Gamma_i}(\Gamma_i, r+s); \gamma \to \gamma\Gamma_i$$

is bijective and thus $\|\eta''\| = \|\eta'\|$. It is then straightforward to check that the image of η' under the map $C_c(\Gamma, B_\Gamma) \to C_c(\Gamma, A_\Gamma)$ induced by the canonical projection $B_\Gamma \to A_\Gamma$ is precisely $x_T.\xi$. The result is then a consequence of the following:

$$\begin{aligned} \|x_T.\xi\| &\leq \|\eta''\| \\ &\leq \|\xi''\| \\ &\leq 1+\varepsilon \end{aligned}$$

□

Proposition 2.10. *The inclusion $\mathcal{K}(\ell^2(X(\Gamma))\otimes H) \hookrightarrow C^*_{max}(X(\Gamma))$ induced an injection $\mathbb{Z} \hookrightarrow K_0(C^*_{max}(X(\Gamma)))$*

Proof. Let p be a projector in $\mathcal{K}(\ell^2(X(\Gamma))\otimes H)$ such that $[p]=0$ in $K_0(C^*_{max}(X(\Gamma)))$. We can assume without loss of generality that p belongs to $\mathcal{K}(\ell^2(\coprod_{i=1}^n \Gamma/\Gamma_i)\otimes H)$ for some n. This means that $\begin{pmatrix} p & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is homotopic to $\begin{pmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in some $M_N(\widetilde{C^*_{max}(X(\Gamma))})$ by a homotopy of projectors. Hence for every positive number $\varepsilon < 1/4$ there exists a real r such that $\begin{pmatrix} p & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are connected by a homotopy $(p_t)_{t\in[0,1]}$ of ε-almost projectors of $M_N(\widetilde{C^*_{max}(X(\Gamma))})$ (i.e selfadjoint elements satisfying $\|p_t^2 - p_t\| \leq \varepsilon$) such that p_t has propagation less than r for all t in $[0,1]$. Let us fix an integer $k \geq \max\{n, r\}$. Then for every $t\in[0,1]$ we can write $p_t = p'_t + p''_t$, where $(p'_t)_{t\in[0,1]}$ is a homotopy of selfadjoint elements in

$$M_N(\mathcal{K}(\ell^2(\coprod_{i=1}^{k-1} \Gamma/\Gamma_i)\otimes H) + \mathbb{C}Id_{\ell^2(\coprod_{i=1}^{k-1}\Gamma/\Gamma_i)\otimes H}))$$

and $(p''_t)_{t\in[0,1]}$ is a homotopy of selfadjoint elements in

$$M_N\left(\prod_{i\geq k}\mathcal{K}(\ell^2(\Gamma/\Gamma_i)\otimes H)+\mathbb{C}Id_{\ell^2(\coprod_{i\geq k}\Gamma/\Gamma_i)\otimes H}\right).$$

Moreover, since p_t can be written diagonally as $p'_t\oplus p''_t$ in the decomposition $\ell^2(X(\Gamma))\otimes H=\ell^2(\coprod_{i=1}^{k-1}\Gamma/\Gamma_i)\otimes H\bigoplus\ell^2(\coprod_{i\geq k}\Gamma/\Gamma_i)\otimes H$, then p'_t is also a ε-projectors with propagation less than r. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\varphi(s)=0$ if $s\leq 1/2$ and $\varphi(s)=1$ if $\frac{\sqrt{1-4\varepsilon}+1}{2}\leq s$. Then $(\varphi(p'_t))_{t\in[0,1]}$ is a homotopy of projectors in $M_N(\mathcal{K}(\ell^2(\coprod_{i=1}^{k-1}\Gamma/\Gamma_i)\otimes H)+\mathbb{C}Id_{\ell^2(\coprod_{i=1}^{k-1}\Gamma/\Gamma_i)\otimes H})$ between $\begin{pmatrix}p&0&0\\0&I_k&0\\0&0&0\end{pmatrix}$ and $\begin{pmatrix}I_k&0&0\\0&0&0\\0&0&0\end{pmatrix}$ and thus $p=0$. □

In conclusion, we get

Corollary 2.11. *With previous notations, we have*

(i) *a short exact sequence*

$$0\longrightarrow\mathbb{Z}\longrightarrow K_0(C^*_{max}(X(\Gamma)))\stackrel{\Psi_{A_\Gamma,\Gamma,max,*}}{\longrightarrow}K_0(A_\Gamma\rtimes_{max}\Gamma)\longrightarrow 0,$$

where the copy of $\mathbb{Z}$ *in* $K_0(C^*_{max}(X(\Gamma)))$ *is generated by the class of a rank one projector on* $\ell^2(X)\otimes H$.

(ii) *an isomorphism*

$$K_1(C^*_{max}(X(\Gamma)))\stackrel{\Psi_{A_\Gamma,\Gamma,max,*}}{\longrightarrow}K_1(A_\Gamma\rtimes_{max}\Gamma)$$

Remark 2.12. *The same proof as for proposition 2.10 applies to show that the injection* $\mathcal{K}(\ell^2(X(\Gamma))\otimes H)\hookrightarrow C^*(X(\Gamma))$ *induces an injection* $\mathbb{Z}\hookrightarrow K_0(C^*(X(\Gamma)))$. *But when the group* Γ *has the property* τ *with respects to the family* $(\Gamma_i)_{i\in\mathbb{N}}$, *then* $X(\Gamma)$ *is a family of expanders and it was proved in* [7] *that the composition*

$$\mathbb{Z}\longrightarrow K_0(C^*(X(\Gamma)))\stackrel{\Psi_{\Gamma,red,*}}{\longrightarrow}K_0(A_\Gamma\rtimes_{red}\Gamma)$$

(and thus the composition $\mathcal{K}(\ell^2(X(\Gamma))\otimes H)\hookrightarrow C^*(X(\Gamma))\stackrel{\Psi_{\Gamma,red}}{\longrightarrow}A_\Gamma\rtimes_{red}\Gamma$*) is not exact in the middle*

2.3. **Assembly map for the Maximal Roe algebra.** Let X be a locally compact metric space, then according to a result of [8] that we shall recall below, the K-theory group $K_*(C^*(X))$ up to canonical isomorphism does not depend on the chosen non-degenerated standard X-module defining $C^*(X)$. In [8] was defined an assembly map for the Roe algebra $\widehat{\mu}_X:K_*(X)\to K_*(C^*(X))$ in the following way. Let H_X be a standard X-Hilbert module with respect to a non-degenerated representation $\rho_X:C_0(X)\to\mathcal{L}(H_X)$. Let us define the following subalgebras of $\mathcal{L}(H_X)$

$$\begin{aligned}\mathcal{D}(X)&=\{T\in\mathcal{L}(H_X)\text{ such that }[f,T]\in\mathcal{K}(H_X)\text{ for all }f\in C_0(X)\},\\ \mathcal{C}(X)&=\{T\in\mathcal{L}(H_X)\text{ such that }f\cdot T\in\mathcal{K}(H_X)\text{ and }T\cdot f\text{ for all }f\in C_0(X)\},\\ D^*(X)&=\{T\in\mathcal{D}(X)\text{ and }T\text{ is in the closure of finite propagation operator}\}.\end{aligned}$$

Then every element in $K_*(X)$ can be represented by a K-cycle (ρ_X, H_X, T), with $T \in \mathcal{D}(X)$. This operator then defines a class $[T]$ in $K_{*+1}(\mathcal{D}(X)/\mathcal{C}(X))$ and we get in this way an isomorphism called the Paschke duality [8]

$$K_*(X) \longrightarrow K_{*+1}(\mathcal{D}(X)/\mathcal{C}(X)); [(\rho_X, H_X, T)] \mapsto [T].$$

According to [8, Lemma 12.3.2], the C^*-algebras inclusions $C^*(X) \hookrightarrow \mathcal{C}(X)$ and $D^*(X) \hookrightarrow \mathcal{D}(X)$ induce an isomorphism

$$D^*(X)/C^*(X) \stackrel{\cong}{\longrightarrow} \mathcal{D}(X)/\mathcal{C}(X). \tag{2.1}$$

Using the inverse of this isomorphism, we get finally an isomorphism

$$K_*(X) \stackrel{\cong}{\longrightarrow} K_{*+1}(D^*(X)/C^*(X))$$

which when composed with the boundary map in K-theory associated to the short exact sequence

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0$$

gives rise to the assembly map

$$\widehat{\mu}_{X,*} : K_*(X) \longrightarrow K_*(C^*(X))$$

Remark 2.13. *Every element x in $K_*(X)$ can be indeed, represented by a K-cycle (ρ_X, H_X, T), with $T \in D(X)$. In this case, $(T, C^*(X))$ is K-cycle for $K_*(C^*(X)) = KK_(\mathbb{C}, C^*(X))$ and thus defines an element of $K_*(C^*(X))$ we shall denote by $\operatorname{Ind}_X T$. It is then straightforward to check that $\widehat{\mu}_{X,*}(x) = \operatorname{Ind}_X T$.*

In order to define the coarse Baum-Connes assembly maps, we shall recall some functoriality results of the Roe algebras under coarse maps.

Let $\phi : X \mapsto Y$ be a coarse map between locally compact metric spaces. Let H_X (resp. H_Y) be a non-degenerated standard X-Hilbert module (resp. Y-Hilbert module) with respect to a representation ρ_X (resp. ρ_Y). Recall from [8] that there is an isometry $V : H_X \to H_Y$ that covers ϕ, i.e there exists a real s such that for any x and y in X with $d(\phi(x), y) > s$, we can find f in $C_0(Y)$ and g in $C_0(X)$ that satisfy $f(y) \neq 0$, $g(x) \neq 0$ and $\rho_Y(f)V\rho_X(g) = 0$. The map $\mathcal{L}(H_X) \to \mathcal{L}(H_Y)$; $T \mapsto VTV^*$ then restricts to a $*$-homomorphism $C[X] \to C[Y]$ and thus to a homomorphism $\operatorname{Ad} V : C^*(X) \to C^*(Y)$. The crucial point, due to [8] is that the homomorphism $\operatorname{Ad}_* V : K_*(C^*(X)) \to K_*(C^*(Y))$ induced in K-theory by $\operatorname{Ad} V$ does not depend on the choice of the isometry V covering ϕ. Hence, we define $\phi_* = \operatorname{Ad}_* V : K_*(C^*(X)) \to K_*(C^*(Y))$, where $V : H_X \to H_Y$ is any isometry covering ϕ.

Remark 2.14.

(i) *If $\phi : X \to X$ is a coarse map such that for some real C, we have $d(x, \phi(x) < C$ for all x in X, then Id_{H_X} covers ϕ and hence $\phi_* = \operatorname{Id}_{K_*(C^*(X))}$.*

(ii) *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two coarse maps, then $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.*

(iii) *In consequence if $\phi : X \to Y$ is a coarse equivalence, then ϕ_* is an isomorphism. Moreover, if $\phi' : X \to Y$ is another coarse equivalence, then $\phi_* = \phi'_*$.*

(iv) *In particular, by chosing for two non-degenerated standard X-modules H_X and H'_X an isometry $V : H_X \to H'_X$ that covers Id_X, we see that up to canonical isomorphisms, the K-group $K_*(C^*(X))$ does not depend on the choice of a non-degenerated standard X-module.*

The previous construction can be extended to maximal Roe algebras by using the following lemma.

Lemma 2.15. *With above notations, assume that X and Y a locally compact metric spaces both containing nets with bounded geometry. For any isometry $V : H_X \to H_Y$ covering a coarse map $\phi : X \to Y$, we have*

(i) *$C[X] \to C[Y]; T \mapsto VTV^*$ extends to a homomorphism $C[X] \to C[Y]$ and thus to a homomorphism $\mathrm{Ad}_{max} V : C^*_{max}(X) \to C^*_{max}(Y)$.*

(ii) *The homomorphism $\mathrm{Ad}_{max,*} V : K_*(C^*_{max}(X)) \to K_*(C^*_{max}(Y))$ induced by $\mathrm{Ad}_{max} V$ in K-theory does not depend on the choice of the isometry $V : H_X \to H_Y$ covering ϕ.*

Proof. The first item is just a consequence of the universal properties for $C^*_{\max}(X)$ and $C^*_{\max}(Y)$. For the second point, assume that $V' : H_X \to H_Y$ is another isometry covering ϕ and let us set $W = V \cdot V'^*$. Then W is a partial isometry of H_Y with finite propagation. Then Wx and $(\mathrm{Id}_{H_X} - W^*W)x$ are in $C[Y]$ for any x in $C[Y]$, and since

$$\begin{aligned} x^*x &= x^*W^*Wx + x^*(\mathrm{Id}_{H_X} - W^*W)^2x \\ &= (Wx)^*Wx + (x(\mathrm{Id}_{H_X} - W^*W))^*(\mathrm{Id}_{H_X} - W^*W)x, \end{aligned}$$

we get that $(Wx)^*Wx \leq x^*x$ in $C^*_{\max}(Y)$ and thus $\|Wx\| \leq \|x\|$. Hence $C[Y] \to C[Y]; x \mapsto W \cdot x$ extends to a bounded linear map $C^*_{\max}(X) \to C^*_{\max}(X)$ we shall denote again by W. But W^* is also a partial isometry of H_Y with finite support and it is straightforward to check that for any x and y in $C^*_{\max}(Y)$ then $W^*(x)y = xW(y)$ and thus W is a multiplier of $C^*_{\max}(Y)$. Since $\mathrm{Ad}_{\max,*} V = W \cdot \mathrm{Ad}_{\max,*} V \cdot W^*$, we get the result by using [8, Lemma 4.6.2]. □

With above notation, this allowed to define for a coarse map $\phi : X \to Y$ with X and Y both containing nets with bounded geometry,

$$\phi_{\max,*} = \mathrm{Ad}_{\max,*} V : K_*(C^*_{\max}(X)) \longrightarrow K_*(C^*_{\max}(Y)),$$

where $V : H_X \to H_Y$ is any isometry covering ϕ. Notice that the remark 2.14 obvioulsy admits a analogous formulation for maximal Roe algebra. In view of remark 2.13 and in order to define the maximal assembly map, we will need the following result.

Lemma 2.16. *Let H_X a standard non-degenerated X-module. Then, for any pseudo-local operator T on H_X with finite propagation, the map*

$$C[X] \to C[X];\ x \mapsto Tx$$

extends in a unique way to a multiplier of $C^*_{max}(X)$ we shall again denote by T. Moreover, for any positive real r, there exists a real constant c_r such that if T has propagation less than r, then

$$\|T\|_{M(C^*_{max}(X))} \leq c_r \|T\|_{\mathcal{L}(H_X)},$$

where $M(C^*_{max}(X))$ stands for the algebra of multiplier of $C^*_{max}(X)$.

Proof. We can assume without loss of generallity that T as an operator on H_X is norm 1.

- Let $(f_i)_{i\in I}$ be a partition of unit, whose supports have uniformally bounded diameters. Since $T-\sum_{i\in I} f_i^{1/2}Tf_i^{1/2}$ is in $C[X]$ and has norm operator for H_X less than 2, then according to lemma 2.6 it is enought to prove the result for $\sum_{i\in I} f_i^{1/2}Tf_i^{1/2}$ instead T. For an element x of $C[X]$, let us set
$$x'=\sum_{i\in I} f_i^{1/2}Tf_i^{1/2}x,$$
$$A=(\mathrm{Id}_{H_X}-T^*T)^{1/2}$$
and
$$y=(\sum_{i\in I} f_i^{1/2}Af_i^{1/2})^*(\sum_{i\in I} f_i^{1/2}Af_i^{1/2})-\sum_{i\in I} f_i^{1/2}A^*Af_i^{1/2}.$$
Then A is a positive pseudo-local operator of H_X and y is a self-adjoint element in $C[X]$ and
$$\begin{aligned} x'^*x'-x^*x &= \sum_{i\in I} x^*f_i^{1/2}(T^*T-\mathrm{Id}_{H_X})f_i^{1/2}x+x^*yx\\ &= -(\sum_{i\in I} f_i^{1/2}Af_i^{1/2}x)^*(\sum_{i\in I} f_i^{1/2}Af_i^{1/2}x)+x^*(y+y')x, \end{aligned}$$
where $y'=(\sum_{i\in I} f_i^{1/2}Af_i^{1/2})^2-\sum_{i\in I} f_i^{1/2}A^2f_i^{1/2}$ lies in $C[X]$. According to lemma 2.6, since y and y' has operator norm on H_X bounded by 2 and have propagation less than r, there exists a real constant c'_r, depending only on H_X and r and such that $\|y+y'\|\leq c'_r$ in $C^*_{\max}(X)$. Hence $x'^*x'-x^*x\leq c_rx^*x$ and hence $\|x'\|\leq(1+c_r)^{1/2}\|x\|$ in $C^*_{\max}(X)$. In consequence, the map $C[X]\to C[X];\, x\mapsto\sum_{i\in I} f_i^{1/2}Tf_i^{1/2}x$, is bounded for the norm of $C^*_{\max}(X)$ and thus extends to a bounded linear map $C^*_{\max}(X)\to C^*_{\max}(X)$.
- Applying the preceding point also to T^*, we get that $(T^*x)^*y=x(Ty)$ for all x and y in $C^*_{\max}(X)$ (check it on $C[X]$) and thus T is a multiplier for $C^*_{\max}(X)$.

□

Remark 2.17. *The set of pseudo-local operator of H_X of finite propagation is a $*$-subalgebra of $\mathcal{L}(H_X)$ which contains $C[X]$ as an ideal. From preceding lemma, we get then a $*$-homomorphism from the algebra of pseudo-local operator of H_X of finite propagation to the multiplier of $C^*_{max}(X)$ whose restriction to $C[X]$ is just the inclusion $*$-homomorphism $C[X]\hookrightarrow C^*_{max}(X)$.*

Corollary 2.18.

- *If (ρ_X,H_X,T) is a K-cycle for $K_*(X)$ with T of finite propagation. Then $(T,C^*_{max}(X))$ is a K-cycle for $K_*(C^*_{max}(X))=KK(\mathbb{C},C^*_{max}(X))$ we shall denote by $\mathrm{Ind}_{X,max}T$.*
- *$(\rho_X,H_X,T)\mapsto\mathrm{Ind}_{X,max}T$ gives rise to a homomorphism*
$$\widehat{\mu}_{X,max,*}:K_*(X)\to K_*(C^*_{max}(X)).$$

Proof. We only have to check that the definition of $\mathrm{Ind}_{X,\max}T$ only depends on the class of (ρ_X,H_X,T) in $K_*(X)$. But if (ρ_X,H_X,T) and (ρ_X,H_X,T') are two K-cycle for $K_*(X)$ with T and T' of finite propagation then

- if $f(T-T')$ is compact for all f in $C_0(X)$, then $\mathrm{Ind}_{\max}T=\mathrm{Ind}_{\max}T'$;

- if T and T' are connected by a homotopy of operators $(T_s)_{s\in[0,1]}$ such that (ρ_X, H_X, T_s) is a K-cycle for all s in $[0,1]$, then according to the preceding point, we can replace $(T_s)_{s\in[0,1]}$ by $(\sum_{i\in\mathbb{N}} f_i^{1/2} T_s f_i^{1/2})_{s\in[0,1]}$, where $(f_i)_{i\in\mathbb{N}}$ is a partition of unit with support of uniformally bounded diameter.

The result is then a consequence of the second item of lemma 2.16 □

Remark 2.19. *Let x be an element of $K_0(X))$ represented by an even K-cycle (ρ_X, H_X, T) as in the previous corollary. Let us set*

$$W = \begin{pmatrix} \mathrm{Id}_{H_X} & T \\ 0 & \mathrm{Id}_{H_X} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{H_X} & 0 \\ -T & \mathrm{Id}_{H_X} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{H_X} & T \\ 0 & \mathrm{Id}_{H_X} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Id}_{H_X} \\ \mathrm{Id}_{H_X} & 0 \end{pmatrix}.$$

Then

$$\left[W \begin{pmatrix} \mathrm{Id}_{H_X} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}\right] - \left[\begin{pmatrix} \mathrm{Id}_{H_X} & 0 \\ 0 & 0 \end{pmatrix}\right]$$

defines an element in $K_0(C^*_{max}(X))$ which is precisely $\widehat{\mu}_{X,*,max}(x)$.

We are now in position to define the coarse Baum-Connes assembly maps. Recall that for a proper metric set Σ and a real r, the Rips complex of order r is the set $P_r(\Sigma)$ of probability measures on Σ with support of diameter less than r. Recall that $P_r(\Sigma)$ is a locally finite simplicial complex that can be provided with a proper metric extending the euclidian metric on each simplex. Moreover, by viewing an element of Σ as a Dirac measure, we get an inclusion $\Sigma \hookrightarrow P_r(\Sigma)$, which turns out to be a coarse equivalence. If we fix for each real r a coarse equivalence $\phi_r : P_r(\Sigma) \to \Sigma$, then the collections of homomorphisms given by the compositions

$$K_*(P_r(\Sigma)) \stackrel{\widehat{\mu}_{P_r(\Sigma),*}}{\longrightarrow} K_*(C^*(P_r(\Sigma))) \stackrel{\phi_{r,*}}{\longrightarrow} K_*(C^*(\Sigma))$$

and

$$K_*(P_r(\Sigma)) \stackrel{\widehat{\mu}_{P_r(\Sigma),\max,*}}{\longrightarrow} K_*(C^*_{\max}(P_r(\Sigma))) \stackrel{\phi_{r,\max,*}}{\longrightarrow} K_*(C^*_{\max}(\Sigma))$$

give rise respectivelly to the the Baum-connes coarse assembly map

$$\mu_{\Sigma,*} : \lim_{r>0} K_*(P_r(\Sigma)) \longrightarrow K_*(C^*(\Sigma))$$

and to the maximal Baum-Connes assembly map

$$\mu_{\Sigma,*,\max} : \lim_{r>0} K_*(P_r(\Sigma)) \longrightarrow K_*(C^*_{\max}(\Sigma)).$$

Moreover, if z in $\lim_r K_*(P_r(\Sigma))$ comes from a K-cycle $(\rho_{P_r(\Sigma)}, H_{P_r(\Sigma)}, T)$ for some $K_*(P_r(\Sigma))$, where T is a finite propagation operator on the non-degenerated standard $P_r(\Sigma)$-module $H_{P_r(\Sigma)}$ then

$$\mu_{\Sigma,*}(z) = \phi_{r,*} \operatorname{Ind}_{P_r(\Sigma)} T$$

and

$$\mu_{\Sigma,\max,*}(z) = \phi_{r,\max,*} \operatorname{Ind}_{P_r(\Sigma),\max} T.$$

Remark 2.20. *Let $\lambda_\Sigma : C^*_{max}(\Sigma) \to C^*(\Sigma)$ be the homomorphism induced from the representation $C[X] \hookrightarrow B(H_\Sigma)$. Then $\mu_{\Sigma,*} = \mu_{\Sigma,max,*} \circ \lambda_{\Sigma,*}$*

3. The Baum-Connes assembly map

We gather this section with result we will need later on concerning the Baum-Connes assembly map and its left hand side. For a proper Γ-space X and a Γ-algebra A, we shall denote for short $KK^\Gamma_*(X, A)$ instead of $KK^\Gamma_*(C_0(X), A)$.

