On Limiting Likelihood Ratio Processes of some Change-Point Type Statistical Models

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Abstract

Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. In this paper we consider two such likelihood ratios. The first one is an exponential functional of a two-sided Poisson process driven by some parameter, while the second one is an exponential functional of a two-sided Brownian motion. We establish that for sufficiently small values of the parameter, the Poisson type likelihood ratio can be approximated by the Brownian type one. As a consequence, several statistically interesting quantities (such as limiting variances of different estimators) related to the first likelihood ratio can also be approximated by those related to the second one. Finally, we discuss the asymptotics of the large values of the parameter and illustrate the results by numerical simulations.

Keywords: non-regularity, change-point, limiting likelihood ratio process, Bayesian estimators, maximum likelihood estimator, limiting distribution, limiting variance, asymptotic efficiency

Mathematics Subject Classification (2000): 62F99, 62M99

1 Introduction

Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. In this paper we consider two of these processes. The first one is the random process Z_ρ on $\mathbb{R}$ defined by

$$\ln Z_\rho(x) = \begin{cases} \rho\,\Pi_+(x) - x, & \text{if } x \geqslant 0, \\ -\,\rho\,\Pi_-(-x) - x, & \text{if } x \leqslant 0, \end{cases} \tag{1}$$

where $\rho > 0$, and Π_+ and Π_- are two independent Poisson processes on $\mathbb{R}_+$ with intensities $1/(e^\rho - 1)$ and $1/(1 - e^{-\rho})$ respectively. We also consider the random variables

$$\zeta_\rho = \frac{\int_{\mathbb{R}} x\, Z_\rho(x)\, dx}{\int_{\mathbb{R}} Z_\rho(x)\, dx} \quad \text{and} \quad \xi_\rho = \operatorname*{argsup}_{x\in\mathbb{R}} Z_\rho(x) \tag{2}$$

related to this process, as well as to their second moments $B_\rho = \mathbf{E}\zeta_\rho^2$ and $M_\rho = \mathbf{E}\xi_\rho^2$.

The process Z_ρ (up to a linear time change) arises in some non-regular, namely change-point type, statistical models as the limiting likelihood ratio process, and the variables ζ_ρ and ξ_ρ (up to a multiplicative constant) as the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator respectively. In particular, B_ρ and M_ρ (up to the square of the above multiplicative constant) are the limiting variances of these estimators, and the Bayesian estimators being asymptotically efficient, the ratio $E_\rho = B_\rho / M_\rho$ is the asymptotic efficiency of the maximum likelihood estimator in these models.

The main such model is the below detailed model of i.i.d. observations in the situation when their density has a jump (is discontinuous). Probably the first general result about this model goes back to Chernoff and Rubin [1]. Later, it was exhaustively studied by Ibragimov and Khasminskii in [10, Chapter 5] (see also their previous works [7] and [8]).

Model 1. Consider the problem of estimation of the location parameter θ based on the observation $X^n = (X_1, \ldots, X_n)$ of the i.i.d. sample from the density $f(x - \theta)$, where the known function f is smooth enough everywhere except at 0, and in 0 we have

$$0 \neq \lim_{x\uparrow 0} f(x) = a \neq b = \lim_{x\downarrow 0} f(x) \neq 0.$$

Denote $\mathbf{P}_\theta^n$ the distribution (corresponding to the parameter θ) of the observation X^n. As $n\to\infty$, the normalized likelihood ratio process of this model defined by

$$Z_n(u)=\frac{d\mathbf{P}^n_{\theta+\frac{u}{n}}}{d\mathbf{P}^n_\theta}(X^n)=\prod_{i=1}^n\frac{f\left(X_i-\theta-\frac{u}{n}\right)}{f(X_i-\theta)}$$

converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ (the Skorohod space of functions on $\mathbb{R}$ without discontinuities of the second kind and vanishing at infinity) to the process $Z_{a,b}$ on $\mathbb{R}$ defined by

$$\ln Z_{a,b}(u)=\begin{cases}\ln(\frac{a}{b})\,\Pi_b(u)-(a-b)\,u, & \text{if } u\geqslant 0,\\[2ex] -\ln(\frac{a}{b})\,\Pi_a(-u)-(a-b)\,u, & \text{if } u\leqslant 0,\end{cases}$$

where Π_b and Π_a are two independent Poisson processes on $\mathbb{R}_+$ with intensities b and a respectively. The limiting distributions of the Bayesian estimators and of the maximum likelihood estimator are given by

$$\zeta_{a,b}=\frac{\int_{\mathbb{R}} u\,Z_{a,b}(u)\,du}{\int_{\mathbb{R}} Z_{a,b}(u)\,du}\quad\text{and}\quad \xi_{a,b}=\operatorname*{argsup}_{u\in\mathbb{R}} Z_{a,b}(u)$$

respectively. The convergence of moments also holds, and the Bayesian estimators are asymptotically efficient. So, $\mathbf{E}\zeta_{a,b}^2$ and $\mathbf{E}\xi_{a,b}^2$ are the limiting variances of these estimators, and $\mathbf{E}\zeta_{a,b}^2/\mathbf{E}\xi_{a,b}^2$ is the asymptotic efficiency of the maximum likelihood estimator.

Now let us note, that up to a linear time change, the process $Z_{a,b}$ is nothing but the process Z_ρ with $\rho=\left|\ln(\frac{a}{b})\right|$. Indeed, by putting $u=\frac{x}{a-b}$ we get

$$\begin{aligned}\ln Z_{a,b}(u)&=\begin{cases}\ln(\frac{a}{b})\,\Pi_b(\frac{x}{a-b})-x, & \text{if } \frac{x}{a-b}\geqslant 0,\\[2ex] -\ln(\frac{a}{b})\,\Pi_a(-\frac{x}{a-b})-x, & \text{if } \frac{x}{a-b}\leqslant 0,\end{cases}\\ &=\ln Z_\rho(x)=\ln Z_\rho\big((a-b)\,u\big).\end{aligned}$$

So, we have

$$\zeta_{a,b}=\frac{\zeta_\rho}{a-b}\quad\text{and}\quad \xi_{a,b}=\frac{\xi_\rho}{a-b}\,,$$

and hence

$$\mathbf{E}\zeta_{a,b}^2=\frac{B_\rho}{(a-b)^2}\,,\quad \mathbf{E}\xi_{a,b}^2=\frac{M_\rho}{(a-b)^2}\quad\text{and}\quad \frac{\mathbf{E}\zeta_{a,b}^2}{\mathbf{E}\xi_{a,b}^2}=E_\rho\,.$$

Some other models where the process Z_ρ arises occur in the statistical inference for inhomogeneous Poisson processes, in the situation when their intensity function has a jump (is discontinuous). In Kutoyants [14, Chapter 5] $\big($see also his previous work [12]$\big)$ one can find several examples, one of which is detailed below.

