On perturbed substochastic semigroups in abstract state spaces

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ABSTRACT. The object of this paper is twofold: In the first part, we unify and extend the recent developments on honesty theory of perturbed substochastic semigroups (on $L^1(\mu)$-spaces or noncommutative L^1 spaces) to general state spaces; this allows us to capture for instance a honesty theory in preduals of abstract von Neumann algebras or subspaces of duals of abstract C^*-algebras. In the second part of the paper, we provide another honesty theory (a semigroup-perturbation approach) independent of the previous resolvent-perturbation approach and show the equivalence of the two approaches. This second viewpoint on honesty is new even in $L^1(\mu)$ spaces. Several fine properties of Dyson-Phillips expansions are given and a classical generation theorem by T. Kato is revisited.

KEYWORDS: Substochastic semigroups; additive norm; total mass carried by a trajectory; Dyson-Phillips expansion.

1. INTRODUCTION

In his famous paper on Kolmogorov's differential equations (for Markov processes with denumerable states) T. Kato [18] introduced the main tools for dealing with positive unbounded perturbations $\mathcal{B}$ of generators $\mathcal{A}$ of substochastic semigroups in $\ell^1(\mathbb{N})$ provided that a suitable dissipation on the positive cone is satisfied. Among other things, he showed that there exists a unique extension $\mathcal{G} \supset \mathcal{B}+\mathcal{A}$ which generates a substochastic semigroup and characterized the closure property $\mathcal{G} = \overline{\mathcal{B}+\mathcal{A}}$ by the fact that $[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^n \to 0$ strongly as $n \to +\infty$ (in general, $\mathcal{G}$ may be a proper extension of $\overline{\mathcal{B}+\mathcal{A}}$). We note that for "formally conservative" equations, such as Kolmogorov's differential equations, the property $\mathcal{G} = \overline{\mathcal{B}+\mathcal{A}}$ is essential (i.e. necessary and sufficient) to assert that the corresponding semigroup is *mass-preserving* on the positive cone. Finally, T. Kato [18] pointed out that his formalism is adapted to general AL–spaces, i.e. Banach lattices $\mathfrak{X}$ whose norm is additive on the positive cone $\mathfrak{X}_+$ i.e. $\|x+y\| = \|x\| + \|y\|\,, \quad x, y \in \mathfrak{X}_+$. Actually, even the lattice assumption is not essential since Kato's ideas were applied by E. B. Davies [11] to quantum dynamical semigroups in the real Banach space of self-adjoint trace class operators; in this case, the closure property $\mathcal{G} = \overline{\mathcal{B}+\mathcal{A}}$ is essential to assert that the corresponding semigroup is *trace-preserving* on the positive cone.

By the end of the 1980's, Kato's paper [18] was revisited by means of Miyadera perturbations in AL–spaces [34, 35, 2] and new functional analytic developments followed also in the 2000's [5, 4, 15] which are known nowadays as the honesty theory of perturbed substochastic semigroups in $L^1(\mu)$ spaces [6, Chapter 6]. Of course, this theory is motivated by various applications to kinetic theory, fragmentation equations, birth-and-death equations and so on; see [6] and references therein. We note also that the analysis of piecewise deterministic Markov processes is nicely related to honesty theory in L^1 spaces [33] (see also [19] for related topics). On the other hand, in a noncommutative context, there exists also an important literature (relying on Kato's paper [18] or some dual version) on quantum dynamical semigroups, e.g. [11, 26, 8, 9, 14, 16, 17, 30]; such semigroups acting on spaces of operators arise in the theory of open quantum systems as models of irreversible (albeit conservative) quantum dynamics. We mention that quantum dynamical semigroups enjoy the complete positivity property (a stronger property than the fact to leave invariant the positive cone) which gives their generators a special structure (see e.g. [14]).

More recently, in [23], the honesty theory of perturbed substochastic semigroups in $L^1(\mu)$ spaces has been improved and extended in different directions while a noncommutative version of [23] was given in [22]. The first goal of the present paper is to provide a general theory in abstract *state spaces* (i.e. real ordered Banach spaces such that the norm is additive on the positive cone) which covers both [23] and [22]. The interest of this abstract approach is not simply motivated by a unified presentation of [23] and [22]: it provides us with an intrinsic treatment of honesty theory in much more general spaces covering in particular preduals of abstract von Neumann algebras or more generally subspaces of duals of abstract C^*-algebras (see for example [32, 24] on measure-valued generalization of Kolmogorov equations on abstract measurable spaces). We refer to E. B. Davies [10, p. 30-31] for the relevance of the concept of abstract state spaces in probability theory, quantum statistical mechanics, etc. For its most part, the general theory we give follows closely [23, 22] but we provide also new informations on the structure of the set of honest trajectories in the Banach space of bounded measures on a measurable space and in the Banach space of trace class operators on a Hilbert space. The second goal of this paper is to provide another approach of honesty theory. This alternative approach of honesty relies on Dyson-Phillips expansions (in contrast to the previous resolvent approach) and is new even in $L^1(\mu)$ spaces. To this end, we give several fine properties of Dyson-Phillips expansions. We also revisit a classical generation theorem by T. Kato [18]. Finally, this alternative viewpoint on honesty presents the great advantage of being adaptable to *nonautonomous* problems [3]

We recall briefly some properties of the class of Banach spaces we shall deal with in this paper (more information on general real ordered Banach space can be recovered from [25, 7]). In all this paper, we shall assume that $\mathfrak{X}$ is a real ordered Banach space with a

generating positive cone $\mathfrak{X}_+$ (i.e. $\mathfrak{X} = \mathfrak{X}_+ - \mathfrak{X}_+$) on which the norm is additive, i.e.

$$\|u+v\| = \|u\| + \|v\| \qquad\qquad u, v \in \mathfrak{X}_+.$$

The additivity of the norm implies that the norm is monotone, i.e.

$$0 \leqslant u \leqslant v \Longrightarrow \|u\| \leqslant \|v\|.$$

In particular, the cone $\mathfrak{X}_+$ is normal [7, Proposition 1.2.1]. It follows easily that any bounded monotone sequence of $\mathfrak{X}_+$ is convergent. A property playing an important role in this paper is the existence of a linear positive functional $\boldsymbol{\Psi}$ on $\mathfrak{X}$ which coincides with the norm on the positive cone (see e.g. [10, p. 30]), i.e.

$$\boldsymbol{\Psi} \in \mathfrak{X}_+^\star, \qquad \langle \boldsymbol{\Psi}, u\rangle = \|u\|, \qquad u \in \mathfrak{X}_+ \tag{1.1}$$

Note that $\|\boldsymbol{\Psi}\| = 1$. Indeed, given $u \in \mathfrak{X}$, one has $u = u_1 - u_2 \in \mathfrak{X}$ with $u_i \in \mathfrak{X}_+$ $(i = 1, 2)$ and $|\langle \boldsymbol{\Psi}, u\rangle| = |\|u_1\| - \|u_2\|| \leqslant \|u\|$. This proves that $\|\boldsymbol{\Psi}\| \leqslant 1$ and the equality sign follows from (1.1). We note also that by a Baire category argument there exists a constant $M > 0$ such that each $u \in \mathfrak{X}$ has a decomposition $u = u_1 - u_2$ where $u_i \in \mathfrak{X}_+$ and $\|u_i\| \leqslant M\|u\|$ $(i = 1, 2)$; i.e. the positive cone $\mathfrak{X}_+$ is non-flat, see [25, Proposition 19.1]. We recall that a C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ of bounded linear operators on $\mathfrak{X}$ is called substochastic (resp. stochastic) if $\mathcal{T}(t)$ is positive (i.e. leaves $\mathfrak{X}_+$ invariant for any $t \geqslant 0$) and $\|\mathcal{T}(t)u\| \leqslant \|u\|$ (resp. $\|\mathcal{T}(t)u\| = \|u\|$) for all $u \in \mathfrak{X}_+$ and $t \geqslant 0$. It is not difficult to see that a positive C_0-semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ with generator $\mathcal{A}$ is substochastic (resp. stochastic) if and only if $\langle \boldsymbol{\Psi}, \mathcal{A}u\rangle \leqslant 0$ (resp. $\langle \boldsymbol{\Psi}, \mathcal{A}u\rangle = 0$) for all $u \in \mathscr{D}(\mathcal{A})_+ = \mathscr{D}(\mathcal{A}) \cap \mathfrak{X}_+$. Because of a lack (a priori) of a lattice structure, $(\mathcal{T}(t))_{t\geqslant 0}$ need not be a contraction semigroup. However, one easily sees that $\|\mathcal{T}(t)\| \leqslant 2M$ for all $t \geqslant 0$; in particular, its type is nonpositive.

The general structure of the paper is the following: our general setting is an abstract state space $\mathfrak{X}$, a substochastic C_0-semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ on $\mathfrak{X}$ with generator $\mathcal{A}$ and a linear operator $\mathcal{B} : \mathscr{D}(\mathcal{A}) \to \mathfrak{X}$ which is assumed to be positive (i.e. $\mathcal{B} : \mathscr{D}(\mathcal{A}) \cap \mathfrak{X}_+ \to \mathfrak{X}_+$) and such that

$$\langle \boldsymbol{\Psi}, \mathcal{A}u + \mathcal{B}u\rangle \leqslant 0 \qquad\qquad u \in \mathscr{D}(\mathcal{A}) \cap \mathfrak{X}_+.$$

In Section 2, we show that there exists a unique minimal substochastic C_0-semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ generated by an extension $\mathcal{G}$ of $\mathcal{A} + \mathcal{B}$. This result was first given by T. Kato [18] under a lattice assumption on $\mathfrak{X}$. Our purpose here is simply to show (by following essentially Kato's ideas) that the lattice assumption is actually unnecessary. We note that this result has been proved differently by means of Miyadera perturbations [32] or by using Desch's theorem [21]. We also show that the corresponding semigroup is given by a (strongly convergent) Dyson-Phillips expansion

$$\mathcal{V}(t)u = \sum_{n=0}^{\infty} \mathcal{V}_n(t)u$$

without using the theory of Miyadera perturbations. It turns out that the resolvent of $\mathcal{G}$ is given by the strongly convergent series

$$(\lambda-\mathcal{G})^{-1}u=\sum_{n=0}^{\infty}(\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u, \qquad \lambda>0.$$

This series (which does *not* converge a priori in operator norm) is the corner-stone of a general honesty theory of the C_0-semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ given in Section 3 in the spirit of the recent results [22, 23]. Besides the functional

$$\mathfrak{a}_0 \,:\, u\in\mathscr{D}(\mathcal{G})\to -\langle\mathbf{\Psi},\mathcal{G}u\rangle$$

and its restriction $\mathfrak{a}$ to $\mathscr{D}(\mathcal{A})$ we build up and study another functional

$$\overline{\mathfrak{a}} \,:\, u\in\mathscr{D}(\mathcal{G})\to\mathbb{R}$$

which has the properties that $\overline{\mathfrak{a}}_{|\mathscr{D}(\mathcal{A})}=\mathfrak{a}$ and $\overline{\mathfrak{a}}\leqslant\mathfrak{a}_0$ on $\mathscr{D}(\mathcal{G})_+=\mathscr{D}(\mathcal{G})\cap\mathfrak{X}_+$. The trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ emanating from $u\in\mathfrak{X}_+$ is said to be *honest* if

$$\|\mathcal{V}(t)u\|=\|u\|-\overline{\mathfrak{a}}\left(\int_0^t\mathcal{V}(r)u\mathrm{d}r\right), \qquad \forall t\geqslant 0$$

or equivalently if

$$\overline{\mathfrak{a}}\left(\int_0^t\mathcal{V}(r)u\mathrm{d}r\right)=\mathfrak{a}_0\left(\int_0^t\mathcal{V}(r)u\mathrm{d}r\right) \qquad \forall t\geqslant 0.$$

Various characterization of honesty are given; in particular we show that $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest if and only if $\lim_{n\to\infty}\|\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\|=0$ which is equivalent to $(\lambda-\mathcal{G})^{-1}u\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$. Under the "conservativity" assumption

$$\langle\mathbf{\Psi},\mathcal{A}u+\mathcal{B}u\rangle=0, \qquad \forall u\in\mathscr{D}(\mathcal{A}),$$

the mass-preservation in time (i.e. $\|\mathcal{V}(t)u\|=\|u\|$ for any $t\geqslant 0$) holds if and only if the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest. The semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is said to be honest if all trajectories are honest. We show that the honesty of $(\mathcal{V}(t))_{t\geqslant 0}$ is equivalent to the identity $\overline{\mathfrak{a}}=\mathfrak{a}_0$ or to the closure property $\mathcal{G}=\overline{\mathcal{A}+\mathcal{B}}$. Actually, we extend most of the results of [22, 23]; in particular we show that the set $\mathcal{H}$ of initial data giving rise to a honest trajectory is a closed hereditary subcone of $\mathfrak{X}_+$ and provide a description of the order ideal $\mathcal{H}-\mathcal{H}$ (induced by it) in the case where $\mathfrak{X}$ is either the Banach space of self-adjoint trace class operators on a Hilbert space or the Banach space of bounded signed measures on a measurable space.

In Section 4, the Dyson-Phillips expansion is the corner-stone of *another* honesty theory of trajectories. To this end, we build up and study a new functional

$$\widehat{\mathfrak{a}}:u\in\mathscr{D}(\mathcal{G})\to\mathbb{R}$$

and show in particular that $\widehat{\mathfrak{a}}_{|\mathscr{D}(\mathcal{A})} = \mathfrak{a}$ and $\widehat{\mathfrak{a}} \leqslant \mathfrak{a}_0$ on $\mathscr{D}(\mathcal{G})_+$. To distinguish a priori the second notion of honesty from the previous one, we say that a trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ emanating from $u \in \mathcal{X}_+$ is mild honest if

$$\|\mathcal{V}(t)u\| = \|u\| - \widehat{\mathfrak{a}}(\int_0^t \mathcal{V}(r)u\mathrm{d}r), \;\; t \geqslant 0.$$

Various characterizations of mild honesty are given; in particular we show that $(\mathcal{V}(t)u)_{t\geqslant 0}$ is mild honest if and only if $\int_0^t \mathcal{V}(r)u\mathrm{d}r \in \mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$ or if and only if the integral $\mathcal{B}\int_0^t \mathcal{V}_n(r)u\mathrm{d}r$ converges strongly to 0 as $n \to \infty$. This mild honesty is based on several new fine properties of the operators $\mathcal{V}_n$. Finally we prove that the functionals $\widehat{\mathfrak{a}}$ and $\overline{\mathfrak{a}}$ coincide showing thus that the notions of honesty and mild honesty are actually equivalent. Moreover, the equivalence of the two viewpoints on honesty theory provides us with nontrivial additional results. As we already said it, a honesty theory in terms of Dyson-Phillips expansions suggests a convenient tool for the study of nonautonomous problems [3].

2. Kato's generation theorem and first consequences

2.1. **Classical Kato's Theorem revisited.** Let $(\mathcal{U}(t))_{t\geqslant 0}$ be a substochastic C_0-semigroup on $\mathfrak{X}$ with generator $\mathcal{A}$. Kato's generation theorem [18] provides a useful sufficient condition ensuring that some extension of $(\mathcal{A}+\mathcal{B}, \mathscr{D}(\mathcal{A}))$ generates a substochastic C_0-semigroup on $\mathfrak{X}$:

Theorem 2.1. *Let $(\mathcal{U}(t))_{t\geqslant 0}$ be a substochastic C_0-semigroup on $\mathfrak{X}$ with generator $\mathcal{A}$. Let $\mathcal{B} : \mathscr{D}(\mathcal{A}) \to \mathfrak{X}$ be a positive linear operator satisfying:*

$$\langle \mathbf{\Psi}, (\mathcal{A}+\mathcal{B})u\rangle \leqslant 0, \qquad \forall u \in \mathscr{D}(\mathcal{A})_+ := \mathscr{D}(\mathcal{A}) \cap \mathfrak{X}_+. \tag{2.1}$$

Then, there exists an extension $\mathcal{G}$ of $(\mathcal{A}+\mathcal{B}, \mathscr{D}(\mathcal{A}))$ that generates a substochastic C_0-semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ on X. Moreover, for any $\lambda > 0$, the resolvent of $\mathcal{G}$ is given by

$$(\lambda - \mathcal{G})^{-1}u = \lim_{n\to\infty} (\lambda - \mathcal{A})^{-1} \sum_{k=0}^{n} \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^k u, \qquad u \in \mathfrak{X}. \tag{2.2}$$

Finally, $(\mathcal{V}(t))_{t\geqslant 0}$ is the smallest substochastic C_0-semigroup whose generator is an extension of $(\mathcal{A}+\mathcal{B}, \mathscr{D}(\mathcal{A}))$.

The general strategy to prove such a result consists in two steps: show that

$$\mathcal{G}_r = \mathcal{A} + r\mathcal{B}, \qquad \mathscr{D}(\mathcal{G}_r) = \mathscr{D}(\mathcal{A})$$

is a generator of a substochastic C_0-semigroup for any $0 < r < 1$ and *then* use a monotonic convergence theorem by letting $r \nearrow 1$. The first step can be dealt with by means of three different arguments: a direct approach via Hille-Yosida estimates; the use of

Miyadera perturbation theory [32] or simply the use of Desch theorem [21]. We revisit here the direct approach via Hille-Yosida estimates by T. Kato [18].