3.1. **Definition of the maximal assembly map.** Let X be a locally compact proper and cocompact Γ-space and let $(\rho, \mathcal{E}, T)$ be a K-cycle for $KK_*^\Gamma(X, A)$. Up to averaging with a cut-off function, we can assume that the operator T is Γ-equivariant. Let $\mathcal{E}_\Gamma$ be the separated completion of $C_c(X) \cdot \mathcal{E}$ with respect to the $A \rtimes_{\max} \Gamma$-valued scalar product defined by $\langle e/e' \rangle_{\mathcal{E}_\Gamma}(\gamma) = \langle \xi / \gamma(\xi') \rangle_{\mathcal{E}}$ for ξ and ξ' in $C_c(X) \cdot \mathcal{E}$ and γ in Γ (recall that the separated completion is obtained by first divide out by the submodule of vanishing elements for the pseudo-norm associated to the inner product and then by completion of the quotient with respect to the induced norm). Up to replace T by $\sum_{\gamma \in \Gamma} \gamma(f^{1/2}) T \gamma(f^{1/2})$, for $f \in C_c(X, [0,1])$ a cut-off function with respect to the action of Γ of X, the map $C_c(X) \cdot \mathcal{E} \to C_c(X) \cdot \mathcal{E}$; $\xi \mapsto T\xi$ extends to an adjointable operator $T_\Gamma : \mathcal{E}_\Gamma \to \mathcal{E}_\Gamma$. Then we can check that $(\mathcal{E}_\Gamma, T_\Gamma)$ is a K-cycle for $KK_*(\mathbb{C}, A \rtimes_{\max} \Gamma) = K_*(A \rtimes_{\max} \Gamma)$ whose class $\mathrm{Ind}_{\Gamma, A, \max} T$ only depends on the class of $(\rho, \mathcal{E}, T)$ in $KK_*^\Gamma(X, A)$. The left hand side of the maximal assembly map is then the topological K-theory for Γ with coefficients in A

$$K_*^{\mathrm{top}}(\Gamma, A) = \lim_{r > 0} KK_*^\Gamma(P_r(\Gamma), A)$$

and the assembly map

$$\mu_{\Gamma, A, \max} : K_*^{\mathrm{top}}(\Gamma, A) \longrightarrow K_*(A \rtimes_{\max} \Gamma)$$

is defined for an element x in $K_*^{\mathrm{top}}(\Gamma, A)$ coming from the class of a K-cycle $(\rho, \mathcal{E}, T)$ for $KK_*^\Gamma(P_r(\Gamma), A)$ by $\mu_{\Gamma, A, \max}(x) = \mathrm{Ind}_{\Gamma, A, \max} T$.

3.2. **Induction.** We recall now from [11] the description of induction to a group, from the action of one of its subgroup on a C^*-algebra, and the behaviour of the left-hand side of the Baum-Connes assembly map under this transformation.

Let Γ' be a subgroup of a discrete group Γ, and let A be a Γ'-C^*-algebra. Define

$$\mathrm{I}_{\Gamma'}^{\Gamma} A = \{ f : \Gamma \to A \text{ such that } \gamma' \cdot f(\gamma \gamma') = f(\gamma) \text{ for all } \gamma \in \Gamma, \gamma' \in \Gamma' \\ \text{and } \gamma \Gamma' \mapsto \| f(\gamma) \| \text{ is in } C_0(\Gamma / \Gamma') \}.$$

Then Γ acts on $\mathrm{I}_{\Gamma'}^{\Gamma} A$ by left translation and it is a standard fact that the dynamical systems $(\mathrm{I}_{\Gamma'}^{\Gamma} A, \Gamma)$ and (A, Γ') have equivalent covariant representations. In particular, the C^*-algebras $A \rtimes_{\max} \Gamma'$ and $\mathrm{I}_{\Gamma'}^{\Gamma} A \rtimes_{\max} \Gamma$ are Morita equivalent (the same holds for reduced crossed products). Notice that if the action of Γ' on A is indeed the restriction of an action of Γ, then

$$\mathrm{I}_{\Gamma'}^{\Gamma} A \to C_0(\Gamma / \Gamma', A); \; f \mapsto [\gamma \Gamma' \mapsto \gamma \cdot f(\gamma)]$$

is a Γ-equivariant isomorphism, where $C_0(\Gamma / \Gamma', A) \cong C_0(\Gamma / \Gamma') \otimes A$ is provided with the diagonal action of Γ. In [11] was defined an induction homomorphism

$$\mathrm{I}_{\Gamma', A, *}^{\Gamma, \mathrm{top}} : K^{\mathrm{top}}(\Gamma', A) \to K^{\mathrm{top}}(\Gamma, \mathrm{I}_{\Gamma'}^{\Gamma} A),$$

which turned out to be an isomorphism. If Γ' has finite index in Γ, then $\mathrm{I}_{\Gamma', A, *}^{\Gamma, \mathrm{top}}$ can be described quite easily as follows. Recall first that if Γ' is a subgroup of Γ with finite index, then the family of inclusions $C_0(P_r(\Gamma')) \hookrightarrow C_0(P_r(\Gamma))$ gives rise to an isomorphism

$$\lim_r KK_*^{\Gamma'}(P_r(\Gamma'), A) = K^{\mathrm{top}}(\Gamma', A) \xrightarrow{\cong} \lim_r KK_*^{\Gamma'}(P_r(\Gamma), A), \tag{3.1}$$

and under this identification, the assembly map is defined as before : if x in an element in $K_*^{\mathrm{top}}(\Gamma, A)$ coming from the class of a K-cycle $(\rho, \mathcal{E}, T)$ for $KK_*^{\Gamma'}(C_0(P_r(\Gamma), A)$, then $\mu_{\Gamma,A,\max,*}(x) = \mathrm{Ind}_{\Gamma',A,\max} T$.

Now let $(\rho, \mathcal{E}, T)$ be a K-cycle for some $KK_*^{\Gamma'}(X, A)$, where X is a proper and cocompact Γ'-space. Let us define

$$\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E} = \{\xi : \Gamma \to A,\ \gamma' \cdot \xi(\gamma\gamma') = \xi(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

The pointwise right A-Hilbert module structure provides a right $\mathrm{I}_{\Gamma'}^{\Gamma} A$-Hilbert module structure for $\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}$ which is covariant for the action of Γ by left translations. If T is chosen Γ'-equivariant, then $\Gamma \to \mathcal{E};\ \gamma \mapsto T.\xi$ lies in $\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}$ for ξ in $\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}$ and we get in this way a Γ-equivariant and adjointable operator $\mathrm{I}_{\Gamma'}^{\Gamma} T : \mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E} \to \mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}$. Finally, for f in $C_0(X)$, pointwise left multiplication by $\gamma \mapsto \gamma(\rho(f))$ defines a covariant representation $\mathrm{I}_{\Gamma'}^{\Gamma} \rho$ of $C_0(X)$ on the Hilbert right $\mathrm{I}_{\Gamma'}^{\Gamma} A$-Hilbert module $\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}$. It is straightforward to check that $(\mathrm{I}_{\Gamma'}^{\Gamma} \rho, \mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}, \mathrm{I}_{\Gamma'}^{\Gamma} T)$ is a K-cycle for $KK^{\Gamma}(X, \mathrm{I}_{\Gamma'}^{\Gamma} A)$ whose class only depends on the class of $(\rho, \mathcal{E}, T)$ in $KK_*^{\Gamma'}(X, A)$. On the other hand, if $(\rho', \mathcal{E}', T')$ is a K-cycle for $KK^{\Gamma}(X, \mathrm{I}_{\Gamma'}^{\Gamma} A)$, and if we consider the Γ'-equivariant homomorphism $\psi : \mathrm{I}_{\Gamma'}^{\Gamma} A \to A;\ f \mapsto f(e)$, then $\mathcal{E} = \mathcal{E}' \otimes_{\psi} A$ is a Γ'-covariant right A-Hilbert module. Let us set $T = T' \otimes_{\psi} \mathrm{Id}_A$ and $\rho(f) = \rho'(f) \otimes_{\psi} \mathrm{Id}_A$ for all f in $C_0(X)$. Then $(\rho, \mathcal{E}, T)$ is a K-cycle for $KK_*^{\Gamma'}(X, A)$ and we can check that $(\mathrm{I}_{\Gamma'}^{\Gamma} \rho, \mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}, \mathrm{I}_{\Gamma'}^{\Gamma} T)$ is a K-cycle unitary equivalent to (ρ', ξ', T'). Finally, we get an isomorphism

$$\mathrm{I}_{\Gamma',A,*}^{\Gamma,X,*} : KK_*^{\Gamma'}(X, A) \xrightarrow{\cong} KK_*^{\Gamma}(X, \mathrm{I}_{\Gamma'}^{\Gamma} A)$$

which maps the class of a K-cycle $(\rho, \mathcal{E}, T)$ for $KK_*^{\Gamma'}(X, A)$ to the class of the K-cycle $(\mathrm{I}_{\Gamma'}^{\Gamma} \rho, \mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E}, \mathrm{I}_{\Gamma'}^{\Gamma} T)$ in $KK_*^{\Gamma}(X, \mathrm{I}_{\Gamma'}^{\Gamma} A)$. Under the identification of equation 3.1, the family of isomorphisms $(\mathrm{I}_{\Gamma',A,*}^{\Gamma,P_r(\Gamma)})_{r>0}$ gives rise to an isomorphism

$$\mathrm{I}_{\Gamma',A,*}^{\Gamma,\mathrm{top}} : K_*^{\mathrm{top}}(\Gamma', A) \xrightarrow{\cong} K_*^{\mathrm{top}}(\Gamma, \mathrm{I}_{\Gamma'}^{\Gamma} A).$$

Moreover, up to the identification $K_*(A \rtimes_{\max} \Gamma') \cong K_*(\mathrm{I}_{\Gamma'}^{\Gamma} A \rtimes_{\max} \Gamma)$ induced by the Morita equivalence, we have

$$\mu_{\Gamma',A,\max} = \mu_{\Gamma,\mathrm{I}_{\Gamma'}^{\Gamma} A,\Gamma,\max} \circ \mathrm{I}_{\Gamma',A,*}^{\Gamma,\mathrm{top}}.$$

Remark 3.1. *Let A be a Γ-algebra, let Γ' be a subgroup of Γ with finite index and let $(\rho, \mathcal{E}, T)$ be a K-cycle for $KK_*^{\Gamma'}(X, A)$, such that the action of Γ' on $\mathcal{E}$ is indeed the restriction of a covariant action of Γ. Under the identification $\mathrm{I}_{\Gamma'}^{\Gamma} A \cong C(\Gamma/\Gamma', A)$ we have seen before, then*

$$\mathrm{I}_{\Gamma'}^{\Gamma} \mathcal{E} \to C(\Gamma/\Gamma', \mathcal{E});\ \xi \mapsto [\gamma\Gamma' \mapsto \gamma \cdot \xi(\gamma)]$$

is a Γ-equivariant isomorphism of right $C(\Gamma/\Gamma', A)$-Hilbert module, where we have equipped $C(\Gamma/\Gamma', \mathcal{E}) \cong C(\Gamma/\Gamma') \otimes \mathcal{E}$ with the diagonal action of Γ. Moreover, under this identification, $\mathrm{I}_{\Gamma'}^{\Gamma} \rho$ is given pointwise by the representation ρ and $\mathrm{I}_{\Gamma'}^{\Gamma} T$ is the pointwise multiplication by $\gamma \mapsto \gamma(T)$.

3.3. **The left hand side for product of stable algebras.** As it was proved in [2], the topological K-theory for a group is a functor with respect to the coefficients which commutes with direct sums, i.e $K^{\mathrm{top}}(G, \oplus_{i\in I} A_i) = \oplus_{i\in I} K^{\mathrm{top}}(G, A_i)$ for every locally compact group G and every family $(A_i)_{i\in I}$ of C^*-algebras A_i equipped with an action of G by automorphisms. The aim of this section is to prove a similar result for product of a family of stable C^*-algebras.

Let us first prove the result for usual K-theory.

Lemma 3.2. *Let $\mathcal{A}=(A_i)_{i\in I}$ be a family of unital C^*-algebras. Let*

$$\Theta_*^{\mathcal{A}}: K_*(\Pi_{i\in I}(A_i\otimes\mathcal{K}(H))\longrightarrow\prod_{i\in I}K_*(A_i\otimes\mathcal{K}(H))\cong\prod_{i\in I}K_*(A_i)$$

be the homomorphism induced on the j-th factor by the projection

$$\prod_{i\in I}(A_i\otimes\mathcal{K}(H))\longrightarrow A_j\otimes\mathcal{K}(H).$$

Then $\Theta_*^{\mathcal{A}}$ is an isomorphism.

Proof. Is clear that $\Theta_*^{\mathcal{A}}$ is onto. The injectivity of $\Theta_*^{\mathcal{A}}$ is then a consequence of the next lemma. ☐

Lemma 3.3. *There exists a map $\phi:(0,+\infty[\to(0,+\infty[$ such that for any unital C^*-algebra A, the following properties hold:*

- (i) *If p and q are projectors in some $M_n(A)$ connected by a homotopy of projectors. Then there exists integers k and N with $n+k\leq N$ and a homotopy of projectors $(p_t)_{t\in[0,1]}$ in $M_N(A)$ connecting $\operatorname{diag}(p,I_k,0)$ and $\operatorname{diag}(q,I_k,0)$ and such that for any positive real ε and any s and t in $[0,1]$ with $|s-t|\leq\phi(\varepsilon)$, then $\|p_s-p_t\|\leq\varepsilon$.*
- (ii) *If u and v are homotopic unitaries in $U_n(A)$, then there exists an integer k and a homotopy $(w_t)_{t\in[0,1]}$ in $U_{n+k}(A)$ connecting $\operatorname{diag}(u,I_k)$ and $\operatorname{diag}(v,I_k)$ such that for any positive real ε and any s and t in $[0,1]$ with $|s-t|\leq\phi(\varepsilon)$, then $\|w_s-w_t\|\leq\varepsilon$.*

Proof. We can assume without loss of generality that $n=1$.

- Let us notice first that using [16, Proposition 5.2.6, page 90], then there exists a positive real α such that for any unital C^*-algebra A and any projectors p and q in A such that $\|p-q\|\leq\alpha$, then $q=u\cdot p\cdot u^*$ for some unitary u of A with $\|u-1\|\leq 1/2$. Hence there is a self-adjoint element h of A with $\|h\|\leq\ln 2$ such that $u=\exp ih$. Considering the homotopy of projectors $(\exp\imath th\cdot p\cdot\exp-\imath th)_{t\in[0,1]}$, we see that there exists a map $\phi_1:(0,+\infty[\to(0,+\infty[$ such that for any C^*-algebra A and any projectors p and q in A such that $\|p-q\|\leq\alpha$, then p and q are connected by a homotopy of projectors $(p_t)_{t\in[0,1]}$ and such that for any positive real ε and any s and t in $[0,1]$ with $|s-t|\leq\phi_1(\varepsilon)$, then $\|p_s-p_t\|\leq\varepsilon$.
- By considering for a projector p in A the homotopy of projectors

$$\begin{pmatrix}\cos^2\pi t/2\cdot p & \sin\pi t/2\cos\pi t/2\cdot p\\ \sin\pi t/2\cos\pi t/2\cdot p & \sin^2\pi t/2\cdot p+1-p\end{pmatrix}_{t\in[0,1]}$$

  in $M_2(A)$, we also get that there exists a map $\phi_2:(0,+\infty[\to(0,+\infty[$ such that for any C^*-algebra A and any projector p in A, then $\operatorname{diag}(1,0)$ and $\operatorname{diag}(p,1-p)$ are connected by a homotopy of projectors $(q_t)_{t\in[0,1]}$ such that for any positive real ε and any s and t in $[0,1]$ with $|s-t|\leq\phi_2(\varepsilon)$, then $\|q_s-q_t\|\leq\varepsilon$.

- To prove the general case, let p and q be two homotopic projectors in a C^*-algebra A, and let $p=p_0\,,p_1,\cdots,p_m=q$ be $m+1$ projectors in A such that $\|p_{i+1}-p_i\|\leq\alpha$ for $i=0,\cdots,m-1$. Let us consider the following projectors in $M_{2m+1}(A)$:

$$\begin{aligned}
q_0 &= \operatorname{diag}(p_0, I_{2k-1}, 0)\\
q_1 &= \operatorname{diag}(p_0, 1, 0, \cdots, 1, 0)\\
q_2 &= \operatorname{diag}(p_0, 1-p_1, p_1, \cdots, 1-p_m, p_m)\\
q_3 &= \operatorname{diag}(p_0, 1-p_0, p_1, 1-p_1, \cdots, p_{m-1}, 1-p_{m-1}, p_m)\\
q_4 &= \operatorname{diag}(1, 0, \cdots, 1, 0, p_m)\\
q_5 &= \operatorname{diag}(0, I_{2k-1}, p_m)\\
q_6 &= \operatorname{diag}(p_m, I_{2k-1}, 0)
\end{aligned}$$

Since $\|q_3-q_2\|\leq\alpha$, if we set $\phi=\min\{\phi_1,\phi_2\}$ and if we use the previous cases, we get for every l in $\{0,5\}$ homotopies $(q_t^l)_{t\in[l,l+1]}$ between q_l and q_{l+1} such that for any positive real ε and any s and t in $[0,1]$ with $|s-t|\leq\phi(\varepsilon)$, then $\|q_s^l-q_t^l\|\leq\varepsilon$. Hence, by considering the total homotopy, we get the result.

The proof for unitaries is similar. □

Proposition 3.4. *Let Γ be a discrete group. Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras equipped with an action of Γ by automorphisms. Let us equip $A_i\otimes\mathcal{K}(H)$ with the diagonal action, the action of Γ on $\mathcal{K}(H)$ being trivial and let us then consider the induced action on $\prod_{i\in I}(A_i\otimes\mathcal{K}(H))$. Let*

$$\Theta_*^{\Gamma,\mathcal{A}}: KK_*^\Gamma(P_r(\Gamma),\Pi_{i\in I}(A_i\otimes\mathcal{K}(H)))\longrightarrow \prod_{i\in I}KK_*^\Gamma(P_r(\Gamma),A_i\otimes\mathcal{K}(H))\cong\prod_{i\in I}KK_*^\Gamma(P_r(\Gamma),A_i)$$

be the homomorphism induced on the k-th factor by the projection

$$\prod_{i\in I}(A_i\otimes\mathcal{K}(H))\to A_k\otimes\mathcal{K}(H).$$

Then $\Theta_*^{\Gamma,\mathcal{A}}$ is an isomorphism.

Proof. Let us set $B_i=A_i\otimes\mathcal{K}(H)$ for i in I. We can define an analogous morphism

$$\Theta_*^X: KK_*^\Gamma(X,\Pi_{i\in I}B_i)\to\prod_{i\geq n}KK_*^\Gamma(X,B_i)$$

for any locally compact space X equipped with an action of Γ. Let us denote by $\Theta_{*,k}^X: KK_*^\Gamma(X,\Pi_{i\in I}B_i)\to KK_*^\Gamma(X,B_k)$ the homomorphism induced by the projection on the k-th factor. Up to take a barycentric subdivision of $P_r(\Gamma)$, we can assume that $P_r(\Gamma)$ is a locally finite and finite dimension typed simplicial complex, equipped with a simplicial and type preserving action of Γ. Let $Z_0,\cdots,Z_n$ be the skeleton decomposition of $P_r(\Gamma)$. Then Z_j is a simplicial complex of dimension j, locally finite and equipped with a proper, cocompact and type preserving simplicial action of Γ. Let us prove by induction on i that $\Theta_*^{Z_j}$ is an isomorphism. The 0-skeletton Z_0 is a finite union of orbits and thus, for $j=0$, it is enought to prove that $\Theta_*^{\Gamma/F}$ is an isomorphism when F is a finite subgroup of Γ. Let us recall from [11] that for every C^*-algebra B equipped with an action of Γ, there is a natural restriction isomorphism $\mathrm{Res}^B_{F,\Gamma} : KK^\Gamma_*(\Gamma/F, B)\longrightarrow KK^F_*(\mathbb{C}, B) \cong K_*(B \rtimes F)$. We get by naturality the following commutative diagram

$$\begin{array}{ccc} KK^\Gamma_*(\Gamma/F, \Pi_{i\in I} B_i) & \xrightarrow{\Theta^{\Gamma/F}_{*,k}} & KK^\Gamma_*(\Gamma/F, B_k) \\ {\scriptstyle \mathrm{Res}^{\Pi_{i\in I}B_i}_{F,\Gamma}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{B_k}_{F,\Gamma}} \\ K_*((\Pi_{i\in I} B_i) \rtimes F) & \longrightarrow & K_*(B_k \rtimes F) \end{array},$$

where the bottom row is induced by the homomorphism $\Pi_{i\in I}(B_i \rtimes F) \to B_k \rtimes F$ arising from the projection on the k-th factor $\Pi_{i\in I} B_i \to B_k$. Since F is finite, $\Pi_{i\in I}(B_i \rtimes F)$ is naturally isomorphic to $(\Pi_{i\in I} B_i) \rtimes F$, and under this identification, the bottom row homomorphism induces by lemma 3.2 an isomorphism

$$K_*((\Pi_{i\in I} B_i) \rtimes F)\longrightarrow \prod_{i\in I} K_*(B_i \rtimes F).$$

Hence $\Theta^{\Gamma/F}_*$ is an isomorphism.