Model 2. Consider the problem of estimation of the location parameter $\theta \in\,]\alpha,\beta[$, $0<\alpha<\beta<\tau$, based on the observation X^T on $[0,T]$ of the Poisson process with τ-periodic strictly positive intensity function $S(t+\theta)$, where the known function S is smooth enough everywhere except at points $t^*+\tau k$, $k\in\mathbb{Z}$, with some $t^*\in[0,\tau]$, in which we have

$$0\neq \lim_{t\uparrow t^*} S(t) = S_- \neq S_+ = \lim_{t\downarrow t^*} S(t)\neq 0.$$

Denote $\mathbf{P}_\theta^T$ the distribution (corresponding to the parameter θ) of the observation X^T. As $T\to\infty$, the normalized likelihood ratio process of this model defined by

$$Z_T(u)=\frac{d\mathbf{P}^T_{\theta+\frac{u}{T}}}{d\mathbf{P}^T_\theta}(X^T)=\exp\biggl\{\int_0^T \ln\frac{S_{\theta+\frac{u}{T}}(t)}{S_\theta(t)}\,dX(t)-\int_0^T\bigl[S_{\theta+\frac{u}{T}}(t)-S_\theta(t)\bigr]\,dt\biggr\}$$

converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ to the process Z_{τ,S_-,S_+} on $\mathbb{R}$ defined by

$$\ln Z_{\tau,S_-,S_+}=\begin{cases}\ln\bigl(\frac{S_+}{S_-}\bigr)\,\Pi_{S_-}\bigl(\frac{u}{\tau}\bigr)-(S_+-S_-)\,\frac{u}{\tau}\,, & \text{if } u\geqslant 0,\\[2ex] -\ln\bigl(\frac{S_+}{S_-}\bigr)\,\Pi_{S_+}\bigl(-\frac{u}{\tau}\bigr)-(S_+-S_-)\,\frac{u}{\tau}\,, & \text{if } u\leqslant 0,\end{cases}$$

where Π_{S_-} and Π_{S_+} are two independent Poisson processes on $\mathbb{R}_+$ with intensities S_- and S_+ respectively. The limiting distributions of the Bayesian estimators and of the maximum likelihood estimator are given by

$$\zeta_{\tau,S_-,S_+}=\frac{\int_{\mathbb{R}} u\,Z_{\tau,S_-,S_+}(u)\,du}{\int_{\mathbb{R}} Z_{\tau,S_-,S_+}(u)\,du}\quad\text{and}\quad \xi_{\tau,S_-,S_+}=\operatorname*{argsup}_{u\in\mathbb{R}} Z_{\tau,S_-,S_+}(u)$$

respectively. The convergence of moments also holds, and the Bayesian estimators are asymptotically efficient. So, $\mathbf{E}\zeta^2_{\tau,S_-,S_+}$ and $\mathbf{E}\xi^2_{\tau,S_-,S_+}$ are the limiting variances of these estimators, and $\mathbf{E}\zeta^2_{\tau,S_-,S_+}/\mathbf{E}\xi^2_{\tau,S_-,S_+}$ is the asymptotic efficiency of the maximum likelihood estimator.

Now let us note, that up to a linear time change, the process Z_{τ,S_-,S_+} is nothing but the process Z_ρ with $\rho = \left|\ln\left(\frac{S_+}{S_-}\right)\right|$. Indeed, by putting $u = \frac{\tau x}{S_+ - S_-}$ we get

$$Z_{\tau,S_-,S_+}(u) = Z_\rho(x) = Z_\rho\left(\frac{S_+ - S_-}{\tau}\, u\right).$$

So, we have

$$\zeta_{\tau,S_-,S_+} = \frac{\tau\,\zeta_\rho}{S_+ - S_-} \quad \text{and} \quad \zeta_{\tau,S_-,S_+} = \frac{\tau\,\xi_\rho}{S_+ - S_-}\,,$$

and hence

$$\mathbf{E}\zeta^2_{\tau,S_-,S_+} = \frac{\tau^2\, B_\rho}{(S_+ - S_-)^2}\,, \quad \mathbf{E}\xi^2_{\tau,S_-,S_+} = \frac{\tau^2\, M_\rho}{(S_+ - S_-)^2} \quad \text{and} \quad \frac{\mathbf{E}\zeta^2_{\tau,S_-,S_+}}{\mathbf{E}\xi^2_{\tau,S_-,S_+}} = E_\rho\,.$$

The second limiting likelihood ratio process considered in this paper is the random process

$$Z_0(x) = \exp\left\{W(x) - \frac{1}{2}\,|x|\right\}, \quad x \in \mathbb{R}, \tag{3}$$

where W is a standard two-sided Brownian motion. In this case, the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator (up to a multiplicative constant) are given by

$$\zeta_0 = \frac{\int_{\mathbb{R}} x\, Z_0(x)\; dx}{\int_{\mathbb{R}} Z_0(x)\; dx} \quad \text{and} \quad \xi_0 = \operatorname*{argsup}_{x\in\mathbb{R}} Z_0(x) \tag{4}$$

respectively, and the limiting variances of these estimators (up to the square of the above multiplicative constant) are $B_0 = \mathbf{E}\zeta_0^2$ and $M_0 = \mathbf{E}\xi_0^2$.

The models where the process Z_0 arises occur in various fields of statistical inference for stochastic processes. A well-known example is the below detailed model of a discontinuous signal in a white Gaussian noise exhaustively studied by Ibragimov and Khasminskii in [10, Chapter 7.2] (see also their previous work [9]), but one can also cite change-point type models of dynamical systems with small noise (see Kutoyants [12] and [13, Chapter 5]), those of ergodic diffusion processes (see Kutoyants [15, Chapter 3]), a change-point type model of delay equations (see Küchler and Kutoyants [11]), an i.i.d. change-point type model (see Deshayes and Picard [3]), a model of a discontinuous periodic signal in a time inhomogeneous diffusion (see Höpfner and Kutoyants [6]), and so on.