Proof. Our proof is inspired by the original one of T. Kato [18] that we adapt here to the more general situation we are dealing with (recall in particular that substochastic semigroups are contracting *only* on $\mathfrak{X}_+$). The proof consists in several steps.
• *Construction of* $(\mathcal{V}(t))_{t\geqslant 0}$*:* For any $\lambda>0$, set $\mathcal{J}(\lambda)=\mathcal{B}(\lambda-\mathcal{A})^{-1}$. Clearly, $\mathcal{J}(\lambda)$ is a bounded linear positive operator on $\mathfrak{X}$ and (2.1) implies that

$$\begin{aligned}\|\mathcal{J}(\lambda)u\| = \langle \mathbf{\Psi}, \mathcal{J}(\lambda)u\rangle &\leqslant -\left\langle \mathbf{\Psi}, \mathcal{A}(\lambda-\mathcal{A})^{-1}u\right\rangle \\ &\leqslant \|u\| - \lambda\|(\lambda-\mathcal{A})^{-1}u\| \leqslant \|u\|, \qquad \text{for any } u\in\mathfrak{X}_+ \text{ and any } \lambda>0.\end{aligned}$$

Iterating such an inequality leads to

$$\|(\mathcal{J}(\lambda))^n u\| \leqslant \|u\|, \qquad \text{for any } u\in\mathfrak{X}_+ \text{ and any } \lambda>0,\ n\in\mathbb{N}$$

which implies that

$$\|(\mathcal{J}(\lambda))^n\| \leqslant 2M, \qquad \forall n\in\mathbb{N},\ \lambda>0$$

where we recall (see the introduction) that $M>0$ is a positive constant such that any $u\in\mathfrak{X}$ admits a decomposition $u=u_1-u_2$ with $u_i\in\mathfrak{X}_+$ and $\|u_i\|\leqslant M\|u\|$ $(i=1,2)$. In particular, the spectral radius $r_\sigma(\mathcal{J}(\lambda))$ of the bounded operator $\mathcal{J}(\lambda)$ is such that

$$r_\sigma(\mathcal{J}(\lambda)) \leqslant 1, \qquad \forall \lambda>0. \tag{2.3}$$

Moreover, the resolvent formula shows that $0\leqslant\mathcal{J}(\mu)\leqslant\mathcal{J}(\lambda)$ for any $0<\lambda<\mu$. Now, for any $0\leqslant r<1$, let us define $\mathcal{G}_r$ as

$$\mathcal{G}_r=\mathcal{A}+r\mathcal{B}, \qquad \mathscr{D}(\mathcal{G}_r)=\mathscr{D}(\mathcal{A}).$$

Eq. (2.3) implies that $(\lambda-\mathcal{G}_r)$ is invertible for any $\lambda>0$ with

$$(\lambda-\mathcal{G}_r)^{-1}=(\lambda-\mathcal{A})^{-1}\sum_{n=0}^{\infty} r^n\left[\mathcal{J}(\lambda)\right]^n, \qquad 0\leqslant r<1 \tag{2.4}$$

where the series converges in $\mathscr{B}(\mathfrak{X})$. For any fixed $f\in\mathfrak{X}_+$, set $v=(\lambda-\mathcal{A})^{-1}f$, $\lambda>0$. One has $v\in\mathscr{D}(\mathcal{A})_+$ and

$$\begin{aligned}\|(\lambda-\mathcal{G}_r)v\| = \|(\lambda-\mathcal{A}-r\mathcal{B})v\| &\geqslant \|(\lambda-\mathcal{A})v\| - r\|\mathcal{B}v\| \\ &= \lambda\langle\mathbf{\Psi},v\rangle - \langle\mathbf{\Psi},\mathcal{A}v\rangle - r\langle\mathbf{\Psi},\mathcal{B}v\rangle \geqslant \lambda\|v\|.\end{aligned}$$

Now given $u\in\mathfrak{X}_+$ and applying the above reasoning with $f=\sum_{n=0}^{\infty} r^n\left[\mathcal{J}(\lambda)\right]^n u$, we deduce from (2.4) that

$$\|(\lambda-\mathcal{G}_r)^{-1}u\| \leqslant \lambda^{-1}\|u\|, \qquad \text{for any } u\in\mathfrak{X}_+. \tag{2.5}$$

Iterating this relation, we see that

$$\|\left[(\lambda-\mathcal{G}_r)^{-1}\right]^n u\| \leqslant \lambda^{-n}\|u\|, \qquad \text{for any } u\in\mathfrak{X}_+ \text{ and any } n\in\mathbb{N}.$$

Then, since $\mathfrak{X}_+$ is non flat, such an estimate extends to the whole space $\mathfrak{X}$ leading to

$$\| \left[(\lambda - \mathcal{G}_r)^{-1}\right]^n \| \leqslant \frac{2M}{\lambda^n}, \qquad \forall \lambda > 0,\ n \in \mathbb{N},$$

and one deduces from Hille-Yosida Theorem that, for any $0 \leqslant r < 1$, $(\mathcal{G}_r, \mathscr{D}(\mathcal{A}))$ generates a C_0-semigroup $(\mathcal{S}_r(t))_{t\geqslant 0}$ in $\mathfrak{X}$. Since $(\lambda - \mathcal{G}_r)^{-1}$ is positive and because of (2.5), $(\mathcal{S}_r(t))_{t\geqslant 0}$ is a substochastic C_0-semigroup in $\mathfrak{X}$. Moreover, the mapping $r \mapsto (\lambda - \mathcal{G}_r)^{-1}u$ is nondecreasing for any fixed $\lambda > 0$ and any $u \in \mathfrak{X}_+$ and one sees from the exponential formula

$$\mathcal{S}_r(t)u = \lim_{n\to\infty} \frac{n}{t}\left[\left(\frac{n}{t} - \mathcal{G}_r\right)^{-1}\right]^n u, \qquad u \in \mathfrak{X}_+,$$

that the mapping $r \in [0,1) \longmapsto \mathcal{S}_r(t)u$ is also nondecreasing for any fixed $t \geqslant 0$ and any $u \in \mathfrak{X}_+$. Since $\sup_{0\leqslant r<1} \|\mathcal{S}_r(t)\| \leqslant 2M$ for any $t \geqslant 0$ and any bounded monotone sequence of $\mathfrak{X}_+$ is convergent, one gets that $\mathcal{S}_r(t)$ converges strongly to some operator $\mathcal{V}(t)$ for any fixed $t \geqslant 0$ as $r \to 1$. Obviously, $\mathcal{V}(t)$ is a positive contraction on $\mathfrak{X}_+$ with $\mathcal{S}_r(t) \leqslant \mathcal{V}(t)$ for any $0 \leqslant r < 1$ and any $t \geqslant 0$.

• *$(\mathcal{V}(t))_{t\geqslant 0}$ is a C_0-semigroup on $\mathfrak{X}$.* Since $\mathcal{S}_r(t+s) = \mathcal{S}_r(t)\mathcal{S}_r(s)$ for any $t, s \geqslant 0$ and any $0 \leqslant r < 1$, one has, at the limit, $\mathcal{V}(t+s) = \mathcal{V}(t)\mathcal{V}(s)$, $\forall t, s \geqslant 0$. Moreover, $\mathcal{V}(0) = Id$. To prove that $(\mathcal{V}(t))_{t\geqslant 0}$ is a C_0-semigroup on $\mathfrak{X}$, it is enough to prove that $t \geqslant 0 \mapsto \mathcal{V}(t)u$ is continuous at $t = 0$ for any $u \in \mathfrak{X}$. Let us fix $\varepsilon > 0$ and $u \in \mathfrak{X}_+$. Since $(\mathcal{U}(t))_{t\geqslant 0}$ is a strongly continuous, there exists $\delta > 0$ such that $\|\mathcal{U}(t)u - u\| < \varepsilon$ for any $0 \leqslant t \leqslant \delta$. For such a t, we see that, for any $r \in [0,1)$, since $\mathcal{S}_r(t) \geqslant \mathcal{U}(t)$, one has

$$\begin{aligned}\|\mathcal{S}_r(t)u - \mathcal{U}(t)u\| &= \langle \mathbf{\Psi}, \mathcal{S}_r(t)u - \mathcal{U}(t)u\rangle = \langle \mathbf{\Psi}, \mathcal{S}_r(t)u\rangle - \langle \mathbf{\Psi}, \mathcal{U}(t)u\rangle \\ &\leqslant \|u\| - \|\mathcal{U}(t)u\| \leqslant \|u - \mathcal{U}(t)u\| < \varepsilon.\end{aligned}$$

One deduces from this estimate that

$$\|\mathcal{S}_r(t)u - u\| \leqslant \|\mathcal{S}_r(t)u - \mathcal{U}(t)u\| + \|\mathcal{U}(t) - u\| \leqslant 2\varepsilon, \qquad \forall 0 < t < \delta.$$

The important fact is that such an estimate is uniform with respect to $r \in [0,1)$ so that, letting $r \nearrow 1$, one deduces that $\|\mathcal{V}(t)u - u\| \leqslant 2\varepsilon$ for any $0 < t < \delta$. This shows that $\lim_{t\to 0} \mathcal{V}(t)u = u$ for any $u \in \mathfrak{X}_+$ and, by linearity, the result is true for any $u \in \mathfrak{X}$ which proves that $\mathcal{V}(t)$ is strongly continuous at $t = 0$. We denote by $\mathcal{G}$ the generator of $(\mathcal{V}(t))_{t\geqslant 0}$. Clearly, $]0,\infty[\subset \varrho(\mathcal{G})$ and

$$(\lambda - \mathcal{G})^{-1} \text{ is positive}, \quad \|(\lambda - \mathcal{G})^{-1}u\| \leqslant \|u\|/\lambda, \qquad u \in \mathfrak{X}_+.$$

Note that, since $\mathcal{S}_r(t) \leqslant \mathcal{V}(t)$ for any $t \geqslant 0$ and any $r \in [0,1)$, one also has $(\lambda - \mathcal{G}_r)^{-1} \leqslant (\lambda - \mathcal{G})^{-1}$ for any $r \in [0,1)$ and any $\lambda > 0$.

• *$(\lambda - \mathcal{G}_r)^{-1}$ converges strongly to $(\lambda - \mathcal{G})^{-1}$ as $r \to 1$.* Since for any $u \in \mathfrak{X}_+$ the mapping $r \mapsto \mathcal{S}_r(t)u$ is nondecreasing, by Dini's Theorem one has for any $T > 0$ and any $u \in \mathfrak{X}_+$:

$$\lim_{r\to 1} \sup_{0\leqslant t\leqslant T} \|\mathcal{S}_r(t)u - \mathcal{V}(t)u\| = 0. \tag{2.6}$$

Now, writing

$$(\lambda-\mathcal{G})^{-1}u-(\lambda-\mathcal{G}_r)^{-1}u=\int_0^T \exp(-\lambda t)\,(\mathcal{V}(t)u-\mathcal{S}_r(t)u)\,\mathrm{d}t+ \int_T^\infty \exp(-\lambda t)\,(\mathcal{V}(t)u-\mathcal{S}_r(t)u)\,\mathrm{d}t, \qquad \forall T\geqslant 0,$$

one sees from the uniform convergence that the first integral converges to 0 as $r\nearrow 1$ for any $T>0$ while the uniform bound $\sup_{t\geqslant 0}\|\mathcal{S}_r(t)u-\mathcal{V}(t)u\|\leqslant 2\|u\|$ allows us to let $T\to\infty$ in the second integral leading to

$$\lim_{r\to 1}\|(\lambda-\mathcal{G}_r)^{-1}u-(\lambda-\mathcal{G})^{-1}u\|=0, \qquad \forall\lambda>0, u\in\mathfrak{X}.$$

• *Proof of Eq.* (2.2). Let us fix $\lambda>0$. From Eq. (2.4) and the fact that $0\leqslant(\lambda-\mathcal{G}_r)^{-1}\leqslant(\lambda-\mathcal{G})^{-1}$ for any $0\leqslant r<1$, one has $\mathcal{R}_r^{(n)}\leqslant(\lambda-\mathcal{G}_r)^{-1}\leqslant(\lambda-\mathcal{G})^{-1}$, for any $n\geqslant 1$ where $\mathcal{R}_r^{(n)}(\lambda)=(\lambda-\mathcal{A})^{-1}\sum_{k=0}^n r^k\,[\mathcal{J}(\lambda)]^k$. Letting $r\nearrow 1$, one gets

$$\mathcal{R}^{(n)}(\lambda):=(\lambda-\mathcal{A})^{-1}\sum_{k=0}^n[\mathcal{J}(\lambda)]^k\leqslant(\lambda-\mathcal{G})^{-1}, \qquad \forall n\geqslant 1.$$

Since the sequence $\left(\mathcal{R}^{(n)}(\lambda)\right)_n$ is nondecreasing, the strong limit

$$\mathcal{R}(\lambda):=\mathrm{s}-\lim_{n\to\infty}\mathcal{R}^{(n)}(\lambda)$$

exists and $\mathcal{R}(\lambda)\leqslant(\lambda-\mathcal{G})^{-1}$. We also have $\mathcal{R}_r^{(n)}(\lambda)\leqslant\mathcal{R}^{(n)}(\lambda)\leqslant\mathcal{R}(\lambda)$ for all $0\leqslant r<1$ and $n\geqslant 1$. Hence, $(\lambda-\mathcal{G}_r)^{-1}=\mathrm{s}-\lim_{n\to\infty}\mathcal{R}_r^{(n)}(\lambda)\leqslant\mathcal{R}(\lambda)$ and $(\lambda-\mathcal{G})^{-1}=\mathrm{s}-\lim_{r\to 1}(\lambda-\mathcal{G}_r)^{-1}\leqslant\mathcal{R}(\lambda)$. This proves finally that $\mathcal{R}(\lambda)=(\lambda-\mathcal{G})^{-1}$ and Eq. (2.2) is proved.

• *$\mathcal{G}$ is a closed extension of $\mathcal{A}+\mathcal{B}$.* With the notation of the previous item, since $\mathcal{J}(\lambda)=\mathcal{B}(\lambda-\mathcal{A})^{-1}$, one has

$$\begin{aligned}\mathcal{R}^{(n)}(\lambda)&=(\lambda-\mathcal{A})^{-1}+(\lambda-\mathcal{A})^{-1}\left(\sum_{k=0}^{n-1}[\mathcal{J}(\lambda)]^k\right)\mathcal{B}(\lambda-\mathcal{A})^{-1}\\&=(\lambda-\mathcal{A})^{-1}+\mathcal{R}^{(n-1)}(\lambda)\mathcal{B}(\lambda-\mathcal{A})^{-1}.\end{aligned}$$

Thus, for any $u\in\mathscr{D}(\mathcal{A})$, $\mathcal{R}^{(n)}(\lambda)(\lambda-\mathcal{A})u=u+\mathcal{R}^{(n-1)}(\lambda)\mathcal{B}u$ for any $n\geqslant 1$. Letting $n\to\infty$, Eq. (2.2) yields $(\lambda-\mathcal{G})^{-1}(\lambda-\mathcal{A})u=u+(\lambda-\mathcal{G})^{-1}\mathcal{B}u$ or equivalently, $(\lambda-\mathcal{G})^{-1}(\lambda-\mathcal{A}-\mathcal{B})u=u$. In particular, $u\in\mathscr{D}(\mathcal{G})$ and $(\lambda-\mathcal{G})u=(\lambda-\mathcal{A}-\mathcal{B})u$. This proves that $\mathcal{G}$ is an extension of $\mathcal{A}$ and $\mathcal{G}$ is closed as the generator of a C_0-semigroup on $\mathfrak{X}$.

• $(\mathcal{V}(t))_{t\geqslant 0}$ *is minimal.* Let $(\mathcal{S}(t))_{t\geqslant 0}$ be a substochastic semigroup in $\mathfrak{X}$ whose generator $\mathcal{G}'$ is a closed extension of $\mathcal{A}+\mathcal{B}$. Let us prove that $\mathcal{S}(t)\geqslant\mathcal{V}(t)$ for any $t\geqslant 0$. Actually, for any $\lambda>0$, one has

$$(\lambda-\mathcal{G}')^{-1}-(\lambda-\mathcal{G}_r)^{-1}=(\lambda-\mathcal{G}')^{-1}(\mathcal{G}'-\mathcal{G}_r)(\lambda-\mathcal{G}_r)^{-1}$$

and, since the range of $(\lambda-\mathcal{G}_r)^{-1}$ is $\mathscr{D}(\mathcal{A})\subset\mathscr{D}(\mathcal{G}')\cap\mathscr{D}(\mathcal{G}_r)$, one has

$$\begin{aligned}(\lambda-\mathcal{G}')^{-1}-(\lambda-\mathcal{G}_r)^{-1}&=(\lambda-\mathcal{G}')^{-1}(\mathcal{A}+\mathcal{B}-\mathcal{A}-r\mathcal{B})(\lambda-\mathcal{G}_r)^{-1}\\&=(1-r)(\lambda-\mathcal{G}')^{-1}\mathcal{B}(\lambda-\mathcal{G}_r)^{-1}\end{aligned}$$

and one sees that, at the (strong) limit, $(\lambda-\mathcal{G}')^{-1}\geqslant(\lambda-\mathcal{G})^{-1}$. From the exponential formula, one obtains $\mathcal{S}(t)\geqslant\mathcal{V}(t)$ for any $t\geqslant 0$. □

2.2. **On Dyson-Phillips expansion series.** It is possible to strengthen the above Theorem 2.1 by proving that the semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is given by a Dyson-Phillips expansion series. Our approach generalizes the result of [27] to the non lattice case and relies on different arguments inspired by [20, Chapter 8]. We first need some preliminary result. Let us define the space $\mathscr{C}_{sb}(\mathbb{R}^+,\mathscr{B}(\mathfrak{X}))$ of strongly continuous and bounded mappings

$$\mathcal{S}\ :\ t\geqslant 0\longmapsto\mathcal{S}(t)\in\mathscr{B}(\mathfrak{X})$$

endowed with the norm

$$\|\mathcal{S}\|_\infty=\sup_{t\geqslant 0}\|\mathcal{S}(t)\|_{\mathscr{B}(\mathfrak{X})}$$

which makes it a Banach space. For any $\mathcal{S}\in\mathscr{C}_{sb}(\mathbb{R}^+,\mathscr{B}(\mathfrak{X}))$, it is possible to define the time-dependent operator $\mathscr{L}(\mathcal{S})(t)$ defined over $\mathscr{D}(\mathcal{A})$ by

$$\mathscr{L}(\mathcal{S})(t)\ :\ u\in\mathscr{D}(\mathcal{A})\longmapsto\int_0^t\mathcal{S}(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s\in\mathfrak{X},\ t\geqslant 0.$$

We shall write that $\mathcal{S}\in\mathscr{C}_{sb}(\mathbb{R}^+,\mathscr{B}^+(\mathfrak{X}))$ if $\mathcal{S}\in\mathscr{C}_{sb}(\mathbb{R}^+,\mathscr{B}(\mathfrak{X}))$ and $\mathcal{S}(t)$ is a positive operator in $\mathfrak{X}$ for any $t\geqslant 0$. One has the following

LEMMA 2.2. *For any $\mathcal{S}\in\mathscr{C}_{sb}(\mathbb{R}^+,\mathscr{B}^+(\mathfrak{X}))$ and any $t\geqslant 0$, $\mathscr{L}(\mathcal{S})(t)$ extends uniquely to a bounded positive operator in $\mathfrak{X}$, still denoted $\mathscr{L}(\mathcal{S})(t)$. Moreover, for any $u\in\mathfrak{X}$, the mapping $t\geqslant 0\mapsto\mathscr{L}(\mathcal{S})(t)u\in\mathfrak{X}$ is continuous.*

Proof. It is clear that $\mathscr{L}(\mathcal{S})(t)$ is a nonnegative operator and, for any $u\in\mathscr{D}(\mathcal{A})_+$ and $\lambda>0$ one has

$$\left\|\int_0^t\mathcal{S}(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s\right\|=\int_0^t\|\mathcal{S}(t-s)\mathcal{B}\mathcal{U}(s)u\|\mathrm{d}s\leqslant\|\mathcal{S}\|_\infty\int_0^t\|\mathcal{B}\mathcal{U}(s)u\|\mathrm{d}s.$$

Now,

$$\begin{aligned}\int_0^t \|\mathcal{BU}(s)u\|\mathrm{d}s &= \int_0^t \langle \mathbf{\Psi}, \mathcal{BU}(s)u\rangle \,\mathrm{d}s \leqslant -\int_0^t \langle \mathbf{\Psi}, \mathcal{AU}(s)u\rangle \,\mathrm{d}s \\ &= -\left\langle \mathbf{\Psi}, \int_0^t \mathcal{AU}(s)u\mathrm{d}s \right\rangle = -\left\langle \mathbf{\Psi}, \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{U}(s)u\mathrm{d}s \right\rangle \\ &= \langle \mathbf{\Psi}, u - \mathcal{U}(t)u\rangle \leqslant \|u\|.\end{aligned} \tag{2.7}$$

Therefore,

$$\left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u\mathrm{d}s \right\| \leqslant \|\mathcal{S}\|_\infty \|u\| \qquad \forall t \geqslant 0,\ \forall u \in \mathscr{D}(\mathcal{A})_+. \tag{2.8}$$

Now, let $u \in \mathscr{D}(\mathcal{A})$ be arbitrary and let $u = u_1 - u_2$ where $u_i \in \mathfrak{X}_+$ are such that $\|u_i\| \leqslant M\|u\|$, $i = 1, 2$. Then, for any $n \geqslant 1$, $u_n^i := n\int_0^{1/n} \mathcal{U}(s)u_i\mathrm{d}s \in \mathscr{D}(\mathcal{A})_+$ with $u_n^i \longrightarrow u_i$ in $\mathfrak{X}$ as $n \to \infty$, while

$$u_n^1 - u_n^2 = n\int_0^{1/n} \mathcal{U}(s)u\mathrm{d}s \longrightarrow u \qquad \text{in } \mathscr{D}(\mathcal{A}), \qquad i = 1, 2.$$

Therefore,

$$\begin{aligned}\left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u\mathrm{d}s \right\| &= \lim_{n\to\infty} \left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)(u_n^1 - u_n^2)\mathrm{d}s \right\| \\ &\leqslant \lim_{n\to\infty} \left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u_n^1\mathrm{d}s \right\| + \\ &\qquad \lim_{n\to\infty} \left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u_n^2\mathrm{d}s \right\|\end{aligned}$$

and Eq. (2.8) yields

$$\left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u\mathrm{d}s \right\| \leqslant \|\mathcal{S}\|_\infty \lim_{n\to\infty} \left(\|u_n^1\| + \|u_n^2\|\right) = \|\mathcal{S}\|_\infty \left(\|u_1\| + \|u_2\|\right).$$

Consequently,

$$\left\| \int_0^t \mathcal{S}(t-s)\mathcal{BU}(s)u\mathrm{d}s \right\| \leqslant 2M\|\mathcal{S}\|_\infty \|u\|, \qquad \forall u \in \mathscr{D}(\mathcal{A}).$$

Since $\mathscr{D}(\mathcal{A})$ is dense in $\mathfrak{X}$, $\mathscr{L}(\mathcal{S})(t)$ extends uniquely to a bounded operator on $\mathfrak{X}$. We still denote $\mathscr{L}(\mathcal{S})(t)$ this extension. Notice that, since $\mathscr{D}(\mathcal{A})_+$ is dense in $\mathfrak{X}_+$, the extension $\mathscr{L}(\mathcal{S})(t)$ is still positive. One notes that, for any $u \in \mathscr{D}(\mathcal{A})$, the mapping $t \mapsto \mathscr{L}(\mathcal{S})(t)u$

is continuous. Now, if $u \in \mathfrak{X}$, considering a sequence $(u_n)_n \subset \mathscr{D}(\mathcal{A})$ which converges to u, one has, for any $T > 0$

$$\sup_{t\in[0,T]} \|\mathscr{L}(\mathcal{S})(t)u_n - \mathscr{L}(\mathcal{S})(t)u_m\| \leqslant 2M\|\mathcal{S}\|_\infty \|u_n - u_m\|, \quad n, m \in \mathbb{N},$$

which implies that the mapping $t \in [0, \infty[\mapsto \mathscr{L}(\mathcal{S})(t)u$ is continuous. □

Arguing as in [20, Lemma 8.4], we prove the following

THEOREM 2.3. *For any $t \geqslant 0$, the following Duhamel formula holds:*

$$\mathcal{V}(t)u = \mathcal{U}(t)u + \int_0^t \mathcal{V}(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s, \qquad t \geqslant 0, \quad u \in \mathscr{D}(\mathcal{A}). \tag{2.9}$$

*Moreover, the semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ defined in Theorem 2.1 is given by the **Dyson-Phillips** expansion series*

$$\mathcal{V}(t) = \sum_{n=0}^{\infty} \mathscr{L}^n(\mathcal{U})(t), \qquad t \geqslant 0 \tag{2.10}$$

where the series converges strongly in $\mathfrak{X}$.