Let us assume that we have proved that $\Theta^{Z_{j-1}}_*$ is an isomorphism. Then the short exact sequence

$$0\longrightarrow C_0(Z_j \setminus Z_{j-1})\longrightarrow C_0(Z_j)\longrightarrow C_0(Z_{j-1})\longrightarrow 0$$

gives rise to an natural long exact sequence

$$\longrightarrow KK^\Gamma_*(Z_{j-1}, \bullet)\longrightarrow KK^\Gamma_*(Z_j, \bullet)\longrightarrow KK^\Gamma_*(Z_j \setminus Z_{j-1}, \bullet)\longrightarrow KK^\Gamma_{*+1}(Z_{j-1}, \bullet)$$

and thus by naturallity, we get a diagram

$$\begin{array}{ccccc} KK^\Gamma_*(Z_{j-1}, \Pi_{i\in I} B_i) & \longrightarrow & KK^\Gamma_*(Z_j, \Pi_{i\in I} B_i) & \longrightarrow & KK^\Gamma_*(Z_j \setminus Z_{j-1}, \Pi_{i\in I} B_i) \\ {\scriptstyle\Theta^{Z_{j-1}}_*}\downarrow & & {\scriptstyle\Theta^{Z_j}_*}\downarrow & & {\scriptstyle\Theta^{Z_{j-1}}_{*+1}}\downarrow \\ \Pi_{i\in I} KK^\Gamma_*(Z_{j-1}, B_i) & \longrightarrow & \Pi_{i\in I} KK^\Gamma_*(Z_j, B_i) & \longrightarrow & \Pi_{i\in I} KK^\Gamma_*(Z_j, B_i) \end{array}$$

$$\begin{array}{cc} \longrightarrow & KK^\Gamma_{*+1}(Z_{j-1}, \Pi_{i\in I} B_i) \\ & {\scriptstyle\Theta^{Z_j\setminus Z_{j-1}}_*}\downarrow \\ \longrightarrow & \Pi_{i\in I} KK^\Gamma_{*+1}(Z_{j-1}, B_i) \end{array},$$

Let $\overset{\circ}{\sigma}_j$ be the interior of the standard j-simplex. Since the action of Γ is type preserving, then $Z_j \setminus Z_{j-1}$ is equivariantly homeomorphic to $\overset{\circ}{\sigma}_j \times \Sigma_j$, where Σ_j is the set of center of j-simplices of Z_j, and where Γ acts trivially on $\overset{\circ}{\sigma}_j$. This identification, together with Bott periodicity provides a commutative diagram

$$\begin{array}{ccc} KK^\Gamma_*(Z_j \setminus Z_{j-1}, \Pi_{i\in I} B_i) & \longrightarrow & KK^\Gamma_{*+1}(\Sigma_j, \Pi_{i\in I} B_i) \\ {\scriptstyle\Theta^{Z_j\setminus Z_{j-1}}_*}\downarrow & & {\scriptstyle\Theta^{\Sigma_j}_{*+1}}\downarrow \\ \prod_{i\in I} KK^\Gamma_*(Z_j \setminus Z_{j-1}, B_i) & \longrightarrow & \prod_{i\in I} KK^\Gamma_{*+1}(\Sigma_j, B_i) \end{array},$$

By the first step of induction, $\Theta^{\Sigma_j}_*$ is an isomorphism, and hence $\Theta^{Z_j\setminus Z_{j-1}}_*$ is an isomorphism. Using the induction hypothesis and the five lemma, we get then that $\Theta^{Z_j}_*$ is an isomorphism. □

Let $(A_i)_{i\in\mathbb{N}}$ be a family of Γ-algebras, let H be an Hilbert space and let x be an element of $KK^*(P_r(\Gamma), \prod_{i\in\mathbb{N}} A_i \otimes \mathcal{K}(H))$ (the action of Γ on H being trivial) represented by a K-cycle $(\phi, \mathcal{E}, T)$. Let $p_k : \prod_{i\in\mathbb{N}} A_i \otimes \mathcal{K}(H) \to A_k \otimes \mathcal{K}(H)$ be the canonical projection on the k-th factor, and let us set $\mathcal{E}_k = \mathcal{E} \otimes_{p_k} A_k \otimes \mathcal{K}(H)$, $T_k =$

$T\otimes_{p_k}\mathrm{Id}_{A_k\otimes\mathcal{K}(H)}$ and let us define the Γ-equivariant representation of $C_0(P_r(\Gamma))$ on $\mathcal{E}_k$ by $\phi_k(f)=\phi(f)\otimes_{p_k}\mathrm{Id}_{A_k\otimes\mathcal{K}(H)}$ for all f in $C_0(P_r(\Gamma))$. Then $\prod_{i\in\mathbb{N}}\mathcal{E}_i$ provided with the diagonal action is a Γ-equivariant right $\prod_{i\in\mathbb{N}}A_i\otimes\mathcal{K}(H)$-Hilbert module. Moreover, if S is a compact operator on $\mathcal{E}$, then for every $\varepsilon>0$, there exists a finite rank operator S' on $\mathcal{E}$ such that $\|S-S'\|\leq\varepsilon$. Then $(S'_i)_{i\in\mathbb{N}}$ provides a finite rank operator on $\prod_{i\in\mathbb{N}}\mathcal{E}_i$ such that $\|S_i-S'_i\|\leq\varepsilon$ for all integer i. Hence $(S_i)_{i\in\mathbb{N}}$ gives rise to a compact operator on $\prod_{i\in\mathbb{N}}\mathcal{E}_i$. Consequently, $((\phi_i)_{i\in\mathbb{N}},\prod_{i\in\mathbb{N}}\mathcal{E}_i,(T_i)_{i\in\mathbb{N}})$ is a K-cycle for $KK^\Gamma_*(P_r(\Gamma),\prod_{i\in\mathbb{N}}A_i\otimes\mathcal{K}(H))$ which in view of the isomorphism of proposition 3.4 represents also x. Using the imprimitivity bimodule implementing the Morita equivalence between A_i and $A_i\otimes\mathcal{K}(H)$, we can actually replace $\mathcal{E}_i$ by $\mathcal{K}(A_i\otimes H,H_i)$ where $H_i=\mathcal{E}_i\otimes_{A_i\otimes\mathcal{K}(H)}\otimes A_i$. Hence we obtain that every element x in $KK^*(P_r(\Gamma),\prod_{i\in\mathbb{N}}A_i\otimes\mathcal{K}(H))$ can be represented by a K-cycle $((\phi_i)_{i\in\mathbb{N}},\prod_{i\in\mathbb{N}}\mathcal{K}_{A_i}(A_i\otimes H,\mathcal{H}_i),(T_i)_{i\in\mathbb{N}})$ such that for every integer i,

- $\mathcal{H}_i$ is a Γ-equivariant right A_i-Hilbert module;
- ϕ_i is a Γ-equivariant representation of $C_0(P_r(\Gamma))$ on $\mathcal{H}_i$;
- T_i is a Γ-equivariant operator on $\mathcal{H}_i$;
- the action of T_i and of $\phi_i(f)$ for f in $C_0(P_r(\Gamma))$ on $\mathcal{K}_{A_i}(A_i\otimes H,\mathcal{H}_i)$ being by left composition.

Moreover, we can assume that $\|T_i\|\leq 1$ for all positive integer i.

As a consequence of proposition 3.4 we get

Corollary 3.5. *If Γ admits a universal example which is a finite dimension and cocompact simplicial complex (equipped with a simplicial action of Γ), then we have an isomorphism*

$$K_*^{top}(\Gamma,\Pi_{i\in I}(A_i\otimes\mathcal{K}(H)))\to\prod_{i\in I}K_*^{top}(\Gamma,A_i\otimes\mathcal{K}(H))\cong\prod_{i\in I}K_*^{top}(\Gamma,A_i)$$

induced on the k-th factor by the projection

$$\Pi_{i\in I}(A_i\otimes\mathcal{K}(H))\to A_k\otimes\mathcal{K}(H).$$

3.4. **The case of coverings.** Recall from [10] that for a cocompact covering $\widetilde{X}\to X$ of group Γ, we have a natural isomorphism

$$\Upsilon^\Gamma_{\widetilde{X},*}:KK^\Gamma_*(\widetilde{X},\mathbb{C})\longrightarrow K_*(X)$$

which can be described as follows. Let (ρ,H,T) be a K-cycle for $KK^\Gamma_*(\widetilde{X},\mathbb{C})$. We can assume without loss of generality that the representation $\rho:C_0(\widetilde{X})\to\mathcal{L}(H)$ is non-degenerated. We can also assume that the operator T of $\mathcal{L}(H)$ is Γ-equivariant and that

$$T\cdot C_c(\widetilde{X})\cdot H\subset C_c(\widetilde{X})\cdot H.$$

If $\langle\bullet,\bullet\rangle$ is the scalar product on the Hilbert space H, then we can define on $C_c(\widetilde{X})\cdot H$ the inner product

$$\langle\langle\xi,\eta\rangle\rangle=\sum_{\gamma\in\Gamma}\langle\xi,\gamma(\eta)\rangle.$$

Then $\langle\langle\bullet,\bullet\rangle\rangle$ is positive and thus by taking the separated completion of $C_c(\widetilde{X})\cdot H$, we get a Hilbert space $\widehat{H}$. The operator T being equivariant, its restriction to $C_c(\widetilde{X})\cdot H$ extends to a continuous operator $\widehat{T}$ on $\widehat{H}$. Since ρ is non-degenerated, it extends to a representation of $C(X)$ (viewed as an algebra of multiplier for $C_0(\widetilde{X})$) on H by equivariant operator. Moreover, since this representation preserves

$C_c(\widetilde{X})\cdot H$, it induces a representation $\widehat{\rho}$ of $C(X)$ on $\widehat{H}$. It is straightforward to check that $(\widehat{\rho},\widehat{H},\widehat{T})$ is a K-cycle for $K_*(X)$ and we get is this way a homomorphism

$$\Upsilon^{\Gamma}_{\widetilde{X},*}: KK^{\Gamma}_*(C_0(\widetilde{X}),\mathbb{C})\longrightarrow K_*(X) \tag{3.2}$$

which maps the class of (ρ,H,T) to the class of $(\widehat{\rho},\widehat{H},\widehat{T})$.

Theorem 3.6. [10] $\Upsilon^{\Gamma}_{\widetilde{X},*}$ *is a isomorphism.*

We want now to study how the propagation behave under the above transformation. So assume that $\widetilde{X}$ is a locally compact metric space equipped with a free, proper, isometric and cocompact action of Γ. Let η be a Γ-invariant measure, and let $\widehat{\eta}$ be the measure induced on $X=\widetilde{X}/\Gamma$. Let us set $H_{\widetilde{X}}=L^2(\eta)\otimes H$ and $H_X=L^2(\widehat{\eta})\otimes H$. We can view $C(X)$ as the algebra of Γ-invariant continuous and bounded functions on $\widetilde{X}$ and according to this, for any continuous and compactly supported function $f:\widetilde{X}\to\mathbb{C}$, then $\widehat{f}=\sum_{\gamma\in\Gamma}\gamma(f)$ belongs to $H_{\widehat{X}}$. It is straightforward to check that $f\mapsto\widehat{f}$ extends to a unitary map $\widehat{H_{\widetilde{X}}}\to H_X$.

Lemma 3.7. *If T is a locally compact equivariant operator on $H_{\widetilde{X}}$ with propagation less than r. Then, under the above identification between $\widehat{H_{\widetilde{X}}}$ and H_X, the operator $\widehat{T}$ is a compact operator with propagation less than r.*

Proof. Since T is equivariant and since $\widetilde{X}$ is cocompact, the operator T is given by a kernel $K:\widetilde{X}\times\widetilde{X}\to\mathcal{K}(H)$ such that $K(\gamma x,\gamma y)=K(x,y)$ for almost all (x,y) in $\widetilde{X}\times\widetilde{X}$ and with cocompact support (for the diagonal action of Γ on $\widetilde{X}\times\widetilde{X}$) of diameter less than r. Under the above identification between $\widehat{H_{\widetilde{X}}}$ and H_X, then for any continuous and compactly supported function $f:\widetilde{X}\to\mathbb{C}$, we have

$$\widehat{T}\cdot\widehat{f}=\sum_{\gamma\in\Gamma}\gamma(T\cdot f)=\sum_{\gamma\in\Gamma}T\cdot\gamma(f).$$

By viewing X as a borelian fundamental domain for the action of Γ on $\widetilde{X}$, we get

$$\begin{aligned}
\widehat{T}\cdot\widehat{f}(x) &= \sum_{\gamma\in\Gamma}\int_{\widetilde{X}}K(x,y)f(\gamma y)d\eta(y)\\
&= \sum_{(\gamma,\gamma')\in\Gamma^2}\int_X K(x,\gamma' y)f(\gamma\gamma' y)d\widehat{\eta}(y)\\
&= \sum_{(\gamma,\gamma')\in\Gamma^2}\int_X K(\gamma'^{-1}x,y)f(\gamma y)d\widehat{\eta}(y)\\
&= \int_X\sum_{\gamma'\in\Gamma}K(\gamma'^{-1}x,y)\widehat{f}(y)d\widehat{\eta}(y)\\
&= \int_X F(x,y)\widehat{f}(y)d\eta(y),
\end{aligned}$$

with

$$F:\widetilde{X}\times\widetilde{X};(x,y)\mapsto\sum_{\gamma\in\Gamma}K(\gamma x,y).$$

The kernel F is $\Gamma\times\Gamma$-invariant and thus can be viewed as a kernel on $X\times X$ and thus we get $\widehat{T}\cdot\widehat{f}(x)=\int_X F(x,y)\widehat{f}(y)d\widehat{\eta}(y)$. Hence $\widehat{T}$ is a compact operator and since $F(x,y)=0$ for almost every (x,y) in $X\times X$ such that $d(x,y)\geq r$, we see that $\widehat{T}$ as propagation less than r. □

The previous lemma can be extended to pseudo-local operators on $H_{\widetilde{X}}$ with finite propagation. Recall from [10] that if T is a pseudo-local equivariant operator on $H_{\widetilde{X}}$ with finite propagation, then $\widehat{T}$ is a pseudo-local operator on $\widehat{H_{\widetilde{X}}}\cong H_X$

Lemma 3.8. *With notation of lemma 3.7, if T is pseudo-local Γ-equivariant operator on $H_{\widetilde{X}}$ with propagation less than r. Then, the operator $\widehat{T}$ is a pseudo-local operator with propagation less than r*

Proof. Let $f_1,\ldots,f_n$ be a partition of unit for X with support of diameter less than r. Let us set $T'=\sum_{i=1}^n f_i^{1/2}\circ q\cdot T\cdot f_i^{1/2}\circ q$, where $q:\tilde{X}\to X$ is the projection map of the covering. Then $T'-T$ is an equivariant locally compact on $H_{\widetilde{X}}$ with propagation less than r and thus, according to lemma 3.7, we get that $\widehat{T}-\widehat{T'}$ is compact and has propagation less than r. Since $\widehat{T'}=\sum_{i=1}^n f_i^{1/2}\cdot\widehat{T}\cdot f_i^{1/2}$, then T' is a pseudo-local operator of propagation less than r and hence we get the result. □

4. The left hand side for the coarse space associated to a residually finite group

The aim of this section is to state for the sources of the assembly maps the analogous of proposition 2.8, i.e the existence of a group homomorphism

$$\Psi_{X(\Gamma),*}:\lim_r KK^\Gamma(P_r(X(\Gamma)),\mathbb{C})\longrightarrow K^{\mathrm{top}}(\Gamma,A_\Gamma),$$

such that

$$\Psi_{\Gamma,A_\Gamma,*}\circ\mu_{X(\Gamma),\max,*}=\mu_{\Gamma,A_\Gamma,\max,*}\circ\Psi_{X(\Gamma),*}. \tag{4.1}$$

4.1. **Rips complexes associated to a residually finite group.** Let Γ be a residually finite group, finitely generated. Let $\Gamma_0\supset\Gamma_1\supset\ldots\Gamma_n\supset\ldots$ be a decreasing sequence of normal finite index subgroups of Γ such that $\bigcap_{i\in\mathbb{N}}\Gamma_i=\{e\}$. Recall notations of section 2.2, if d be a left invariant metric associated to any finite set of generators for Γ, then we endow Γ/Γ_i with the metric $d(a\Gamma_i,b\Gamma_i)=\min\{d(a\gamma_1,b\gamma_2),\gamma_1$ and γ_2 in $\Gamma_i\}$. We set $X(\Gamma)=\coprod_{i\in\mathbb{N}}\Gamma/\Gamma_i$ and we equip $X(\Gamma)$ with a metric d such that on Γ/Γ_i, then d is the metric defined above and $d(\Gamma/\Gamma_i,\Gamma/\Gamma_i)\geq i+j$ if $i\neq j$.

For every integer n such that $r>n$, then

$$P_r(X(\Gamma))=P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i)\coprod\left(\coprod_{i\geq n}P_r(\Gamma/\Gamma_i)\right),$$

where $P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i)$ and $\coprod_{i\geq n}P_r(\Gamma/\Gamma_i)$ can be viewed as distinct open subsets of $P_r(X(\Gamma))$. Hence we have a splitting

$$\begin{aligned}K_*(P_r(X(\Gamma)))&\cong K_*(P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i))\bigoplus K_*(\coprod_{i\geq n}P_r(\Gamma/\Gamma_i))\\ &\cong K_*(P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i))\bigoplus\prod_{i\geq n}K_*(P_r(\Gamma/\Gamma_i))\end{aligned} \tag{4.2}$$

corresponding to the inclusion of the disjoint open subsets $P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i)$ and $\coprod_{i\geq n} P_r(\Gamma/\Gamma_i)$ into $P_r(X(\Gamma))$.

Let us show that in the inductive limit when r runs through positive real, $K_*(P_r(\Gamma/\Gamma_i))$, behave like $K_*(P_r(\Gamma)/\Gamma_i)$ (recall that Γ_i acts properly on $P_r(\Gamma)$). For f in $P_r(\Gamma)$, let us define

$$\widetilde{f}:\Gamma/\Gamma_i\to[0,1];\,\gamma\Gamma_i\mapsto\sum_{g\in\Gamma_i}f(\gamma g).$$

Then $\widetilde{f}$ is a probability on Γ/Γ_i. Let γ and γ' be elements of Γ such that $\widetilde{f}(\gamma\Gamma_i)\neq 0$ and $\widetilde{f}(\gamma'\Gamma_i)\neq 0$. Then there exists g and g' in Γ_i such that $f(\gamma g)\neq 0$ and $f(\gamma' g)\neq 0$. Since f is in $P_r(\Gamma)$, we get that $d(\gamma g,\gamma' g)\leq r$ and hence $d(\gamma\Gamma_i,\gamma'\Gamma_i)\leq r$. Thus $\widetilde{f}$ belongs to $P_r(\Gamma/\Gamma_i)$, and since $\widetilde{\gamma\cdot f}=\widetilde{f}$ for any γ in Γ_i, we finally obtain a continuous map $\upsilon_{r,i}:P_r(\Gamma)/\Gamma_i\to P_r(\Gamma/\Gamma_i);\dot{f}\mapsto\widetilde{f}$, where $\dot{f}$ is the class in $P_r(\Gamma)/\Gamma_i$ of f in $P_r(\Gamma)$. For a positive real r and an integer n, let

$$\Lambda_{*,r,n}:\prod_{k\geq n}K_*(P_r(\Gamma)/\Gamma_k)\longrightarrow\prod_{k\geq n}K_*(P_r(\Gamma/\Gamma_k))$$

be the homomorphism induced on the k-th factor by the map

$$P_r(\Gamma)/\Gamma_k\to P_r(\Gamma/\Gamma_k);\,\dot{f}\mapsto\widetilde{f}.$$

Lemma 4.1. *Let i be an integer such that $B_\Gamma(e,2r)\cap\Gamma_i=\{e\}$. Let $\{\gamma_1,\cdots,\gamma_n\}$ and $\{\gamma'_1,\cdots,\gamma'_n\}$ be subsets of Γ of diameter less than r and such that $\gamma_j{\gamma'}_j^{-1}$ is in Γ_i for all j in $\{1,\cdots,n\}$, then $\gamma_j{\gamma'}_j^{-1}=\gamma_k{\gamma'}_k^{-1}$ for all j and k in $\{1,\cdots,n\}$.*

Proof. We have $d(\gamma_1,\gamma_j)\leq r$ and $d(\gamma'_1,\gamma'_j)\leq r$ for all j in $\{1,\cdots,n\}$. Let us set $g=\gamma_1{\gamma'}_1^{-1}$. Then

$$\begin{aligned}d(\gamma_j,g\gamma'_j)&\leq d(\gamma_j,\gamma_1)+d(\gamma_1,g\gamma'_j)\\&\leq d(\gamma_j,\gamma_1)+d(g\gamma'_1,g\gamma'_j)\\&\leq d(\gamma_j,\gamma_1)+d(\gamma'_1,\gamma'_j)\\&\leq 2r.\end{aligned}$$

Hence, since Γ_i is normal, $\gamma_j^{-1}g\gamma'_j=(\gamma_j^{-1}g\gamma_j)\gamma_j^{-1}(\gamma'_j\gamma_j^{-1})\gamma_j$ belongs to $B_\Gamma(e,2r)\cap\Gamma_i$ and thus $\gamma_j=g\gamma'_j$. □

Let i be an integer such that $B_\Gamma(e,4r)\cap\Gamma_i=\{e\}$. Let h be an element of $P_r(\Gamma/\Gamma_i)$. We can choose a finite subset $\{\gamma_1,\cdots,\gamma_n\}$ of diameter less than $2r$ such that the support of h lies in $\{\gamma_1\Gamma_i,\cdots,\gamma_n\Gamma_i\}$. According to lemma 4.1, a such subset is unique up to left translations by an element of Γ_i. Let us define $\hat{h}$ in $P_{2r}(\Gamma)/\Gamma_i$ as the class of the probability of $P_r(\Gamma)$ with support in $\{\gamma_1,\cdots,\gamma_n\}$ with value on an element γ in that set $\hat{h}(\gamma)=h(\gamma\Gamma_i)$. It is straightforward to check that if h is in $P_r(\Gamma/\Gamma_i)$, then $\widetilde{\hat{h}}$ is the image of h under the inclusion map $P_r(\Gamma/\Gamma_i)\hookrightarrow P_{2r}(\Gamma/\Gamma_i)$. If f is an element of $P_r(\Gamma)$, then since $B_\Gamma(e,r)\cap\Gamma_i=\{e\}$, the intersection of the support of f with any $\gamma\Gamma_i$ for γ in Γ has at most one element. Hence, according to lemma 4.1, $\hat{\widetilde{f}}$ is the image of the class of f in $P_r(\Gamma)/\Gamma_i$ under the inclusion $P_r(\Gamma)/\Gamma_i\hookrightarrow P_{2r}(\Gamma)/\Gamma_i$.

Lemma 4.2. *Let r be a positive real and let i be an integer such that $B_\Gamma(e,r)\cap\Gamma_i=\{e\}$. Then the action of Γ_i on $P(\Gamma)$ is free.*

Proof. Let f be an element of $P_r(\Gamma)$. If $\gamma\cdot f=f$ with γ in Γ_i, then the support of f is invariant under the action of γ. In particular, $\gamma=g_1\cdot g_2^{-1}$ with g_1 and g_2 in the support of f. Hence γ is in $B_\Gamma(e,r)\cap\Gamma_i$ and thus $\gamma=e$. □

In consequence, with condition of the lemma above, $P_r(\Gamma)\to P_r(\Gamma)/\Gamma_i$ is a covering map and since Γ_i has finite index in Γ, this covering is cocompact.