Model 3. Consider the problem of estimation of the location parameter $\theta \in \,]\alpha,\beta[$, $0 < \alpha < \beta < 1$, based on the observation X^ε on $[0,1]$ of the random process satisfying the stochastic differential equation

$$dX^\varepsilon(t) = \frac{1}{\varepsilon}\, S(t-\theta)\,dt + dW(t),$$

where W is a standard Brownian motion, and S is a known function having a bounded derivative on $]-1,0[\,\cup\,]0,1[$ and satisfying

$$\lim_{t\uparrow 0} S(t) - \lim_{t\downarrow 0} S(t) = r \neq 0.$$

Denote $\mathbf{P}_\theta^\varepsilon$ the distribution (corresponding to the parameter θ) of the observation X^ε. As $\varepsilon \to 0$, the normalized likelihood ratio process of this model defined by

$$\begin{aligned} Z_\varepsilon(u) &= \frac{d\mathbf{P}^\varepsilon_{\theta+\varepsilon^2 u}}{d\mathbf{P}^\varepsilon_\theta}(X^\varepsilon) \\ &= \exp\biggl\{\frac{1}{\varepsilon}\int_0^1 \bigl[S(t-\theta-\varepsilon^2 u) - S(t-\theta)\bigr]\,dW(t) \\ &\qquad\qquad - \frac{1}{2\,\varepsilon^2}\int_0^1 \bigl[S(t-\theta-\varepsilon^2 u) - S(t-\theta)\bigr]^2\,dt\biggr\} \end{aligned}$$

converges weakly in the space $\mathcal{C}_0(-\infty,+\infty)$ (the space of continuous functions vanishing at infinity equipped with the supremum norm) to the process $Z_0(r^2 u)$, $u \in \mathbb{R}$. The limiting distributions of the Bayesian estimators and of the maximum likelihood estimator are $r^{-2}\zeta_0$ and $r^{-2}\xi_0$ respectively. The convergence of moments also holds, and the Bayesian estimators are asymptotically efficient. So, $r^{-4}B_0$ and $r^{-4}M_0$ are the limiting variances of these estimators, and E_0 is the asymptotic efficiency of the maximum likelihood estimator.

Let us also note that Terent'yev in [20] determined explicitly the distribution of ξ_0 and calculated the constant $M_0 = 26$. These results were taken up by Ibragimov and Khasminskii in [10, Chapter 7.3], where by means of numerical simulation they equally showed that $B_0 = 19.5 \pm 0.5$, and so $E_0 = 0.73 \pm 0.03$. Later in [5], Golubev expressed B_0 in terms of the second derivative (with respect to a parameter) of an improper integral of a composite function of modified Hankel and Bessel functions. Finally in [18], Rubin and Song obtained the exact values $B_0 = 16\,\zeta(3)$ and $E_0 = 8\,\zeta(3)/13$, where ζ is Riemann's zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ .$$

The random variables ζ_ρ and ξ_ρ and the quantities B_ρ, M_ρ and E_ρ, $\rho > 0$, are much less studied. One can cite Pflug [16] for some results about the distribution of the random variables

$$\operatorname*{argsup}_{x\in\mathbb{R}_+} Z_\rho(x) \quad \text{and} \quad \operatorname*{argsup}_{x\in\mathbb{R}_-} Z_\rho(x)$$

related to ξ_ρ.

In this paper we establish that the limiting likelihood ratio processes Z_ρ and Z_0 are related. More precisely, we show that as $\rho \to 0$, the process $Z_\rho(y/\rho)$, $y \in \mathbb{R}$, converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ to the process Z_0. So, the random variables $\rho\,\zeta_\rho$ and $\rho\,\xi_\rho$ converge weakly to the random variables ζ_0 and ξ_0 respectively. We show equally that the convergence of moments of these random variables holds, that is, $\rho^2 B_\rho \to 16\,\zeta(3)$, $\rho^2 M_\rho \to 26$ and $E_\rho \to 8\,\zeta(3)/13$.

These are the main results of the present paper, and they are presented in Section 2, where we also briefly discuss the second possible asymptotics $\rho \to +\infty$. The necessary lemmas are proved in Section 3. Finally, some numerical simulations of the quantities B_ρ, M_ρ and E_ρ for $\rho \in \,]0,\infty[$ are presented in Section 4.

2 Main results

Consider the process $X_\rho(y) = Z_\rho(y/\rho)$, $y \in \mathbb{R}$, where $\rho > 0$ and Z_ρ is defined by (1). Note that

$$\frac{\int_{\mathbb{R}} y\, X_\rho(y)\, dy}{\int_{\mathbb{R}} X_\rho(y)\, dy} = \rho\,\zeta_\rho \quad \text{and} \quad \operatorname*{argsup}_{y\in\mathbb{R}} X_\rho(y) = \rho\,\xi_\rho\,,$$

where the random variables ζ_ρ and ξ_ρ are defined by (2). Remind also the process Z_0 on $\mathbb{R}$ defined by (3) and the random variables ζ_0 and ξ_0 defined by (4). Recall finally the quantities $B_\rho = \mathbf{E}\zeta_\rho^2$, $M_\rho = \mathbf{E}\xi_\rho^2$, $E_\rho = B_\rho/M_\rho$, $B_0 = \mathbf{E}\zeta_0^2 = 16\,\zeta(3)$, $M_0 = \mathbf{E}\xi_0^2 = 26$ and $E_0 = B_0/M_0 = 8\,\zeta(3)/13$. Now we can state the main result of the present paper.

Theorem 1 *The process X_ρ converges weakly in the space $\mathcal{D}_0(-\infty,+\infty)$ to the process Z_0 as $\rho \to 0$. In particular, the random variables $\rho\,\zeta_\rho$ and $\rho\,\xi_\rho$*

converge weakly to the random variables ζ_0 *and* ξ_0 *respectively. Moreover, for any* $k>0$ *we have*

$$\rho^k\,\mathbf{E}\zeta_\rho^k \to \mathbf{E}\zeta_0^k \quad \textit{and} \quad \rho^k\,\mathbf{E}\xi_\rho^k \to \mathbf{E}\xi_0^k,$$

and in particular $\rho^2 B_\rho \to 16\,\zeta(3)$, $\rho^2 M_\rho \to 26$ *and* $E_\rho \to 8\,\zeta(3)/13$.

The results concerning the random variable ζ_ρ are direct consequence of Ibragimov and Khasminskii [10, Theorem 1.10.2] and the following three lemmas.

Lemma 2 *The finite-dimensional distributions of the process* X_ρ *converge to those of* Z_0 *as* $\rho \to 0$.

Lemma 3 *For all* $\rho>0$ *and all* $y_1, y_2 \in \mathbb{R}$ *we have*

$$\mathbf{E}\left|X_\rho^{1/2}(y_1)-X_\rho^{1/2}(y_2)\right|^2 \leqslant \frac{1}{4}\,|y_1-y_2|\,.$$

Lemma 4 *For any* $c \in \,]\,0\,,\,1/8\,[$ *we have*

$$\mathbf{E}X_\rho^{1/2}(y) \leqslant \exp\bigl(-c\,|y|\bigr)$$

for all sufficiently small ρ *and all* $y \in \mathbb{R}$.