Proof. Let us first establish Duhamel formula. We use the ideas of [27, Lemma 1.4]. Let $u \in \mathscr{D}(\mathcal{A})$ and $\lambda > 0$. We see from (2.2) that

$$(\lambda - \mathcal{G})^{-1}u - (\lambda - \mathcal{A})^{-1}u = (\lambda - \mathcal{G})^{-1}\mathcal{B}(\lambda - \mathcal{A})^{-1}u. \tag{2.11}$$

Moreover, since $\mathcal{B}$ is $\mathcal{A}$-bounded, the mapping $t \in [0, \infty) \mapsto \mathcal{B}\mathcal{U}(t)u \in \mathfrak{X}$ is continuous for all $u \in \mathscr{D}(\mathcal{A})$ and

$$\mathcal{B}(\lambda - \mathcal{A})^{-1}u = \mathcal{B}\int_0^\infty \exp(-\lambda t)\mathcal{U}(t)u\mathrm{d}t = \int_0^\infty \exp(-\lambda t)\mathcal{B}\mathcal{U}(t)u\mathrm{d}t.$$

Since $(\lambda - \mathcal{G})^{-1}$ is the Laplace transform of $(\mathcal{V}(t))_{t\geqslant 0}$, one gets from (2.11)

$$\begin{aligned}\int_0^\infty \exp(-\lambda t)\left(\mathcal{V}(t)u - \mathcal{U}(t)u\right)\mathrm{d}t &= \int_0^\infty \mathrm{d}t \int_0^\infty \exp(-\lambda(t+s))\mathcal{V}(t)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s \\ &= \int_0^\infty \exp(-\lambda t)\left(\int_0^t \mathcal{V}(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

Finally, the uniqueness theorem for the Laplace transform provides the conclusion. Let us prove now that $(\mathcal{V}(t))_{t\geqslant 0}$ is given by the Dyson–Phillips expansion (2.10). Duhamel formula (2.9) reads

$$\mathcal{V}(t)u = \mathcal{U}(t)u + \mathscr{L}(\mathcal{T})(t)u, \qquad \forall t \geqslant 0, \quad u \in \mathfrak{X}$$

and, by iteration,

$$\mathcal{V}(t)u = \sum_{k=0}^{n} \mathscr{L}^k(\mathcal{U})(t)u + \mathscr{L}^{n+1}(\mathcal{T})(t)u, \qquad t \geqslant 0, \quad n \geqslant 1, \quad u \in \mathfrak{X}.$$

In particular, for any $u \in \mathfrak{X}_+$, one has

$$\sum_{k=0}^{n} \mathscr{L}^k(\mathcal{U})(t)u \leqslant \mathcal{V}(t)u, \qquad n \geqslant 1, \quad u \in \mathfrak{X}_+ \tag{2.12}$$

and the series $\sum_{n=0}^{\infty} \mathscr{L}^n(\mathcal{U})u$ is convergent towards a limit that we denote $\widetilde{\mathcal{T}}(t)u$. Notice that, for a given $u \in \mathfrak{X}_+$, the mapping $t \in [0,\infty[\mapsto \widetilde{\mathcal{T}}(t)u$ is measurable. One has

$$\widetilde{\mathcal{T}}(t)u \leqslant \mathcal{V}(t)u, \qquad \forall u \in \mathfrak{X}_+,\ t \geqslant 0. \tag{2.13}$$

Now, it is not difficult to check by induction that

$$\int_0^{\infty} \exp(-\lambda t)\mathscr{L}^n(\mathcal{U})(t)u\mathrm{d}t = (\lambda - \mathcal{A})^{-1}\left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n u \tag{2.14}$$

so that,

$$\sum_{n=0}^{\infty}(\lambda - \mathcal{A})^{-1}\left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n u = \int_0^{\infty} \exp(-\lambda t)\widetilde{\mathcal{T}}(t)u\mathrm{d}t$$

and Eq. (2.13) together with Eq. (2.2) yield

$$\int_0^{\infty} \exp(-\lambda t)\widetilde{\mathcal{T}}(t)u\mathrm{d}t = \int_0^{\infty} \exp(-\lambda t)\mathcal{V}(t)u\mathrm{d}t, \qquad \forall u \in \mathfrak{X}_+, \quad \lambda > 0.$$

The uniqueness theorem for the Laplace transform implies then $\widetilde{\mathcal{T}}(t)u = \mathcal{V}(t)u$ for any $t \geqslant 0$ and any $u \in \mathfrak{X}_+$ so that

$$\sum_{n=0}^{\infty} \mathscr{L}^n(\mathcal{U})(t)u = \mathcal{V}(t)u, \qquad \forall u \in \mathfrak{X}_+, \quad t \geqslant 0.$$

Note that, according to Dini's convergence theorem, the series converges uniformly in bounded time. One extends then the convergence to arbitrary $u \in \mathfrak{X}$ by linearity. □

REMARK 2.4. *Notice that the family of operators $\mathcal{V}_n(t) = \mathscr{L}^n(\mathcal{U})(t)$ $(n \in \mathbb{N}, t \geqslant 0)$, is nothing but the classical Dyson-Phillips iterated usually defined by induction* [20, Chapter 7]*:*

$$\mathcal{V}_{n+1}(t)u = \int_0^t \mathcal{V}_n(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s, \qquad \forall n \in \mathbb{N},\ u \in \mathscr{D}(\mathcal{A}). \tag{2.15}$$

Notice that, according to (2.12)*, one sees easily that*

$$\sum_{k=0}^{n} \|\mathcal{V}_k(t)u\| \leqslant \|u\| \quad \textit{for any} \quad t \geqslant 0,\ u \in \mathfrak{X}_+. \tag{2.16}$$

Moreover, for any $n \in \mathbb{N}$, the mapping $t \in [0,\infty) \mapsto \mathcal{V}_n(t)u$ is continuous for any $u \in \mathfrak{X}$. Finally, arguing as in [6, p. 129]*, it is not difficult to prove that, for any $n \in \mathbb{N}$, the following relation holds:*

$$\mathcal{V}_n(t+s)u = \sum_{k=0}^{n} \mathcal{V}_k(t)\mathcal{V}_{n-k}(s)u \quad \textit{for any } u \in \mathfrak{X},\ \ t,s \geqslant 0. \tag{2.17}$$

3. On honesty theory: resolvent approach

From now, in all the paper, we assume that the assumptions of Theorem 2.1 are met.

3.1. **About some useful functionals.** Since the C_0-semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is substochastic, one has, for any $u \in \mathfrak{X}_+$,

$$\langle \mathbf{\Psi}, \mathcal{V}(t)u - u\rangle = \|\mathcal{V}(t)u\| - \|u\| \leqslant 0, \qquad \forall t \geqslant 0, \quad u \in \mathfrak{X}_+.$$

In particular, if one chooses $u \in \mathscr{D}(\mathcal{G})_+$ here above, since,

$$\langle \mathbf{\Psi}, \mathcal{G}u\rangle = \lim_{t\searrow 0} t^{-1} \langle \mathbf{\Psi}, \mathcal{V}(t)u - u\rangle$$

one gets

$$\langle \mathbf{\Psi}, \mathcal{G}u\rangle \leqslant 0, \qquad\qquad u \in \mathscr{D}(\mathcal{G})_+. \tag{3.1}$$

Because of this elementary but fundamental inequality, a crucial role in the present approach will be based on the properties of the following functional:

$$\mathfrak{a}_0 \ : \ u \in \mathscr{D}(\mathcal{G}) \mapsto \mathfrak{a}_0(u) = -\langle \mathbf{\Psi}, \mathcal{G}u\rangle \in \mathbb{R}.$$

Because of (3.1), this functional $\mathfrak{a}_0$ is nondecreasing, i.e. $\mathfrak{a}_0(u) \geqslant \mathfrak{a}_0(v)$ for any $u, v \in \mathscr{D}(\mathcal{G})$ with $u \geqslant v$. Moreover, since $\|\mathbf{\Psi}\| \leqslant 1$, one has $\mathfrak{a}_0(u) \leqslant \|\mathcal{G}u\|$ for any $u \in \mathscr{D}(\mathcal{G})$. We denote by $\mathfrak{a}$ its restriction to $\mathscr{D}(\mathcal{A})$, i.e.

$$\mathfrak{a} \ : \ u \in \mathscr{D}(\mathcal{A}) \mapsto \mathfrak{a}(u) = -\langle \mathbf{\Psi}, \mathcal{A}u + \mathcal{B}u\rangle \in \mathbb{R}.$$

Let $\lambda > 0$ be *fixed*. The following obvious identity

$$-\mathfrak{a}((\lambda - \mathcal{A})^{-1}u) = \lambda\|(\lambda-\mathcal{A})^{-1}u\| + \|\mathcal{B}(\lambda-\mathcal{A})^{-1}u\| - \|u\|, \tag{3.2}$$

is valid for any $u \in \mathfrak{X}_+$. Moreover, the sequence $\left(\sum_{k=0}^{n}(\lambda-\mathcal{A})^{-1}[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^k u\right)_n$ is nondecreasing and convergent to $(\lambda-\mathcal{G})^{-1}u$. Since $\mathfrak{a}(\cdot)$ is nondecreasing, one gets

$$\mathfrak{a}\left(\sum_{k=0}^{n}(\lambda-\mathcal{A})^{-1}[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^k u\right) \leqslant \mathfrak{a}_0((\lambda-\mathcal{G})^{-1}u),$$

for all $u \in \mathfrak{X}_+$ and any $n \in \mathbb{N}$. The bounded and nondecreasing real sequence

$$\left(\mathfrak{a}\left(\sum_{k=0}^{n}(\lambda-\mathcal{A})^{-1}[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^k u\right)\right)_n$$

is therefore convergent. This convergence holds for any $u \in \mathfrak{X} = \mathfrak{X}_+ - \mathfrak{X}_+$ and therefore defines a functional $\overline{\mathfrak{a}}_\lambda$ (that depends *a priori* on $\lambda > 0$) on the domain of $\mathcal{G}$ by

$$\overline{\mathfrak{a}}_\lambda\left((\lambda-\mathcal{G})^{-1}u\right) = \sum_{n=0}^{\infty} \mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u\right), \qquad u \in \mathfrak{X}.$$

Following [23], we derive another expression for $\overline{\mathfrak{a}}_\lambda$ from the identity

$$(\lambda-\mathcal{G}_r)^{-1}u = \sum_{n=0}^{\infty} r^n(\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u, \qquad u \in \mathfrak{X}_+$$

established in the proof of Theorem 2.1. We recall that, denoting $\mathscr{D}_{\mathcal{A}}$ and $\mathscr{D}_{\mathcal{G}}$ the domain of $\mathcal{A}$ and $\mathcal{G}$ equipped with their respective graph norm, the series is convergent in $\mathscr{D}_{\mathcal{A}}$ and, since $(\lambda-\mathcal{A})^{-1} \leqslant (\lambda-\mathcal{G})^{-1}$, the embedding $\mathscr{D}_{\mathcal{A}} \hookrightarrow \mathscr{D}_{\mathcal{G}}$ is continuous. Therefore,

$$\mathfrak{a}((\lambda-\mathcal{G}_r)^{-1}u) = \sum_{n=0}^{\infty} r^n \mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u\right), \qquad u \in \mathfrak{X}_+.$$

Letting now $r \to 1$, one gets

$$\overline{\mathfrak{a}}_\lambda\left((\lambda-\mathcal{G})^{-1}u\right) = \lim_{r\nearrow 1} \mathfrak{a}((\lambda-\mathcal{G}_r)^{-1}u) = \sum_{n=0}^{\infty} \mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u\right).$$

One has the following basic result which can be proved exactly as [23, Prop. 1.1] (see also an alternative proof at the end of the paper, Theorem 4.9):

PROPOSITION 3.1. *Let $0 < \lambda < \mu$. Then,*

(1) $\overline{\mathfrak{a}}_{\lambda|\mathscr{D}(\mathcal{A})} = \mathfrak{a}$;
(2) $\overline{\mathfrak{a}}_\lambda = \overline{\mathfrak{a}}_\mu$.

This defines a functional $\overline{\mathfrak{a}} := \overline{\mathfrak{a}}_\lambda$ for any λ.

REMARK 3.2. *Let us point out that $\overline{\mathfrak{a}}$ is continuous with respect to the graph norm of $\mathcal{G}$.*

The above definitions of functionals $\overline{\mathfrak{a}}$ and $\mathfrak{a}_0$ lead to the following:

DEFINITION 3.3. *For any $\lambda > 0$, we define the functional $\Xi_\lambda \in \mathfrak{X}^\star$ by*

$$\langle \Xi_\lambda, u\rangle = \mathfrak{a}_0\left((\lambda-\mathcal{G})^{-1}u\right) - \overline{\mathfrak{a}}\left((\lambda-\mathcal{G})^{-1}u\right), \qquad u \in \mathfrak{X}.$$

One has the following Lemma:

LEMMA 3.4. *For any $\lambda > 0$ and $u \in \mathfrak{X}$*

$$\langle \Xi_\lambda, u\rangle = \lim_{n\to\infty} \left\langle \Psi, \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^n u\right\rangle = \lim_{r\nearrow 1}(1-r)\left\langle \Psi, \mathcal{B}(\lambda-\mathcal{G}_r)^{-1}u\right\rangle$$

Proof. One has to compute $\langle \boldsymbol{\Xi}_\lambda, u\rangle = \mathfrak{a}_0\left((\lambda-\mathcal{G})^{-1}u\right) - \overline{\mathfrak{a}}\left((\lambda-\mathcal{G})^{-1}u\right)$. First,

$$\begin{aligned}\overline{\mathfrak{a}}\left((\lambda-\mathcal{G})^{-1}u\right) &= \sum_{n=0}^{\infty}\mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right)\\ &= \sum_{n=0}^{\infty}\left\langle \boldsymbol{\Psi}, -(\mathcal{A}+\mathcal{B})(\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle.\end{aligned}$$

Now, the latter is equal to

$$\sum_{n=0}^{\infty}\left(\left\langle \boldsymbol{\Psi}, \left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u - \left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^{n+1} u - \lambda(\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle\right).$$

Thus

$$\begin{aligned}\overline{\mathfrak{a}}\left((\lambda-\mathcal{G})^{-1}u\right) &= \langle \boldsymbol{\Psi}, u\rangle - \lim_{n\to\infty}\left\langle \boldsymbol{\Psi}, \left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle -\\ &\qquad\qquad \lambda\left\langle \boldsymbol{\Psi}, \sum_{n=0}^{\infty}(\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle\\ &= \langle \boldsymbol{\Psi}, u\rangle - \lim_{n\to\infty}\left\langle \boldsymbol{\Psi}, \left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle - \lambda\left\langle \boldsymbol{\Psi}, (\lambda-\mathcal{G})^{-1}u\right\rangle\\ &= \mathfrak{a}_0\left((\lambda-\mathcal{G})^{-1}u\right) - \lim_{n\to\infty}\left\langle \boldsymbol{\Psi}, \left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^n u\right\rangle\end{aligned}$$

which proves the first assertion. On the other hand,

$$\begin{aligned}\overline{\mathfrak{a}}\left((\lambda-\mathcal{G})^{-1}u\right) &= \lim_{r\nearrow 1}\mathfrak{a}\left((\lambda-\mathcal{G}_r)^{-1}u\right)\\ &= \lim_{r\nearrow 1}\left\langle \boldsymbol{\Psi}, (\lambda-\mathcal{A}-r\mathcal{B}-\lambda-(1-r)\mathcal{B})(\lambda-\mathcal{G}_r)^{-1}u\right\rangle\\ &= \lim_{r\nearrow 1}\left(\langle \boldsymbol{\Psi}, u\rangle - \lambda\left\langle \boldsymbol{\Psi}, (\lambda-\mathcal{G}_r)^{-1}u\right\rangle - (1-r)\left\langle \boldsymbol{\Psi}, \mathcal{B}(\lambda-\mathcal{G}_r)^{-1}u\right\rangle\right)\\ &= \langle \boldsymbol{\Psi}, u\rangle - \lambda\left\langle \boldsymbol{\Psi}, (\lambda-\mathcal{G})^{-1}u\right\rangle - \lim_{r\nearrow 1}(1-r)\left\langle \boldsymbol{\Psi}, \mathcal{B}(\lambda-\mathcal{G}_r)^{-1}u\right\rangle\end{aligned}$$

provides the second assertion. □

We end this section with the following fundamental result:

THEOREM 3.5. *Let $\lambda > 0$ and $u \in \mathfrak{X}_+$ be fixed. The following assertions are equivalent:*

(i) *the set $\{[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^n u\}_n$ is relatively weakly compact;*
(ii) $\lim_{n\to\infty}\|[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^n u\| = 0$;
(iii) $\langle \boldsymbol{\Xi}_\lambda, u\rangle = 0$;
(iv) $(\lambda-\mathcal{G})^{-1}u \in \mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$.

Proof. It is clear from the definition of $\mathbf{\Xi}_\lambda$ that *(ii)* $\Longrightarrow$ *(iii)* and that *(iii)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(i)*.

(i) $\Longrightarrow$ *(ii)* and *(iv)*. Let $v_n := \sum_{k=0}^{n}(\lambda-\mathcal{A})^{-1}\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^k u$. Clearly, $v_n \in \mathscr{D}(\mathcal{A}+\mathcal{B})$ and v_n converges to $v=(\lambda-\mathcal{G})^{-1}u$ in $\mathfrak{X}$ as n goes to infinity. Moreover, it is not difficult to see that

$$(\lambda-\mathcal{A}-\mathcal{B})v_n = u - \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n+1} u.$$

If some subsequence $\left([\mathcal{B}(\lambda-\mathcal{A})^{-1}]^{n_k}u\right)_k$ converges weakly in $\mathfrak{X}$ to some $z\in\mathfrak{X}$, then $(\lambda-\mathcal{A}-\mathcal{B})v_{n_k}$ converges weakly to $u-\mathcal{B}(\lambda-\mathcal{A})^{-1}z$ as $k\to\infty$. It follows from the weak closedness of the graph $\overline{\mathcal{A}+\mathcal{B}}$ that $v\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$ and

$$(\lambda-\overline{\mathcal{A}+\mathcal{B}})v = u-\mathcal{B}(\lambda-\mathcal{A})^{-1}z.$$

Since $\mathcal{G}$ is a closed extension of $\mathcal{A}+\mathcal{B}$ and $v=(\lambda-\mathcal{G})^{-1}u$, the latter reads

$$u = u-\mathcal{B}(\lambda-\mathcal{A})^{-1}z$$

so that $\mathcal{B}(\lambda-\mathcal{A})^{-1}z=0$. Hence, $[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^{n_k+1}u$ converges weakly to 0 as $k\to\infty$. In particular,

$$\lim_{k\to\infty}\left\langle \mathbf{\Psi}, \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n_k+1} u\right\rangle = 0 \quad \text{and} \quad \lim_{n\to\infty}\left\langle \mathbf{\Psi}, \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n} u\right\rangle = 0$$

since the whole sequence is always convergent. This proves *(ii)*. Notice also that $v=(\lambda-\mathcal{G})^{-1}u\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$ and *(iv)* is proved.