4.2. **Construction of $\Psi_{X(\Gamma)}$.** For a positive real r and an integer n, such that $r\geq n$ and $B_\Gamma(e,4r)\cap\Gamma_n=\{e\}$ let us define

- $\Psi^1_{*,r,n}: K_*(P_r(X(\Gamma)))\to\prod_{i\geq n}K_*(P_r(\Gamma/\Gamma_i))$ the projection homomorphism corresponding to the decomposition in equation 4.2 of section 4.1.
- $\Psi^2_{*,r,n}:\prod_{k\geq n}K_*(P_r(\Gamma/\Gamma_k))\longrightarrow\prod_{k\geq n}K_*(P_{2r}(\Gamma)/\Gamma_k)$ the homomorphism induced on the k-th factor by the maps $P_r(\Gamma/\Gamma_k)\to P_{2r}(\Gamma)/\Gamma_k$; $h\mapsto\hat{h}$.
- $$\Psi^3_{*,r,n}:\prod_{k\geq n}K_*(P_r(\Gamma)/\Gamma_k)\longrightarrow\prod_{k\geq n}KK_*^{\Gamma_k}(P_r(\Gamma),\mathbb{C})$$
  the homomorphism given on the k-th factor by the inverse of the isomorphism $\Upsilon^{\Gamma_k}_{P_r(\Gamma),*}:KK_*^{\Gamma_k}(P_r(\Gamma),\mathbb{C})\stackrel{\cong}{\longrightarrow}K_*(P_r(\Gamma)/\Gamma_k)$ (see section 3.4).
- $\Psi^4_{*,r,n}:\prod_{k\geq n}KK_*^{\Gamma_k}(P_r(\Gamma),\mathbb{C})\longrightarrow\prod_{k\geq n}KK_*^{\Gamma}(P_r(\Gamma),C(\Gamma/\Gamma_k))$ the homomorphism given on the k-th factor by the induction homomorphism $\mathrm{I}^{\Gamma,P_r(\Gamma)}_{\Gamma_k,*}$ (see section 3.2).
- $\Psi^5_{*,r,n}:\prod_{i\geq n}KK_*^{\Gamma}(P_r(\Gamma),C(\Gamma/\Gamma_i))\longrightarrow KK_*^{\Gamma}(P_r(\Gamma),\ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i,\mathcal{K}(H)))$ the inverse of the isomorphism $\Theta^{\Gamma,\mathcal{A}}_*$ of proposition 3.4 applied to the family $\mathcal{A}=(C(\Gamma/\Gamma_i))_{i\in\mathbb{N}}$;
- $\Psi^6_{*,r,n}:KK_*^{\Gamma}(P_r(\Gamma),\ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i,\mathcal{K}(H)))\longrightarrow KK_*^{\Gamma}(P_r(\Gamma),A_\Gamma)$ the homomorphism induced by the Γ-equivariant epimorphism
  $$\ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i,\mathcal{K}(H))\longrightarrow A_\Gamma.$$

Remark 4.3. (i) *Let us also define for any real r and r' such that $0\leq r\leq r'$ the homomorphisms*

$$\iota^{k\geq n}_{*,r,r'}:\prod_{k\geq n}K_*(P_r(\Gamma/\Gamma_k))\longrightarrow\prod_{k\geq n}K_*(P_{r'}(\Gamma/\Gamma_k))$$

and

$$\iota'^{k\geq n}_{*,r,r'}:\prod_{k\geq n}K_*(P_r(\Gamma/)\Gamma_k)\longrightarrow\prod_{k\geq n}K_*(P_{r'}(\Gamma/\Gamma_k)$$

respectively induced on the k-th factor by the inclusions $P_r(\Gamma/\Gamma_k)\hookrightarrow P_{r'}(\Gamma/\Gamma_k)$ and $P_r(\Gamma)/\Gamma_k\hookrightarrow P_{r'}(\Gamma)/\Gamma_k$. According to the discussion that follows lemma 4.1 and with notations of section 4.1, if n is chosen such that $n\leq r$ and $B_\Gamma(e,4r)\cap\Gamma_n=\{e\}$, then

$$\Psi^2_{*,r,n}\circ\Lambda_{*,r,n}=\iota'^{k\geq n}_{*,r,2r}\tag{4.3}$$

and

$$\Lambda_{*,2r,n}\circ\Psi^2_{*,r,n}=\iota^{k\geq n}_{*,r,2r}.\tag{4.4}$$

(ii) *Using the same argument as in the proof of proposition 3.4, we get that* $\Psi^5_{*,r,n}$ *restricts to an isomorphism*

$$\bigoplus_{i\geq n} KK^\Gamma_*(P_r(\Gamma), C(\Gamma/\Gamma_i)) \xrightarrow{\cong} KK^\Gamma_*(P_r(\Gamma), C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))).$$

According to the next lemma, $\Psi^6_{*,r,n}$ is an epimorphism.

Proposition 4.4. *The equivariant short exact sequence*

$$0 \longrightarrow C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H)) \longrightarrow \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H)) \longrightarrow A_\Gamma \longrightarrow 0$$

gives rise to a short exact sequence

$$0 \longrightarrow KK^\Gamma_*(P_r(\Gamma), C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^\Gamma_*(P_r(\Gamma), \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^\Gamma_*(P_r(\Gamma), A_\Gamma) \longrightarrow 0.$$

Proof. Using the six-term exact sequence associated to an equivariant short exact sequence of C^*-algebras, this amounts to show that the inclusion

$$\iota : C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H)) \hookrightarrow \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))$$

induces a monomorphism

$$\iota_* : KK^\Gamma_*(P_r(\Gamma), C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \hookrightarrow KK^\Gamma_*(P_r(\Gamma), \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))).$$

According to remark 4.3 and to proposition 3.4, we have a splitting

$$KK^\Gamma_*(P_r(\Gamma), C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \cong \bigoplus_{i\geq n} KK^\Gamma_*(P_r(\Gamma), C(\Gamma/\Gamma_i))$$

and an isomorphism

$$KK^\Gamma_*(P_r(\Gamma), \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \cong \prod_{i\geq n} KK^\Gamma_*(P_r(\Gamma), C(\Gamma/\Gamma_i)).$$

Up to these identifications, ι_* is the inclusion

$$\bigoplus_{i\geq n} KK^\Gamma_*(P_r(\Gamma), C(\Gamma/\Gamma_i)) \hookrightarrow \prod_{i\geq n} KK^\Gamma_*(P_r(\Gamma), C(\Gamma/\Gamma_i)).$$

□

Remark 4.5. *Taking the inductive limit over all the* $P_r(\Gamma)$*, we get a short exact sequence*

$$0 \longrightarrow K^{top}_*(\Gamma, C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow K^{top}_*(\Gamma, \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow K^{top}_*(\Gamma, A_\Gamma) \longrightarrow 0$$

In the same way, since the composition

$$\bigoplus_{i\geq n} K_*(C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{max} \Gamma) \to K_*(\prod_{i\geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{max} \Gamma) \to \prod_{i\geq n} K_*(C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{max} \Gamma)$$

is injective, where the second map is induced on the k-th factor by the projection $\prod_{i\geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \to C(\Gamma/\Gamma_k, \mathcal{K}(H))$*, the exact sequence for maximal cross product*

$$0 \longrightarrow C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{max} \Gamma \longrightarrow \ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{max} \Gamma \longrightarrow A_\Gamma \rtimes_{max} \Gamma \longrightarrow 0$$

gives rise to a short exact sequence

$$0\longrightarrow \oplus_{i\geq n} K_*(C_0(\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{max}\Gamma)\longrightarrow K_*(\ell^\infty(\sqcup_{i\geq n}\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{max}\Gamma)\longrightarrow K_*(A_\Gamma\rtimes_{max}\Gamma)\longrightarrow 0$$

and moreover, the assembly maps intertwin the corresponding above exact sequences

Let r be a positive real and let n be an integer such that $n\leq r$ and $B_\Gamma(e,4r)\cap\Gamma_n=\{e\}$. Let us define

$$\Psi_{*,r,n}: KK_*(P_r(X(\Gamma)),\mathbb{C})\longrightarrow KK_*^\Gamma(P_{2r}(\Gamma),A_\Gamma)$$

by

$$\Psi_{*,r}=\Psi^6_{*,2r,n}\circ\Psi^5_{*,2r,n}\circ\Psi^4_{*,2r,n}\circ\Psi^3_{*,2r,n}\circ\Psi^2_{*,r,n}\circ\Psi^1_{*,r,n}.$$

Notice that $\Psi_{*,r,n}$ does not depend on the choice of the integer n such that $n\leq r$ and $B_\Gamma(e,4r)\cap\Gamma_n=\{e\}$.

For any positive real r and r' such that $r\leq r'$, let

$$\iota^{\Gamma,A_\Gamma}_{*,r,r'}: KK_*^\Gamma(P_r(\Gamma),A_\Gamma)\to KK_*^\Gamma(P_{r'}(\Gamma),A_\Gamma)$$

be the homomorphism induced by the inclusion $P_r(\Gamma)\subset P_{r'}(\Gamma)$.

Lemma 4.6. *For every element y in $KK_*^\Gamma(P_r(\Gamma),A_\Gamma)$, there exists an element x in $KK_*(P_r(X(\Gamma)),\mathbb{C})$ such that $\Psi_{*,r}(x)=\iota^{\Gamma,A_\Gamma}_{*,r,2r}(y)$.*

Proof. According to proposition 4.4, the homomorphism $\Psi^6_{*,2r,n}$ is onto. Since $\Psi^5_{*,2r,n}$,$\Psi^4_{*,2r,n}$ and $\Psi^3_{*,2r,n}$ are isomorphisms, there exists a z in $\prod_{i\geq n}K_*(P_r(\Gamma)/\Gamma_i)$ such that $y=\Psi^6_{*,r,n}\circ\Psi^5_{*,r,n}\circ\Psi^4_{*,r,n}\circ\Psi^3_{*,r,n}(z)$, Using equation 4.3, we have

$$\Psi^2_{*,r,n}\circ\Lambda_{*,r,n}(z)=\iota'^{k\geq n}_{*,r,2r}(z)$$

and since $\Psi^1_{*,r,n}$ is onto, there exists an element x in $K_*(P_r(X(\Gamma)))$ such that $\Lambda_{*,r,n}(z)=\Psi^1_{*,r,n}(x)$. The lemma is then a consequence of the equality

$$\Psi^6_{*,2r,n}\circ\Psi^5_{*,2r,n}\circ\Psi^4_{*,2r,n}\circ\Psi^3_{*,2r,n}\circ\iota'^{k\geq n}_{*,r,2r}=\iota^{\Gamma,A_\Gamma}_{*,r,r'}\circ\Psi^6_{*,2r,n}\circ\Psi^5_{*,2r,n}\circ\Psi^4_{*,2r,n}\circ\Psi^3_{*,2r,n}.$$

□

Let us denote for a pair of real r and r' such that $0\leq r\leq r'$ by $\iota^{X(\Gamma)}_{*,r,r'}$: $K_*(P_r(X(\Gamma)))\longrightarrow K_*(P_{r'}(X(\Gamma)))$ the morphism induced by the inclusion $P_r(X(\Gamma))\subset P_{r'}(X(\Gamma))$. The class Γ_0 of Γ/Γ_0 can be viewed as an element of $P_r(X(\Gamma))$ and this inclusion induces a homomorphism

$$\kappa_{*,r}:\mathbb{Z}\cong K_*(\{[\Gamma_0]\})\longrightarrow K_*(P_r(X(\Gamma))).$$

Lemma 4.7. *Let x be an element of $K_*(P_r(X(\Gamma)))$ such that $\Psi_{*,r}(x)=0$, then there exists a real r' such that $r\leq r'$ such that $\iota^{X(\Gamma)}_{*,r,r'}(x)$ is in the range of $\kappa_{*,r'}$.*

Proof. Let us fix a integer n such that $n\geq r$ and $B_\Gamma(e,4r)\cap\Gamma_n=\{e\}$. According to proposition 4.4,

$$\Psi^5_{*,2r,n}\circ\Psi^4_{*,2r,n}\circ\Psi^3_{*,2r,n}\circ\Psi^2_{*,r,n}\circ\Psi^1_{*,r,n}(x)\in KK_*^\Gamma(P_{2r}(\Gamma),C_0(\sqcup_{i\geq n}\Gamma/\Gamma_i,\mathcal{K}(H)))\subset KK_*^\Gamma(P_{2r}(\Gamma),\ell^\infty(X(\Gamma),\mathcal{K}(H))).$$

In view of remark 4.3, we get that

$$\Psi^4_{*,2r,n}\circ\Psi^3_{*,2r,n}\circ\Psi^2_{*,r,n}\circ\Psi^1_{*,r,n}(x)\in\bigoplus_{k\geq n}KK_*^\Gamma(P_{2r}(\Gamma),C(\Gamma/\Gamma_k)).$$

Since $\Psi^4_{*,2r,n}$ and $\Psi^3_{*,2r,n}$ restricts on direct summands to isomorphisms, then we get that $\Psi^2_{*,r,n}\circ\Psi^1_{*,r,n}(x)$ lies in $\bigoplus_{k\geq n} KK_*(P_{2r}(\Gamma)/\Gamma_k,\mathbb{C})$. According to equation 4.4,

$$\iota^{k\geq n}_{*,r,2r}\circ\Psi^1_{*,r,n}(x)=\Psi^1_{*,2r,n}\circ\iota^{X(\Gamma)}_{*,r,2r}(x)\in\bigoplus_{k\geq n} KK_*(P_{2r}(\Gamma/\Gamma_k),\mathbb{C}).$$

But then $\iota^{X(\Gamma)}_{*,r,2r}(x)$ lies in a finite sum of summands of $\bigoplus_{k\geq n} KK_*(P_{2r}(\Gamma)/\Gamma_k,\mathbb{C})$ and thus we get that for some integer m and some real s with $s\geq 2r$ and $m\geq\sup\{n,s\}$, then $\iota^{X(\Gamma)}_{*,r,s}(x)$ belongs to $KK_*(P_s(\sqcup_{0\leq k\leq m}\Gamma/\Gamma_k),\mathbb{C})$. But since $\coprod_{0\leq k\leq m}\Gamma/\Gamma_k$ is finite, $P_s(\sqcup_{0\leq k\leq m}\Gamma/\Gamma_k)$ is compact and up to choose a bigger s is also convex. Hence, $\iota^{X(\Gamma)}_{*,r,r'}(x)$ lies in the range of $\kappa_{*,r'}$ for r' big enough. □

It is straightforward to check that $\Psi_{*,r'}\circ\iota^{X(\Gamma)}_{*,r,r'}=\iota^{\Gamma,A_\Gamma}_{*,r,r'}\circ\Psi_{*,r}$ and thus the family of homomorphism $(\Psi_{*,r})_{r\leq 0}$ gives rise to a homomorphism

$$\Psi_{X(\Gamma),*}:\lim_r KK_*(P_r(X(\Gamma)),\mathbb{C})\longrightarrow K^{\mathrm{top}}_*(\Gamma,A_\Gamma).$$

Let x_0 be the image of (any) $\kappa_{*,r}(1)$ in $\lim_r KK_*(P_r(X(\Gamma)),\mathbb{C})$. As a consequence of lemmas 4.6 and 4.7, we get

Theorem 4.8.

- $\Psi_{X(\Gamma),*}$ *is onto;*
- *In odd degree,* $\Psi_{X(\Gamma),*}$ *is an isomorphism;*
- *In even degree,* $\ker\Psi_{X(\Gamma),*}$ *is the infinite cyclic group generated by* x_0*;*

4.3. **Compatibility of $\Psi_{X(\Gamma),*}$ with the assembly maps.** The proof of equation 4.1 require some preliminary work. For a locally compact and proper Γ-space X, the notion of standard-X-module, was extended to the equivariant case in [15] as follows.

Definition 4.9. *Let X be a locally compact and proper Γ-space, let H be a Γ-Hilbert space. A non-degenarated Γ-equivariant representation $\rho: C_0(X)\to\mathcal{L}(H)$ is called X-Γ-ample if when extended to $C_0(X)\rtimes\Gamma$, then $\rho(C_0(X)\rtimes\Gamma)\cap\mathcal{K}(H)=\{0\}$.*

Example 4.10. *If η is a Γ-invariant measure on $P_r(\Gamma)$ fully supported i.e with support $P_r(\Gamma)$ and if H is a separable Hilbert space, then $L^2(\eta)\otimes H$ equipped with the diagonal action of Γ, trivial on H together with the representation*

$$\rho_r: C_0(P_r(\Gamma))\to\mathcal{L}(L^2(\eta)\otimes H);\ f\mapsto f\otimes \mathrm{Id}_H$$

is an X-Γ-ample representation. The reason is that $P_r(\Gamma)$ contains as a Γ-space a copy of $\Gamma\times Y$, where Y is a open subset of $P_r(\Gamma)$, and where Γ acts diagonally, by left translations on Γ and trivially on Y.

Lemma 4.11. *Let X be locally compact and proper Γ-space, let H_0 and H_1 be two Γ-Hilbert spaces and let $\rho_i: C_0(X)\to\mathcal{L}(H_i)$ for $i=0,1$ be two non-degenerated and Γ-equivariant representation. Assume that ρ_0 is X-Γ-ample. Then there exists*

- *H_2 a Γ-Hilbert space;*
- *$\rho_2: C_0(X)\to\mathcal{L}(H_2)$ a non-degenerated Γ-equivariant representation;*
- *$U: H_1\oplus H_2\to H_0$ a unitary*

such that for every f in $C_0(X)$,

$$U\cdot(\rho_1\oplus\rho_2)(f)-\rho_0(f)\cdot U\in\mathcal{K}(H_1\oplus H_2,H_0).$$

Proof. Up to replace ρ_1 by$\rho_0\oplus\rho_1$, we can assume without loss of generality that ρ_1 is also X-Γ-ample. Then, according to [15], there exists an Γ-equivariant isometry $W: H_1\to H_0$ such that $W\cdot\rho_1(f)-\rho_0(f)\cdot W$ is in $\mathcal{K}(H_1,H_0)$ for every f in $C_0(X)$. Let us set $P=\mathrm{Id}_{H_0}-W\cdot W^*$. Now, by using the completly positive map

$$C_0(X)\to\mathcal{L}(P\cdot H_0);\ f\mapsto P\cdot\rho_0(f)\cdot P,$$

we can use the proof of [8, Theorem 3.4.6] to conclude. □

From this and by using next lemma, we can prove that if X is a locally compact and proper Γ-space, then every element in $KK_*^\Gamma(X,\mathbb{C})$ can be represented by a K-cycle supported on a prescribed X-Γ-ample and non-degenerated representation.

Lemma 4.12. *Let G be a locally compact group and let A and B be two G-algebras. Let $(\rho,\mathcal{E},T)$ be a K-cycle for $KK_*^G(A,B)$ and let ρ' be an equivariant representation of A on the right B-Hilbert module $\mathcal{E}$ such that $\rho(a)-\rho'(a)$ is compact for all a in A. Then $(\rho',\mathcal{E},T)$ is a K-cycle for $KK_*^G(A,B)$ equivalent to $(\rho,\mathcal{E},T)$.*

Proof. It is clear that $(\rho',\mathcal{E},T)$ is a K-cycle for $KK_*^G(A,B)$. Then

$$\begin{pmatrix}\cos t\pi/2 & -\sin t\pi/2\\ \sin t\pi/2 & \cos t\pi/2\end{pmatrix}\cdot\begin{pmatrix}T & 0\\ 0 & \mathrm{Id}_E\end{pmatrix}\cdot\begin{pmatrix}\cos t\pi/2 & \sin t\pi/2\\ -\sin t\pi/2 & \cos t\pi/2\end{pmatrix}_{t\in[0,1]}$$

provides a homotopy between the K-cycles $(\rho\oplus\rho',\mathcal{E}\oplus\mathcal{E},T\oplus\mathrm{Id}_E)$ and $(\rho\oplus\rho',\mathcal{E}\oplus\mathcal{E},\mathrm{Id}_E\oplus T)$. □

Corollary 4.13. *Let X be a locally compact and proper Γ-space and let ρ_X be a X-Γ-ample representation of $C_0(X)$ on a Γ-Hilbert space H_X. Then every element of $KK_*^\Gamma(X,\mathbb{C})$ can be represented by a K-cycle (ρ_X,H_X,T) where T is a Γ-equivariant operator on H_X.*

We fix once for all a separable Hilbert space H and for each real r a Γ-invariant measure η_r on $P_r(\Gamma)$ fully supported. Let us consider $H_{P_r(\Gamma)}=L^2(\eta_r)\otimes H$ with the X-Γ-ample representation ρ_r defined in example 4.10. Define $\Psi^{\Gamma_i}(H_{P_r(\Gamma)})$ as the $*$-algebra of pseudo-local, Γ_i-equivariant and finite propagation operators on $H_{P_r(\Gamma)}$. An element of $\Psi^{\Gamma_i}(H_{P_r(\Gamma)})$ is called a K-cycle if it satisfies the K-cycle condition with respect to ρ_r.

Lemma 4.14. *Let x be an element of $KK^\Gamma(P_r(\Gamma),\ell^\infty(X(\Gamma),\mathcal{K}(H)))$. Then there exists a real s and a family $(T_i)_{i\in\mathbb{N}}$ of bounded operators on $H_{P_r(\Gamma)}$ such that*

(i) *T_i is a K-cycle of $\Psi^{\Gamma_i}(P_r(\Gamma))$ of propagation less than s and $\|T_i\|\leq 1$ for every integer i;*
(ii) *Under the identification*

$$\ell^\infty(X,\mathcal{K}(H,H_{P_r(\Gamma)}))\cong\prod_{i\in\mathbb{N}}C(\Gamma/\Gamma_i,\mathcal{K}(H,H_{P_r(\Gamma)}))\cong\prod_{i\in\mathbb{N}}\mathrm{I}_{\Gamma_i}^{\Gamma}\ \mathcal{K}(H,H_{P_r(\Gamma)}),$$

then

$$((\mathrm{I}_{\Gamma_i}^{\Gamma}\ \rho_r)_{i\in\mathbb{N}},\ \ell^\infty(X,\mathcal{K}(H,H_{P_r(\Gamma)})),\ (\mathrm{I}_{\Gamma_i}^{\Gamma}\ T_i)_{i\in\mathbb{N}})$$

is a K-cycle that represents x.
(iii) *If x_i is the class of $(\rho_r,H_{P_r(\Gamma)},T_i)$ in $KK_*^{\Gamma_i}(P_r(\Gamma),\mathbb{C})$, then $\Theta_*^{\Gamma,\mathcal{A}}(x)=(\mathrm{I}_{\Gamma_i}^{\Gamma}\ x_i))_{i\in\mathbb{N}}$, with $\mathcal{A}=(C(\Gamma/\Gamma_i))_{i\in\mathbb{N}}$,*

Proof. Item (iii) is a consequence of item (ii) together with proposition 3.4. According to the discussion following propositon 3.4, we can assume that x is represented by a K-cycle $((\phi_i)_{i\in\mathbb{N}}, \prod_{i\in\mathbb{N}}\mathcal{K}_{C(\Gamma/\Gamma_i)}(C(\Gamma/\Gamma_i,H),\mathcal{E}_i),(T_i)_{i\in\mathbb{N}})$ such that for all integer i

- $\mathcal{E}_i$ is a Γ-equivariant $C(\Gamma/\Gamma_i)$-Hilbert module;
- T_i is a Γ-equivariant adjointable operator on $\mathcal{E}_i$ with $\|T_i\|\leq 1$;
- ϕ_i is a Γ-equivariant representation of $C_0(P_r(\Gamma))$ on $\mathcal{E}_i$;
- $C_0(P_r(\Gamma))$ and T_i act then on $\mathcal{K}_{C(\Gamma/\Gamma_i)}(H,\mathcal{E}_i)$ by left composition.