Note that these lemmas are not sufficient to establish the weak convergence of the process X_ρ in the space $\mathcal{D}_0(-\infty,+\infty)$ and the results concerning the random variable ξ_ρ. However, the increments of the process $\ln X_\rho$ being independent, the convergence of its restrictions (and hence of those of X_ρ) on finite intervals $[A,B] \subset \mathbb{R}$ $\bigl($that is, convergence in the Skorohod space $\mathcal{D}[A,B]$ of functions on $[A,B]$ without discontinuities of the second kind$\bigr)$ follows from Gihman and Skorohod [4, Theorem 6.5.5], Lemma 2 and the following lemma.

Lemma 5 *For any* $\varepsilon>0$ *we have*

$$\lim_{h\to 0}\,\lim_{\rho\to 0}\,\sup_{|y_1-y_2|<h}\mathbf{P}\Bigl\{\bigl|\ln X_\rho(y_1)-\ln X_\rho(y_2)\bigr|>\varepsilon\Bigr\}=0.$$

Now, Theorem 1 follows from the following estimate on the tails of the process X_ρ by standard argument.

Lemma 6 *For any $b \in \,]\,0\,,3/40\,[$ we have*

$$\mathbf{P}\left\{\sup_{|y|>A} X_\rho(y) > e^{-bA}\right\} \leqslant 2\,e^{-bA}$$

for all sufficiently small ρ and all $A > 0$.

All the above lemmas will be proved in the next section, but before let us discuss the second possible asymptotics $\rho \to +\infty$. One can show that in this case, the process Z_ρ converges weakly in the space $\mathcal{D}_0(-\infty, +\infty)$ to the process $Z_\infty(u) = e^{-u}\,\mathbb{1}_{\{u>\eta\}}$, $u \in \mathbb{R}$, where η is a negative exponential random variable with $\mathbf{P}\{\eta < t\} = e^t$, $t \leqslant 0$. So, the random variables ζ_ρ and ξ_ρ converge weakly to the random variables

$$\zeta_\infty = \frac{\int_{\mathbb{R}} u\, Z_\infty(u)\; du}{\int_{\mathbb{R}} Z_\infty(u)\; du} = \eta + 1 \quad \text{and} \quad \xi_\infty = \operatorname*{argsup}_{u\in\mathbb{R}} Z_\infty(u) = \eta$$

respectively. One can equally show that, moreover, for any $k > 0$ we have

$$\mathbf{E}\zeta_\rho^k \to \mathbf{E}\zeta_\infty^k \quad \text{and} \quad \mathbf{E}\xi_\rho^k \to \mathbf{E}\xi_\infty^k,$$

and in particular, denoting $B_\infty = \mathbf{E}\zeta_\infty^2$, $M_\infty = \mathbf{E}\xi_\infty^2$ and $E_\infty = B_\infty/M_\infty$, we finally have $B_\rho \to B_\infty = \mathbf{E}(\eta+1)^2 = 1$, $M_\rho \to M_\infty = \mathbf{E}\eta^2 = 2$ and $E_\rho \to E_\infty = 1/2$.

Let us note that these convergences are natural, since the process Z_∞ can be considered as a particular case of the process Z_ρ with $\rho = +\infty$ if one admits the convention $+\infty \cdot 0 = 0$.

Note also that the process Z_∞ (up to a linear time change) is the limiting likelihood ratio process of Model 1 (Model 2) in the situation when $a \cdot b = 0$ ($S_- \cdot S_+ = 0$). In this case, the variables $\zeta_\infty = \eta + 1$ and $\xi_\infty = \eta$ (up to a multiplicative constant) are the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator respectively. In particular, $B_\infty = 1$ and $M_\infty = 2$ (up to the square of the above multiplicative constant) are the limiting variances of these estimators, and the Bayesian estimators being asymptotically efficient, $E_\infty = 1/2$ is the asymptotic efficiency of the maximum likelihood estimator.

3 Proofs of the lemmas

First we prove Lemma 2. Note that the restrictions of the process $\ln X_\rho$ (as well as those of the process $\ln Z_0$) on $\mathbb{R}_+$ and on $\mathbb{R}_-$ are mutually independent processes with stationary and independent increments. So, to obtain the convergence of all the finite-dimensional distributions, it is sufficient to show the convergence of one-dimensional distributions only, that is,

$$\ln X_\rho(y) \Rightarrow \ln Z_0(y) = W(y) - \frac{|y|}{2} = \mathcal{N}\Big(-\frac{|y|}{2}\,, |y|\Big)$$

for all $y \in \mathbb{R}$. Here and in the sequel "$\Rightarrow$" denotes the weak convergence of the random variables, and $\mathcal{N}(m, V)$ denotes a "generic" random variable distributed according to the normal law with mean m and variance V.

Let $y > 0$. Then, noting that $\Pi_+\Big(\dfrac{y}{\rho}\Big)$ is a Poisson random variable of parameter $\lambda = \dfrac{y}{\rho\,(e^\rho - 1)} \to \infty$, we have

$$\begin{aligned}\ln X_\rho(y) &= \rho\,\Pi_+\Big(\frac{y}{\rho}\Big) - \frac{y}{\rho} = \rho\sqrt{\frac{y}{\rho\,(e^\rho - 1)}}\;\frac{\Pi_+\big(\frac{y}{\rho}\big) - \lambda}{\sqrt{\lambda}} + \frac{y}{e^\rho - 1} - \frac{y}{\rho}\\ &= \sqrt{y}\sqrt{\frac{\rho}{e^\rho - 1}}\;\frac{\Pi_+\big(\frac{y}{\rho}\big) - \lambda}{\sqrt{\lambda}} - y\,\frac{e^\rho - 1 - \rho}{\rho\,(e^\rho - 1)} \Rightarrow \mathcal{N}\Big(-\frac{y}{2}\,, y\Big),\end{aligned}$$

since

$$\frac{\rho}{e^\rho - 1} = \frac{\rho}{\rho + o(\rho)} \to 1, \qquad \frac{e^\rho - 1 - \rho}{\rho\,(e^\rho - 1)} = \frac{\rho^2/2 + o(\rho^2)}{\rho\,\big(\rho + o(\rho)\big)} \to \frac{1}{2}$$

and

$$\frac{\Pi_+\big(\frac{y}{\rho}\big) - \lambda}{\sqrt{\lambda}} \Rightarrow \mathcal{N}(0, 1).$$

Similarly, for $y < 0$ we have

$$\begin{aligned}\ln X_\rho(y) &= -\rho\,\Pi_-\Big(\frac{-y}{\rho}\Big) - \frac{y}{\rho} = \rho\sqrt{\frac{-y}{\rho\,(1 - e^{-\rho})}}\;\frac{\lambda' - \Pi_-\big(\frac{-y}{\rho}\big)}{\sqrt{\lambda'}} - \frac{-y}{1 - e^{-\rho}} - \frac{y}{\rho}\\ &= \sqrt{-y}\sqrt{\frac{\rho}{1 - e^{-\rho}}}\;\frac{\lambda' - \Pi_-\big(\frac{-y}{\rho}\big)}{\sqrt{\lambda'}} + y\,\frac{e^{-\rho} - 1 + \rho}{\rho\,(1 - e^{-\rho})} \Rightarrow \mathcal{N}\Big(\frac{y}{2}\,, -y\Big),\end{aligned}$$

and so, Lemma 2 is proved.