(iv) $\Longrightarrow$ *(iii)*. One can assume without loss of generality that $\mathbf{\Xi}_\lambda\neq 0$. Assume that $(\lambda-\mathcal{G})^{-1}u\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$. According to the following identity (see [6, Lemma 4.5, p. 117])

$$\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}}) = (\lambda-\mathcal{G})^{-1}\overline{(I-\mathcal{B}(\lambda-\mathcal{A})^{-1})\mathfrak{X}} \tag{3.3}$$

one sees that there exists a sequence $(u_n)_n\subset(I-\mathcal{B}(\lambda-\mathcal{A})^{-1})\mathfrak{X}$ such that $\lim_n u_n = u$. It is easy to see that $\langle\mathbf{\Xi}_\lambda,u_n\rangle=0$ for any $n\in\mathbb{N}$ so that $\langle\mathbf{\Xi}_\lambda,u\rangle=0$. □

One deduces from the above result that $\mathscr{D}(\mathcal{A}+\mathcal{B})$ is a core for $\mathcal{G}$ if and only if $\mathbf{\Xi}_\lambda=0$:

COROLLARY 3.6. *One has $\mathcal{G}=\overline{\mathcal{A}+\mathcal{B}}$ if and only if $\mathbf{\Xi}_\lambda=0$ for some (or equivalently for all) $\lambda>0$.*

REMARK 3.7. *For $v\in\mathscr{D}(\mathcal{G})_+$ one can show as in* [23, Proposition 1.6] *that $v\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$ if and only if $\mathfrak{a}_0(v)=\overline{\mathfrak{a}}(v)$ which strengthens Proposition 3.1.*

3.2. **On honest trajectories.** We note that, for any $u \in \mathfrak{X}_+$ and any $t \geqslant 0$, one has

$$\int_0^t \mathcal{V}(s)u\mathrm{d}s \in \mathscr{D}(\mathcal{G}) \quad \text{with} \quad \mathcal{V}(t)u - u = \mathcal{G}\int_0^t \mathcal{V}(s)u\mathrm{d}s.$$

Since the semigroup is positive, one has

$$\|\mathcal{V}(t)u\| - \|u\| = -\mathfrak{a}_0\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right). \tag{3.4}$$

DEFINITION 3.8. *Let $u \in \mathfrak{X}_+$ be given. Then, the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is said to be* ***honest*** *if and only if*

$$\|\mathcal{V}(t)u\| = \|u\| - \overline{\mathfrak{a}}\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right), \quad \textit{for any } t \geqslant 0.$$

The whole C_0-semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ will be said to be honest if all trajectories are honest.

REMARK 3.9. *Note that, in the spirit of* [23]*, it is possible to define a more general concept of local honest trajectory on an interval $\mathcal{I} \subset [0,\infty)$ by*

$$\overline{\mathfrak{a}}\left(\int_s^t \mathcal{V}(r)u\mathrm{d}r\right) = \mathfrak{a}_0\left(\int_s^t \mathcal{V}(r)u\mathrm{d}r\right), \qquad \textit{for any } t,s \in \mathcal{I},\ t \geqslant s.$$

We do not try to elaborate on this point here.

REMARK 3.10. *One can deduce from Theorem 3.5 and Corollary 3.6 the following: given $u \in \mathfrak{X}_+$, one sees from* (3.4) *that $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest if and only if*

$$\overline{\mathfrak{a}}\left(\int_s^t \mathcal{V}(r)u\mathrm{d}r\right) = \mathfrak{a}_0\left(\int_s^t \mathcal{V}(r)u\mathrm{d}r\right) \textit{ for any } t \geqslant s \geqslant 0.$$

Moreover, it is easy to see that this is equivalent to $\overline{\mathfrak{a}}(\int_0^t \mathcal{V}(r)u\mathrm{d}r) = \mathfrak{a}_0(\int_0^t \mathcal{V}(r)u\mathrm{d}r)$ for any $t \geqslant 0$.

The link between honest trajectory and the functional Ξ_λ given by Definition 3.3 is provided by the following:

THEOREM 3.11. *Let $u \in \mathfrak{X}_+$. The trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest if and only if $\langle \Xi_\lambda, u\rangle = 0$ for all/some $\lambda > 0$.*

Proof. We recall that, for any $\lambda > 0$,

$$(\lambda - \mathcal{G})^{-1}u = \int_0^\infty \exp(-\lambda t)\mathcal{V}(t)u\mathrm{d}t = \lambda\int_0^\infty \exp(-\lambda t)\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right)\mathrm{d}t. \tag{3.5}$$

Moreover, the function $t \longmapsto \int_0^t \mathcal{V}(s)uds$ is continuous and linearly bounded as a $\mathscr{D}_{\mathcal{G}}$-function. This means that the above outer integral in (3.5) is convergent in $\mathscr{D}_{\mathcal{G}}$ and commute with $\mathfrak{a}_0$. Moreover, according to Prop. 3.1, it also commutes with $\overline{\mathfrak{a}}$ so that

$$\mathfrak{a}_0\bigg((\lambda-\mathcal{G})^{-1}u\bigg) = \lambda\int_0^\infty \exp(-\lambda t)\mathfrak{a}_0\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg)\mathrm{d}t$$

and

$$\overline{\mathfrak{a}}\bigg((\lambda-\mathcal{G})^{-1}u\bigg) = \lambda\int_0^\infty \exp(-\lambda t)\overline{\mathfrak{a}}\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg)\mathrm{d}t.$$

One sees therefore that $\mathfrak{a}_0\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg) = \overline{\mathfrak{a}}\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg)$ for any $t \geqslant 0$ is equivalent to $\mathfrak{a}_0\bigg((\lambda-\mathcal{G})^{-1}u\bigg) = \overline{\mathfrak{a}}\bigg((\lambda-\mathcal{G})^{-1}u\bigg)$ for any $\lambda > 0$ and proves the Theorem. □

REMARK 3.12. *Notice that the whole semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is honest if and only $\mathcal{G} = \overline{\mathcal{A}+\mathcal{B}}$ and this is also equivalent to $\Xi_\lambda = 0$ for some / all $\lambda > 0$.*

3.3. **On an order ideal invariant under $(\mathcal{V}(t))_{t\geqslant 0}$.** We already know that, for any $u \in \mathfrak{X}_+$, the property $\langle \Xi_\lambda, u\rangle = 0$ is independent of the choice of $\lambda > 0$. This allows us to define the set

$$\mathcal{H} = \bigg\{u \in \mathfrak{X}_+\,;\, \langle \Xi_\lambda, u\rangle = 0 \text{ for any } \lambda > 0\bigg\}. \tag{3.6}$$

Notice that, by virtue of Theorem 3.11, $\mathcal{H}$ is precisely the set of initial positive data u giving rise to honest trajectories:

$$\mathcal{H} = \bigg\{u \in \mathfrak{X}_+\,;\, (\mathcal{V}(t)u)_{t\geqslant 0} \text{ is honest}\bigg\}.$$

One has the following

PROPOSITION 3.13. *The set $\mathcal{H}$ is invariant under $(\mathcal{V}(t))_{t\geqslant 0}$ and $(\lambda-\mathcal{G})^{-1}$ $(\lambda > 0)$. Moreover, for any $u \in \mathcal{H}$, if $\mathcal{I}_u = \{z \in \mathfrak{X}_+\,;\, \exists p \in \mathbb{R}_+$ such that $pu - z \in \mathfrak{X}_+\}$ then $\overline{\mathrm{span}(\mathcal{I}_u)} \cap \mathfrak{X}_+ \subset \mathcal{H}$.*

Proof. Let $u \in \mathcal{H}$. This means that

$$\|\mathcal{V}(t)u\| - \|u\| = -\overline{\mathfrak{a}}\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg), \qquad \forall t \geqslant 0.$$

Let $t_0 > 0$ be fixed and set $v = \mathcal{V}(t_0)u$. One has $\|v\| - \|u\| = -\overline{\mathfrak{a}}\bigg(\int_0^{t_0} \mathcal{V}(s)u\mathrm{d}s\bigg)$ and, for any $t \geqslant t_0$,

$$\|\mathcal{V}(t-t_0)v\| - \|u\| = -\overline{\mathfrak{a}}\bigg(\int_0^t \mathcal{V}(s)u\mathrm{d}s\bigg) = -\overline{\mathfrak{a}}\bigg(\int_0^{t_0} \mathcal{V}(s)u\mathrm{d}s\bigg) - \overline{\mathfrak{a}}\bigg(\int_{t_0}^t \mathcal{V}(s)u\mathrm{d}s\bigg)$$

so that

$$\|\mathcal{V}(t-t_0)v\| = \|v\| - \overline{\mathfrak{a}}\left(\int_{t_0}^{t}\mathcal{V}(s)u\mathrm{d}s\right) = \|v\| - \overline{\mathfrak{a}}\left(\int_{0}^{t-t_0}\mathcal{V}(s)v\mathrm{d}s\right), \qquad \forall t \geqslant t_0.$$

In other words, $v \in \mathcal{H}$ and $\mathcal{H}$ is invariant under the action of $(\mathcal{V}(t))_{t\geqslant 0}$. Let $\lambda > 0$ and $u \in \mathcal{H}$ be fixed. One has $\mathfrak{a}_0((\lambda-\mathcal{G})^{-1}u) = \overline{\mathfrak{a}}((\lambda-\mathcal{G})^{-1}u)$ and $\mathfrak{a}_0((\mu-\mathcal{G})^{-1}u) = \overline{\mathfrak{a}}((\mu-\mathcal{G})^{-1}u)$ for any $\mu > 0$. One sees as a direct application of the resolvent formula that

$$\mathfrak{a}_0\Big((\mu-\mathcal{G})^{-1}(\lambda-\mathcal{G})^{-1}u\Big) = \overline{\mathfrak{a}}\Big((\mu-\mathcal{G})^{-1}(\lambda-\mathcal{G})^{-1}u\Big), \quad \forall \mu > 0$$

which amounts to $(\lambda-\mathcal{G})^{-1}u \in \mathcal{H}$. Finally, let $u \in \mathcal{H}$ and $z \in \mathcal{I}_u$ be fixed, there is some nonnegative real number p such that $pu - z \in \mathfrak{X}_+$. Then, for any $n \in \mathbb{N}$,

$$[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^{n+1}z \leqslant p[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^{n+1}u.$$

Since $\langle \Xi_\lambda, u\rangle = 0$, Lemma 3.4 clearly implies that

$$\lim_{n\to\infty}\left\langle \mathbf{\Psi}, \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n+1} z\right\rangle = 0$$

and $(\mathcal{V}(t)z)_{t\geqslant 0}$ is honest according to Theorem 3.5. This proves that $\mathcal{I}_u \subset \mathcal{H}$ and, since Ξ_λ is a continuous and positive linear form on $\mathfrak{X}$, one deduces easily that $\overline{\mathrm{span}(\mathcal{I}_u)}\cap\mathfrak{X}_+ \subset \mathcal{H}$. □

Thanks to the above structure of $\mathcal{H}$, it is possible to provide sufficient conditions ensuring that the whole semigroup is honest.

THEOREM 3.14.

(1) If $\mathcal{H}$ contains a quasi-interior element u, then the whole semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is honest.

(2) Assume $(\mathcal{V}(t))_{t\geqslant 0}$ to be irreducible. Let there exists $u \in \mathfrak{X}_+ \setminus \{0\}$ such that $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest. Then, the whole semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is honest.

Proof. (1) If $\mathfrak{X}$ contains a quasi-interior element u, then [29, 25] $\overline{\mathrm{span}(\mathcal{I}_u)} = \mathfrak{X}_+$. One sees then that, if $u \in \mathcal{H}$, Proposition 3.13 implies $\mathcal{H} = \mathfrak{X}_+$.

(2) According to Proposition 3.13, $\mathcal{H}$ is invariant by $(\lambda-\mathcal{G})^{-1}$ for any $\lambda > 0$. Therefore, $v = (\lambda-\mathcal{G})^{-1}u$ is a quasi-interior element of $\mathcal{H}$ and we conclude by the first point. □

Before giving some more precise properties of $\mathcal{H}$ let us introduce the notions of ideal and hereditary subcone:

DEFINITION 3.15. *A subcone $\mathcal{C}$ of $\mathfrak{X}_+$ is said to be **hereditary** if $0 \leqslant u \leqslant v$ and $v \in \mathcal{C}$ imply $u \in \mathcal{C}$. An **order ideal** of $\mathfrak{X}$ is a linear subspace $\mathscr{A}$ of $\mathfrak{X}$ such that $u_1 \leqslant v \leqslant u_2$ and*

$u_i \in \mathscr{A}$, $i = 1, 2$ *imply* $v \in \mathscr{A}$. *An order ideal* $\mathscr{A}$ *of* $\mathfrak{X}$ *is said to be* ***positively generated*** *if* $\mathscr{A} = (\mathscr{A} \cap \mathfrak{X}_+) - (\mathscr{A} \cap \mathfrak{X}_+)$.

REMARK 3.16. *Notice that, if* $\mathscr{A}$ *is a positively generated order ideal of* $\mathfrak{X}$ *then*

$$u \in \mathscr{A} \qquad \Longrightarrow \qquad |u| \in \mathscr{A}.$$

Indeed, since $\mathscr{A}$ *is positively generated one has* $u = u_1 - u_2$ *with* $u_i \in \mathscr{A} \cap \mathfrak{X}_+$. *Moreover, according to* [28, Lemma 2], $\mathscr{A} \cap \mathfrak{X}_+$ *is an hereditary subcone of* $\mathfrak{X}_+$. *In particular, since* $0 \leqslant |u| \leqslant u_1 + u_2$ *one gets* $|u| \in \mathscr{A} \cap \mathfrak{X}_+$.

The subset

$$\mathscr{H} := \mathcal{H} - \mathcal{H}$$

enjoys the following properties:

THEOREM 3.17. *Let* $\mathcal{H}$ *be defined by* (3.6). *Then,* $\mathcal{H}$ *is a closed hereditary subcone of* $\mathfrak{X}_+$ *and* $\mathscr{H}$ *is an order ideal with induced positive cone* $\mathscr{H}_+$ *equal to* $\mathcal{H}$. *Moreover,* $\mathscr{H}$ *is invariant under* $(\mathcal{V}(t))_{t \geqslant 0}$.

Proof. We first note that, since $\mathbf{\Xi}_\lambda$ is a positive and continuous linear form over $\mathfrak{X}$,

$$\mathcal{H} = \left\{ u \in \mathfrak{X} \, ; \, \langle \mathbf{\Xi}_\lambda, u \rangle = 0 \text{ for any } \lambda > 0 \right\} \cap \mathfrak{X}_+$$

is clearly a closed convex subcone of $\mathfrak{X}_+$. Moreover, if $0 \leqslant u \leqslant v$ with $v \in \mathcal{H}$ then, for any $\lambda > 0$, $\langle \mathbf{\Xi}_\lambda, v \rangle = 0$ and consequently $\langle \mathbf{\Xi}_\lambda, u \rangle = 0$ since $\mathbf{\Xi}_\lambda$ is positive, i.e. $\mathcal{H}$ is a closed hereditary subcone of $\mathfrak{X}_+$. It is easy to see that $\mathscr{H} := \mathcal{H} - \mathcal{H}$ is the linear space generated by $\mathcal{H}$. Then, by [28, Lemma 2], $\mathscr{H}$ is an order ideal with positive cone $\mathcal{H}$. The fact that $\mathscr{H}$ is invariant under the semigroup $(\mathcal{V}(t))_{t \geqslant 0}$ follows from the previous Proposition. □

A priori, in the general setting above, it is *not clear* that $\mathscr{H}$ is closed in $\mathfrak{X}$. However, we have more precise results in AL-spaces (i.e. Banach lattices with additive norm) and in preduals of von Neumann algebras.

PROPOSITION 3.18. *(i) If* $\mathfrak{X}$ *is a* AL*-space then* $\mathscr{H}$ *is a closed lattice ideal (and therefore a projection band) of* $\mathfrak{X}$. *In particular, there exists a band projection* $\mathbf{P}$ *onto* $\mathscr{H}$ *such that* $\mathscr{H} = \mathbf{P}\mathfrak{X}$ *and* $\mathfrak{X} = \mathscr{H} \oplus \mathscr{H}_d$ *where the disjoint complement* $\mathscr{H}_d$ *of* $\mathscr{H}$ *is given by* $\mathscr{H}_d = (I - \mathbf{P})\mathfrak{X}$.

(ii) Let $\mathfrak{X}$ *be the predual of a von Neumann algebra. Then,* $\mathscr{H}$ *is a closed order ideal.*

Proof. (i) Let $(u_n)_n \subset \mathscr{H}$ be such that $u_n \to u$ in $\mathfrak{X}$. By assumption, $u_n = v_n - w_n$ with $v_n, \, w_n \in \mathcal{H}$. In particular, $|u_n| \leqslant v_n + w_n$ and $\langle \mathbf{\Xi}_\lambda, |u_n| \rangle \leqslant \langle \mathbf{\Xi}_\lambda, v_n \rangle + \langle \mathbf{\Xi}_\lambda, w_n \rangle = 0$ whence $|u_n| \in \mathcal{H}$. It follows that the negative and positive parts u_n^- and u_n^+ both belong to $\mathcal{H}$. Since $\mathfrak{X}$ is a vector lattice, the mappings $v \in \mathfrak{X} \mapsto v^\pm \in \mathfrak{X}_+$ are continuous

[29, Proposition 5.2], one has $u_n^{\pm} \to u^{\pm}$ and u^+, u^- belong to $\mathcal{H}$. This proves that $u = u^+ - u^- \in \mathscr{H}$.