But for every integer i, there exist a Γ_i-Hilbert space H_i, a Γ_i-equivariant representation ψ_i of $C_0(P_r(\Gamma))$ on H_i and a Γ_i-equivariant bounded operator F_i on H_i such that $\mathcal{E}_i=\mathrm{I}_{\Gamma_i}^{\Gamma}H_i$, $\phi_i=\mathrm{I}_{\Gamma_i}^{\Gamma}\psi_i$ and $T_i=\mathrm{I}_{\Gamma_i}^{\Gamma}F_i'$. Up to replace H_i by the Γ-Hilbert space induced by the inclusion $\Gamma_i\hookrightarrow\Gamma$, we can assume that H_i is a Γ-Hilbert which is up to add the degenerated K-cycle $(\rho_r,H_{P_r(\Gamma)},\mathrm{Id}_{H_{P_r(\Gamma)}})$ can be chosen X-Γ-ample. By adding the degenerated K-cycle

$$(\bigoplus_{k\in\mathbb{N},\,k\neq i}\psi_k,\ \bigoplus_{k\in\mathbb{N},\,k\neq i}H_k,\ \bigoplus_{k\in\mathbb{N},\,k\neq i}\mathrm{Id}_{H_k}),$$

we can assume that $H_i=H_0$ and $\psi_i=\psi_0$. According to lemma 4.11, by taking an unitary equivalence of K-cycle, we can assume that $H_0=H_{P_r(\Gamma)}$ and that $\psi_0(f)-\rho_r(f)\in\mathcal{K}(H_{P_r(\Gamma)})$ for every integer f in $C_0(P_r(\Gamma))$. It is straightforward to check that

$$((\mathrm{I}_{\Gamma_i}^{\Gamma}\rho_r)_{i\in\mathbb{N}},\ \ell^\infty(X,\mathcal{K}(H,H_{P_r(\Gamma)})),\ (\mathrm{I}_{\Gamma_i}^{\Gamma}F_i)_{i\in\mathbb{N}})$$

is a K-cycle for $KK_*^G(P_r(\Gamma),\ell^\infty(X(\Gamma),\mathcal{K}(H)))$, which is by proposition 3.4 and lemma 4.12 equivalent to

$$((\mathrm{I}_{\Gamma_i}^{\Gamma}\ \psi_0)_{i\in\mathbb{N}},\ \ell^\infty(X,\mathcal{K}(H,H_{P_r(\Gamma)})),\ (\mathrm{I}_{\Gamma_i}^{\Gamma}\ F_i)_{i\in\mathbb{N}}).$$

If $f\in C_c(P_r(\Gamma),[0,1])$ is a cut-off function for the action of Γ on $P_r(\Gamma)$, then up to replace $F=(\mathrm{I}_{\Gamma_i}^{\Gamma}\ F_i)_{i\in\mathbb{N}}$ by $\sum_{\gamma\in\Gamma}\gamma(f)F\gamma(f)$, we can assume that there exists a real s such that for all integer i the operator F_i has propagation less than s. □

Set $\zeta=\ell^\infty(X(\Gamma),\mathcal{K}(H,\ell^2(\Gamma\otimes H))$ and let ζ_Γ be the right $\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$-Hilbert module constructed from ζ in section 3.1. Viewing $\ell^2(\Gamma)\otimes\ell^\infty(X(\Gamma),\mathcal{K}(H))$ as a right $\ell^\infty(X(\Gamma),\mathcal{K}(H))$-Hilbert submodule of $\ell^\infty(X(\Gamma),\mathcal{K}(H,\ell^2(\Gamma)\otimes H))$, we see that

$$\begin{array}{rcl} C_c(\Gamma)\cdot\ell^\infty(X(\Gamma),\mathcal{K}(H,\ell^2(\Gamma)\otimes H)) & = & C_c(\Gamma)\cdot\left(\ell^2(\Gamma)\otimes\ell^\infty(X(\Gamma),\mathcal{K}(H))\right)\\ & \cong & C_c(\Gamma)\otimes\ell^\infty(X(\Gamma),\mathcal{K}(H)) \end{array}$$

and under this identification, we get that

$$C_c(\Gamma)\otimes\ell^\infty(X(\Gamma),\mathcal{K}(H))\to\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma;\ \delta\gamma\otimes a\mapsto\gamma^{-1}(a)\delta\gamma^{-1}$$

extends to isomorphism of right $\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$-Hilbert module

$$\zeta_\Gamma\xrightarrow{\cong}\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma. \tag{4.5}$$

Define for i integer $\Psi^{\Gamma_i}(\Gamma)$ as the $*$-algebras of pseudo-local, Γ_i-equivariant and finite propagation operators on $\ell^2(\Gamma)\otimes H$.

Lemma 4.15. *Let $(S_i)_{i\in\mathbb{N}}$ be a family in $\prod_{i\in\mathbb{N}}\Psi^{\Gamma_i}(\Gamma)$ uniformally bounded and with propagation uniformally bounded by a real s. Then under the identification*

$$\zeta\cong\prod_{i\in\mathbb{N}}C(\Gamma/\Gamma_i,\mathcal{K}(H,\ell^2(\Gamma)\otimes H))\cong\prod_{i\in\mathbb{N}}\mathrm{I}_{\Gamma_i}^{\Gamma}\,\mathcal{K}(H,\ell^2(\Gamma)\otimes H)),$$

(i) *there exists a unique multiplier $\lambda_\Gamma(S_i)_{i\in\mathbb{N}}$ of $\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{max}\Gamma$ which under the identification $\zeta_\Gamma\cong\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{max}\Gamma$ restricts to $(\mathrm{I}_{\Gamma_i}^{\Gamma}\,S_i)_{i\in\mathbb{N}}$ on $C_c(\Gamma)\cdot\zeta$.*

(ii) *The multiplier image of $\lambda_\Gamma(S_i)_{i\in\mathbb{N}}$ under the canonical projection*

$$\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{max}\Gamma\longrightarrow A_\Gamma\rtimes_{max}\Gamma$$

and the multiplier image of $\bigoplus_{i\in\mathbb{N}}\widehat{S}_i$ under the map

$$C^*_{max}(X(\Gamma))\stackrel{\Psi_\Gamma}{\longrightarrow}A_\Gamma\rtimes_{max}\Gamma$$

coincide.

Proof. Let us prove first the lemma for a family $(S_i)_{i\in\mathbb{N}}$ of locally compact operators. Since such families are algebraically generated by families

- $(f_i)_{i\in\mathbb{N}}$ uniformally bounded with f_i in $C(\Gamma/\Gamma_i,\mathcal{K}(H))$ acting by pointwise multiplication;
- $(R_\gamma)_{i\in\mathbb{N}}$, for γ in Γ, where R_γ is induced by the right regular representation on $\ell^2(\Gamma)\otimes H$,

this amounts to prove the lemma for these families.

Then the elements of $\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$ given by

- the image of $(f_i)_{i\in\mathbb{N}}$ viewed as element of $\ell^\infty(X(\Gamma),\mathcal{K}(H))$ under the inclusion $\ell^\infty(X(\Gamma),\mathcal{K}(H))\hookrightarrow\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$ for the first case.
- the element $\delta_\gamma\in\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$ for the family $(R_\gamma)_{i\in\mathbb{N}}$, where δ_γ be the Dirac function at γ.

satisfy the required property. For a family $(S_i)_{i\in\mathbb{N}}$ of pseudo-local operators, let us set for i integer $S'_i=\sum_{\gamma\in\Gamma}\delta_\gamma S_i\delta_\gamma$. Then S'_i-S_i is Γ_i-equivariant and locally compact for all integer i. Moreover, as already mention in subsection 3.1, $(\mathrm{I}_{\Gamma_i}^{\Gamma}\,S'_i)_{i\in\mathbb{N}}=\sum_{\gamma\in\Gamma}\delta_\gamma(\mathrm{I}_{\Gamma_i}^{\Gamma}\,S_i)_{i\in\mathbb{N}}\delta_\gamma$ extends to an adjointable operator of ζ_Γ and thereby to a multiplier $\lambda_\Gamma(S'_i)_{i\in\mathbb{N}}$ of $\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma$. We then set $\lambda_\Gamma(S_i)_{i\in\mathbb{N}}=\lambda_\Gamma(S'_i)_{i\in\mathbb{N}}+\lambda_\Gamma(S_i-S'_i)_{i\in\mathbb{N}}$. Unicity is quite obvious. Since pseudo-local operator on $\ell^2(\Gamma)\otimes H$ are multiplier for locally compact operator, we get item (ii) by multiplicativity of $(S_i)_{i\in\mathbb{N}}\mapsto\lambda_\Gamma(S_i)_{i\in\mathbb{N}}$ and of $(S_i)_{i\in\mathbb{N}}\mapsto\bigoplus_{i\in\mathbb{N}}\widehat{S}_i$. □

we are now in position to prove the main theorem of the section.

Theorem 4.16. *Let Γ be a residually finite and finitely generated discrete group with respect to a sequence $\Gamma_0\subset\cdots\Gamma_n\subset\cdots$ of finite index normal subgroups. Then if we set $X(\Gamma)=\coprod_{n\in\mathbb{N}}\Gamma/\Gamma_i$, we have*

$$\Psi_{\Gamma,A_\Gamma,*}\circ\mu_{X(\Gamma),max,*}=\mu_{\Gamma,A_\Gamma,max,*}\circ\Psi_{X(\Gamma),*}.$$

Proof. Let r be a real and let n be any integer such that $2r\leq n$ and $B(e,4r)\cap\Gamma_n=\{e\}$. Then

$$K_*(P_r(X(\Gamma)))\cong K_*(P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i))\oplus\prod_{i\geq n}K_*(P_r(\Gamma/\Gamma_i)).\tag{4.6}$$

If under this identification, x comes from $K_*(P_r(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i))$, then $\Psi_{X(\Gamma),*}(x)=0$ and using the naturality of the assembly map, we can see that $\mu_{X(\Gamma),\max,*}$ lies in $K_*(C^*_{max}(\coprod_{i=1}^{n-1}\Gamma/\Gamma_i))\subset K_*(C^*_{max}(X(\Gamma)))$ and hence $\Psi_{\Gamma,A_\Gamma,*}\circ\mu_{X(\Gamma),\max,*}(x)=0$. Thereby, we have to prove that $\Psi_{\Gamma,*}\circ\mu_{X(\Gamma),\max,*}(x)=\mu_{\Gamma,A_\Gamma,\max,*}\circ\Psi_*(x)$, for element x coming under the identification of equation 4.6 from x' in $\prod_{i\geq n}K_*(P_r(\Gamma/\Gamma_i))$. According to equation 4.4, and up to replace r by $2r$, we can assume that $x'=\Lambda_{*,r,n}(y)$, with y in $\prod_{i\geq n}K_*(P_r(\Gamma)/\Gamma_i)$. We can assume indeed without loss of generality that $n=0$ and that the action of Γ_i on $P_r(\Gamma)$ is free for all integer i. From now on, we will write Λ_* (resp. Ψ^i_*, $i=1,\cdots,6$) instead of $\Lambda_{*,r,0}$ (resp. $\Psi^i_{*,r,0}$, $i=1,\cdots,6$). Let us set $z=\Psi^3_*(y)$ in $\prod_{i\in\mathbb{N}}KK^{\Gamma_i}_*(C_0(P_r(\Gamma)),\mathbb{C})$. The proof of the theorem is divided in the following steps.

First Step: Assume that z is given by a family of K-cycles $(\rho_{P_r(\Gamma)},H_{P_r(\Gamma)},T_i)_{i\in\mathbb{N}}$ such that for a real s, then for all integer i the operator T_i is Γ_i-equivariant with $\|T_i\|\leq 1$ and has propagation less than s. Let us set $H_{P_r(\Gamma)/\Gamma_i}=L^2(\eta_{r,i})\otimes H$ and $\rho_{P_r(\Gamma)/\Gamma_i}:C_0(P_r(\Gamma)/\Gamma_i)\to\mathcal{L}(L^2(\eta_{r,i})\otimes H);\ f\mapsto f\otimes \mathrm{Id}_H$ where for all integer i, the measure $\eta_{r,i}$ is induced by η_r on $P_r(\Gamma)/\Gamma_i$. Let us choose a Γ-equivariant coarse map $\widetilde{\phi_r}:P_r(\Gamma)\to\Gamma$. Then $\widetilde{\phi_r}$ is a coarse equivalence and induces a coarse equivalence

$$\phi_r:\coprod_{i\in\mathbb{N}}P_r(\Gamma)/\Gamma_i\longrightarrow\coprod_{i\in\mathbb{N}}\Gamma/\Gamma_i=X(\Gamma).$$

Let us show that with notations of section 2.3,

$$\mu_{X(\Gamma),\max,*}(x)=\phi_{r,\max,*}\operatorname{Ind}_{\max,X(\Gamma)}\oplus_{k\in\mathbb{N}}\widehat{T^k},\tag{4.7}$$

where $\oplus_{k\in\mathbb{N}}\widehat{T^k}$ is viewed as an operator on the non-degenerated standard $\coprod_{k\in\mathbb{N}}P_r(\Gamma)/\Gamma_k$-module $\bigoplus_{k\in\mathbb{N}}H_{P_r(\Gamma)/\Gamma_k}$ (for the representation $\oplus_{i\in\mathbb{N}}\rho_{H_{P_r(\Gamma)/\Gamma_i}}$).

Let $\upsilon_{r,k}:P_r(\Gamma)/\Gamma_k\to P_r(\Gamma/\Gamma_k);\dot{h}\mapsto\widetilde{h}$ be the map defined in section 4. Notice that the family $(\upsilon_{r,k})_{\in\mathbb{N}}$ induces a coarse equivalence $\upsilon_r:\coprod_{k\in\mathbb{N}}P_r(\Gamma)/\Gamma_k\longrightarrow\coprod_{k\in\mathbb{N}}P_r(\Gamma/\Gamma_k)$. Moreover, if we set

$$\phi_k:C_0(P_r(\Gamma/\Gamma_k))\to\mathcal{L}(H_{P_r(\Gamma)/\Gamma_k});\ f\mapsto\rho_{P_r(\Gamma)/\Gamma_k}(f\circ\upsilon_{r,k}),$$

then $x'=\Lambda_*(y)$ is the class of the K-cycle

$$\left(\bigoplus_{k\in\mathbb{N}}\phi_k,\bigoplus_{k\in\mathbb{N}}H_{P_r(\Gamma)/\Gamma_k},\bigoplus_{k\in\mathbb{N}}\widehat{T_k}\right)$$

in $K_*(\coprod_{k\in\mathbb{N}}(P_r(\Gamma/\Gamma_k))$. For any non-degenerated standard $P_r(\Gamma/\Gamma_k)$-module H_k given by a representation ρ_k, then $\phi_k\oplus\rho_k$ also provides a non-degenerated standard $P_r(\Gamma/\Gamma_k))$-Hilbert module structure for $H_{P_r(\Gamma)/\Gamma_k}\oplus H_k$. Since the K-cycles

$$\left(\bigoplus_{k\in\mathbb{N}}\phi_k,\bigoplus_{k\in\mathbb{N}}H_{P_r(\Gamma)/\Gamma_k},\bigoplus_{k\in\mathbb{N}}\widehat{T_k}\right)$$

and

$$\left(\bigoplus_{k\in\mathbb{N}}\phi_k\oplus\rho_k,\bigoplus_{k\in\mathbb{N}}H_{P_r(\Gamma)/\Gamma_k}\oplus H_k,\bigoplus_{k\in\mathbb{N}}\widehat{T_k}\oplus\mathrm{Id}_{H_k}\right)$$

are equivalent, we get that

$$\mu_{X(\Gamma),\max,*}(x) = \psi_{r,\max,*}\operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}P_r(\Gamma/\Gamma_k)}\bigoplus_{k\in\mathbb{N}}\widehat{T_k}\oplus\mathrm{Id}_{H_k},$$

where $\psi_r : \coprod_{i\in\mathbb{N}} P_r(\Gamma/\Gamma_k) \to X(\Gamma)$ is any coarse equivalence. Since the inclusion $\bigoplus_{k\in\mathbb{N}} H_{P_r(\Gamma)/\Gamma_k} \hookrightarrow \bigoplus_{k\in\mathbb{N}} H_{P_r(\Gamma)/\Gamma_k} \oplus H_k$ covers the coarse map $\upsilon_r : \coprod_{k\in\mathbb{N}} P_r(\Gamma)/\Gamma_k \to \coprod_{k\in\mathbb{N}} P_r(\Gamma/\Gamma_k)$, we get that

$$\operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}P_r(\Gamma/\Gamma_k)}\left(\oplus_{k\in\mathbb{N}}\widehat{T_k}\oplus\mathrm{Id}_{H_k}\right) = \upsilon_{r,\max,*}\operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}P_r(\Gamma)/\Gamma_k}\left(\oplus_{k\in\mathbb{N}}\widehat{T_k}\right).$$

Notice that since ϕ_r and $\psi_r \circ \upsilon_r$ are both coarse equivalence between $\coprod_{k\in\mathbb{N}} P_r(\Gamma)/\Gamma_k$ and $X(\Gamma)$, then $\phi_{r,\max,*} = \psi_{r,\max,*} \circ \upsilon_{r,\max,*}$ and hence we get the equality of equation 4.7.

Second step: According to [15], there exists an Γ-equivariant isometrie $W_r : H_{P_r(\Gamma)} \to \ell^2(\Gamma)\otimes H$ that covers $\widehat{\phi_r} : P_r(\Gamma) \to \Gamma$. Then, if $W_{r,k} : H_{P_r(\Gamma/\Gamma_k)} \to \ell^2(\Gamma/\Gamma_k) \otimes H$ stands for the isometry induced by W_r for all integer k, then

$$\bigoplus_{k\in\mathbb{N}} W_{r,k} : \bigoplus_{k\in\mathbb{N}} H_{P_r(\Gamma/\Gamma_k)} \longrightarrow \bigoplus_{k\in\mathbb{N}} \ell^2(\Gamma/\Gamma_k)\otimes H$$

is an isometrie that covers ϕ_r and thus

$$\phi_{r,\max,*}\operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}P_r(\Gamma)/\Gamma_k}\bigoplus_{k\in\mathbb{N}}\widehat{T_k} = \operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}\Gamma/\Gamma_k}\bigoplus_{k\in\mathbb{N}} W_{r,k}\widehat{T_k}W_{r,k}^* + \mathrm{Id}_{\ell^2(\Gamma/\Gamma_k)\otimes H} - W_{r,k}W_{r,k}^*.$$

Finally we get that

$$\begin{aligned}\mu_{X(\Gamma),\max,*}(x) &= \operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}\Gamma/\Gamma_k}\bigoplus_{k\in\mathbb{N}} W_{r,k}\widehat{T_k}W_{r,k}^* + \mathrm{Id}_{\ell^2(\Gamma/\Gamma_k)\otimes H} - W_{r,k}W_{r,k}^*.\\ &= \operatorname{Ind}_{\max,\coprod_{k\in\mathbb{N}}\Gamma/\Gamma_k}\bigoplus_{k\in\mathbb{N}} \widehat{W_rT_kW_r^*} + \mathrm{Id}_{\ell^2(\Gamma/\Gamma_k)\otimes H} - \widehat{W_rW_r^*}\end{aligned}$$

Third step: By naturallity of the assembly map, we get that

$$\mu_{\Gamma,A_\Gamma,\max,*}\circ\Psi_*(x) = \Psi^6_{\Gamma,*}\circ\mu_{\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)),\max,*}(z'),$$

where

- z' is the element in $K_*^{top}(\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)))$ coming from $\Psi^5_*\circ\Psi^4_*(z) \in KK^\Gamma_*(P_r(\Gamma),\ell^\infty(X(\Gamma),\mathcal{K}(H)))$;
- $\Psi^6_\Gamma : \ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma \to A_\Gamma\rtimes_{\max}\Gamma$ is induced by the projection $\Psi^6 : \ell^\infty(X(\Gamma),\mathcal{K}(H)) \to A_\Gamma$.

Let us compute $\mu_{\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)),\max,*}(z')$.

According to lemma 4.11, under the identification

$$\ell^\infty(X(\Gamma),\mathcal{K}(H,H_{P_r(\Gamma)})) \cong \prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i,\mathcal{K}(H,H_{P_r(\Gamma)})) \cong \prod_{i\in\mathbb{N}} \mathrm{I}^\Gamma_{\Gamma_i}\ \mathcal{K}(H,H_{P_r(\Gamma)}),$$

the element $\Psi^5_*\circ\Psi^4_*(z)$ of $KK^\Gamma_*(P_r(\Gamma),\ell^\infty(X(\Gamma),\mathcal{K}(H)))$ can be represented by a K-cycle

$$\left((\mathrm{I}^\Gamma_{\Gamma_i}\ \rho_r)_{i\in\mathbb{N}},\ \ell^\infty(X,\mathcal{K}(H,H_{P_r(\Gamma)})),\ (\mathrm{I}^\Gamma_{\Gamma_i}\ F_i)_{i\in\mathbb{N}}\right)$$

where

- F_i is a K-cycle of $\Psi^{\Gamma_i}(P_r(\Gamma))$ with $\|F_i\| \leq 1$ for all integer i;
- there exists a real s such that F_i has propagation less than s for all integer i;
- if x_i is the class of $(\rho_r, H_{P_r(\Gamma)}, F_i)$ in $KK_*^{\Gamma_i}(P_r(\Gamma), \mathbb{C})$ then $z = (x_i)_{i\in\mathbb{N}}$.