Now we turn to the proof of Lemma 4 (we will prove Lemma 3 just after). For $y > 0$ we can write

$$\mathbf{E} X_\rho^{1/2}(y) = \mathbf{E}\exp\Big(\frac{\rho}{2}\,\Pi_+\Big(\frac{y}{\rho}\Big) - \frac{y}{2\rho}\Big) = \exp\Big(-\frac{y}{2\rho}\Big)\;\mathbf{E}\exp\Big(\frac{\rho}{2}\,\Pi_+\Big(\frac{y}{\rho}\Big)\Big).$$

Note that $\Pi_+\Big(\dfrac{y}{\rho}\Big)$ is a Poisson random variable of parameter $\lambda = \dfrac{y}{\rho\,(e^\rho - 1)}$ with moment generating function $M(t) = \exp\big(\lambda\,(e^t-1)\big)$. So, we get

$$
\begin{aligned}
\mathbf{E}X_\rho^{1/2}(y) &= \exp\left(-\frac{y}{2\rho}\right)\,\exp\left(\frac{y}{\rho\,(e^\rho-1)}\,(e^{\rho/2}-1)\right)\\
&= \exp\left(-\frac{y}{2\rho}+\frac{y}{\rho\,(e^{\rho/2}+1)}\right) = \exp\left(-y\,\frac{e^{\rho/2}-1}{2\rho\,(e^{\rho/2}+1)}\right)\\
&= \exp\left(-y\,\frac{e^{\rho/4}-e^{-\rho/4}}{2\rho\,(e^{\rho/4}+e^{-\rho/4})}\right) = \exp\left(-y\,\frac{\tanh(\rho/4)}{2\rho}\right).
\end{aligned}
$$

For $y < 0$ we obtain similarly

$$
\begin{aligned}
\mathbf{E}X_\rho^{1/2}(y) &= \mathbf{E}\exp\left(-\frac{\rho}{2}\,\Pi_-\Big(\frac{-y}{\rho}\Big)-\frac{y}{2\rho}\right)\\
&= \exp\left(-\frac{y}{2\rho}\right)\,\exp\left(\frac{-y}{\rho\,(1-e^{-\rho})}\,(e^{-\rho/2}-1)\right)\\
&= \exp\left(-\frac{y}{2\rho}+\frac{y}{\rho\,(1+e^{-\rho/2})}\right) = \exp\left(y\,\frac{1-e^{-\rho/2}}{2\rho\,(1+e^{-\rho/2})}\right)\\
&= \exp\left(y\,\frac{\tanh(\rho/4)}{2\rho}\right).
\end{aligned}
$$

Thus, for all $y \in \mathbb{R}$ we have

$$
\mathbf{E}X_\rho^{1/2}(y) = \exp\left(-\,|y|\,\frac{\tanh(\rho/4)}{2\rho}\right), \tag{5}
$$

and since

$$
\frac{\tanh(\rho/4)}{2\rho} = \frac{\rho/4+o(\rho)}{2\rho} \to \frac{1}{8}
$$

as $\rho \to 0$, for any $c \in \,]\,0\,,1/8\,[$ we have $\mathbf{E}X_\rho^{1/2}(y) \leqslant \exp\big(-c\,|y|\big)$ for all sufficiently small ρ and all $y \in \mathbb{R}$. Lemma 4 is proved.

Further we verify Lemma 3. We first consider the case $y_1, y_2 \in \mathbb{R}_+$ (say $y_1 \geqslant y_2$). Using (5) and taking into account the stationarity and the independence of the increments of the process $\ln X_\rho$ on $\mathbb{R}_+$, we can write

$$
\begin{aligned}
\mathbf{E}\,\big|X_\rho^{1/2}(y_1)-X_\rho^{1/2}(y_2)\big|^2 &= \mathbf{E}X_\rho(y_1)+\mathbf{E}X_\rho(y_2)-2\,\mathbf{E}X_\rho^{1/2}(y_1)X_\rho^{1/2}(y_2)\\
&= 2-2\,\mathbf{E}X_\rho(y_2)\,\mathbf{E}\frac{X_\rho^{1/2}(y_1)}{X_\rho^{1/2}(y_2)}
\end{aligned}
$$

$$
\begin{aligned}
&= 2 - 2\,\mathbf{E}X_\rho^{1/2}(y_1 - y_2) \\
&= 2 - 2\exp\left(-|y_1 - y_2|\,\frac{\tanh(\rho/4)}{2\rho}\right) \\
&\leqslant |y_1 - y_2|\,\frac{\tanh(\rho/4)}{\rho} \leqslant \frac{1}{4}\,|y_1 - y_2|\,.
\end{aligned}
$$

The case $y_1, y_2 \in \mathbb{R}_-$ can be treated similarly.

Finally, if $y_1 y_2 \leqslant 0$ (say $y_2 \leqslant 0 \leqslant y_1$), we have

$$
\begin{aligned}
\mathbf{E}\left|X_\rho^{1/2}(y_1) - X_\rho^{1/2}(y_2)\right|^2 &= 2 - 2\,\mathbf{E}X_\rho^{1/2}(y_1)\,\mathbf{E}X_\rho^{1/2}(y_2) \\
&= 2 - 2\exp\left(-|y_1|\,\frac{\tanh(\rho/4)}{2\rho} - |y_2|\,\frac{\tanh(\rho/4)}{2\rho}\right) \\
&= 2 - 2\exp\left(-|y_1 - y_2|\,\frac{\tanh(\rho/4)}{2\rho}\right) \\
&\leqslant \frac{1}{4}\,|y_1 - y_2|\,,
\end{aligned}
$$

and so, Lemma 3 is proved.