(ii) If $\mathfrak{A}$ is a von Neumann algebra and $\mathfrak{X} = \mathfrak{A}_\star$ is its predual, then the mapping $u \in \mathfrak{X} \mapsto |u| \in \mathfrak{X}_+$ is continuous (see e.g. [31, Proposition 4.10, p. 415]) and then, arguing as in (i), one gets the conclusion. □

REMARK 3.19. *In the above case (i), the positive cone of the disjoint complement $\mathscr{H}_d$ does not contain non-trivial elements with a honest trajectory. In particular, dishonest trajectories are all emanating from elements of the positive cone of $\mathfrak{X} = \mathscr{H} \oplus \mathscr{H}_d$ having a non-trivial component over $\mathscr{H}_d$.*

We now deal with two practical examples for concrete spaces:

Example 1: The space of bounded signed measures. Let $(\Sigma, \mathcal{F})$ be a measure space and $\mathfrak{X} = \mathcal{M}(\Sigma, \mathcal{F})$ denote the Banach space of all bounded signed measures over $(\Sigma, \mathcal{F})$ endowed with the total variation norm:

$$\|\mu\| = |\mu|(\Sigma), \qquad \forall \mu \in \mathcal{M}.$$

We recall here that $\mathfrak{X} = \mathcal{M}(\Sigma, \mathcal{F})$ is a AL-space [29, Example 3, p. 114] and every $\mu \in \mathfrak{X}$ splits as $\mu = \mu_+ - \mu$ where $\mu_\pm \in \mathfrak{X}_+$ and $|\mu| = \mu_+ + \mu_-$. Given two measures μ and ν of $\mathfrak{X}$, we shall denote $\nu \prec \mu$ if ν is absolutely continuous with respect to $|\mu|$. Using the terminology of [1], we shall say that a closed subspace $\mathscr{A}$ of $\mathfrak{X} = \mathcal{M}(\Sigma, \mathcal{F})$ is a M-ideal if, for any $\mu \in \mathscr{A}$ and any $\nu \in \mathfrak{X}$, $\nu \prec \mu$ implies $\nu \in \mathscr{A}$. Then, one has the following

PROPOSITION 3.20. *A subspace $\mathscr{A}$ of $\mathcal{M}(\Sigma, \mathcal{F})$ is a M-ideal of $\mathcal{M}$ if and only if $\mathscr{A}$ is a closed and positively generated order ideal of $\mathcal{M}(\Sigma, \mathcal{F})$.*

Proof. Let us first assume that $\mathscr{A}$ is a closed and positively generated order ideal of $\mathfrak{X}$ and let $\mu \in \mathscr{A}$ and $\nu \in \mathfrak{X}$ such that $\nu \prec \mu$. From Radon-Nikodym Theorem, there is some $h \in L^1(\Sigma, \mathcal{F}), \mathrm{d}\,|\mu|)$ such that $\nu = h\,|\mu|$. Thus, $|\nu| = |h|\,|\mu|$ and

$$\lim_{n \to \infty} \| |\nu| - \beta_n \| = 0$$

where $\beta_n := (|h| \wedge n)\,|\mu|$. Indeed $\beta_n \leqslant |\nu|$ for any $n \in \mathbb{N}$ and

$$\| |\nu| - \beta_n \| = |\nu|\,(\Sigma) - \beta_n(\Sigma) = \int_\Sigma [|h| - (|h| \wedge n)]\,\mathrm{d}\,|\mu|$$

goes to zero as $n \to \infty$ according to the dominated convergence theorem. Now, $\beta_n \leqslant n\,|\mu|$ with $|\mu| \in \mathscr{A}$ (see Remark 3.16) and, from the ideal property, $\beta_n \in \mathcal{A}$. From the closedness of $\mathscr{A}$, one gets that $|\nu| \in \mathcal{A}$. Since $-|\nu| \leqslant \nu \leqslant |\nu|$, one finally obtains $\nu \in \mathcal{A}$ and $\mathscr{A}$ is a M-ideal. Conversely, let $\mathscr{A}$ be a M-ideal. By definition, if $\mu \in \mathscr{A}$ then $|\mu| \in \mathscr{A}$ and $\mu_\pm \in \mathscr{A}$. In particular, $\mathscr{A} = (\mathscr{A} \cap \mathfrak{X}_+) - (\mathscr{A} \cap \mathfrak{X}_+)$. Moreover, since

$0 \leqslant \mu \leqslant \nu \implies \mu \prec \nu$, one sees that $\mathscr{A} \cap \mathfrak{X}_+$ is an hereditary subcone of $\mathfrak{X}_+$ and $\mathscr{A}$ is an order ideal of $\mathfrak{X}$ according to [28, Lemma 2]. □

One deduces from this the following which allows to give a complete description of the state μ leading to a dishonest trajectory (see Remark 3.19):

PROPOSITION 3.21. *Under the assumptions of Theorem 3.17 with $\mathfrak{X} = \mathcal{M}(\Sigma, \mathcal{F})$, one has $\mathscr{H}$ is a M-ideal of $\mathfrak{X}$ and $\mathfrak{X} = \mathscr{H} \oplus \mathscr{H}_d$ where*

$$\mathscr{H}_d = \{\mu \in \mathfrak{X} = \mathcal{M}(\Sigma, \mathcal{F}) \ \text{ such that } \nu \prec \mu \text{ and } \nu \in \mathscr{H} \implies \nu = 0\}. \tag{3.7}$$

Proof. We saw in Theorem 3.17 that $\mathscr{H}$ is a closed lattice ideal of $\mathfrak{X}$. In particular, one can define a band projection $\mathbf{P}$ onto $\mathscr{H}$ such that $\mathscr{H} = \mathbf{P}\mathfrak{X}$ and the disjoint complement $\mathscr{H}_d$ of $\mathscr{H}$ given by $\mathscr{H}_d = (I - \mathbf{P})\mathfrak{X}$ are such that $\mathfrak{X} = \mathscr{H} \oplus \mathscr{H}_d$ [29]. Since, according to Prop. 3.20, $\mathscr{H}$ is a M-ideal of $\mathfrak{X}$, one deduces from [1] that $\mathscr{H}_d = \mathscr{H}^\perp$ where $\mathscr{H}^\perp$ is given by (3.7). □

Example 2: The space of trace class operators. We assume here that $\mathfrak{X} = \mathscr{T}_s(\mathfrak{h})$ is the Banach space of all linear self-adjoint trace class operators on some separable Hilbert space $\mathfrak{h}$ endowed with the trace norm $\|\varrho\| = \mathrm{Trace}[\,|\varrho|\,]$ for any $\varrho \in \mathfrak{X}$ (see [22] for details). The scalar product of $\mathfrak{h}$ shall be denoted by $(\cdot, \cdot)$. Under the assumptions of the present section, one deduces from [22, Theorem 5] that, for any $\lambda > 0$, there exists $\beta_\lambda \in \mathscr{L}_s^+(\mathfrak{h})$ such that

$$\langle \Xi_\lambda, u \rangle = \mathrm{Trace}[\beta_\lambda \varrho] \qquad \forall \varrho \in \mathfrak{X}_+$$

where $\mathscr{L}_s^+(\mathfrak{h})$ is the space of all positive bounded self-adjoint operators on $\mathfrak{h}$. One has the following

THEOREM 3.22. *The null space of β_λ is independent of λ and*

$$\mathcal{H} = \{\varrho \in \mathfrak{X}_+ \,;\, \varrho = \mathbf{P}\varrho = \varrho\mathbf{P}\} = \{\varrho \in \mathfrak{X}_+ \,;\, \mathbf{Q}\varrho = \varrho\mathbf{Q} = 0\}$$

where $\mathbf{P}$ is the projection of $\mathfrak{h}$ onto $\mathrm{Null}(\beta_\lambda)$ while $\mathbf{Q} = \mathbf{Id}_\mathfrak{h} - \mathbf{P}$.

Proof. Let $\lambda > 0$ be fixed. According to Theorem 3.15, $\mathcal{H}$ is a closed hereditary subcone of $\mathfrak{X}_+$. On the other hand, closed hereditary cones of $\mathfrak{X}$ are characterized in [10, Lemma 3.2, P. 54-55] which tells us that the set

$$\mathfrak{h}_0 = \{h \in \mathfrak{h} \,;\, |h\rangle\langle h| \in \mathcal{H}\}$$

is a closed linear subspace[1] of $\mathfrak{h}$ and

$$\mathcal{H} = \{\varrho \in \mathfrak{X}_+ \,;\, \varrho = \mathbf{P}\varrho = \varrho\mathbf{P}\}$$

where $\mathbf{P}$ is the orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{h}_0$ while $|h\rangle\langle h|$ denotes the one-dimensional trace class operator : $x \mapsto (x, h)h$. The proof consists in showing that $\mathrm{Null}(\beta_\lambda) = \mathfrak{h}_0$ for

[1]Notice that, in [10, Lemma 3.2, P. 54-55], Davies calls ideal what we call closed hereditary subcone

any $\lambda > 0$. First, let $h \in \mathfrak{h}_0$, $h \neq 0$ and let $\varrho = |h\rangle\langle h|$. For any orthonormal basis $(e_n)_n$ of $\mathfrak{h}$ we have

$$\begin{aligned}\mathrm{Trace}[\beta_\lambda \varrho] &= \sum_n (\beta_\lambda \varrho(e_n), e_n) = \sum_n (\varrho e_n, \beta_\lambda(e_n)) \\ &= \sum_n (h, e_n)\,(h, \beta_\lambda(e_n)) = \sum_n (h, e_n)\,(\beta_\lambda(h), e_n).\end{aligned}$$

Choosing in particular a basis $(e_n)_n$ with $e_0 = h/\|h\|$, one gets that

$$\mathrm{Trace}[\beta_\lambda \varrho] = 0 \Longleftrightarrow (\beta_\lambda(h), h) = 0 \Longleftrightarrow h \in \mathrm{Null}(\beta_\lambda)$$

since $\beta_\lambda \geqslant 0$. This proves that $\mathfrak{h}_0 = \mathrm{Null}(\beta_\lambda)$ which, in particular, turns out to be independent of $\lambda > 0$. Finally, since $\mathbf{PQ} = 0$ and $\mathbf{P} + \mathbf{Q} = \mathbf{Id}$, we see that $\varrho = \mathbf{P}\varrho = \varrho\mathbf{P}$ amounts to $\mathbf{Q}\varrho = \varrho\,\mathbf{Q} = 0$. This is equivalent to $\mathbf{Q}\varrho\,\mathbf{Q} = 0$. □

This allows to provide a full characterization of $\mathscr{H}$:

COROLLARY 3.23. *One has* $\mathscr{H} = \mathcal{H} - \mathcal{H} = \{\varrho \in \mathfrak{X}\,;\, \varrho = \mathbf{P}\varrho = \varrho\mathbf{P}\}$.

Proof. The fact that $\mathscr{H} \subset \{\varrho \in \mathfrak{X}\,;\, \varrho = \mathbf{P}\varrho = \varrho\mathbf{P}\}$ is clear. Conversely, let $\varrho \in \mathfrak{X}$ be such that $\varrho = \mathbf{P}\varrho = \varrho\,\mathbf{P}$. Since $\varrho \in \mathscr{T}_s(\mathfrak{h})$, one has

$$\varrho = \sum_n \alpha_n |e_n\rangle\langle e_n|$$

where $(e_n)_n$ is an orthonormal basis of $\mathfrak{h}$ made of eigenvectors of ϱ associated to the real eigenvalues $(\alpha_n)_n$, i.e. $\varrho(h) = \sum_n \alpha_n (h, e_n) e_n$ for any $h \in \mathfrak{h}$. Since $\varrho = \varrho\,\mathbf{P}$, one has

$$\varrho(h) = \sum_n \alpha_n (h, e_n) e_n = \sum_n \alpha_n (\mathbf{P}h, e_n) e_n = \sum_n \alpha_n (h, \mathbf{P}e_n) e_n \qquad \forall h \in \mathfrak{h}$$

while, since $\mathbf{P}\varrho = \varrho$, one has $\varrho(h) = \sum_n \alpha_n (h, \mathbf{P}e_n)\mathbf{P}e_n$ for any $h \in \mathfrak{h}$. In particular,

$$\varrho = \sum_n \alpha_n |\mathbf{P}e_n\rangle\langle \mathbf{P}e_n|.$$

As we saw in the proof of the above theorem, $|\mathbf{P}e_n\rangle\langle\mathbf{P}e_n| \in \mathcal{H}$ for any $n \in \mathbb{N}$ so that, writing $\alpha_n = \alpha_n^+ - \alpha_n^-$ with $\alpha_n^\pm \geqslant 0$, we see that $\varrho = \varrho^+ - \varrho^-$ with $\varrho^\pm \in \mathcal{H}$. □

3.4. **Sufficient conditions of honesty.** We provide here sufficient conditions of honesty based on the above Theorem 3.11 and on a new derivation of the functional $\mathbf{\Xi}_\lambda$

THEOREM 3.24. *For any* $\lambda > 0$, *let* $(\psi_n(\lambda))_n \subset \mathfrak{X}^\star$ *be defined inductively by*

$$\psi_{n+1}(\lambda) = \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^\star \psi_n(\lambda), \qquad \psi_0(\lambda) = \mathbf{\Psi}$$

where we recall that $\mathbf{\Psi}$ is the positive functional defined in (1.1). Then, $(\psi_n(\lambda))_n$ is nonincreasing and converges in the weak-$\star$ topology of $\mathfrak{X}$ to $\psi(\lambda)$ such that

$$\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \psi(\lambda) = \psi(\lambda). \tag{3.8}$$

Moreover, $\psi(\lambda) = \mathbf{\Xi}_\lambda$ for all $\lambda > 0$ and $\mathbf{\Xi}_\lambda$ is the maximal element of $\{\psi \in \mathfrak{X}^\star,\ \psi \leqslant \mathbf{\Psi}\}$ satisfying (3.8).

Proof. It is clear that $[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^\star$ is a positive contraction in $\mathfrak{X}^\star$. Then, for all $\psi \in \mathfrak{X}^\star_+$ with $\|\psi\| \leqslant 1$,

$$\left\| \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \psi \right\| \leqslant 1$$

or, in an equivalent way,

$$\left\langle \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \psi, u \right\rangle \leqslant \|u\| = \langle \mathbf{\Psi}, u\rangle\,, \qquad \forall u \in \mathfrak{X}_+,$$

i.e. $\mathbf{\Psi} - [\mathcal{B}(\lambda-\mathcal{A})^{-1}]^\star \psi$ is an element of the positive cone of $\mathfrak{X}^\star$. Actually, it is straightforward to see that, for any given $u \in \mathfrak{X}_+$, the sequence $\left(\langle \psi_n(\lambda), u\rangle\right)_n$ is bounded and nonincreasing in $\mathbb{R}_+$. This means that $(\psi_n(\lambda))_n$ converges in the weak-$\star$ topology to some $\psi(\lambda) \leqslant \mathbf{\Psi}$. Let $u \in \mathfrak{X}_+$ be given. Then,

$$\langle \psi_{n+1}(\lambda), u\rangle = \left\langle \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \psi_n(\lambda), u \right\rangle = \left\langle \psi_n(\lambda), \mathcal{B}(\lambda-\mathcal{A})^{-1} u \right\rangle$$

so, letting $n \to \infty$,

$$\langle \psi(\lambda), u\rangle = \left\langle \psi(\lambda), \mathcal{B}(\lambda-\mathcal{A})^{-1} u \right\rangle$$

which shows (3.8). Now, since

$$\begin{aligned}
\langle \mathbf{\Xi}_\lambda, u\rangle &= \lim_{n\to\infty} \left\langle \mathbf{\Psi}, \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n+1} u \right\rangle \\
&= \lim_{n\to\infty} \left\langle \left(\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^{n+1} \right)^\star \mathbf{\Psi}, u \right\rangle \\
&= \lim_{n\to\infty} \langle \psi_{n+1}(\lambda), u\rangle = \langle \psi(\lambda), u\rangle
\end{aligned}$$

one sees that $\psi(\lambda) = \mathbf{\Xi}_\lambda$. Let us now prove that $\psi(\lambda) = \mathbf{\Xi}_\lambda$ is the maximal element of $\{\psi \in \mathfrak{X}^\star,\ 0 \leqslant \psi \leqslant \mathbf{\Psi}\}$ satisfying (3.8) ($\lambda > 0$). To do so, let ψ be in the positive cone of $\mathfrak{X}^\star$, $\psi \leqslant \mathbf{\Psi}$ be such that $[\mathcal{B}(\lambda-\mathcal{A})^{-1}]^\star \psi = \psi$. Then,

$$\psi = \left(\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \right)^n \psi \leqslant \left(\left[\mathcal{B}(\lambda-\mathcal{A})^{-1}\right]^\star \right)^n \mathbf{\Psi}$$

which proves, letting n go to infinity, that $\psi \leqslant \mathbf{\Xi}_\lambda$. □

As a consequence, one has

COROLLARY 3.25. *Assume there exists $\lambda > 0$ such that $\mathcal{B}(\lambda-\mathcal{A})^{-1}$ is irreducible. Then, the whole semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is honest if and only if there is some $u \in \mathfrak{X}_+$, $u \neq 0$, for which the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest.*

Proof. We give two proofs of this result. The first one uses Theorem 3.14 and the second one the spectral interpretation of the functional Ξ_λ.

Proof 1. Let $u \in \mathfrak{X}_+ \setminus \{0\}$ and $\omega \in \mathfrak{X}_+^\star \setminus \{0\}$. Then, $(\lambda - \mathcal{A}^\star)^{-1}\omega \in \mathfrak{X}_+^\star \setminus \{0\}$ and

$$\begin{aligned}\left\langle \omega, (\lambda - \mathcal{G})^{-1} u \right\rangle &= \sum_{k=0}^{\infty} \left\langle \omega, (\lambda - \mathcal{A})^{-1} \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^k u \right\rangle \\ &= \sum_{k=0}^{\infty} \left\langle (\lambda - \mathcal{A}^\star)^{-1}\omega, \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^k u \right\rangle > 0\end{aligned}$$

where we used the fact that there exists $k_0 > 0$ such that

$$\left\langle (\lambda - \mathcal{A}^\star)^{-1}\omega, \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^{k_0} u \right\rangle > 0.$$

One obtains then that $\langle \omega, (\lambda - \mathcal{G})^{-1} u\rangle > 0$ for any $\omega \in \mathfrak{X}_+^\star \setminus \{0\}$, i.e. $(\lambda - \mathcal{G})^{-1}u$ is quasi-interior for any $u \in \mathfrak{X}_+ \setminus \{0\}$. Thus, $(\mathcal{V}(t))_{t\geqslant 0}$ is irreducible and Theorem 3.14 leads to the conclusion.

Proof 2. Let $\mathcal{B}(\lambda - \mathcal{A})^{-1}$ be irreducible and assume there exists some honest trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ with $u \in \mathfrak{X}_+ \setminus \{0\}$. Then, from Theorem 3.11, $\langle \Xi_\lambda, u\rangle = 0$. Assume that $\Xi_\lambda \neq 0$. Then, for any $z \in \mathfrak{X}_+ \setminus \{0\}$, there exists an integer $n \geqslant 0$ such that

$$\left\langle \Xi_\lambda, \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n z \right\rangle > 0.$$

According to Theorem 3.24, it is clear that

$$\langle \Xi_\lambda, z\rangle = \left\langle \left(\left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^\star\right)^n \Xi_\lambda, z \right\rangle = \left\langle \Xi_\lambda, \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n z \right\rangle$$

i.e. $\langle \Xi_\lambda, z\rangle > 0$ for any $z \in \mathfrak{X}_+ \setminus \{0\}$. This is a contradiction and, necessarily, $\Xi_\lambda = 0$. Thus, the whole semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is honest. □

We end this section with two practical sufficient conditions ensuring the existence of honest trajectories:

THEOREM 3.26. *Let $\lambda > 0$ and $u \in \mathfrak{X}_+$ be such that*

$$\mathcal{B}(\lambda - \mathcal{A})^{-1} u \leqslant u, \tag{3.9}$$

then the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest.

Proof. Since $\mathcal{B}(\lambda - \mathcal{A})^{-1}$ is positive, our assumption (3.9) implies that the sequence $([\mathcal{B}(\lambda - \mathcal{A})^{-1}]^n u)_n$ is nonincreasing in $\mathfrak{X}$ and

$$\left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n u \leqslant u, \qquad \forall n \geqslant 1.$$

Therefore the whole sequence $\left([\mathcal{B}(\lambda - \mathcal{A})^{-1}]^n u\right)_n$ is convergent in $\mathfrak{X}$ which ends the proof because of Theorem 3.5. □

This provides another honesty criterion in terms of sub-eigenvalues of $\mathcal{A} + \mathcal{B}$.