Moreover, if we set $\mathcal{E} = \ell^\infty(X(\Gamma), \mathcal{K}(H, H_{P_r(\Gamma)}))$ and $F = (\mathrm{I}_{\Gamma_i}^{\Gamma}\ F_i)_{i\in\mathbb{N}}$, we can assume by averaging by a cut-of function for the action of Γ on $P_r(\Gamma)$ that $F \cdot C_c(P_r(\Gamma)) \cdot \mathcal{E} \subset C_c(P_r(\Gamma)) \cdot \mathcal{E}$. Let us also set

$$\zeta = \ell^\infty(X(\Gamma), \mathcal{K}(H, \ell^2(\Gamma) \otimes H))$$

and let $\mathcal{E}_\Gamma$ and ζ_Γ be the right $\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma$-module constructed in section 3.1 respectively from $\mathcal{E}$ and ζ. Since the isometrie

$$W_\Gamma : H_{P_r(\Gamma)} \to \ell^2(\Gamma) \otimes H$$

has finite propagation, it induces a map

$$C_c(P_r(\Gamma)) \cdot \mathcal{E} \to C_c(\Gamma) \cdot \zeta; f \mapsto W_r \circ f,$$

which extends to an isometrie $W_\Gamma : \mathcal{E}_\Gamma \to \zeta_\Gamma$. As we have seen before, ζ_Γ is a right-$\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma$ module isomorphic to $\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma$ and in view of this, $\mathcal{E}_\Gamma$ is a direct factor of $\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma$. Moreover, if F_Γ is the operator of $\mathcal{E}_\Gamma$ extending $C_c(\Gamma) \cdot \mathcal{E} \to C_c(\Gamma) \cdot \mathcal{E}; f \mapsto T \circ f$, then we get with notations of lemma 4.15, that

$$\lambda_\Gamma(W_r F_i W_r^* + \mathrm{Id}_{\ell^2(\Gamma)\otimes H} - W_r W_r^*)_{i\in\mathbb{N}} = W_\Gamma F_\Gamma \cdot W_\Gamma^* + \mathrm{Id}_{\zeta_\Gamma} - W_\Gamma W_\Gamma^*.$$

Hence $\mu_{\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H)),*}(z')$ is the class in $K_*(\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma)$ of the K-cycle $(\ell^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\max} \Gamma, \lambda_\Gamma(W_r F_i W_r^* + \mathrm{Id}_{\ell^2(\Gamma)\otimes H} - W_r W_r^*)_{i\in\mathbb{N}})$. The theorem is then a consequence of lemma 4.15.

□

4.4. **Applications.** We end this section with application concerning injectivity and bijectivity of the maximal coarse Baum-Connes assembly map. Let Γ be a residually finite and finitely generated discrete group with respect to a fixed sequence $\Gamma_0 \subset \cdots \Gamma_n \subset \cdots$ of finite index normal subgroups. Recall that we have defined $X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$ and $A_\Gamma = \ell^\infty(X(\Gamma), \mathcal{K}(H))/C_0(X(\Gamma), \mathcal{K}(H))$. We can formulate corollary 2.11, theorem 4.8 and theorem 4.16 together as follows: We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \lim_r K_0(P_r(X(\Gamma))) & \xrightarrow{\Psi_{X(\Gamma),*}} & K_0^{\mathrm{top}}(\Gamma, A_\Gamma) & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{\mu_{X(\Gamma),\max,*}} & & \Big\downarrow{\mu_{\Gamma,A_\Gamma,\max,*}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(C^*_{max}(X(\Gamma))) & \xrightarrow{\Psi_{\Gamma,A_\Gamma,*}} & K_0(A_\Gamma \rtimes_{\max} \Gamma) & \longrightarrow & 0
\end{array}$$

with exact rows and a commutative diagram

$$\begin{array}{ccc}
\lim_r KK_1(P_r(X(\Gamma))) & \xrightarrow[\cong]{\Psi_{X(\Gamma),*}} & K_1^{\mathrm{top}}(\Gamma, A_\Gamma) \\
\Big\downarrow{\mu_{X(\Gamma),\max,*}} & & \Big\downarrow{\mu_{\Gamma,A_\Gamma,\max,*}} \\
K_1(C^*_{max}(X(\Gamma))) & \xrightarrow[\cong]{\Psi_{\Gamma,A_\Gamma,*}} & K_1(A_\Gamma \rtimes_{\max} \Gamma)
\end{array}$$

From this commutative diagram, we can deduce the following series of results concerning injectivity and bijectivity of assembly maps.

Theorem 4.17. *The following assertions are equivalent:*

(i) *The maximal coarse assembly map*

$$\mu_{X(\Gamma),max,*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*_{max}(X(\Gamma)))$$

is an isomorphism.

(ii) *the maximal assembly map*

$$\mu_{\Gamma,A_\Gamma,max}:K_*^{top}(\Gamma,A_\Gamma)\to K_*(A_\Gamma\rtimes_{max}\Gamma)$$

is an isomorphism.

Example of groups that satisfies item (ii) of the theorem are provided by groups that satisfy the so called *strong Baum-Connes conjecture.* Recall first that a Γ-algebra D is said to be a *proper* Γ-algebra, if D is a $C_0(Z)$-algebra for some proper Γ-space Z in such a way that the structure map $\Phi : C_0(Z) \to ZM(D)$ is Γ-equivariant. A group Γ satisfies the strong Baum-Connes conjecture if there exist, a proper Γ-algebra D, an element α in $KK_*^\Gamma(D,\mathbb{C})$ and a element β in $KK_*^\Gamma(\mathbb{C},D)$ such that $\beta\otimes_D\alpha$ is the unit of $KK_*^\Gamma(\mathbb{C},\mathbb{C})$. It is well know (see [14] for instance) that if Γ satisfies the strong Baum-Connes conjecture, then $\mu_{\Gamma,B,\max}:K_*^{\mathrm{top}}(\Gamma,B)\to K_*(B\rtimes_{\max}\Gamma)$ is an isomorphism for every Γ-algebra B. As a consequence, we get

Corollary 4.18. *If Γ satisfies the strong Baum-Connes conjecture, then*

$$\mu_{X(\Gamma),max}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*_{max}(X(\Gamma)))$$

is an isomorphism.

As a particular case, we obtain

Corollary 4.19. *If Γ is a group with the Haagerup property, then*

$$\mu_{X(\Gamma),max,*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*_{max}(X(\Gamma)))$$

is an isomorphism.

Example 4.20. *If $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_k = \ker : SL_2(\mathbb{Z})\longrightarrow SL_2(\mathbb{Z}/k\mathbb{Z})$, then $\mu_{X(\Gamma),max,*}$ is an isomorphism while $\mu_{X(\Gamma),*}$ is not surjective.*

Remark 4.21. *If the group Γ has the Kazdhan property (T), then the family of projectors corresponding to the 0-eigenvalue of the Laplacians of the family $(\Gamma/\Gamma_i)_{i\in\mathbb{N}}$ provides a projector p in $C^*_{max}(X(\Gamma))$ [9]. We know from [7] that the image in $K_0(C^*(X(\Gamma)))$ of the class of p under the homomorphism*

$$\lambda_{X(\Gamma),*}:K_0(C^*_{max}(X(\Gamma)))\to K_0(C^*(X(\Gamma)))$$

is not in the range of the coarse assembly map

$$\mu_{X(\Gamma),*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*(X(\Gamma))).$$

Hence, according to remark 2.20, the assembly map $\mu_{X(\Gamma),max,*}$ is not surjective.

Recall from [14] that if the group Γ satisfies the strong Baum-Connes conjecture, then Γ is K-amenable and in particular, the K-theory of reduced and maximal crossed product coincide. This allows to get explicit computation for $K_*(C^*_{max}(X(\Gamma)))$ in the following situation.

Corollary 4.22. *If Γ satisfies the strong Baum-Connes conjecture and admits a universal example for proper action which is simplicial, with simplicial and cocompact action of Γ, then we have a short exact sequence*

$$0 \to \mathbb{Z} \to K_0(C^*_{max}(X(\Gamma))) \to \prod_{i\in\mathbb{N}} K_0(C^*_{red}(\Gamma_i))/\bigoplus_{i\in\mathbb{N}} K_0(C^*_{red}(\Gamma_i)) \to 0$$

and an isomorphism

$$K_1(C^*_{max}(X(\Gamma))) \xrightarrow{\cong} \prod_{i\in\mathbb{N}} K_1(C^*_{red}(\Gamma_i))/\bigoplus_{i\in\mathbb{N}} K_1(C^*_{red}(\Gamma_i)).$$

Proof. First notice that since Γ is K-amenable, then

$$\lambda_{\Gamma,A_\Gamma,*} : K_*(A_\Gamma \rtimes_{\max}\Gamma) \to K_*(A_\Gamma \rtimes_{\text{red}} \Gamma).$$

Let us show that we have an isormorphim

$$K_*(A_\Gamma \rtimes_{\text{red}} \Gamma) \xrightarrow{\cong} \prod_{i\in\mathbb{N}} K_*(C^*_{\text{red}}(\Gamma_i))/\bigoplus_{i\in\mathbb{N}} K_*(C^*_{\text{red}}(\Gamma_i)).$$

Let us consider the following commutative diagram

$$\begin{array}{ccc} K^{\text{top}}_*(\Gamma, \prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, \mathcal{K}(H))) & \xrightarrow{\mu_{\Gamma,\prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i)\Gamma),\text{red},*}} & K_*((\prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, \mathcal{K}(H))) \rtimes_{\text{red}} \Gamma) \\ \downarrow & & \downarrow \\ \prod_{i\in\mathbb{N}} K^{\text{top}}_*(\Gamma, C(\Gamma/\Gamma_i)) & \xrightarrow{\prod_{i\in\mathbb{N}} \mu_{\Gamma,C(\Gamma/\Gamma_i,\mathcal{K}(H))\Gamma),\text{red},*}} & \prod_{i\in\mathbb{N}} K_*(C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rtimes_{\text{red}} \Gamma) \end{array},$$

where the vertical arrow are induced on the k-th factor by the projection

$$\prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \to C(\Gamma/\Gamma_k, \mathcal{K}(H)).$$

But since the group Γ admits a universal example for proper action which is simplicial, with simplicial and cocompact action of Γ, then the left vertical arrow is an isomorphism. Since Γ satisfies the Baum-Connes conjecture, then the horizontal map are also isomorphism. Hence the right vertical map is also an isomorphism and hence the result is a consequence of remark 4.5 and of the Morita equivalence between $C(\Gamma/\Gamma_i) \rtimes_{\text{red}} \Gamma$ and $C^*_{\text{red}}(\Gamma_i)$. □

Regarding injectivity, we have similar results.

Theorem 4.23. *The following assertions are equivalent:*

(i) *The maximal coarse assembly map*

$$\mu_{X(\Gamma),max,*} : \lim_r K_*(P_r(X(\Gamma)), \mathbb{C}) \to K_*(C^*_{max}(X(\Gamma)))$$

is injective.

(ii) *the maximal assembly map*

$$\mu_{\Gamma,A_\Gamma,max,*} : K^{top}_*(\Gamma, A_\Gamma) \to K_*(A_\Gamma \rtimes_{max}\Gamma)$$

is injective.

We can also deduce the following result concerning the (usual) coarse Baum-Connes conjecture.

Theorem 4.24. *Assume that the assembly map*

$$\mu_{\Gamma,A_\Gamma,red,*}:K_*^{top}(\Gamma,A_\Gamma)\to K_*(A_\Gamma\rtimes_{red}\Gamma)$$

is injective . Then the coarse assembly map

$$\mu_{X(\Gamma),*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*(X(\Gamma)))$$

is also injective.

Proof. In the even case, let us consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \lim_r KK_0^\Gamma(P_r(X(\Gamma)),\mathbb{C}) & \xrightarrow{\Psi_{X(\Gamma),*}} & K_0^{\mathrm{top}}(\Gamma,A_\Gamma) & \longrightarrow & 0\\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle \mu_{X(\Gamma),\max,*}}\downarrow & & {\scriptstyle \mu_{\Gamma,A_\Gamma,\max,*}}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(C^*_{max}(X(\Gamma))) & \xrightarrow{\Psi_{\Gamma,A_\Gamma,\max,*}} & K_0(A_\Gamma\rtimes_{\max}\Gamma) & \longrightarrow & 0\\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle \lambda_{X(\Gamma),*}}\downarrow & & {\scriptstyle \lambda_{\Gamma,A_\Gamma,*}}\downarrow & & \\
 & & \mathbb{Z} & \longrightarrow & K_0(C^*(X(\Gamma))) & \xrightarrow{\Psi_{\Gamma,A_\Gamma,\mathrm{red},*}} & K_0(A_\Gamma\rtimes_{\mathrm{red}}\Gamma) & &
\end{array}$$

where the bottom left corner horizontal arrow is induced by the inclusion

$$\mathcal{K}(\ell^2(X(\Gamma))\otimes H)\hookrightarrow C^*(X(\Gamma))$$

and is according to remark 2.12 injective. Thereby, since the top row is exact, we get that injectivity of $\mu_{\Gamma,A_\Gamma,\max,*}=\lambda_{\Gamma,A_\Gamma}\circ\mu_{\Gamma,A_\Gamma,\max,*}$ implies injectivity of $\mu_{X(\Gamma),*}=\lambda_{X(\Gamma)}\circ\mu_{X(\Gamma),\max,*}$ ☐

It was proved in [13] that for a group Γ which embeds uniformally in a Hilbert space, then $\mu_{\Gamma,B,*}$ is injective for any Γ-algebra B. As a consequence we obtain

Corollary 4.25. *Let Γ be a group uniformally embeddable in a Hilbert space, then the coarse assembly map*

$$\mu_{X(\Gamma),*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*(X(\Gamma)))$$

is injective.

The last application is to rational injectivity of $\mu_{X(\Gamma),\max,*}$. Theorem 4.24 admits an obvious rational version. This allowed to recover the following result of [4]

Theorem 4.26. *Assume that Γ admits a universal example for proper action which is simplicial and with simplicial and cocompact action of Γ. If $\mu_{\Gamma,\mathbb{C},max,*}$ is rationnaly injective, then $\mu_{X(\Gamma),max,*}$ is also rationnaly injective.*

Proof. Since rationnal injectivity of $\mu_{\bullet,\mathbb{C},\max,*}$ is inherited by finite index subgroups, we get under the hypothesis of the theorem that $\mu_{\Gamma_i,\mathbb{C},\max,*}$ is rationnaly injective for all integer i. Since assembly maps are compatible with induction, we get that $\mu_{\Gamma,C(\Gamma/\Gamma_i),\max,*}$ is rationnaly injective. According to corollary 3.5 and since we have the commutative diagram

(4.8)
$$\begin{array}{ccc}
K_*^{\mathrm{top}}(\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H))) & \xrightarrow[\mu_{\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)),\max,*}]{} & K_*(\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)\\
\downarrow & & \downarrow\\
\prod_{i\in\mathbb{N}}K_*^{\mathrm{top}}(\Gamma,C(\Gamma/\Gamma_i)) & \xrightarrow{(\mu_{\Gamma_i,C(\Gamma/\Gamma_i),\max,*})_{i\in\mathbb{N}}} & \prod_{i\in\mathbb{N}}K_*(C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma)
\end{array}$$

where the vertical arrows are induced on the k-th factor up to Morita equivalence by the projection $\Pi_{i\in\mathbb{N}}\ell^\infty(X(\Gamma),\mathcal{K}(H))\to C(\Gamma/\Gamma_k,\mathcal{K}(H))$, we see that

$$\mu_{\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)),\max,*}:K_*^{\mathrm{top}}(\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)))\to K_*(\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)$$

is rationnaly injective. As we have already seen before, the assembly is also compatible with direct sum of coefficients. Hence we get that

$$\mu_{\Gamma,C_0(X(\Gamma),\mathcal{K}(H)),\max,*}:K_*^{\mathrm{top}}(\Gamma,C_0(X(\Gamma),\mathcal{K}(H)))\to K_*(C_0(X(\Gamma),\mathcal{K}(H)\rtimes_{\max}\Gamma)$$

is also rationnaly injective. By using the maps induced for each integer k by the k-th factor $\Pi_{i\in\mathbb{N}}\ell^\infty(X(\Gamma),\mathcal{K}(H))\to C(\Gamma/\Gamma_k,\mathcal{K}(H))$, with see that the inclusion $C_0(X(\Gamma),\mathcal{K}(H))\hookrightarrow\ell^\infty(X(\Gamma),\mathcal{K}(H))$ induces inclusions

$$K_*^{\mathrm{top}}(\Gamma,C_0(X(\Gamma),\mathcal{K}(H)))\hookrightarrow K_*^{\mathrm{top}}(\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H)))$$

and

$$K_*(C_0(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)\hookrightarrow K_*(\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)$$

and thus we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_*^{\mathrm{top}}(\Gamma,C_0(X(\Gamma),\mathcal{K}(H))) & \longrightarrow & K_*^{\mathrm{top}}(\Gamma,\ell^\infty(X(\Gamma),\mathcal{K}(H))) & \longrightarrow & K_*^{\mathrm{top}}(\Gamma,A_\Gamma) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_*(C_0(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma) & \longrightarrow & K_*(\ell^\infty(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma) & \longrightarrow & K_*(A_\Gamma\rtimes_{\max}\Gamma) & \longrightarrow & 0
\end{array}$$

with exact rows and where the vertical arrows are given by the assembly maps. Using once again the commutativity of diagram 4.8, we get that if $\mu_{\Gamma,A_\Gamma,\max,*}(x)$ comes rationnally from an element in $K_*(C_0(X(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)$, then x comes rationnaly from an element in $K_*^{\mathrm{top}}(\Gamma,C_0(X(\Gamma),\mathcal{K}(H)))$. Hence

$$\mu_{\Gamma,A_\Gamma,\max,*}:K_*^{\mathrm{top}}(\Gamma,A_\Gamma)\to K_*(A_\Gamma\rtimes_{\max}\Gamma)$$

is rationally injective. □

5. Asymptotic quantitative Novikov/Baum-Connes conjecture

Corollary 4.22 suggest that the property of the coarse assembly map

$$\mu_{X(\Gamma),*}:\lim_r K_*(P_r(X(\Gamma)),\mathbb{C})\to K_*(C^*(X(\Gamma)))$$

should be closely related to the family fo assembly maps

$$\left(\mu_{\Gamma_i,\mathbb{C},\max,*}:K_*^{\mathrm{top}}(\Gamma_i,\mathbb{C})\to K_*(C^*_{\max}(\Gamma_i)\right)_{i\in\mathbb{N}}.$$

We this, we introduce some quantitative assembly maps which take into account the propagation. The relevant propagation here is indeed the one induced by Γ under the Morita equivalence between $C^*_{\max}(\Gamma_i)$ and $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$. We then give asymptotic statements for these quantitative assembly maps and give examples of group for which they are satisfied.

5.1. Almost projectors, almost unitaries and propagation.

Definition 5.1. *Let A be a unital C^*-algebra and let ε in $(0,1/4)$.*

- *An element p in A is called* **an ε-projector** *if $p=p^*$ and $\|p^2-p\|<\varepsilon$.*
- *An element u in A is called* **an ε-unitary** *if $\|u^*u-1\|<\varepsilon$ and $\|uu^*-1\|<\varepsilon$*

Notice that if p is an ε-projector of a C^*-algebra A and $\phi_\varepsilon:\mathbb{R}\to\mathbb{R}$ is any continuous function such that $\phi_\varepsilon(t)=0$ for $t<\frac{1-\sqrt{1-4\varepsilon}}{2}$ and $\phi_\varepsilon(t)=1$ for $t>\frac{1+\sqrt{1-4\varepsilon}}{2}$, then $\phi_\varepsilon(p)$ is a projector. Moreover, we have $\|\phi_\varepsilon(p)-p\|\leq 2\varepsilon$.

Remark 5.2. *If A is a C^*-algebra, then for every ε in $(0,1/4)$*

- *if p is an ε-projector of A, then any element q in A such that $\|p-q\|<\frac{\varepsilon-\|p^2-p\|}{4}$ is an ε-projector. In this case $(tp+(1-t)q)_{t\in[0,1]}$ is a homotopy of ε-projectors between p and q and in consequence $\phi_\varepsilon(p)$ and $\phi_\varepsilon(q)$ are homotopic projectors.*
- *if A is unital and if u is an ε-unitary of A, then any element v such that $\|u-v\|<\frac{\varepsilon-\max\{\|u^*u-1\|,\|uu^*-1\|\}}{3}$ is ε-unitary and $(tu+(1-t)v)_{t\in[0,1]}$ is a homotopy of ε-unitary connecting u and v.*

Definition 5.3. *Let A be a Γ-algebra. An element x of $A\rtimes_{max}\Gamma$ is said to be of finite propagation if x lies in $C_c(\Gamma,A)$. We say that x has propagation less than r if the support of x as an element of $C_c(\Gamma,A)$ is in $B_\Gamma(e,s)$. These definitions have an obvious extension to $\widetilde{A\rtimes_{max}\Gamma}$ by requiring the unit to be of propagation zero.*

For ε in $(0,1/4)$, A a unital Γ-algebra, and p_0 and p_1 two ε-projectors of $A\rtimes_{\max}\Gamma\otimes\mathcal{K}(H)$ with propagation less than s and n_0 and n_1 positive integers, we write $(p_0,n_0)\sim_{s,\varepsilon}(p_1,n_1)$ if there is an integer k and a ε-projector homotopy $(q_t)_{t\in[0,1]}$ in $C([0,1],(A\rtimes_{\max}\Gamma)\otimes\mathcal{K}(H))$ between $\begin{pmatrix} p_0 & 0\\ 0 & I_{k+n_1}\end{pmatrix}$ and $\begin{pmatrix} p_1 & 0\\ 0 & I_{k+n_0}\end{pmatrix}$ such that q_t has propagation less than s for every t in $[0,1]$. Similarly if u_0 and u_1 are ε-unitaries in $A\rtimes_{\max}\widetilde{\Gamma\otimes}\mathcal{K}(H)$ with propagation less than s, we write $u_0\sim_{s,\varepsilon}u_1$ if there is a ε-unitary homotopy $(v_t)_{t\in[0,1]}$ in $C([0,1],(A\rtimes_{\max}\widetilde{\Gamma)\otimes}\mathcal{K}(H))$ between u_0 and u_1 and such that v_t has propagation less than s for every t in $[0,1]$.

Notice that if p and q are two ε-projectors in $A\rtimes_{\max}\Gamma\otimes\mathcal{K}(H)$ such that $p_0\sim_{s,\varepsilon}p_1$ then $\phi_\varepsilon(p_0)$ and $\phi_\varepsilon(p_1)$ are homotopic projectors.

5.2. **Propagation and assembly map.** As before Γ is a finitely generated group which is residually finite with respect to a family $\Gamma_0\supset\Gamma_1\supset\ldots\Gamma_n\supset\ldots$ of normal finite index subgroups of Γ.