Now let us check Lemma 5. First let $y_1, y_2 \in \mathbb{R}_+$ (say $y_1 \geqslant y_2$) such that $\Delta = |y_1 - y_2| < h$. Then

$$
\begin{aligned}
\mathbf{P}\Big\{\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big| > \varepsilon\Big\} &\leqslant \frac{1}{\varepsilon^2}\,\mathbf{E}\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big|^2 \\
&= \frac{1}{\varepsilon^2}\,\mathbf{E}\big|\ln X_\rho(\Delta)\big|^2 \\
&= \frac{1}{\varepsilon^2}\,\mathbf{E}\left|\rho\,\Pi_+\Big(\frac{\Delta}{\rho}\Big) - \frac{\Delta}{\rho}\right|^2 \\
&= \frac{1}{\varepsilon^2}\left(\rho^2(\lambda + \lambda^2) + \frac{\Delta^2}{\rho^2} - 2\lambda\Delta\right) \\
&= \frac{1}{\varepsilon^2}\big(\beta(\rho)\,\Delta + \gamma(\rho)\,\Delta^2\big) \\
&< \frac{1}{\varepsilon^2}\big(\beta(\rho)\,h + \gamma(\rho)\,h^2\big)\,,
\end{aligned}
$$

where $\lambda = \frac{\Delta}{\rho\,(e^\rho - 1)}$ is the parameter of the Poisson random variable $\Pi_+\big(\frac{\Delta}{\rho}\big)$,

$$
\beta(\rho) = \frac{\rho}{(e^\rho - 1)} = \frac{\rho}{\rho + o(\rho)} \to 1
$$

and

$$\gamma(\rho) = \frac{1}{(e^\rho - 1)^2} + \frac{1}{\rho^2} - \frac{2}{\rho\,(e^\rho - 1)} = \left(\frac{1}{\rho} - \frac{1}{e^\rho - 1}\right)^2$$
$$= \left(\frac{e^\rho - 1 - \rho}{\rho\,(e^\rho - 1)}\right)^2 = \left(\frac{\rho^2/2 + o(\rho^2)}{\rho\,\big(\rho + o(\rho)\big)}\right)^2 \to \frac{1}{4}$$

as $\rho \to 0$. So, we have

$$\lim_{\rho\to 0} \sup_{|y_1 - y_2| < h} \mathbf{P}\Big\{\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big| > \varepsilon\Big\} \leqslant \lim_{\rho\to 0} \frac{1}{\varepsilon^2}\left(\beta(\rho)\,h + \gamma(\rho)\,h^2\right)$$
$$= \frac{1}{\varepsilon^2}\left(h + \frac{h^2}{4}\right),$$

and hence

$$\lim_{h\to 0} \lim_{\rho\to 0} \sup_{|y_1 - y_2| < h} \mathbf{P}\Big\{\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big| > \varepsilon\Big\} = 0,$$

where the supremum is taken only over $y_1, y_2 \in \mathbb{R}_+$.

For $y_1, y_2 \in \mathbb{R}_-$ such that $\Delta = |y_1 - y_2| < h$ one can obtain similarly

$$\mathbf{P}\Big\{\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big| > \varepsilon\Big\} \leqslant \frac{1}{\varepsilon^2}\,\mathbf{E}\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big|^2$$
$$= \frac{1}{\varepsilon^2}\left(\beta'(\rho)\,\Delta + \gamma'(\rho)\,\Delta^2\right)$$
$$< \frac{1}{\varepsilon^2}\left(\beta'(\rho)\,h + \gamma'(\rho)\,h^2\right),$$

where

$$\beta'(\rho) = \frac{\rho}{(1 - e^{-\rho})} = \frac{\rho}{\rho + o(\rho)} \to 1$$

and

$$\gamma'(\rho) = \left(\frac{e^{-\rho} - 1 + \rho}{\rho\,(1 - e^\rho)}\right)^2 = \left(\frac{\rho^2/2 + o(\rho^2)}{\rho\,\big(\rho + o(\rho)\big)}\right)^2 \to \frac{1}{4}$$

as $\rho \to 0$, which will yield the same conclusion as above, but with the supremum taken over $y_1, y_2 \in \mathbb{R}_-$.

Finally, for $y_1 y_2 \leqslant 0$ (say $y_2 \leqslant 0 \leqslant y_1$) such that $|y_1 - y_2| < h$, using the elementary inequality $(a-b)^2 \leqslant 2(a^2 + b^2)$ we get

$$\mathbf{P}\Big\{\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big| > \varepsilon\Big\} \leqslant \frac{1}{\varepsilon^2}\,\mathbf{E}\big|\ln X_\rho(y_1) - \ln X_\rho(y_2)\big|^2$$

$$\leqslant \frac{2}{\varepsilon^2}\Big(\mathbf{E}\big|\ln X_\rho(y_1)\big|^2+\mathbf{E}\big|\ln X_\rho(y_2)\big|^2\Big)$$
$$= \frac{2}{\varepsilon^2}\left(\beta(\rho)y_1+\gamma(\rho)y_1^2+\beta'(\rho)|y_2|+\gamma'(\rho)|y_2|^2\right)$$
$$\leqslant \frac{2}{\varepsilon^2}\Big(\big(\beta(\rho)+\beta'(\rho)\big)\,h+\big(\gamma(\rho)+\gamma'(\rho)\big)\,h^2\Big),$$

which again will yield the desired conclusion. Lemma 5 is proved.

It remains to verify Lemma 6. Clearly,

$$\mathbf{P}\Big\{\sup_{|y|>A} X_\rho(y)>e^{-bA}\Big\}\leqslant \mathbf{P}\Big\{\sup_{y>A} X_\rho(y)>e^{-bA}\Big\}+\mathbf{P}\Big\{\sup_{y<-A} X_\rho(y)>e^{-bA}\Big\}.$$

In order to estimate the first term, we need two auxiliary results.

Lemma 7 *For any $c\in\,]\,0\,,\,3/32\,[$ we have*

$$\mathbf{E}X_\rho^{1/4}(y)\leqslant \exp\big(-c\,|y|\big)$$

for all sufficiently small ρ and all $y\in\mathbb{R}$.