COROLLARY 3.27. *Assume that there exists* $\lambda > 0$ *and* $u \in \mathscr{D}(\mathcal{A})_+$ *such that* $(\mathcal{A}+\mathcal{B})u \leqslant \lambda u$*, Then,* $(\mathcal{V}(t)u)_{t\geqslant 0}$ *is honest.*

Proof. Define $z = (\lambda - \mathcal{A})u$. One has $z \geqslant \mathcal{B}u \geqslant 0$ and z satisfies (3.9). The trajectory $(\mathcal{V}(t)z)_{t\geqslant 0}$ is therefore honest from Theorem 3.26. Defining $v = (\lambda - \mathcal{G})^{-1}z$, one has also that $(\mathcal{V}(t)v)_{t\geqslant 0}$ is honest (see Proposition 3.13). Since $0 \leqslant u = (\lambda - \mathcal{A})^{-1}z \leqslant v$, $(\mathcal{V}(t)u)_{t\geqslant 0}$ is honest since $\mathcal{H}$ is a closed hereditary subcone of $\mathfrak{X}_+$ (see Theorem 3.17). □

3.5. **Instantaneous dishonesty.** According to Definition 3.8, if a trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is not honest, then there exists $t_0 \geqslant 0$ such that

$$\|\mathcal{V}(t_0)u\| < \|u\| - \overline{\mathfrak{a}}\left(\int_0^{t_0} \mathcal{V}(s)u\mathrm{d}s\right) \tag{3.10}$$

This suggests to introduce the following mass loss functional

$$\Delta_u(t) = \|\mathcal{V}(t)u\| - \|u\| + \overline{\mathfrak{a}}\left(\int_0^{t} \mathcal{V}(s)u\mathrm{d}s\right), \qquad t \geqslant 0.$$

One has the following property:

LEMMA 3.28. *For any* $u \in \mathfrak{X}_+$*, the mapping* $t \geqslant 0 \mapsto \Delta_u(t)$ *is nonincreasing.*

Proof. Let $t_2 \geqslant t_1 \geqslant 0$ be fixed. Then,

$$\begin{aligned}\Delta_u(t_2) - \Delta_u(t_1) &= \|\mathcal{V}(t_2)u\| - \|\mathcal{V}(t_1)u\| + \overline{\mathfrak{a}}(\int_{t_1}^{t_2} \mathcal{V}(s)u\mathrm{d}s)\\ &= \langle \mathbf{\Psi}, \mathcal{V}(t_2)u - \mathcal{V}(t_1)u\rangle + \overline{\mathfrak{a}}(\int_{t_1}^{t_2} \mathcal{V}(s)u\mathrm{d}s).\end{aligned}$$

Since $\mathcal{V}(t_2)u - \mathcal{V}(t_1)u = \mathcal{G}\int_{t_1}^{t_2} \mathcal{V}(s)u\mathrm{d}s$, one sees that

$$\Delta_u(t_2) - \Delta_u(t_1) = \overline{\mathfrak{a}}(\int_{t_1}^{t_2} \mathcal{V}(s)u\mathrm{d}s) - \mathfrak{a}_0(\int_{t_1}^{t_2} \mathcal{V}(s)u\mathrm{d}s) \geqslant 0,$$

since $\overline{\mathfrak{a}}$ always dominate $\mathfrak{a}_0$. □

LEMMA 3.29. *Let* $u \in \mathfrak{X}_+$*. If the trajectory* $(\mathcal{V}(t)u)_{t\geqslant 0}$ *is dishonest, then there exists* $t_0 > 0$ *such that* $\Delta_u(t) < 0$ *for any* $t > t_0$ *and* $\Delta_v(t) < 0$ *for any* $t > 0$ *where* $v = \mathcal{V}(t_0)u$*.*

Proof. By definition of dishonest trajectory and since $\Delta_u(t) \leqslant 0$ for any $t \geqslant 0$, one has

$$t_0 := \inf\{t > 0 \text{ such that } \Delta_u(t) < 0\}$$

is well-defined. Since $\Delta_u(\cdot)$ is nonincreasing, one has $\Delta_u(t) < 0$ for any $t > t_0$. Moreover, since the mapping $t \mapsto \Delta_u(t) \in (-\infty, 0]$ is clearly continuous, one has $\Delta_u(t) = 0$ for any $t \in [0, t_0]$. Set $v = \mathcal{V}(t_0)u$. For any $t > 0$, since $\Delta_u(t + t_0) < 0$ one has

$$\|\mathcal{V}(t)v\| = \|\mathcal{V}(t + t_0)u\| < \|u\| - \overline{\mathfrak{a}}(\int_0^{t+t_0} \mathcal{V}(s)u\mathrm{d}s)$$

while the identity $\Delta_u(t_0) = 0$ reads $\|u\| = \|v\| + \overline{\mathfrak{a}}(\int_0^{t_0} \mathcal{V}(s)u\mathrm{d}s)$. Consequently,

$$\begin{aligned}\|\mathcal{V}(t)v\| &< \|\mathcal{V}(t_0)u\| + \overline{\mathfrak{a}}\left(\int_0^{t_0} \mathcal{V}(s)u\mathrm{d}s\right) - \overline{\mathfrak{a}}\left(\int_0^{t+t_0} \mathcal{V}(s)u\mathrm{d}s\right)\\ &= \|v\| - \overline{\mathfrak{a}}\left(\int_{t_0}^{t+t_0} \mathcal{V}(s)u\mathrm{d}s\right) = \|v\| - \overline{\mathfrak{a}}\left(\int_0^{t} \mathcal{V}(r)v\mathrm{d}r\right)\end{aligned}$$

i.e. $\Delta_v(t) < 0$ for all $t > 0$. □

To summarize, when the semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is dishonest, that is, if some trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is not honest, then it is possible to find some $z \in \mathfrak{X}_+ \setminus \{0\}$ such that the trajectory emanating from z is *instantaneously* dishonest, i.e. $\Delta_z(t) < 0$ for any $t > 0$. In particular, whenever

$$\langle \mathbf{\Psi}, (\mathcal{A} + \mathcal{B})u\rangle = 0 \qquad \forall u \in \mathscr{D}_+,$$

the semigroup $(\mathcal{V}(t))_{t\geqslant 0}$ is dishonest if and only if there exists some $z \in \mathfrak{X}_+ \setminus \{0\}$ such that

$$\|\mathcal{V}(t)z\| < \|z\|, \qquad \forall t > 0.$$

THEOREM 3.30. *Assume that* $\langle \mathbf{\Psi}, (\mathcal{A} + \mathcal{B})u\rangle = 0$ *for any* $u \in \mathscr{D}_+$. *If* $\mathcal{G} \neq \overline{\mathcal{A} + \mathcal{B}}$ *then*

$$\|\mathcal{V}(t)u\| < \|u\|, \qquad t > 0$$

for any quasi-interior $u \in \mathfrak{X}_+$.

Proof. If $(\mathcal{V}(t))_{t\geqslant 0}$ is dishonest, then, according to Lemma 3.29, there exists $z \in \mathfrak{X}_+ \setminus \{0\}$ such that $\|\mathcal{V}(t)z\| < \|z\|$ for all $t > 0$. In particular,

$$\langle \mathbf{\Psi}, \mathcal{V}(t)z - z\rangle < 0, \qquad \forall t > 0.$$

Define $\mathcal{Z}_t := \mathbf{\Psi} - \mathcal{V}^\star(t)\mathbf{\Psi} \in \mathfrak{X}^\star$, for any $t > 0$ where $(\mathcal{V}^\star(t))_{t\geqslant 0}$ is the dual contractions semigroup of $(\mathcal{V}(t))_{t\geqslant 0}$. Since $\langle \mathbf{\Psi}, \mathcal{V}(t)u - u\rangle \leqslant 0$ for any $u \in \mathfrak{X}_+$, $\mathcal{Z}_t$ belongs to the positive cone $\mathfrak{X}^\star_+$ of $\mathfrak{X}^\star$ for any $t \geqslant 0$ while

$$\langle \mathcal{Z}_t, z\rangle > 0, \qquad \forall t > 0.$$

Therefore, $\mathcal{Z}_t$ belongs to $\mathfrak{X}^\star_+ \setminus \{0\}$ for any $t > 0$. Therefore, for any quasi-interior $u \in \mathfrak{X}_+$ one has

$$\langle \mathcal{Z}_t, u\rangle > 0, \qquad \forall t > 0.$$

This proves the result. □

REMARK 3.31. *Whenever $\mathfrak{X}$ **is an** AL**-space**, it is possible to prove a more general result of immediate dishonesty by resuming in a straightforward way the arguments of* [23, Corollary 2.12]. *Precisely, recall that, if $\mathfrak{X}$ is an AL-space, $\mathscr{H}$ is a projection band of $\mathfrak{X}$ (see Prop. 3.18) and let $\mathbf{P}$ be the band projection onto $\mathscr{H}$. Then, one can prove the following: let us assume that $(\mathcal{V}(t))_{t\geqslant 0}$ is not honest and let $u \in \mathfrak{X}_+$ be such that $v = (I-\mathbf{P})u$ is a quasi-interior element of the disjoint complement of $\mathscr{H}$. Then, the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is **immediately dishonest**, i.e. $\|\mathcal{V}(t)u\| < \|u\| - \overline{\mathfrak{a}}(\int_0^t \mathcal{V}(s)u\mathrm{d}s)$ for any $t>0$. However, from the technical point of view, the formal arguments need the use of the concept of local honesty as in* [23].

4. ON HONESTY THEORY: DYSON-PHILLIPS APPROACH

We establish here an alternative of concept of honesty of the trajectory in terms of the Dyson-Phillips iterated defined by (2.15). To do so, we have first to investigate several fine properties of these iteration terms.

4.1. Fine properties of the Dyson-Phillips iterations.

The various terms of the Dyson-Phillips series (2.15) enjoy the following properties:

PROPOSITION 4.1. *For any $n \in \mathbb{N}$, $n \geqslant 1$, the Dyson-Phillips iterated defined in* (2.15) *satisfy:*

(1) For any $u \in \mathscr{D}(\mathcal{A})$, the mapping $t \in (0,\infty) \longmapsto \mathcal{V}_n(t)u$ is continuously differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}_n(t)u = \mathcal{V}_n(t)\mathcal{A}u + \mathcal{V}_{n-1}(t)\mathcal{B}u.$$

(2) For any $u \in \mathscr{D}(\mathcal{A})$, $\mathcal{V}_n(t)u \in \mathscr{D}(\mathcal{A})$, the mapping $t \in (0,\infty) \longmapsto \mathcal{A}\mathcal{V}_n(t)u$ is continuous and

$$\mathcal{A}\mathcal{V}_n(t)u = \mathcal{V}_n(t)\mathcal{A}u + \mathcal{V}_{n-1}(t)\mathcal{B}u - \mathcal{B}\mathcal{V}_{n-1}(t)u$$

(3) For any $u \in \mathfrak{X}$ and any $t \geqslant 0$, $\displaystyle\int_0^t \mathcal{V}_n(s)u\mathrm{d}s \in \mathscr{D}(\mathcal{A})$, the mapping $t \in (0,\infty) \longmapsto \mathcal{A}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s$ is continuous with

$$\mathcal{A}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s = \mathcal{V}_n(t)u - \mathcal{B}\int_0^t \mathcal{V}_{n-1}(s)u\mathrm{d}s. \tag{4.1}$$

(4) For any $u \in \mathfrak{X}_+$ and any $t \geqslant 0$,

$$\left\langle \mathbf{\Psi}, \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s \right\rangle \leqslant -\langle \mathbf{\Psi}, \mathcal{V}_n(t)u\rangle + \left\langle \mathbf{\Psi}, \mathcal{B}\int_0^t \mathcal{V}_{n-1}(s)u\mathrm{d}s \right\rangle. \tag{4.2}$$

(5) For any $u \in \mathfrak{X}$, and $\lambda > 0$, the limit

$$\lim_{t\to\infty} \int_0^t \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s =: \int_0^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s$$

exists in the graph norm of $\mathcal{A}$ and

$$(\lambda - \mathcal{A}) \int_0^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s = \mathcal{B} \int_0^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s. \tag{4.3}$$

REMARK 4.2. *Notice that, in Eq. (4.3), $\mathcal{B}\int_0^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s$ is well-defined since $\mathcal{B}$ is $\mathcal{A}$-bounded and the integral converges in the graph norm of $\mathcal{A}$. Moreover, it is easily deduced from (4.3) that*

$$\mathcal{B} \int_0^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s = \left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^{n+1} u, \qquad \forall u \in \mathfrak{X},\ n \geqslant 1. \tag{4.4}$$

Proof. We first recall that the formula (2.15) reads on $\mathscr{D}(\mathcal{A})$ as:

$$\mathcal{V}_{n+1}(t)u = \int_0^t \mathcal{V}_n(t-s)\mathcal{B}\mathcal{U}(s)u\mathrm{d}s, \qquad \forall u \in \mathscr{D}(\mathcal{A}), \qquad t \geqslant 0, \qquad n \in \mathbb{N}.$$

Then

$$h^{-1}\mathcal{V}_{n+1}(h)u = h^{-1} \int_0^h \mathcal{V}_n(h-s)\mathcal{B}\mathcal{U}(s)u \longrightarrow \mathcal{V}_n(0)\mathcal{B}\mathcal{U}(0)u \qquad \text{as} \quad h \to 0^+$$

because the mapping $(s,h) \mapsto \mathcal{V}_n(h-s)\mathcal{B}\mathcal{U}(s)u$ is strongly continuous on $\{(s,h) \in \mathbb{R}_+ \times \mathbb{R}_+\,;\, 0 \leqslant s \leqslant h\}$. Since $\mathcal{V}_0(0) = \mathcal{U}_0(0) = \mathrm{Id}$ while $\mathcal{V}_n(0) = 0$ for any $n \geqslant 1$, we see that

$$\lim_{h\to 0} h^{-1}\mathcal{V}_{n+1}(h)u = \begin{cases} \mathcal{B}u & \text{when } n = 0 \\ 0 & \text{when } n \geqslant 1, \end{cases} \qquad \forall u \in \mathscr{D}(\mathcal{A}). \tag{4.5}$$

(1) Let $n \geqslant 1$ be fixed. Let $u \in \mathscr{D}(\mathcal{A})$ and $t, h > 0$ be fixed. One deduces from (2.17) that, given $t \geqslant 0$ and $h > 0$,

$$\begin{aligned} \mathcal{V}_n(t+h)u - \mathcal{V}_n(t)u &= \sum_{k=0}^{n-1} \mathcal{V}_k(t)\mathcal{V}_{n-k}(h)u + \mathcal{V}_n(t)(\mathcal{V}_0(h)u - u) \\ &= \sum_{k=1}^{n} \mathcal{V}_{n-k}(t)\mathcal{V}_k(h)u + \mathcal{V}_n(t)(\mathcal{V}_0(h)u - u). \end{aligned}$$

Thus,

$$h^{-1}(\mathcal{V}_n(t+h)u - \mathcal{V}_n(t)u) = \sum_{k=1}^{n} \mathcal{V}_{n-k}(t)\left(\frac{1}{h}\mathcal{V}_k(h)u\right) + \mathcal{V}_n(t)\frac{\mathcal{V}_0(h)u - u}{h}$$

which yields, since $u \in \mathscr{D}(\mathcal{A})$,

$$\lim_{h\to 0^+} \frac{\mathcal{V}_n(t+h)u - \mathcal{V}_n(t)u}{h} = \mathcal{V}_{n-1}(t)\mathcal{B}u + \mathcal{V}_n(t)\mathcal{A}u.$$

Similarly, it is easy to prove that, for any $t > 0$ and any $0 < h < t$,

$$\mathcal{V}_n(t)u - \mathcal{V}_n(t-h)u = \sum_{k=1}^{n} \mathcal{V}_{n-k}(t-h)\mathcal{V}_k(h)u + \mathcal{V}_n(t-h)(\mathcal{V}_0(h)u - u) \tag{4.6}$$

and therefore

$$\lim_{h\to 0^+} \frac{\mathcal{V}_n(t)u - \mathcal{V}_n(t-h)u}{h} = \mathcal{V}_{n-1}(t)\mathcal{B}u + \mathcal{V}_n(t)\mathcal{A}u.$$

Since, for any $u \in \mathscr{D}(\mathcal{A})$, the mapping $t \mapsto \mathcal{V}_{n-1}(t)\mathcal{B}u + \mathcal{V}_n(t)\mathcal{A}u$ is continuous (see Remark 2.4), property *(1)* holds true.

(2) Let $u \in \mathscr{D}(\mathcal{A})$. It is clear that the two properties

$$\mathcal{V}_k(t)u \in \mathscr{D}(\mathcal{A}) \qquad \text{and} \qquad t \geqslant 0 \mapsto \mathcal{A}\mathcal{V}_k(t)u \text{ is continuous}$$

hold true for $k = 0$. Let $n \geqslant 1$ be fixed and assume the above properties hold true for any $k \leqslant n$ and prove they still hold for $k = n+1$. For any $t, h > 0$, Eq. (2.17) yields

$$\mathcal{V}_{n+1}(t+h)u = \mathcal{V}_{n+1}(h+t)u = \sum_{k=0}^{n+1} \mathcal{V}_k(h)\mathcal{V}_{n+1-k}(t)u$$

so that

$$\mathcal{V}_0(h)\mathcal{V}_{n+1}(t)u - \mathcal{V}_{n+1}(t)u = (\mathcal{V}_{n+1}(t+h)u - \mathcal{V}_{n+1}(t)u) - \sum_{k=1}^{n+1} \mathcal{V}_k(h)\mathcal{V}_{n+1-k}(t)u. \tag{4.7}$$

Assume now $u \in \mathscr{D}(\mathcal{A})$, by virtue of point *(1)* and (4.5), we have

$$\lim_{h\to 0^+} \frac{\mathcal{V}_0(h)\mathcal{V}_{n+1}(t)u - \mathcal{V}_{n+1}(t)u}{h} = \mathcal{V}_n(t)\mathcal{B}u + \mathcal{V}_{n+1}(t)\mathcal{A}u - \mathcal{B}\mathcal{V}_n(t)u.$$

This shows that $\mathcal{V}_{n+1}(t)u \in \mathscr{D}(\mathcal{A})$ for any $u \in \mathscr{D}(\mathcal{A})$ with

$$\mathcal{A}\mathcal{V}_{n+1}(t)u = \mathcal{V}_{n+1}(t)\mathcal{A}u + \mathcal{V}_n(t)\mathcal{B}u - \mathcal{B}\mathcal{V}_n(t)u$$

and proves *(2)* since the continuity of the mapping $t \geqslant 0 \mapsto \mathcal{A}\mathcal{V}_{n+1}(t)u$ is easy to prove.

(3) The first part of point *(3)* clearly holds for $n = 0$. Let $u \in \mathfrak{X}$ and $n \in \mathbb{N}$ be fixed. Assume that, for any $t \geqslant 0$ and any $k \leqslant n$, $\displaystyle\int_0^t \mathcal{V}_k(s)u\mathrm{d}s \in \mathscr{D}(\mathcal{A})$, the mapping

$t \in (0,\infty) \longmapsto \mathcal{A}\int_0^t \mathcal{V}_k(s)u\mathrm{d}s$ is continuous. Let us prove the result for $k = n+1$. Let $t, h > 0$. From (4.7) we have

$$\begin{aligned}(\mathcal{V}_0(h) - \mathrm{Id}) \int_0^t \mathcal{V}_{n+1}(s)u\mathrm{d}s &= \int_0^t (\mathcal{V}_0(h)\mathcal{V}_{n+1}(s)u - \mathcal{V}_{n+1}(s)u)\mathrm{d}s \\ &= \int_0^t (\mathcal{V}_{n+1}(s+h)u - \mathcal{V}_{n+1}(s)u)\mathrm{d}s - \sum_{k=1}^{n+1} \mathcal{V}_k(h) \int_0^t \mathcal{V}_{n+1-k}(s)u\mathrm{d}s \\ &= \int_t^{t+h} \mathcal{V}_{n+1}(r)u\mathrm{d}r - \int_0^h \mathcal{V}_{n+1}(r)u\mathrm{d}r - \sum_{k=1}^{n+1} \mathcal{V}_k(h) \int_0^t \mathcal{V}_{n+1-k}(s)u\mathrm{d}s.\end{aligned}$$

Since we assumed that $\int_0^t \mathcal{V}_j(s)u\mathrm{d}s \in \mathscr{D}(\mathcal{A}) \subset \mathscr{D}(\mathcal{B})$ for any $0 \leqslant j \leqslant n$, we deduce immediately

$$\lim_{h\to 0^+} h^{-1}(\mathcal{V}_0(h) - \mathrm{Id}) \int_0^t \mathcal{V}_{n+1}(s)u\mathrm{d}s = \mathcal{V}_{n+1}(t)u - \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s$$

where we used (4.5) and the fact that $h^{-1}\int_t^{t+h} \mathcal{V}_{n+1}u\mathrm{d}s \to \mathcal{V}_{n+1}(t)u$ as $h \to 0^+$. Therefore, property *(3)* holds true for $n+1$.