Recall that $\Psi^{\Gamma_i}(\Gamma)$ and $\Psi^{\Gamma_i}(P_r(\Gamma))$ are respectively the $*$-algebras of pseudo-local, Γ_i-equivariant and finite propagation operators on $\ell^2(\Gamma)\otimes H$ and $H_{P_r(\Gamma)}$. Recall from section 3.2 and corollary 4.13 that any element of $KK^\Gamma_*(P_r(\Gamma),C(\Gamma/\Gamma_i))$ can be represented by K-cycle $(\mathrm{I}^\Gamma_{\Gamma_i}\,\rho_r,C(\Gamma/\Gamma_i,H_{P_r(\Gamma)}),\mathrm{I}^\Gamma_{\Gamma_i}\,F)$, where

- ρ_r is the standard representation of $C_0(P_r(\Gamma))$ on $H_{P_r(\Gamma)}=L^2(P_r(\Gamma),\eta_r)\otimes H$;
- F is a K-cycle of $\Psi^{\Gamma_i}(H_{P_r(\Gamma)})$.
- We have identified $\mathrm{I}^\Gamma_{\Gamma_i}\,H_{P_r(\Gamma)}$ with $C(\Gamma/\Gamma_i,H_{P_r(\Gamma)})\cong C(\Gamma/\Gamma_i)\otimes H_{P_r(\Gamma)}$ provided with the diagonal action of Γ;
- Under this identification, $\mathrm{I}^\Gamma_{\Gamma_i}\,\rho_r$ is the pointwise representation ρ_r on $H_{P_r(\Gamma)}$ and $\mathrm{I}^\Gamma_{\Gamma_i}\,F$ is the pointwise action by $\Gamma/\Gamma_i\to B(H_{P_r(\Gamma)})$; $\gamma\Gamma_i\mapsto\gamma(F)$.

For a K-cycle F of $\Psi^{\Gamma_i}(P_r(\Gamma))$, let us denote by x_F the corresponding element in $K^{\text{top}}(\Gamma, C(\Gamma/\Gamma_i))$ coming from the K-cycle $(\mathrm{I}_{\Gamma_i}^{\Gamma}\ \rho_r, C(\Gamma/\Gamma_i, H_{P_r(\Gamma)}), \mathrm{I}_{\Gamma_i}^{\Gamma}\ T_i)$.

Let us set $\zeta_i = \mathrm{I}_{\Gamma_i}^{\Gamma}\ \ell^2(\Gamma)\otimes H \cong C(\Gamma/\Gamma_i, \ell^2(\Gamma)\otimes H)$ and let $\zeta_{i,\Gamma}$ be the right $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$-Hilbert module constructed from ζ_i in section 3.1. Notice that $\zeta_{i,\Gamma}$ as a right $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$-Hilbert module is isomorphic to $H\otimes C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$ (compare with isomorphism of equation 4.5).

Proceeding as we did for proving lemma 4.15 and denoting the multiplier algebra of $C(\Gamma/\Gamma_i, \mathcal{K}(H))\rtimes_{\max}\Gamma$ by $M(C(\Gamma/\Gamma_i, \mathcal{K}(H))\rtimes_{\max}\Gamma)$, we get with notations of section 4.3

Lemma 5.4. *For every integer i, there is a $*$-homomorphism*

$$\lambda_i : \Psi^{\Gamma_i}(\Gamma) \to M(C(\Gamma/\Gamma_i, \mathcal{K}(H)\rtimes_{max}\Gamma))$$

such that

- *Under the identification $\zeta_{i,\Gamma} \cong H\otimes C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma$, then $\lambda_i(S)$ restricts to $\mathrm{I}_{\Gamma_i}^{\Gamma}\ S$ on $C_c(\Gamma)\cdot\zeta_{i,\Gamma}$;*
- *For any f in $C(\Gamma/\Gamma_i, \mathcal{K}(H))$, viewed as a locally compact operator on $\ell^2(\Gamma)\otimes H$, then $\lambda_i(f)$ is the image of f under the inclusion $C(\Gamma/\Gamma_i, \mathcal{K}(H)) \hookrightarrow C(\Gamma/\Gamma_i, \mathcal{K}(H))\rtimes_{max}\Gamma$;*
- *For every γ in Γ, then $\lambda_i(R_\gamma)$ is the left multiplication by $\delta_\gamma \in C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma$ (viewed as a multiplier of $C(\Gamma/\Gamma_i, \mathcal{K}(H)\rtimes_{max}\Gamma)$);*

Remark 5.5. *Let us denote by $C[\Gamma]^{\Gamma_i}$ the set of Γ_i-equivariant operators of $C[\Gamma]$. Then $C[\Gamma]^{\Gamma_i}$ is a $*$-algebra isomorphic to $C_c(\Gamma, C(\Gamma/\Gamma_i))$ (equiped with convolution product) and thus λ_i induces by restriction a homomorphism*

$$C((\Gamma/\Gamma_i)\rtimes_{max}\Gamma \to M(C(\Gamma/\Gamma_i, \mathcal{K}(H)\rtimes_{max}\Gamma)),$$

which is in fact the natural inclusion. According to [4, lemma 4.13]*, lemma 2.1 can be generalised to the equivariant case and hence for every real t and any integer i, there exists a positive real $C_{t,i}$ such that for any element S of $C[\Gamma]^{\Gamma_i}$ with propagation less than t, then $\|\lambda_i(S)\|_{C(\Gamma/\Gamma_i,\mathcal{K}(H)\rtimes_{max}\Gamma)} \leq C_{t,i}\|S\|_{\ell^2(\Gamma)\otimes H}$.*

Let us fix until the end of this subsection

- a Γ-equivariant coarse equivalence $\widehat{\phi_r} : P_r(\Gamma) \to \Gamma$;
- a isometry $W_r : H_r \to \ell^2(\Gamma)\otimes H$ that covers $\widehat{\phi_r}$.

By using the same argument as in the third step of the proof of theorem 4.16 we get the following result.

Proposition 5.6. *Let x be in $K_*^{top}(\Gamma, C(\Gamma/\Gamma_i))$ coming from an element x_F in $KK_*^{\Gamma}(P_r(\Gamma), C(\Gamma/\Gamma_i)))$ for a K-cycle F in some $\Psi^{\Gamma_i}(P_r(\Gamma))$. Then $\mu_{\Gamma,C(\Gamma/\Gamma_i),max,*}(x)$ is the class in $K_*(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$ of the K-cycle*

$$(H\otimes C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma, \lambda_i(W_rFW_r^* + \mathrm{Id}_{\ell^2(\Gamma)\otimes H} - W_rW_r^*)).$$

Define $F_{r,i} = \lambda_i(W_rFW_r^* + \mathrm{Id}_{\ell^2(\Gamma)\otimes H} - W_rW_r^*)$ for F a K-cycle of $\Psi^{\Gamma_i}(P_r(\Gamma))$ and let us set in the even case

$$V_F = \begin{pmatrix} \mathrm{Id}_H & F_{r,i} \\ 0 & \mathrm{Id}_H \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_H & 0 \\ -F_{r,i} & \mathrm{Id}_H \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_H & F_{r,i} \\ 0 & \mathrm{Id}_H \end{pmatrix} \cdot \begin{pmatrix} 0 & -\mathrm{Id}_H \\ \mathrm{Id}_H & 0 \end{pmatrix},$$

where Id_H is viewed as the unit of $C(\Gamma/\Gamma_i,\mathcal{K}(H))\widetilde{\rtimes_{\max}}\Gamma$. Since λ_i is a $*$-homomorphism, we see that the matrix

$$V_F\begin{pmatrix}\mathrm{Id}_H & 0\\ 0 & 0\end{pmatrix}V_F^{-1}-\begin{pmatrix}\mathrm{Id}_H & 0\\ 0 & 0\end{pmatrix}$$

has coefficients in $C(\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{\max}\Gamma$ and moreover we get

Proposition 5.7. *With notations of proposition 5.6, if x in $K_*^{top}(\Gamma,C(\Gamma/\Gamma_i))$ comes from an element x_F in $KK_*^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))$ for a K-cycle F in some $\Psi^{\Gamma_i}(P_r(\Gamma))$ and if we set $e_F=V_F\begin{pmatrix}\mathrm{Id}_H & 0\\ 0 & 0\end{pmatrix}V_F^{-1}$, then*

$$\mu_{\Gamma,C(\Gamma/\Gamma_i),max,0}(x)=[e_F]-\left[\left(\begin{smallmatrix}\mathrm{Id}_H & 0\\ 0 & 0\end{smallmatrix}\right)\right].$$

The crucial point is that with notations of proposition 5.7, then the coefficients of the idempotent e_F have indeed finite propagation depending only on the propagation of F. Since with notation of lemma 5.4, the algebra $C[\Gamma]^{\Gamma_i}$ is generated by R_γ for γ in Γ an by functions f in $C(\Gamma/\Gamma_i,\mathcal{K}(H))$, it is straightforward to check that λ_i is propagation preserving. Using this, we obtain for every positive real r the existence of a non-decreasing function $h_r:\mathbb{R}^+\to\mathbb{R}^+$,(which is in fact affine), independant on i, such that for every s and every K-cycle F in $\Psi^{\Gamma_i}(\Gamma)$ with propagation less than s, then with notation of lemma 5.7, the idempotent e_F has propagation less than $h_r(s)$. Notice that e_F has operator norm less than $\alpha_{r,i}=(1+\|F_{r,i}\|)^6$. Recall that if we set $e'_F=(1+(2e_F-1)(2e_F^*-1))^{-1/2}e(1+(2e_F-1)(2e_F^*-1))^{1/2}$, then e'_F is a projector equivalent to e_F. Fix once for all two sequences of real polynomial functions $(P_j)_{j\in\mathbb{N}}$ and $(Q_j)_{j\in\mathbb{N}}$ such that P_j and Q_j have degre j for all $j\in\mathbb{N}$ and on every compact subset of $\mathbb{R}^+$,

- $(P_j)_{j\in\mathbb{N}}$ converges uniformally to $t\mapsto\sqrt{1+t}$;
- $(Q_j)_{j\in\mathbb{N}}$ converges uniformally to $t\mapsto\frac{1}{\sqrt{1+t}}$.

Let us define $\Psi_1^{\Gamma_i}(P_r(\Gamma))=\{T\in\Psi^{\Gamma_i}(P_r(\Gamma))$ such that $\|T\|\leq 1\}$. For F a K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$, a positive real r and ε in $(0,1/4)$ let $j_{\varepsilon,F,i}$ be the smallest integer such that

$$|P_j(t)-\sqrt{1+t}|\leq((8+4\alpha_{r,i}\|)\|+2)^{-2}\text{ and }\left|Q_j(t)-\frac{1}{\sqrt{1+t}}\right|\leq((8+4\alpha_{r,i}))^{-2}$$

for all integer $j\geq j_{\varepsilon,F,i}$ and all $t\in[0,4\alpha_{r,i}]$. For F a K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ with propagation less than s, let us set

$$\tilde{p}_{F,\varepsilon}=1/2Q_{j_{\varepsilon,F,i}}((2e_F-1)(2e_F^*-1))(e_F+e_F^*)P_{j_{\varepsilon,F,i}}((2e_F-1)(2e_F^*-1)).$$

Then $\|e'_F-\tilde{p}_{F,\varepsilon}\|<\varepsilon/8$ and according to remark 5.2, then $\tilde{p}_{F,\varepsilon}$ is a ε-projection and has propagation less than $(2j_{\varepsilon,F,i}+1)h_r(s)$. Moreover, for any continuous function $\phi_\varepsilon:\mathbb{R}\to\mathbb{R}$ such that $\phi_\varepsilon(t)=0$ for $t<\frac{1-\sqrt{1-4\varepsilon}}{2}$ and $\phi_\varepsilon(t)=1$ for $t>\frac{1+\sqrt{1-4\varepsilon}}{2}$, then $\phi_\varepsilon(\tilde{p}_\varepsilon)$ is a projector equivalent to e_F. Now fix an identification between $\mathcal{K}(H)$ and the closure of $\cup_{n\in\mathbb{N}}M_n(\mathbb{C})$ and consider q_n the rank $2n$ projector of $M_2(C(\Gamma/\Gamma_i,\mathcal{K}(H))\widetilde{\rtimes_{\max}}\Gamma))$ corresponding to the identity of $M_2(M_n(\mathbb{C}))$. For a K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$, let $n_{F,\varepsilon}$ be the smaller integer n such that $\|q_n\tilde{p}_{F,\varepsilon/2}q_n-\tilde{p}_{F,\varepsilon/2}+\left(\begin{smallmatrix}\mathrm{Id}_H-I_n & 0\\ 0 & 0\end{smallmatrix}\right)\|<\varepsilon/8$ and set $p_{F,\varepsilon}=q_{n_{F,\varepsilon}}\tilde{p}_{F,\varepsilon/2}q_{n_{F,\varepsilon}}$. Then $\|p_{F,\varepsilon}+\left(\begin{smallmatrix}\mathrm{Id}_H-I_n & 0\\ 0 & 0\end{smallmatrix}\right)-e'_F\|<\varepsilon/4$, and according to remark 5.2, $p_{F,\varepsilon}$ is as a summand of $p_{F,\varepsilon}+\left(\begin{smallmatrix}\mathrm{Id}_H-I_n & 0\\ 0 & 0\end{smallmatrix}\right)$

a ε-projector in $M_{2n_{F,\varepsilon}}(C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma)$. Moreover, we have

$$\mu_{\Gamma,C(\Gamma/\Gamma_i),\max,0}(x_F)=[\phi_\varepsilon(p_{F,\varepsilon})]-[I_{n_{F,\varepsilon}}].$$

In the odd case, if F is a K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ with propagation less than s, let us set using the notations of the discussion following proposition 5.6 $q_F=1/2(F_{r,i}+\mathrm{Id}_H)$. For ε in $(0,1/4)$ and r positive, let $l_{\varepsilon,F,i}$ be the smallest integer such that $\sum_{l=l_{\varepsilon,F,i}+1}^{+\infty}(\alpha_{r,i}+2))^l/l!<\varepsilon/(3\alpha_{r,i}+6)$. Let us define

$$u_{F,\varepsilon}=\sum_{l=0}^{l_{\varepsilon,F,i}}(2\imath\pi q_F)^l/l!-q_F\sum_{l=1}^{l_{\varepsilon,F,i}}(2\imath\pi)^l/l!.$$

It is straightforward to check that

- $u_{F,\varepsilon}-\mathrm{Id}_H$ is indeed an element of $C(\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{\max}\Gamma$ with propagation less than $l_{\varepsilon,F,i}h_r(s)$.
- $\|u_{F,\varepsilon}-e^{2\imath\pi q_F}\|\leq\varepsilon/3$.

In view of remark 5.2, $u_{F,\varepsilon}$ is a ε-unitary. Moreover, if x in $K_1^{\mathrm{top}}(\Gamma,C(\Gamma/\Gamma_i))$ comes from x_F in $KK_1^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))$ for F in some $\Psi^{\Gamma_i}(P_r(\Gamma))$, then $\mu_{\Gamma,C(\Gamma/\Gamma_i),\max,1}(x)$ is the class of $u_{F,\varepsilon}$ in $K_1(C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma)$.

Remark 5.8. *According to remark 5.5, for all ε in $(0,1/4)$, i positive integer and s positive real, then the sets $\{j_{\varepsilon,F,i};\ F$ K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ of propagation less than $s\}$ and $\{l_{\varepsilon,F,i};\ F$ K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ of propagation less than $s\}$ are bounded. Thereby, if $J_{\varepsilon,i,s}$ and $L_{\varepsilon,i,s}$ are respectively their suppremium, then for all K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ with propagation less than s, we get that $\widetilde{p_{F,\varepsilon}}$ and $p_{F,\varepsilon}$ have propagation less than $(2J_{\varepsilon,i,s}+1)h_r(s)$ and $u_{F,\varepsilon}$ has propagation less than $L_{\varepsilon,i,s}h_r(s)$*

With notations of lemma 2.2, let x be an element of $K_*^{\mathrm{top}}(\Gamma,B_\Gamma)$. Under the identification $KK_*^\Gamma(P_r(\Gamma),B_\Gamma)\cong\prod_{i\in\mathbb{N}}KK_*^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))$ of proposition 3.4, we can assume that x comes from an element $(x_{F_i})_{i\in\mathbb{N}}$, where

- F_i is K-cycle of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ for every positive integer i;
- there exists a real s such that F_i has propagation less than s for every positive integer i.

By viewing $B_\Gamma\rtimes_{\max}\Gamma=\ell^\infty(\cup_{i\in\mathbb{N}}\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{\max}\Gamma$ as an algebra of multipliers of $\oplus_{i\in\mathbb{N}}C(\Gamma/\Gamma_i,\mathcal{K}(H))\rtimes_{\max}\Gamma$, we see that $B_\Gamma\rtimes_{\max}\Gamma$ is indeed a closed $*$-subalgebra of $\prod_{i\in\mathbb{N}}(C(\Gamma/\Gamma_i,\mathcal{K}(H)\rtimes_{\max}\Gamma)$. In particular, with above notations, if x is even and since $\|\lambda(S_j)\|\leq\|\lambda_\Gamma(S_i)_{i\in\mathbb{N}}\|$ for all uniformally bounded family S_i in $\prod_{i\in\mathbb{N}}\Psi^{\Gamma_i}(\Gamma_i)$ with propagation uniformally bounded and all integer j, then the idempotent $(e_{F_i})_{i\in\mathbb{N}}$ and hence the projector $(e'_{F_i})_{i\in\mathbb{N}}$ belong to $M_2(B_\Gamma\rtimes_{\max}\Gamma)$ and moreover, $\mu_{\Gamma,B_\Gamma,\max,0}(x)=[(e'_{F_i})_{i\in\mathbb{N}}]-\left[\begin{pmatrix}\mathrm{Id}_H&0\\0&0\end{pmatrix}\right]$. Furthermore, the family of integers $(j_{\varepsilon,F_i,i})_{i\in\mathbb{N}}$ is bounded and hence $(\widetilde{p_{F_i,\varepsilon}})_{i\in\mathbb{N}}$ and $(p_{F_i,\varepsilon})_{i\in\mathbb{N}}$ are ε-projector in $M_2(B_\Gamma\rtimes_{\max}\Gamma)$. Since $\|\widetilde{p_{F_i,\varepsilon}}-e'_{F_i}\|<\varepsilon$ for all integer i, we finally get that $(\phi_\varepsilon(\widetilde{p_{F_i,\varepsilon}}))_{i\in\mathbb{N}}$ is a projector of $M_2(B_\Gamma\rtimes_{\max}\Gamma)$ homotopic to $(e'_{F_i})_{i\in\mathbb{N}}$ and hence

$$\begin{aligned}\mu_{\Gamma,B_\Gamma,\max,0}(x)&=[(\phi_\varepsilon(\widetilde{p_{F_i,\varepsilon}}))_{i\in\mathbb{N}}]-[(\mathrm{Id}_H)_{i\in\mathbb{N}}]\\&=[(\phi_\varepsilon(p_{F_i,\varepsilon}))_{i\in\mathbb{N}}]-[(I_{n_{F_i,\varepsilon}})_{i\in\mathbb{N}}].\end{aligned}$$

In the same way, in the odd case we get that $(u_{F_i})_{i\in\mathbb{N}}$ is a ε-unitary of $\widetilde{B_\Gamma\rtimes_{\max}\Gamma}$ and

$$\mu_{\Gamma,B_\Gamma,\max,1}(x)=[(u_{F_i})_{i\in\mathbb{N}}].$$

5.3. **Asymptotic statements.** For any integer i and any positive real r,r',s,s' and any ε in $(0,1/72)$, let us consider the following statements

$QI_0(i,r,r',s,\varepsilon)$: for any (even) K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$, then $(p_{F,\varepsilon},n_{F,\varepsilon})\sim_{18\varepsilon,s}(0,0)$ implies that x_F lies in the kernel of the homomorphism

$$KK_0^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))\longrightarrow KK_0^\Gamma(P_{r'}(\Gamma),C(\Gamma/\Gamma_i))$$

induced by the inclusion $P_r(\Gamma)\hookrightarrow P_{r'}(\Gamma)$.

$QI_1(i,r,r',s,\varepsilon)$: for any (odd) K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$, then $u_{F,\varepsilon}\sim_{\varepsilon,s}\mathrm{Id}_H$ implies that x_F lies in the kernel of the homomorphism

$$KK_1^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))\longrightarrow KK_1^\Gamma(P_{r'}(\Gamma),C(\Gamma/\Gamma_i))$$

induced by the inclusion $P_r(\Gamma)\hookrightarrow P_{r'}(\Gamma)$.

$QS_0(i,r,s,s',\varepsilon)$: For any ε-projector p in some $M_k(C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma)$ with propagation less than s, and any integer n, there exists a (even) K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ such that $(p_{F,\varepsilon},n_{F,\varepsilon})\sim_{18\varepsilon,s}(p,n)$.

$QS_1(i,r,s,s',\varepsilon)$: For any ε-unitary u in $C(\Gamma/\Gamma_i,\widetilde{\mathcal{K}(H)\rtimes_{\max}}\Gamma$ with propagation less than s, there exists a (odd) K-cycle F of $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ such that $u_{F,\varepsilon}\sim_{\varepsilon,s}u$.