Lemma 8 *For all $\rho>0$ the random variable*

$$\eta_\rho=\sup_{t\in\mathbb{R}_+}\big(\Pi_\lambda(t)-t\big),$$

where Π_λ is a Poisson process on $\mathbb{R}_+$ with intensity $\lambda=\rho/(e^\rho-1)\in\,]0,1[$, verifies

$$\mathbf{E}\exp\Big(\frac{\rho}{4}\,\eta_\rho\Big)\leqslant 2.$$

The first result can be easily obtained following the proof of Lemma 4, so we prove the second one only. For this, let us remind that according to Shorack and Wellner [19, Proposition 1 on page 392] (see also Pyke [17]), the distribution function $F_\rho(x)=\mathbf{P}\{\eta_\rho<x\}$ of η_ρ is given by

$$1-F_\rho(x)=\mathbf{P}\{\eta_\rho\geqslant x\}=(1-\lambda)\,e^{\lambda x}\sum_{n>x}\frac{(n-x)^n}{n!}\,\big(\lambda\,e^{-\lambda}\big)^n$$

for $x>0$, and is zero for $x\leqslant 0$. Hence, for $x>0$ we have

$$1-F_\rho(x)\leqslant(1-\lambda)\,e^{\lambda x}\sum_{n>x}\frac{(n-x)^n}{\sqrt{2\pi n}\,n^n\,e^{-n}}\,\big(\lambda\,e^{-\lambda}\big)^n$$

$$
\begin{aligned}
&= \frac{1-\lambda}{\sqrt{2\pi}}\, e^{\lambda x} \sum_{n>x} \frac{1}{\sqrt{n}} \Big(1-\frac{x}{n}\Big)^n \big(\lambda\, e^{1-\lambda}\big)^n \\
&\leqslant \frac{1-\lambda}{\sqrt{2\pi}}\, e^{\lambda x} \sum_{n>x} e^{-x}\, \frac{\big(\lambda\, e^{1-\lambda}\big)^n}{\sqrt{n}} \\
&\leqslant \frac{1-\lambda}{\sqrt{2\pi}}\, e^{(\lambda-1)x} \big(\lambda\, e^{1-\lambda}\big)^x \sum_{n>x} \frac{\big(\lambda\, e^{1-\lambda}\big)^{n-x}}{\sqrt{n-x}} \\
&= \frac{1-\lambda}{\sqrt{2\pi}}\, \lambda^x \sum_{k>0} \frac{\big(\lambda\, e^{1-\lambda}\big)^k}{\sqrt{k}} \leqslant \frac{1-\lambda}{\sqrt{2\pi}}\, \lambda^x \int_{\mathbb{R}_+} \frac{\big(\lambda\, e^{1-\lambda}\big)^t}{\sqrt{t}}\, dt \\
&= \frac{1-\lambda}{\sqrt{2\pi}}\, \lambda^x \frac{\Gamma(1/2)}{\sqrt{-\ln\big(\lambda\, e^{1-\lambda}\big)}} = \frac{1-\lambda}{\sqrt{-2\ln\big(\lambda\, e^{1-\lambda}\big)}} \Big(\frac{\rho}{e^\rho - 1}\Big)^x \\
&\leqslant \Big(\frac{\rho\, e^{-\rho/2}}{e^{\rho/2} - e^{-\rho/2}}\Big)^x = \Big(\frac{\rho\, e^{-\rho/2}}{2\sinh(\rho/2)}\Big)^x \leqslant e^{-\rho x/2},
\end{aligned}
$$

where we used Stirling inequality and the inequality $1-\lambda \leqslant \sqrt{-2\ln\big(\lambda\, e^{1-\lambda}\big)}$, which is easily reduced to the elementary inequality $\ln(1-\mu) \leqslant -\mu - \mu^2/2$ by putting $\mu = 1-\lambda$. So, we can finish the proof of Lemma 8 by writing

$$
\begin{aligned}
\mathbf{E}\exp\Big(\frac{\rho}{4}\,\eta_\rho\Big) &= \int_{\mathbb{R}} e^{\rho x/4}\, dF_\rho(x) \\
&= \Big[e^{\rho x/4}\big(F_\rho(x)-1\big)\Big]_{-\infty}^{+\infty} - \frac{\rho}{4}\int_{\mathbb{R}} e^{\rho x/4}\big(F_\rho(x)-1\big)\, dx \\
&= \frac{\rho}{4}\int_{\mathbb{R}_-} e^{\rho x/4}\, dx + \frac{\rho}{4}\int_{\mathbb{R}_+} e^{\rho x/4}\big(1-F_\rho(x)\big)\, dx \\
&\leqslant 1 + \frac{\rho}{4}\int_{\mathbb{R}_+} e^{-\rho x/4}\, dx = 2.
\end{aligned}
$$

Now, let us get back to the proof of Lemma 6. Using Lemma 8 and taking into account the stationarity and the independence of the increments of the process $\ln X_\rho$ on $\mathbb{R}_+$, we obtain

$$
\begin{aligned}
\mathbf{P}\Big\{\sup_{y>A} X_\rho(y) > e^{-bA}\Big\} &\leqslant e^{bA/4}\, \mathbf{E}\sup_{y>A} X_\rho^{1/4}(y) \\
&= e^{bA/4}\, \mathbf{E}X_\rho^{1/4}(A)\, \mathbf{E}\sup_{y>A} \frac{X_\rho^{1/4}(y)}{X_\rho^{1/4}(A)} \\
&= e^{bA/4}\, \mathbf{E}X_\rho^{1/4}(A)\, \mathbf{E}\sup_{z>0} X_\rho^{1/4}(z)
\end{aligned}
$$

$$= e^{bA/4}\,\mathbf{E}X_\rho^{1/4}(A)\,\mathbf{E}\sup_{z>0}\left(\exp\Big(\frac{\rho}{4}\,\Pi_+(z/\rho)-\frac{z}{4\rho}\Big)\right)$$
$$= e^{bA/4}\,\mathbf{E}X_\rho^{1/4}(A)\,\mathbf{E}\exp\left(\sup_{t>0}\Big(\frac{\rho}{4}\big(\Pi_{\frac{\rho}{e^\rho-1}}(t)-t\big)\Big)\right)$$
$$= e^{bA/4}\,\mathbf{E}X_\rho^{1/4}(A)\,\mathbf{E}\exp\left(\frac{\rho}{4}\,\eta_\rho\right)\leqslant 2\,e^{bA/4}\,\mathbf{E}X_\rho^{1/4}(A).$$

Hence, taking $b\in\,]\,0\,,3/40\,[$, we have $5b/4\in\,]\,0\,,3/32\,[$ and, using Lemma 7, we finally get

$$\mathbf{P}\Big\{\sup_{y>A}X_\rho(y)>e^{-bA}\Big\}\leqslant 2\,e^{bA/4}\,\exp\Big(-\frac{5b}{4}A\Big)=2\,e^{-bA}$$

for all sufficiently small ρ and all $A>0$, and so the first term is estimated.