(4) Let $u \in \mathfrak{X}$ and $t \geqslant 0$ be fixed. Applying (2.1) to $v = \displaystyle\int_0^t \mathcal{V}_n(s)u\mathrm{d}s$ (which belongs to $\mathscr{D}(\mathcal{A})$ from point *(3)*), one deduces easily (4.2) from (4.1).

(5) It is clear that the definition of $\mathcal{V}_n(t)$ given in (2.15) is equivalent to

$$\exp(-\lambda t)\mathcal{V}_{n+1}(t)u = \int_0^t \exp\left(-\lambda(t-s)\right)\mathcal{V}_n(t-s)\mathcal{B}\left[\exp\left(-\lambda s\right)\mathcal{U}(s)\right]u\mathrm{d}s$$

for any $u \in \mathscr{D}(\mathcal{A})$, $n \in \mathbb{N}$ and any $\lambda > 0$. Moreover, for any $\lambda > 0$, the operators $\mathcal{A}_\lambda := \mathcal{A} - \lambda$ (with domain $\mathscr{D}(\mathcal{A})$) and $\mathcal{B}$ satisfy the assumptions of Theorems 2.1 since

$$\langle \mathbf{\Psi}, (\mathcal{A}_\lambda + \mathcal{B})u\rangle \leqslant -\lambda\langle \mathbf{\Psi}, u\rangle \leqslant 0, \qquad \forall u \in \mathscr{D}(\mathcal{A})_+,\ \lambda > 0.$$

One sees then that there is an extension of $\mathcal{A}_\lambda + \mathcal{B}$ that generates a C_0-semigroup $(\mathcal{V}_\lambda(t))_{t\geqslant 0}$ in $\mathfrak{X}$. Clearly, the family $(\exp(-\lambda t)\mathcal{V}_n(t))_{n\in\mathbb{N}}$ is the family of Dyson-Phillips iterated associated to $\mathcal{A}_\lambda$, $\mathcal{B}$ and $\mathcal{V}_\lambda(t)$. In particular, applying Formula (4.1) to $\mathcal{A}_\lambda$, $\mathcal{B}$ and $(\exp(-\lambda t)\mathcal{V}_n(t))_{n\in\mathbb{N}}$, one gets

$$\begin{aligned}\mathcal{A}\int_0^t \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s &= \exp(-\lambda t)\mathcal{V}_n(t)u + \lambda\int_0^t \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \\ &\quad - \mathcal{B}\int_0^t \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s, \quad \forall\lambda > 0, \quad \forall u \in \mathfrak{X},\ n \geqslant 1. \quad (4.8)\end{aligned}$$

Notice that, since for any $n \in \mathbb{N}$, $\mathcal{V}_n(t) = \mathscr{L}^n(\mathcal{U})(t)$, we already saw in the proof of Theorem 2.3 that, for any $u \in \mathfrak{X}$ and any $\lambda > 0$, the limit

$$\lim_{t\to\infty} \int_0^t \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s$$

exists in $\mathfrak{X}$ and

$$\int_0^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s = (\lambda - \mathcal{A})^{-1}\left[\mathcal{B}(\lambda - \mathcal{A})^{-1}\right]^n u, \qquad \forall n \in \mathbb{N}. \tag{4.9}$$

Now, for $n = 0$, since

$$\mathcal{A}\int_0^t \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s = \exp(-\lambda t)\mathcal{U}(t)u - u + \lambda\int_0^t \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s$$

one easily sees that the limit $\lim_{t\to\infty}\mathcal{A}\int_0^t \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s$ exists in $\mathfrak{X}$ with

$$\lim_{t\to\infty}\mathcal{A}\int_0^t \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s = -u + \lambda\int_0^\infty \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s,$$

i.e. $\displaystyle\int_0^\infty \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s$ converges in the graph norm of $\mathcal{A}$. Since $\mathcal{B}$ is $\mathcal{A}$-bounded, the limit

$$\lim_{t\to\infty}\mathcal{B}\int_0^t \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s = \mathcal{B}\int_0^\infty \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s = \mathcal{B}(\lambda - \mathcal{A})^{-1}u$$

exists in $\mathfrak{X}$. Now, applying (4.8) to $n = 1$, the integral $\displaystyle\int_0^\infty \exp(-\lambda s)\mathcal{V}_1(s)u\mathrm{d}s$ converges in the graph norm of $\mathcal{A}$ with

$$\mathcal{A}\int_0^\infty \exp(-\lambda s)\mathcal{V}_1(s)u\mathrm{d}s = \lambda\int_0^\infty \exp(-\lambda s)\mathcal{V}_1(s)u\mathrm{d}s - \mathcal{B}\int_0^\infty \exp(-\lambda s)\mathcal{U}(s)u\mathrm{d}s$$

and, as above, since $\mathcal{B}$ is $\mathcal{A}$-bounded,

$$\lim_{t\to\infty}\mathcal{B}\int_0^t \exp(-\lambda s)\mathcal{V}_1(s)u\mathrm{d}s = \mathcal{B}\int_0^\infty \exp(-\lambda s)\mathcal{V}_1(s)u\mathrm{d}s$$

converges in $\mathfrak{X}$. A simple induction leads to the result for any $n \in \mathbb{N}$. □

REMARK 4.3. *Note that $\mathcal{A}$ is closed but a priori $\mathcal{A}+\mathcal{B}$ is not; however for $u \in \mathcal{D}(\mathcal{A})$*

$$\begin{aligned}(\mathcal{A}+\mathcal{B})\int_0^t \mathcal{V}_k(r)u\mathrm{d}r &= (\overline{\mathcal{A}+\mathcal{B}})\int_0^t \mathcal{V}_k(r)u\mathrm{d}r = \int_0^t (\overline{\mathcal{A}+\mathcal{B}})\mathcal{V}_k(r)u\mathrm{d}r \\ &= \int_0^t (\overline{\mathcal{A}+\mathcal{B}})\mathcal{V}_k(r)u\mathrm{d}r = \int_0^t (\mathcal{A}+\mathcal{B})\mathcal{V}_k(r)u\mathrm{d}r \\ &= \int_0^t (\mathcal{A}+\mathcal{B})\mathcal{V}_k(r)u\mathrm{d}r = \int_0^t \mathcal{A}\mathcal{V}_k(r)u\mathrm{d}r + \int_0^t \mathcal{B}\mathcal{V}_k(r)u\mathrm{d}r;\end{aligned} \tag{4.10}$$

in particular

$$\mathcal{B}\int_0^t \mathcal{V}_k(r)u\mathrm{d}r = \int_0^t \mathcal{B}\mathcal{V}_k(r)u\mathrm{d}r.$$

From the above Proposition, $\lim_{t\to\infty} \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s$ converges to zero for any $n \in \mathbb{N}$ and any $u \in \mathfrak{X}_+$. Actually, this convergence is uniform with respect to n:

PROPOSITION 4.4. *For any $\lambda > 0$ and any $u \in \mathfrak{X}$, one has*

$$\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \left\| \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \right\| = 0.$$

Proof. The combination of (4.8) and (4.3) gives

$$\begin{aligned}\mathcal{A}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s &= -e^{-\lambda t}\mathcal{V}_n(t)u + \lambda\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \\ &\quad - \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s\end{aligned}$$

so that

$$\begin{aligned}\left\langle \mathbf{\Psi}, \mathcal{A}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\, ds \right\rangle &= \left\langle \mathbf{\Psi}, \lambda\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \right\rangle \\ &\quad - \left\langle \mathbf{\Psi}, e^{-\lambda t}\mathcal{V}_n(t)u \right\rangle - \left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s \right\rangle.\end{aligned}$$

Since $\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \in \mathscr{D}(\mathcal{A})_+$ for $u \in \mathfrak{X}_+$ then by (2.1)

$$\left\langle \mathbf{\Psi}, \mathcal{A}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \right\rangle \leqslant -\left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s \right\rangle$$

whence

$$\begin{aligned}&\left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s\right\rangle + \left\langle \mathbf{\Psi}, \lambda\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s\right\rangle\\ \leqslant\ & \left\langle \mathbf{\Psi}, e^{-\lambda t}\mathcal{V}_n(t)u\right\rangle + \left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s\right\rangle.\end{aligned}$$

In particular for all n

$$\left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s\right\rangle \leqslant \left\langle \mathbf{\Psi}, e^{-\lambda t}\mathcal{V}_n(t)u\right\rangle + \left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_{n-1}(s)u\mathrm{d}s\right\rangle$$

and it follows by induction that

$$\begin{aligned}\left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s\right\rangle &\leqslant \sum_{j=1}^n \left\langle \mathbf{\Psi}, e^{-\lambda t}\mathcal{V}_j(t)u\right\rangle + \\ &\left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_0(s)u\mathrm{d}s\right\rangle\\ &\leqslant \left\langle \mathbf{\Psi}, e^{-\lambda t}\mathcal{V}(t)u\right\rangle + \left\langle \mathbf{\Psi}, \mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_0(s)u\mathrm{d}s\right\rangle\end{aligned}$$

and then

$$\left\|\mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_n(s)u\mathrm{d}s\right\| \leqslant e^{-\lambda t}\left\|u\right\| + \left\|\mathcal{B}\int_t^\infty \exp(-\lambda s)\mathcal{V}_0(s)u\mathrm{d}s\right\|$$

which ends the proof since $\mathfrak{X} = \mathfrak{X}_+ - \mathfrak{X}_+$. □

4.2. **A new functional.** While, in Section 3, we introduced a functional $\overline{\mathfrak{a}}$ related to $\mathfrak{a}$ through the resolvent $(\lambda - \mathcal{A})^{-1}$, we introduce here a new functional $\widehat{\mathfrak{a}}$ constructed through the Dyson-Phillips iteration terms:

PROPOSITION 4.5. *Under the assumption of Theorem 2.1, for any $v \in \mathscr{D}(\mathcal{G})$, there exists*

$$\lim_{t\to 0^+}\frac{1}{t}\sum_{n=0}^\infty \mathfrak{a}\left(\int_0^t \mathcal{V}_n(s)v\mathrm{d}s\right) =: \widehat{\mathfrak{a}}(v) \tag{4.11}$$

with $|\widehat{\mathfrak{a}}(v)| \leqslant 4M\left(\|v\| + \|\mathcal{G}v\|\right)$. Furthermore, for $v \in \mathscr{D}(\mathcal{G})_+$, $\widehat{\mathfrak{a}}(v) \leqslant \mathfrak{a}_0(v) \leqslant \|\mathcal{G}v\|$.

Proof. First, one notices that, for any $u \in \mathfrak{X}_+$, $n \in \mathbb{N}$ and any $t > 0$, one has

$$\sum_{k=0}^n \mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right) \leqslant \mathfrak{a}_0\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right) = -\left\langle \mathbf{\Psi}, \mathcal{G}\int_0^t \mathcal{V}(s)u\mathrm{d}s\right\rangle.$$

In particular, the series $\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)u\mathrm{d}s\right)$ converges with

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)u\mathrm{d}s\right)\leqslant-\left\langle\mathbf{\Psi},\mathcal{G}\int_0^t\mathcal{V}(s)u\mathrm{d}s\right\rangle\leqslant\|u\|. \tag{4.12}$$

Now, for any integers $0<n_1<n_2<n_3$, since, for any $s,r\geqslant 0$

$$\begin{aligned}\sum_{k=0}^{n_1}\mathcal{V}_k(s)\left(\sum_{p=0}^{n_2}\mathcal{V}_p(r)u\right)\leqslant\sum_{k=0}^{2n_2}\sum_{p=0}^{2n_2-k}\mathcal{V}_k(s)\mathcal{V}_p(r)u=\sum_{k=0}^{2n_2}\mathcal{V}_k(s+r)u\\ \leqslant\sum_{k=0}^{2n_2}\mathcal{V}_k(s)\left(\sum_{p=0}^{2n_3}\mathcal{V}_p(r)u\right),\qquad\forall u\in\mathfrak{X}_+\end{aligned}$$

we get, for any $t,\,\tau>0$

$$\begin{aligned}\sum_{k=0}^{n_1}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\sum_{p=0}^{n_2}\mathcal{V}_p(r)u\mathrm{d}r\right]\mathrm{d}s\right)\leqslant\sum_{k=0}^{2n_2}\mathfrak{a}\left(\int_0^t\mathrm{d}s\int_0^\tau\mathcal{V}_k(s+r)u\mathrm{d}r\right)\\ \leqslant\sum_{k=0}^{2n_2}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\sum_{p=0}^{2n_3}\mathcal{V}_p(r)u\mathrm{d}r\right]\mathrm{d}s\right)\qquad\forall u\in\mathfrak{X}_+.\end{aligned}$$

Letting first n_3 then n_2 and finally n_1 go to infinity, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right)\leqslant\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathrm{d}s\int_0^\tau\mathcal{V}_k(s+r)u\mathrm{d}r\right)\\ \leqslant\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right)\end{aligned}$$

i.e.

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right)=\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathrm{d}s\int_0^\tau\mathcal{V}_k(s+r)u\mathrm{d}r\right),\quad\forall u\in\mathfrak{X}_+.$$

In particular, for any $t,\tau>0$

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)\left[\int_0^\tau\mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right)=\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^\tau\mathcal{V}_k(s)\left[\int_0^t\mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right). \tag{4.13}$$

From Eq. (4.12)

$$\left|\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)u\mathrm{d}s\right)\right|\leqslant 2M\|u\|\qquad\forall u\in\mathfrak{X}.$$

Since $\lim_{\tau\to 0^+}\tau^{-1}\int_0^\tau \mathcal{V}(s)u\mathrm{d}s = u$, one gets that

$$\lim_{\tau\to 0^+}\frac{1}{\tau}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)\left[\int_0^\tau \mathcal{V}(r)u\mathrm{d}r\right]\mathrm{d}s\right) = \sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right) \qquad \forall t>0. \tag{4.14}$$

Now, for $u\in\mathscr{D}(\mathcal{G})_+$, Eq. (4.12) reads

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right) \leqslant -\left\langle \mathbf{\Psi}, \int_0^t \mathcal{V}(s)\mathcal{G}u\mathrm{d}s\right\rangle = \left\|\int_0^t \mathcal{V}(s)\mathcal{G}u\mathrm{d}s\right\| \leqslant t\|\mathcal{G}u\| \tag{4.15}$$

since $\|\mathbf{\Psi}\|\leqslant 1$. One extends this estimate to $\mathscr{D}(\mathcal{G})$ in the following way: let $u\in\mathscr{D}(\mathcal{G})$ be given and let $v=u-\mathcal{G}u\in\mathfrak{X}$. Then, there exist v_1,v_2 in $\mathfrak{X}_+$ with $v=v_1-v_2$ and $\|v_i\|\leqslant M\|v\|$, $i=1,2$. Set $u_i=(1-\mathcal{G})^{-1}v_i$, $i=1,2$. Then, $u_i\in\mathscr{D}(\mathcal{G})_+$, $\|u_i\|\leqslant\|v_i\|$, $i=1,2$ and

$$\|\mathcal{G}u_1\|+\|\mathcal{G}u_2\| \leqslant 2\left(\|v_1\|+\|v_2\|\right)\leqslant 4M\|v\|\leqslant 4M\left(\|u\|+\|\mathcal{G}u\|\right).$$

Now, from (4.15),

$$\left|\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right)\right| = \left|\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)(u_1-u_2)\mathrm{d}s\right)\right| \leqslant t\left(\|\mathcal{G}u_1\|+\|\mathcal{G}u_2\|\right)$$

i.e.

$$\left|\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right)\right| \leqslant 4Mt\left(\|u\|+\|\mathcal{G}u\|\right) \qquad \forall u\in\mathscr{D}(\mathcal{G}),\ t>0. \tag{4.16}$$

For any $v\in\mathfrak{X}$ and any $t_1,t_2>0$ fixed, applying the above estimate to $u=\frac{1}{t_1}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r - \frac{1}{t_2}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\in\mathscr{D}(\mathcal{G})$ we get

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)\left[\frac{1}{t_1}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right]\mathrm{d}s\right)\right| \leqslant \\ 4Mt\left\|\frac{1}{t_1}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\| + \\ 4Mt\left\|\frac{1}{t_1}\mathcal{G}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\mathcal{G}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\|\end{aligned}$$

which, by virtue of (4.13), reads

$$\left|\frac{1}{t_1}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_1}\mathcal{V}_k(s)z\mathrm{d}s\right)-\frac{1}{t_2}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_2}\mathcal{V}_k(s)z\mathrm{d}s\right)\right|\leqslant$$
$$4M\left\|\frac{1}{t_1}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\|+4M\left\|\frac{1}{t_1}\mathcal{G}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\mathcal{G}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\|$$

where $z=t^{-1}\int_0^t\mathcal{V}(r)v\mathrm{d}r$. Letting now $t\to 0^+$ one deduces from (4.14) that

$$\left|\frac{1}{t_1}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_1}\mathcal{V}_k(s)v\mathrm{d}s\right)-\frac{1}{t_2}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_2}\mathcal{V}_k(s)v\mathrm{d}s\right)\right|\leqslant$$
$$4M\left\|\frac{1}{t_1}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\|+4M\left\|\frac{1}{t_1}\mathcal{G}\int_0^{t_1}\mathcal{V}(r)v\mathrm{d}r-\frac{1}{t_2}\mathcal{G}\int_0^{t_2}\mathcal{V}(r)v\mathrm{d}r\right\|$$

If $v\in\mathscr{D}(\mathcal{G})$ then $\frac{1}{t_i}\mathcal{G}\int_0^{t_i}\mathcal{V}(r)v\mathrm{d}r=\frac{1}{t_i}\int_0^{t_i}\mathcal{V}(r)\mathcal{G}v\mathrm{d}r$, $i=1,2$ and it is easy to see that, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\left|\frac{1}{t_1}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_1}\mathcal{V}_k(s)v\mathrm{d}s\right)-\frac{1}{t_2}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^{t_2}\mathcal{V}_k(s)v\mathrm{d}s\right)\right|\leqslant 4M\varepsilon,\qquad\forall 0<t_1<t_2<\delta.$$

This achieves to prove that, for any $v\in\mathscr{D}(\mathcal{G})$, the limit $\lim_{t\to0^+}\frac{1}{t}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)v\mathrm{d}s\right)$ exists. We denote this limit by $\widehat{\mathfrak{a}}(v)$ and the first part of the Theorem is proved. The first estimate $|\widehat{\mathfrak{a}}(v)|\leqslant 4M\left(\|v\|+\|\mathcal{G}v\|\right)$ is a direct consequence of (4.16). Finally, since

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)v\mathrm{d}s\right)\leqslant\mathfrak{a}_0\left(\int_0^t\mathcal{V}(s)v\mathrm{d}s\right)$$

one gets that

$$\widehat{\mathfrak{a}}(v)=\lim_{t\to0^+}t^{-1}\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)v\mathrm{d}s\right)\leqslant\lim_{t\to0^+}t^{-1}\mathfrak{a}_0\left(\int_0^t\mathcal{V}(s)v\mathrm{d}s\right)=\mathfrak{a}_0(v)$$

since $\lim_{t\to0^+}\frac{1}{t}\int_0^t\mathcal{V}(s)v\mathrm{d}s=v$ in the graph norm of $\mathcal{G}$ and $\mathfrak{a}_0(\cdot)$ is continuous with respect to the graph norm of $\mathscr{D}(\mathcal{G})$. The fact that $\mathfrak{a}_0(v)\leqslant\|\mathcal{G}v\|$ is a direct consequence of the estimate $\|\mathbf{\Psi}\|\leqslant 1$. □

Before investigating further properties of the functional $\hat{\mathfrak{a}}$ we need to establish several properties of the various terms $\mathcal{V}_n(t)$ appearing in (2.15).

4.3. **Further properties of $\widehat{\mathfrak{a}}$.** We are now in position to establish very useful properties of the functional $\widehat{\mathfrak{a}}$ complementing Proposition 4.5.