Remark 5.9. *It is straightforward to check that if two ε-projectors are ε-closed, then they are homotopic as 18ε-projectors and hence conditions QI_0 and QS_0 do not depend on a particular choice of sequences of polynomial functions $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ used in the definition of $p_{F,\varepsilon}$. Moreover, replacing $n_{F,\varepsilon}$ by any integer n with $n\geq n_{F,\varepsilon}$ and $p_{F,\varepsilon}$ by $q_n\widetilde{p_{F,\varepsilon/2}}q_n$ does not either affect conditions QI_0 and QS_0.*

Theorem 5.10. *Let Γ be a finitely generated group residually finite with respect to a family $\Gamma_0\supset\Gamma_1\supset\ldots\Gamma_n\supset\ldots$ of normal finite index subgroups and let l be in $\{0,1\}$. The following statements are equivalent:*

(i) *For any positive real r the following condition holds : there is an ε in $(0,1/72)$ such that for any positive real s, there exists an integer j and a positive real r' for which $QI_l(i,r,r',s,\varepsilon)$ is true for all $i\geq j$.*
(ii) *The maximal coarse assembly map $\mu_{X(\Gamma),max,l}$ is injective.*

Proof. Let us give the proof in the even case, the odd one been quite similar (even simpler). In view of theorem 4.16, condition (ii) is equivalent to injectivity of $\mu_{\Gamma,A_\Gamma,\max,0}$. Let us prove that condition (i) implies injectivity of $\mu_{\Gamma,A_\Gamma,\max,0}$. According to remark 4.5, this amounts to prove that for any x in $K_0^{\mathrm{top}}(\Gamma,B_\Gamma)$, then the condition $\mu_{\Gamma,B_\Gamma,\max,0}(x)\in K_0(B_{\Gamma,0}\rtimes_{\max}\Gamma)$ implies that x belongs indeed to $K_0^{\mathrm{top}}(\Gamma,B_{\Gamma,0})$. Up to replace $\cup_{i\in\mathbb{N}}\Gamma/\Gamma_i$ by $\cup_{i\geq i_0}\Gamma/\Gamma_i$ for some integer i_0, we can actually assume that $\mu_{\Gamma,B_\Gamma,\max,0}(x)=0$. Suppose that x comes from an element $(x_{F_i})_{i\in\mathbb{N}}$ in some $KK_0^\Gamma(P_r(\Gamma),B_\Gamma)\cong\prod_{i\in\mathbb{N}}KK_0^\Gamma(P_r(\Gamma),C(\Gamma/\Gamma_i))$ for r positive real, where $(F_i)_{i\in\mathbb{N}}$ is a family of K-cycles in $\prod_{i\in\mathbb{N}}\Psi_1^{\Gamma_i}(P_r(\Gamma))$ with propagation uniformally bounded. Then there exist integers k and n and a projector homotopy in $M_{n+k+2}(\widetilde{B_\Gamma\rtimes_{\max}}\Gamma)$ between $\begin{pmatrix}(\phi_\varepsilon(p_{F_i,\varepsilon}))_{i\in\mathbb{N}} & 0\\ 0 & p_{n,k}\end{pmatrix}$ and $\begin{pmatrix}(I_{n_{F_i,\varepsilon}})_{i\in\mathbb{N}} & 0\\ 0 & p_{n,k+1}\end{pmatrix}$, where

$p_{n,k}$ is the projector $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ of $M_{n+k}(\widetilde{B_\Gamma \rtimes_{\max} \Gamma})$. Hence we can find a homotopy of 18ε-projector $P : [0,1] \to M_{n+k+2}(\widetilde{B_\Gamma \rtimes_{\max} \Gamma})$ between $\begin{pmatrix} (p_{F_i,\varepsilon})_{i\in\mathbb{N}} & 0 \\ 0 & p_{n,k} \end{pmatrix}$ and $\begin{pmatrix} (I_{n_{F_i,\varepsilon}})_{i\in\mathbb{N}} & 0 \\ 0 & p_{n,k+1} \end{pmatrix}$ such that for some s real, $P(t)$ has propagation less than s for every t in $[0,1]$. From this, by using for every integer j the projection

$$B_\Gamma \rtimes_{\max} \Gamma = \ell^\infty(\cup_{i\in\mathbb{N}}, \mathcal{K}(H)) \rtimes_{\max} \Gamma \to C(\Gamma/\Gamma_j, \mathcal{K}(H)) \rtimes_{\max} \Gamma$$

and proceeding as we did in section 5.2 to obtain $p_{F,\varepsilon}$ from $\widetilde{p_{F,\varepsilon/2}}$, we get that $(p_{F_j,\varepsilon}, n_{F_j,\varepsilon}) \sim_{18\varepsilon,s} (0,0)$. If ε is in $(0,1/72)$ and j is an integer satisfy the assumptions of the theorem for s as above, then there exists a r' such that x_{F_i} lies in the kernel of $KK_0^\Gamma(P_r(\Gamma), C(\Gamma/\Gamma_i)) \to KK_0^\Gamma(P_{r'}(\Gamma), C(\Gamma/\Gamma_i))$ for all integer $i \geq j$. This implies that x comes indeed from an element in $\bigoplus_{i=0}^{j-1} KK_0^\Gamma(P_{r'}(\Gamma), C(\Gamma/\Gamma_i))$ and hence belongs to $\bigoplus_{i\in\mathbb{N}} K_0^{\text{top}}(\Gamma, C(\Gamma/\Gamma_i))$.

Conversely, assume that for some positive real r, then for any ε in $(0,1/72)$ there exists a positive real s such that for every integer j and positive real r', there exists an integer i with $i \geq j$ for which $QI_0(i,r,r',s,\varepsilon)$ does not hold. Let us prove that $\mu_{\Gamma,A_\Gamma,\max,0}$ is not injective. If r is as above, let us fix ε in $(0,1/72)$ and $(r'_n)_{n\in\mathbb{N}}$ an increasing and unbounded sequence of positive reals. Then we can find an increasing sequence $(j_i)_{i\in\mathbb{N}}$ of integers, and for each integer i a K-cycle F_{j_i} in $\Psi_1^{\Gamma_{j_i}}(P_r(\Gamma))$ such that $(p_{F_{j_i},\varepsilon}, n_{F_{j_i},\varepsilon}) \sim_{18\varepsilon,s} (0,0)$ and $x_{F_{j_i}}$ does not belong to the kernel of $KK_0^\Gamma(P_r(\Gamma), C(\Gamma/\Gamma_{j_i})) \to KK_0^\Gamma(P_{r'_i}(\Gamma), C(\Gamma/\Gamma_{j_i}))$. By using a cut-off function for the action of Γ on $P_r(\Gamma)$ and in view of remark 5.9, we can actually assume that the family $(F_{j_i})_{i\in\mathbb{N}}$ as propagation uniformally bounded. Define for any integer k the K-cycle F_k of $\Psi_1^{\Gamma_k}(P_r(\Gamma))$ to be F_{j_i} if $k = j_i$ for some integer i and $\mathrm{Id}_{H_{P_r(\Gamma)}}$ otherwise. Let x be the element of $K_0^{\text{top}}(\Gamma, B_\Gamma)$ arising from $(x_{F_i})_{i\in\mathbb{N}}$. We clearly have $\mu_{\Gamma,B_\Gamma,\max,0}(x) = 0$ and x does not sit in $\bigoplus_{i\in\mathbb{N}} K_0^{\text{top}}(\Gamma, C(\Gamma/\Gamma_i))$. Hence, the image of x under the epimorphism $K_0^{\text{top}}(\Gamma, B_\Gamma) \to K_0^{\text{top}}(\Gamma, A_\Gamma)$ is a non vanishing element of the kernel of $\mu_{\Gamma,A_\Gamma,\max,0}$. □

Corollary 5.11. *If Γ is residually finite, finitely generated and uniformally embeddable into a Hilbert space, then Γ satisfies condition (i) of theorem 5.10.*

Remark 5.12. *As already mentionned, under the assumption of corollary 5.11, the reduced assembly map $\mu_{\Gamma,A_\Gamma,red,*}$ is injective. Moreover, the group Γ is K-exact and hence, in view of the proof of theorem 5.10 and if we consider conditions QI_0 and QI_1 with reduced assembly maps instead of maximal ones, we get in this setting an analogue of corollary 5.11 for Γ.*

Theorem 5.13. *Let Γ be a finitely generated group residually finite with respect to a family $\Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots$ of normal finite index subgroups and let l be in $\{0,1\}$. Assume that there exists ε in $(0,1/72)$ such that the following condition holds: for every positive real s, there exist positive real r and s' and an integer j such that $QS_l(i,r,s,s',\varepsilon)$ is true for all integer $i \geq j$. Then $\mu_{P_r(\Gamma),max,l}$ is surjective.*

Proof. As for injectivity, it is enought in view of theorem 4.16 to prove that $\mu_{\Gamma,A_\Gamma,\max,l}$ is surjective and according to remark 4.5, this amounts to prove that for any z in $K_l(B_\Gamma \rtimes_{\max} \Gamma)$, there exists an element x in $K_l^{\text{top}}(\Gamma, B_\Gamma)$ such that $\mu_{\Gamma,B_\Gamma,\max,l}(x) - z$ belongs to $K_l(B_{\Gamma,0} \rtimes_{\max} \Gamma)$. As before, we give the proof in the even case, the odd case being quite similar. Recall that we have fixed an identification $\mathcal{K}(H) \cong \overline{\cup_{n\in\mathbb{N}} M_n(\mathbb{C})}$. It is then straightforward to check that every element in $K_0(B_\Gamma \rtimes_{\max} \Gamma)$ can be written down as the difference of the classes of projector that belongs to $\left(\prod_{i\in\mathbb{N}} M_{n_i}(C(\Gamma/\Gamma_i))\right) \rtimes_{\max} \Gamma$ for some sequence of integers $(n_i)_{i\in\mathbb{N}}$. Let $p = (p_i)_{i\in\mathbb{N}}$ be a such projector viewed as an element of $\prod_{i\in\mathbb{N}} (M_{n_i}(C(\Gamma/\Gamma_i)) \rtimes_{\max} \Gamma)$. Let ε be as in the assumption of the theorem. We can indeed assume that there exists a positive real s and $(q_i)_{i\in\mathbb{N}}$ an ε-projector of $\left(\prod_{i\in\mathbb{N}} M_{n_i}(C(\Gamma/\Gamma_i)\right) \rtimes_{\max} \Gamma \subset \prod_{i\in\mathbb{N}} (M_{n_i}(C(\Gamma/\Gamma_i)) \rtimes_{\max} \Gamma)$ with propagation less than s and such that $p = (\phi_\varepsilon(q_i))_{i\in\mathbb{N}}$. Let r and s' be positive reals and let j be a positive integer such that $QS_0(i, r, s, s', \varepsilon)$ is true for every integer $i \geq j$, i.e there exists a K-cycle F_i in $\Psi_1^{\Gamma_i}(P_r(\Gamma))$ such that $(p_{F_i,\varepsilon}, n_{F_i,\varepsilon}) \sim_{18\varepsilon,s} (q_i, 0)$. By using a cut-off function for the action of Γ on $P_r(\Gamma)$ and in view of remark 5.2, we can actually assume that the family $(F_i)_{i\in\mathbb{N}}$ as propagation uniformally bounded. If we set $F_i = \mathrm{Id}_{H_{P_r(\Gamma)}}$ for every positive integer i with $i \leq j-1$ and then consider the element x of $K_0^{\mathrm{top}}(\Gamma, B_\Gamma)$ coming from $(x_{F_i})_{i\in\mathbb{N}} \in KK_0^\Gamma(P_r(\Gamma), B_\Gamma)$, we get that $\mu_{\Gamma,A_\Gamma,\max,0}(x) - [p]$ belongs to $\bigoplus_{i\in\mathbb{N}} K_0(C(\Gamma/\Gamma_i) \rtimes_{\max} \Gamma)$. □

As we shall see, up to a slight modification in the sequence of finite index normal subgroups in Γ, we get a converse result for theorem 5.13. This allows in particular to deal at least with group that satisfies the strong Baum-Connes conjecture. Let us set for any i integer $X_i(\Gamma) = \coprod_{j\geq i} \Gamma/\Gamma_j$ and $X^\infty(\Gamma) = \coprod_{i\in\mathbb{N}} X_i(\Gamma)$ provided with the action of Γ inherited by the action on $X(\Gamma)$. Let us equip $X^\infty(\Gamma)$ with a Γ-invariant metric d such that the restriction of d to each $X_i(\Gamma)$ coincides with the metric on $X(\Gamma)$ and $d(X_i(\Gamma), X_j(\Gamma)) \geq i + j$ for every integer i and j. Let us set $A_\Gamma^\infty = \ell^\infty(X^\infty(\Gamma), \mathcal{K}(H))/C_0(X^\infty(\Gamma), \mathcal{K}(H))$. The space $X^\infty(\Gamma)$ is indeed construct in the same way as $X(\Gamma)$ by considering the sequence of finite index normal subgroups $\Gamma_0 \supset \Gamma_1 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_2 \supset \Gamma_2 \supset \Gamma_3 \ldots$ and hence, according to theorem 4.17, we get

Proposition 5.14. *The following assertions are equivalent*

(i) *The maximal coarse assembly map*

$$\mu_{X^\infty(\Gamma),max,*} : \lim_r K_*(P_r(X^\infty(\Gamma)), \mathbb{C}) \to K_*(C_{max}(X^\infty(\Gamma)))$$

is surjective.

(ii) *the maximal assembly map*

$$\mu_{\Gamma,A_\Gamma^\infty,max,*} : K_*^{top}(\Gamma, A_\Gamma^\infty) \to K_*(A_\Gamma^\infty \rtimes_{max} \Gamma)$$

is surjective.

We have of course analogous statements for injectivity and isomorphism. We are now in position to give a weak converse result for theorem 5.13.

Theorem 5.15. *Let Γ be a finitely generated group, residually finite with respect to a family $\Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots$ of normal finite index subgroups and let l be in $\{0, 1\}$. Assume that the maximal coarse assembly map*

$$\mu_{X^\infty(\Gamma),max,l} : \lim_r K_l(P_r(X^\infty(\Gamma)), \mathbb{C}) \to K_l(C_{max}(X^\infty(\Gamma)))$$

is onto. Then there exists ε in $(0, 1/72)$ such that the following condition is satisfied: for every positive reals s, there exist positive reals r and s' and an integer j such that $QS_l(i, r, s, s', \varepsilon)$ is true for all integer $i \geq j$.

Proof. As before we give the prove for the even case. Assume that for all ε in $(0,1/72)$ there exists a positive real s such that for all positive reals r and s' and integer j there exists an integer i with $i\geq j$ for which $QS_0(i,r,s,s',\varepsilon)$ does not holds. In view of proposition 5.14, let us show that $\mu_{\Gamma,A_\Gamma^\infty,\max,0}$ is not onto. Let us fix ε in $(0,1/72)$ and $(s'_i)_{i\in\mathbb{N}}$ and $(r_i)_{i\in\mathbb{N}}$ increasing and unbounded sequences of positive reals. Then for each integer k, there exist an increasing sequence of integers $(j_i^k)_{i\in\mathbb{N}}$, an ε-projector $q_{j_i^k,k}$ with propagation less than s in some $M_{n_{j_i^k,k}}(C(\Gamma/\Gamma_{j_i^k})\rtimes_{\max}\Gamma)$ and an integer $m_{j_i^k,k}$ such that there is no K-cycle F in $\Psi^{\Gamma_{j_i^k}}(P_{r_k}(\Gamma))$ for which $(x_{F,\varepsilon},n_{F,\varepsilon})\sim_{18\varepsilon,s'_i}(q_{j_i^k,k},m_{j_i^k,k})$. For j and k integers such that $j\geq k$, define

- $q_{j,k}$ to be $q_{j_i^k,k}$ if $j=j_i^k$ for some integer i and $q_{j,k}=0$ otherwise.
- $m_{j,k}$ to be $m_{j_i^k,k}$ if $j=j_i^k$ for some integer i and $m_{j,k}=0$ otherwise.

Let us set $B_\Gamma^\infty=\ell^\infty(X^\infty(\Gamma),\mathcal{K}(H))$ and $B_{\Gamma,0}^\infty=C_0(X^\infty(\Gamma),\mathcal{K}(H))$. As in the proof of theorem 5.13 surjectivity will fail if there is no x in $K_0^{\mathrm{top}}(\Gamma,B_\Gamma^\infty)$ such that $\mu_{\Gamma,B_\Gamma^\infty,\max,0}(x)-[(\phi_\varepsilon(q_{j,k})_{j\in\mathbb{N},\,k\leq j}]+[(I_{m_{j,k}})_{j\in\mathbb{N},\,k\leq j}]$ lies in $K_0(B_{\Gamma,0}^\infty\rtimes_{\max}\Gamma)$. Suppose that such an x exists, coming from an element y in $KK_0^\Gamma(P_r(\Gamma),B_\Gamma^\infty)$ and let us fix k an integer such that $r_k\geq r$. Define then y_k as the image of y under the composition

$$KK^\Gamma(P_r(\Gamma),B_\Gamma^\infty)\to KK^\Gamma(P_{r_k}(\Gamma),B_\Gamma^\infty)\to KK^\Gamma(P_{r_k}(\Gamma),A_\Gamma^\infty),$$

where the first map is induced by the inclusion $P_r(\Gamma)\hookrightarrow P_{r_k}(\Gamma)$ and the second by the projection homomorphism $A_\Gamma^\infty\to\ell^\infty(X_k(\Gamma),\mathcal{K}(H))$. We can assume that $y_k=(x_{F_{j,k}})_{j\in\mathbb{N},\,k\leq j}$ where $(F_{j,k})_{j\in\mathbb{N},\,k\leq j}$ is a family of K-cycles in $\prod_{j\in\mathbb{N},\,k\leq j}\Psi_1^{\Gamma^j}(P_{r_k}(\Gamma))$ with propagation uniformally bounded. Then $(\phi_\varepsilon(p_{F_{j,k},\varepsilon}))_{j\in\mathbb{N},\,k\leq j}$ is a projector in $K_0(\ell^\infty(X_k(\Gamma))\rtimes_{\max}\Gamma)$ and by naturality of the assembly map, we get that $[(\phi_\varepsilon(p_{F_{j,k},\varepsilon}))_{j\in\mathbb{N},\,k\leq j}]-[(I_{n_{F_{j,k},\varepsilon}})_{j\in\mathbb{N},\,k\leq j}]-[(\phi_\varepsilon(q_{j,k}))_{j\in\mathbb{N},\,k\leq j}]+[(I_{m_{j,k}})_{j\in\mathbb{N},\,k\leq j}]$ lies in $K_0(C_0(X_k(\Gamma),\mathcal{K}(H))\rtimes_{\max}\Gamma)$. By taking k big enought, we can indeed assume that

$$[(\phi_\varepsilon(p_{F_{j,k},\varepsilon}))_{j\in\mathbb{N},\,k\leq j}]-[(I_{n_{F_{j,k},\varepsilon}})_{j\in\mathbb{N},\,k\leq j}]=[(\phi_\varepsilon(q_{j,k})_{j\in\mathbb{N},\,k\leq j}]-[(I_{m_{j,k}})_{j\in\mathbb{N},\,k\leq j}].$$

Thus, up to stabilisation, there is a homotopy in some $M_n\left(\ell^\infty(X_k(\Gamma),\mathcal{K}(H))\widetilde{\rtimes_{\max}\Gamma}\right)$ of 18ε-projectors $\begin{pmatrix}p_{F_{j,k},\varepsilon}&0\\0&I_{m_{j,k}}\end{pmatrix}_{j\in\mathbb{N},\,k\leq j}$ and $\begin{pmatrix}q_{j,k}&0\\0&I_{n_{F_{j,k},\varepsilon}}\end{pmatrix}_{j\in\mathbb{N},\,k\leq j}$ with finite propagation between. Proceeding as we did in section 5.2 to obtain $p_{F,\varepsilon}$ from $\tilde{p}_{F,\varepsilon/2}$, we actually get that there exists

- a positive real s';
- two sequences of integers $(i_j)_{j\in\mathbb{N},\,k\leq j}$ and $(i'_j)_{j\in\mathbb{N},\,k\leq j}$;
- a homotopy of 18ε-projector $P:[0,1]\to\left(\prod_{j\in\mathbb{N},\,k\leq j}M_{i'_j}(C(\Gamma/\Gamma_j))\right)\rtimes_{\max}\Gamma$ between $\begin{pmatrix}p_{F_{j,k},\varepsilon}&0\\0&I_{i_j+m_{j,k}}\end{pmatrix}_{j\in\mathbb{N},\,k\leq j}$ and $\begin{pmatrix}q_{j,k}&0\\0&I_{i_j+n_{F_{j,k},\varepsilon}}\end{pmatrix}_{j\in\mathbb{N},\,k\leq j}$ such that $P(t)$ has propagation less than s' for every t in $[0,1]$.

If i is an integer such that $s'_i\geq s'$ and $j_i^k\geq k$, then $(p_{F_{j_k^i,k},\varepsilon},n_{F_{j_k^i,k},\varepsilon})\sim_{18\varepsilon,s'_i}(q_{j_k^i,k},m_{j_i^k,k})$, which is in contradiction with the way we have chosen $(q_{j_i^k,k})_{(i,k)\in\mathbb{N}^2}$ and $(m_{j_i^k,k})_{(i,k)\in\mathbb{N}^2}$. □

Corollary 5.16. *Let Γ be a finitely generated group residually finite with respect to a family $\Gamma_0\supset\Gamma_1\supset\dots\Gamma_n\supset\dots$ of normal finite index subgroups and let l be in*

$\{0,1\}$. *If* Γ *satisfies the strong Baum-Connes conjecture (for example if* Γ *has the Haagerup property) then there exists* ε *in* $(0,1/72)$ *such that the following condition is satisfied: for every positive reals* s*, there exist positive real* r *and* s' *and an integer* j *such that* $QS_l(i,r,s,s',\varepsilon)$ *is true for all integer* $i \geq j$.

Since group satisfying the strong Baum-Connes conjecture are K-amenable [14], the same result holds if we replace in the definition of conditions QS_0 and QS_1 maximal assembly maps by the reduced one. More generally, in this setting, the analogue of the hypothesis of theorem 5.15 implies the surjectivity of the reduced Baum-Connes assembly map $\mu_{\Gamma,B_\Gamma^\infty,\mathrm{red},l} : K_l^{\mathrm{top}}(\Gamma,B_\Gamma^\infty) \to K_l(B_\Gamma^\infty \rtimes_{\max}\Gamma)$ for l in $\{0,1\}$ for K-exact groups (in particular for groups that embed uniformly in a Hilbert space).

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References

[1] Paul Baum and Alain Connes. K-theory for discrete groups. *Operator Algebras and Applications, Cambridge Univ. Press*, pages 1–20, 1989.

[2] Jérôme Chabert and Siegfried Echterhoff. Permanence properties of the Baum-Connes conjecture. *Doc. Math.*, 6:127–183 (electronic), 2001.

[3] Xiaoman Chen, Romain Tessera, Xianjin Wang, and Guoliang Yu. Metric sparsification and operator norm localization. *Adv. Math.*, 218:1496–1511, 2008.

[4] Guihua Gong, Qin Wang, and Guoliang Yu. Geometrization of the strong Novikov conjecture for residually finite groups. *J. Reine Angew. Math.*, 621:159–189, 2008.

[5] Misha Gromov. Spaces and questions. *Geom. Funct. Anal.*, Part I(2):118–161, 2000.

[6] Erik Guentner, Romain Tessera, and Guoliang Yu. Operator norm localization for linear groups and its applications to K-theory. *preprint*, 2008.

[7] Nigel Higson, Vincent Lafforgue, and Georges Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[8] Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[9] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.

[10] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[11] Hervé Oyono-Oyono. Baum-Connes conjecture and group actions on trees. *K-Theory*, 24(2):115–134, 2001.

[12] John Roe. Coarse cohomology and index theory on complete riemannian manifolds. *Mem. Amer. Math. Soc.*, 104(497), 1993.

[13] Georges Skandalis, Jean-Louis. Tu, and Guoliang Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.

[14] Jean-Louis Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.

[15] Kroum Tzanev. *K-théorie des C*-algèbres de groupes, Théorème de l'indice, Géométrie co-uniforme.* PhD thesis, Université Paris 7-Denis Diderot, 2000.

[16] N. E. Wegge-Olsen. *K-theory and C^*-algebras*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1993. A friendly approach.
[17] Guoliang Yu. The coarse baum-connes conjecture for spaces which admit a uniform embedding into hilbert space. *Invent. Math.*, 139:201–240, 2000.

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