The second term can be estimated in the same way, if we show that for all $\rho>0$ the random variable

$$\eta'_\rho=\sup_{t\in\mathbb{R}_+}\big(-\Pi_{\lambda'}(t)+t\big)=-\inf_{t\in\mathbb{R}_+}\big(\Pi_{\lambda'}(t)-t\big),$$

where $\Pi_{\lambda'}$ is a Poisson process on $\mathbb{R}_+$ with intensity $\lambda'=\rho/(1-e^{-\rho})\in\,]0,1[$, verifies

$$\mathbf{E}\exp\left(\frac{\rho}{4}\,\eta'_\rho\right)\leqslant 2.$$

For this, let us remind that according to Pyke [17] (see also Cramér [2]), η'_ρ is an exponential random variable with parameter r, where r is the unique positive solution of the equation

$$\lambda'(e^{-r}-1)+r=0.$$

In our case, this equation becomes

$$\frac{\rho}{1-e^{-\rho}}\,(e^{-r}-1)+r=0,$$

and $r=\rho$ is clearly its solution. Hence η'_ρ is an exponential random variable with parameter ρ, which yields

$$\mathbf{E}\exp\left(\frac{\rho}{4}\,\eta'_\rho\right)=\frac{4}{3}<2,$$

and so, Lemma 6 is proved.

4 Numerical simulations

In this section we present some numerical simulations of the quantities B_ρ, M_ρ and E_ρ for $\rho \in \,]0,\infty[$. Besides giving approximate values of these quantities, the simulation results illustrate both the asymptotics

$$B_\rho \sim \frac{B_0}{\rho^2}\,, \quad M_\rho \sim \frac{M_0}{\rho^2} \quad \text{and} \quad E_\rho \to E_0 \quad \text{as} \quad \rho \to 0,$$

with $B_0 = 16\,\zeta(3) \approx 19.2329$, $M_0 = 26$ and $E_0 = 8\,\zeta(3)/13 \approx 0.7397$, and

$$B_\rho \to B_\infty, \quad M_\rho \to M_\infty \quad \text{and} \quad E_\rho \to E_\infty \quad \text{as} \quad \rho \to \infty,$$

with $B_\infty = 1$, $M_\infty = 2$ and $E_\infty = 0.5$.

First, we simulate the events $x_1, x_2, \ldots$ of the Poisson process Π_+ $\big($with the intensity $1/(e^\rho - 1)\big)$, and the events $x_1', x_2', \ldots$ of the Poisson process Π_- $\big($with the intensity $1/(1 - e^{-\rho})\big)$.

Then we calculate

$$\zeta_\rho = \frac{\displaystyle\int_{\mathbb{R}} x\, Z_\rho(x)\; dx}{\displaystyle\int_{\mathbb{R}} Z_\rho(x)\; dx} = \frac{\displaystyle\sum_{i=1}^{\infty} x_i\, e^{\rho i - x_i} + \sum_{i=1}^{\infty} e^{\rho i - x_i} - \sum_{i=1}^{\infty} x_i'\, e^{\rho - \rho i + x_i'} + \sum_{i=1}^{\infty} e^{\rho - \rho i + x_i'}}{\displaystyle\sum_{i=1}^{\infty} e^{\rho i - x_i} + \sum_{i=1}^{\infty} e^{\rho - \rho i + x_i'}}$$

and

$$\xi_\rho = \operatorname*{argsup}_{x\in\mathbb{R}} Z_\rho(x) = \begin{cases} x_k, & \text{if } \rho k - x_k > \rho - \rho\ell + x_\ell', \\ -x_\ell', & \text{otherwise,} \end{cases}$$

where

$$k = \operatorname*{argmax}_{i\geqslant 1}\,(\rho i - x_i) \quad \text{and} \quad \ell = \operatorname*{argmax}_{i\geqslant 1}\,(\rho - \rho i + x_i'),$$

so that

$$x_k = \operatorname*{argsup}_{x\in\mathbb{R}_+} Z_\rho(x) \quad \text{and} \quad -x_\ell' = \operatorname*{argsup}_{x\in\mathbb{R}_-} Z_\rho(x).$$

Finally, repeating these simulations 10^7 times (for each value of ρ), we approximate $B_\rho = \mathbf{E}\zeta_\rho^2$ and $M_\rho = \mathbf{E}\xi_\rho^2$ by the empirical second moments, and $E_\rho = B_\rho/M_\rho$ by their ratio.

The results of the numerical simulations are presented in Figures 1 and 2. The $\rho \to 0$ asymptotics of B_ρ and M_ρ can be observed in Figure 1, where besides these functions we also plotted the functions $\rho^2 B_\rho$ and $\rho^2 M_\rho$, making apparent the constants $B_0 \approx 19.2329$ and $M_0 = 26$.

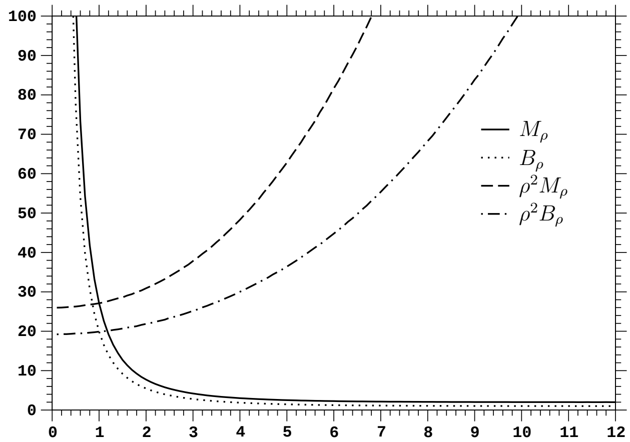

Figure 1: B_ρ and M_ρ ($\rho \to 0$ asymptotics)

In Figure 2 we use a different scale on the vertical axis to better illustrate the $\rho \to \infty$ asymptotics of B_ρ and M_ρ, as well as both the asymptotics of E_ρ. Note that the function E_ρ appear to be decreasing, so we can conjecture that bigger is ρ, smaller is the efficiency of the maximum likelihood estimator, and so, this efficiency is always between $E_\infty = 0.5$ and $E_0 \approx 0.7397$.

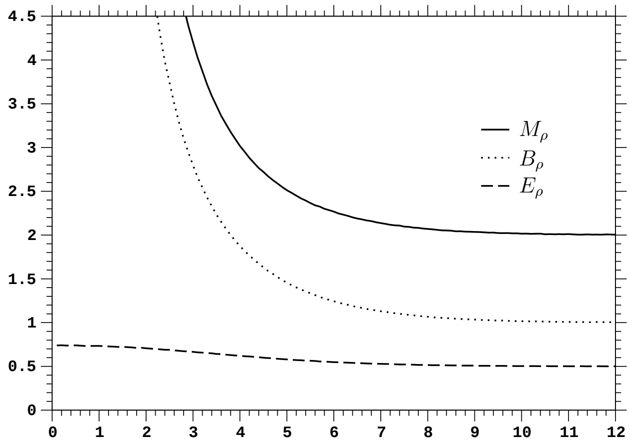

Figure 2: B_ρ and M_ρ ($\rho \to \infty$ asymptotics)
E_ρ (both asymptotics)

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