PROPOSITION 4.6. *The functional $\widehat{\mathfrak{a}}(\cdot) : \mathscr{D}(\mathcal{G}) \to \mathbb{R}$ defined by (4.11) is such that $\widehat{\mathfrak{a}}(v) = \mathfrak{a}_0(v)$ for any $v \in \mathscr{D}(\mathcal{A})$. Consequently,*

$$\widehat{\mathfrak{a}}(u) = \mathfrak{a}_0(u), \qquad \forall u \in \mathscr{D}(\overline{\mathcal{A}+\mathcal{B}}).$$

Proof. From (4.1) one sees that, for any $n \geqslant 1$ and any $u \in \mathfrak{X}_+$

$$\sum_{k=0}^{n}(\mathcal{A}+\mathcal{B})\int_0^t \mathcal{V}_k(s)u\mathrm{d}s = \sum_{k=0}^{n}\mathcal{V}_k(s)u - u + \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s, \qquad \forall n \in \mathbb{N}. \quad (4.17)$$

In particular,

$$-\left\langle \mathbf{\Psi}, \sum_{k=0}^{n}(\mathcal{A}+\mathcal{B})\int_0^t \mathcal{V}_k(s)u\mathrm{d}s \right\rangle = \langle \mathbf{\Psi}, u\rangle - \sum_{k=0}^{n}\langle \mathbf{\Psi}, \mathcal{V}_k(s)u\rangle - \left\langle \mathbf{\Psi}, \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s \right\rangle.$$

Letting n go to infinity, we see that $\lim_{n\to\infty}\left\|\mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s\right\|$ exists and

$$\sum_{k=0}^{\infty}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right) = \langle \mathbf{\Psi}, u - \mathcal{V}(t)u\rangle - \lim_{n\to\infty}\left\langle \mathbf{\Psi}, \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s\right\rangle, \qquad \forall t > 0. \quad (4.18)$$

Now, for any $u \in \mathscr{D}(\mathcal{A})_+$ and any $k \geqslant 1$, one deduces from Proposition 4.1, *(2)* that

$$\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_k(s)u\rangle \leqslant -\langle \mathbf{\Psi}, \mathcal{A}\mathcal{V}_k(s)u\rangle = \langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_{k-1}(s)u\rangle - \langle \mathbf{\Psi}, \mathcal{V}_k(s)\mathcal{A}u\rangle - \langle \mathbf{\Psi}, \mathcal{V}_{k-1}(s)\mathcal{B}u\rangle$$

and

$$\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_k(s)u\rangle - \langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_{k-1}(s)u\rangle \leqslant -\langle \mathbf{\Psi}, \mathcal{V}_k(s)\mathcal{A}u\rangle \qquad \forall s \geqslant 0. \quad (4.19)$$

Since, for any $u \in \mathscr{D}(\mathcal{A})$ the series $\sum_{k=0}^{\infty}\mathcal{V}_k(t)\mathcal{A}u$ converges to $\mathcal{V}(t)\mathcal{A}u$ uniformly on every bounded time interval, for any $T > 0$ and any $\varepsilon > 0$, there exists $N \geqslant 1$ such that, for any $s \in (0,T)$ and any $n \geqslant N$, $|\sum_{k=N}^{n}\langle \mathbf{\Psi}, \mathcal{V}_k(s)\mathcal{A}u\rangle| \leqslant \varepsilon$. From (4.19), one gets

$$\sum_{k=N}^{n}\left(\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_k(s)u\rangle - \langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_{k-1}(s)u\rangle\right) \leqslant \varepsilon, \qquad \forall s \in (0,T)$$

i.e. $\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_n(s)u\rangle \leqslant \langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_{N-1}(s)u\rangle + \varepsilon$ for any $s \in (0,T)$. Fixed $N > 1$ and $u \in \mathscr{D}(\mathcal{A})$, the mapping $s \in (0,T) \mapsto \mathcal{B}\mathcal{V}_{N-1}(s)u$ being continuous and converging to zero as $s \to 0^+$, there exists $t > 0$ such that $\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_{N-1}(s)u\rangle < \varepsilon$ for any $0 < s < t$ and consequently $\langle \mathbf{\Psi}, \mathcal{B}\mathcal{V}_n(s)u\rangle < 2\varepsilon$ for any $n > N$ and any $0 < s < t$. Now, from Eq. (4.10) one has

$$\left\langle \mathbf{\Psi}, \mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s\right\rangle = \left\langle \mathbf{\Psi}, \int_0^t \mathcal{B}\mathcal{V}_n(s)u\mathrm{d}s\right\rangle \leqslant 2\varepsilon t \qquad \forall n > N.$$

Then, one deduces from (4.18) that

$$\left|\sum_{k=0}^{\infty} t^{-1}\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right) - \left\langle \mathbf{\Psi}, \frac{u-\mathcal{V}(t)u}{t}\right\rangle\right| \leqslant 2\varepsilon \qquad \forall t>0.$$

Letting $t\to 0^+$, since $\lim_{t\to 0^+} t^{-1}\left\langle \mathbf{\Psi}, \frac{u-\mathcal{V}(t)u}{t}\right\rangle = -\langle \mathbf{\Psi}, \mathcal{G}u\rangle = \mathfrak{a}_0(u)$, we get that $|\widehat{\mathfrak{a}}(u)-\mathfrak{a}_0(u)|\leqslant 2\varepsilon$. This proves that $\widehat{\mathfrak{a}}$ coincides with $\mathfrak{a}_0$ on $\mathscr{D}(\mathcal{A})$ since ε is arbitrary. Finally, if $u\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$, there exists a sequence $(u_n)_n\subset\mathscr{D}(\mathcal{A})$ with $u_n\to u$ and $(\mathcal{A}+\mathcal{B})u_n\to\mathcal{G}u$ as $n\to\infty$. Since $\widehat{\mathfrak{a}}(u_n)=\mathfrak{a}_0(u_n)$ for any $n\in\mathbb{N}$, one deduces easily that $\widehat{\mathfrak{a}}(u)=\mathfrak{a}_0(u)$. □

4.4. **Mild honesty.** We introduce now another concept of honest trajectories. To distinguish it *a priori* from the previous one, we will speak rather of mild honesty.

DEFINITION 4.7. *Let $u\in\mathfrak{X}_+$ be given. Then, the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is said to be **mild honest** if and only if*

$$\|\mathcal{V}(t)u\| = \|u\| - \widehat{\mathfrak{a}}\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right), \qquad \text{for any } t\geqslant 0.$$

We are now in position to state the main result of this section, reminiscent to Theorem 3.5:

THEOREM 4.8. *Given $u\in\mathfrak{X}_+$, the following statements are equivalent*

(1) the trajectory $(\mathcal{V}(t)u)_{t\geqslant 0}$ is mild honest;
(2) $\lim_{n\to\infty}\|\mathcal{B}\int_0^t\mathcal{V}_n(s)u\mathrm{d}s\|=0$ for any $t>0$;
(3) $\int_0^t\mathcal{V}(s)u\mathrm{d}s\in\mathscr{D}(\overline{\mathcal{A}+\mathcal{B}})$ for any $t>0$;
(4) the set $\left(\mathcal{B}\int_0^t\mathcal{V}_n(s)u\mathrm{d}s\right)_n$ is relatively weakly compact in $\mathfrak{X}$ for any $t>0$.

Proof. Let $u\in\mathfrak{X}_+$ and $t>0$ be fixed. One has $\int_0^t\mathcal{V}(s)u\mathrm{d}s\in\mathscr{D}(\mathcal{G})$ and

$$\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right) = \lim_{\tau\to 0^+}\tau^{-1}\sum_{n=0}^{\infty}\mathfrak{a}\left(\int_0^\tau\mathcal{V}_n(s)\mathrm{d}s\left[\int_0^t\mathcal{V}(r)u\mathrm{d}r\right]\right).$$

From (4.13) and (4.14), it is easy to deduce that

$$\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right) = \sum_{n=0}^{\infty}\mathfrak{a}\left(\int_0^t\mathcal{V}_n(s)u\mathrm{d}s\right), \qquad \forall u\in\mathfrak{X}_+,\ t>0. \tag{4.20}$$

Thus, Eq. (4.18) can be rewritten as

$$\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right) = \langle\mathbf{\Psi}, u-\mathcal{V}(t)u\rangle - \lim_{n\to\infty}\left\|\mathcal{B}\int_0^t\mathcal{V}_n(s)u\mathrm{d}s\right\|.$$

This proves immediately that *(1)* $\Longleftrightarrow$ *(2)*. Let us prove that *(2)* $\Longrightarrow$ *(3)*. Observe that, according to (4.17)

$$(\mathcal{A}+\mathcal{B})\left(\sum_{k=0}^{n}\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right)=\sum_{k=0}^{n}\mathcal{V}_k(t)u-u+\mathcal{B}\int_0^t\mathcal{V}_n(s)u\mathrm{d}s$$

so that, from *(2)* we deduce that the right-hand side converges to $\mathcal{V}(t)u-u$ as n goes to infinity. Since $\sum_{k=0}^n\int_0^t\mathcal{V}_k(s)u\mathrm{d}s$ converges to $\int_0^t\mathcal{V}(s)u\mathrm{d}s$ as n goes to infinity, one gets immediately that *(3)* holds with $\left(\overline{\mathcal{A}+\mathcal{B}}\right)\int_0^t\mathcal{V}(s)u\mathrm{d}s=\mathcal{V}(t)u-u$. Let us now assume that *(3)* holds. Then, from (4.6),

$$\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right)=\mathfrak{a}_0\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right)$$

i.e. $\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right)=\|u\|-\|\mathcal{V}(t)u\|$ which is nothing but *(1)*. Assume now *(4)* to hold. Then, up to extracting a subsequence, we may assume that $\mathcal{B}\int_0^t\mathcal{V}_n(s)u\mathrm{d}s$ converges weakly to some $v\in\mathfrak{X}$. Then, $\sum_{k=0}^n\int_0^t\mathcal{V}_k(s)u\mathrm{d}s$ converges weakly to $\int_0^t\mathcal{V}(s)u\mathrm{d}s$ while

$$(\mathcal{A}+\mathcal{B})\sum_{k=0}^{n}\int_0^t\mathcal{V}_k(s)u\mathrm{d}s \text{ converges weakly to } \left(\mathcal{V}(t)u-u-v\right).$$

In particular, $\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s,\mathcal{V}(t)u-u-v\right)$ belongs to the weak closure (and thus the strong closure) of the graph of $\mathcal{A}+\mathcal{B}$. In particular, *(3)* holds. Finally, it is clear that *(2)* $\Longrightarrow$ *(4)*. □

The following result proves that the two notions of honesty and mild honesty are equivalent:

THEOREM 4.9. *The two functionals $\widehat{a}$ and $\overline{a}$ coincide and consequently the notions of honest or mild honest trajectories are equivalent.*

Proof. Let $u\in\mathfrak{X}_+$ and $\lambda>0$ be given. One deduces from (4.20) that

$$\int_0^\infty\exp(-\lambda t)\widehat{\mathfrak{a}}\left(\int_0^t\mathcal{V}(s)u\mathrm{d}s\right)\mathrm{d}t=\sum_{k=0}^{\infty}\int_0^\infty\exp(-\lambda t)\mathfrak{a}\left(\int_0^t\mathcal{V}_k(s)u\mathrm{d}s\right)\mathrm{d}t$$

because all the functions involved are positive. On the other hand, since the mapping $t\geqslant 0\mapsto\int_0^t\mathcal{V}(s)u\mathrm{d}s\in\mathscr{D}(\mathcal{G})$ is continuous as well as the mapping $t\geqslant 0\mapsto\int_0^t\mathcal{V}_k(s)u\mathrm{d}s\in$

$\mathscr{D}(\mathcal{A}) \subset \mathscr{D}(\mathcal{G})$, we have

$$\int_0^\infty \exp(-\lambda t)\widehat{\mathfrak{a}}\left(\int_0^t \mathcal{V}(s)u\mathrm{d}s\right)\mathrm{d}t = \widehat{\mathfrak{a}}\left(\int_0^\infty \exp(-\lambda t)\mathrm{d}t\int_0^t \mathcal{V}(s)u\mathrm{d}s\right)$$
$$= \frac{1}{\lambda}\widehat{\mathfrak{a}}\left(\int_0^\infty \exp(-\lambda s)\mathcal{V}(s)u\mathrm{d}s\right) = \frac{1}{\lambda}\widehat{\mathfrak{a}}\left((\lambda-\mathcal{A})^{-1}u\right)$$

since $\widehat{\mathfrak{a}}$ is $\mathscr{D}(\mathcal{G})$-continuous. We also have, for any $k \in \mathbb{N}$

$$\int_0^\infty \exp(-\lambda t)\mathfrak{a}\left(\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right)\mathrm{d}t = \mathfrak{a}_0\left(\int_0^\infty \exp(-\lambda t)\mathrm{d}t\int_0^t \mathcal{V}_k(s)u\mathrm{d}s\right)$$
$$= \frac{1}{\lambda}\mathfrak{a}_0\left(\int_0^\infty \exp(-\lambda s)\mathcal{V}_k(s)u\mathrm{d}s\right) = \frac{1}{\lambda}\mathfrak{a}_0\left((\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^k u\right)$$
$$= \frac{1}{\lambda}\mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^k u\right)$$

where we used (4.9) and the fact that $\mathfrak{a}_0$ is $\mathscr{D}(\mathcal{G})$-continuous. Hence

$$\widehat{\mathfrak{a}}\left((\lambda-\mathcal{A})^{-1}u\right) = \sum_{k=0}^{\infty}\mathfrak{a}\left((\lambda-\mathcal{A})^{-1}\left(\mathcal{B}(\lambda-\mathcal{A})^{-1}\right)^k u\right)$$

which proves (see Subsection 3.1) that $\widehat{\mathfrak{a}} = \overline{\mathfrak{a}}$. □

REMARK 4.10. *The above provides an alternative proof of Proposition 3.1.*

REMARK 4.11. *The equivalence between the two notions of honesty established here above has some unsuspected consequences. For instance, one notes that Theorems 3.5 and 4.8 imply that, for a given $u \in \mathfrak{X}_+$,*

$$\int_0^t \mathcal{V}(s)u\mathrm{d}s \in \mathscr{D}(\overline{\mathcal{A}+\mathcal{B}}) \quad \forall t>0 \Longleftrightarrow \left[\mathcal{B}(\lambda-\mathcal{A})^{-1}u\right]^n \to 0 \text{ as } n\to\infty.$$

Notice also that, for any $u \in \mathfrak{X}_+$, the mass loss functional $\Delta_u(t)$ defined in Section 3.5 is given by $\Delta_u(t) = \lim_{n\to\infty}\left\|\mathcal{B}\int_0^t \mathcal{V}_n(s)u\mathrm{d}s\right\|$.

REFERENCES

[1] I. ANTONIOU, C. KARANIKAS & S. SHKARIN, Decomposition of spaces of measures, *Infinite Dim. Anal. Quant. Prob. Related Topics*, **11** (2008), 119–126.

[2] L. ARLOTTI, A perturbation theorem for positive contraction semigroups on L^1-spaces with applications to transport equation and Kolmogorov's differential equations, *Acta Appl. Math.*, **23** (1991), 129–144.

[3] L. ARLOTTI, B. LODS & M. MOKHTAR–KHARROUBI, Honesty theory of perturbed evolution families in state spaces, *work in progress.*

[4] L. ARLOTTI & J. BANASIAK, Strictly substochastic semigroups with application to conservative and shattering solution to fragmentation equation with mass loss, *J. Math. Anal. Appl.*, **293** (2004), 673–720.
[5] J. BANASIAK, On an extension of Kato-Voigt perturbation theorem for substochastic semigroups and its applications, *Taiwanese J. Math.*, **5** (2001), 169–191.
[6] J. BANASIAK & L. ARLOTTI, **Perturbations of positive semigroups with applications**, Springer-Verlag, 2005.
[7] C. J. BATTY & D. W. ROBINSON, Positive one-parameter semigroups on ordered Banach spaces, *Acta Appl. Math.* **1** (1984), 221–296.
[8] A. M. CHEBOTAREV & F. FAGNOLA, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, *J. Funct. Anal.* **153** (1998), 382–404.
[9] A. M. CHEBOTAREV & S. Y. SHUSTIKOV, Conditions sufficient for the conservativity of a minimal quantum dynamical semigroup, *Math Notes* **71** (2002), 692–710.
[10] E. B. DAVIES, **Quantum Theory of Open Systems**, Academic Press, 1976.
[11] E. B. DAVIES, Quantum dynamical semigroups and the neutron diffusion equation, *Rep. Math. Phys.*, *11* (1977), 169–188.
[12] W. DESCH, Perturbations of positive semigroups in AL–spaces, unpublished manuscript, 1988.
[13] K. J. ENGEL & R. NAGEL, **One-parameter semigroups for linear evolution equations**, Springer, New-York, 2000.
[14] F. FAGNOLA, Quantum Markov semigroups and quantum flows, *Proyecciones* **18** (1999)? 1–144.
[15] G. FROSALI, C. VAN DER MEE & F. MUGELLI, A characterization theorem for the evolution semigroup generated by the sum of two unbounded operators, *Math. Meth. Appl. Sci.* **27**, (2004), 669–685.
[16] J. C. GARCIA & R. QUEZADA, Hille-Yosida estimates and nonconservativity criteria for quantum dynamical semigroups, *Infinite Dim. Anal. Quant. Prob. Related Topics*, **7** (2004), 383–394.
[17] D. GOSWAMI & K. B. SINHA, Minimal quantum dynamical semigroups on a von Neumann algebra, *Infinite Dim. Anal. Quant. Prob. Related Topics*, **2** (1999), 221–239.
[18] T. KATO, On the semi-groups generated by Kolmogoroff's differential equations, *J. Math. Soc. Jap.*, **6** (1954), 1–15.
[19] M. C. MACKEY & M. TYRAN-KAMINSKA, Dynamics and density evolution in piecewise deterministic growth processes. Preprint 2008.
[20] M. MOKHTAR-KHARROUBI, **Mathematical topics in neutron transport theory, new aspects,** World Scientific, Vol. 46, 1997.
[21] M. MOKHTAR-KHARROUBI, On resolvent positive operators in ordered Banach spaces with additive norm and application to semigroup theory, *Prépublications du laboratoire de mathématiques de Besançon*, **36**, 2007.
[22] M. MOKHTAR-KHARROUBI, On perturbed positive C_0-semigroups on the Banach space of trace class operators, *Infinite Dim. Anal. Quant. Prob. Related Topics*, **11** (2008), 1–21.
[23] M. MOKHTAR-KHARROUBI & J. VOIGT, On honesty of perturbed substochastic C_0-semigroups in L^1-spaces, *J. Operator Theory*, 2009, in press.
[24] M. MOKHTAR-KHARROUBI, On (measure-valued generalization of) Kolmogorov equations, Work in preparation.
[25] B. DE PAGTER, Ordered Banach spaces, **One-parameter semigroups**, (Ph. Clément *et al.* Eds.), 265–279, North-Holland, Amserdam, 1987.
[26] R. REBOLLEDO, Sur les semigroupes dynamiques quantiques, *Annales Math. Blaise Pascal*, **3** (1996), 125–142.

[27] A. RHANDI, Dyson-Phillips expansion ad unbounded perturbations of linear c_0-semigroups, *J. Comput. Appl. Math.*, **44** (1992), 339–349.
[28] D. W. ROBINSON & S. YAMAMURO, Hereditary cones, order ideals and half-norms, *Pacific J. Math.*, **110** (1984), 335–343.
[29] H. H. SCHAEFER (with M. Wolff), **Topological vector spaces**, Springer-Verlag, 1999.
[30] K. B. SINHA, Quantum dynamical semigroups, *Operator Th. Adv. Appl.* **70** (1994), 161–169.
[31] M. TAKESAKI, **Theory of operator algebra I**. Springer, 2002.
[32] H. THIEME & J. VOIGT, Stochastic semigroups: their construction by perturbation and approximation. Proceedings Positivity IV- Theory and Applications, Dresden (Germany), 135–146 (2006).
[33] M. TYRAN-KAMINSKA, Substochastic semigroups and densities of piecewise deterministic Markov processes. *J. Math. Anal. Appl.*, (2009), in press.
[34] J. VOIGT, On the perturbation theory for strongly continuous semigroups, *Math. Ann.*, **229** (1977), 163–171.
[35] J. VOIGT, On substochastic C_0-semigroups and their generators, *Transp. Theory. Stat. Phys*, **16** (1987), 453–466.
[36] J. VOIGT, On resolvent positive operators and positive C_0-semigroups on AL-spaces. *Semigroup Forum*, **38** (1989), 263–266.

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