Fixed point properties in the space of marked groups

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Abstract. We explain, following Gromov, how to produce uniform isometric actions of groups starting from isometric actions without fixed point, using common ultralimits techniques. This gives in particular a simple proof of a result by Shalom: Kazhdan's property (T) defines an open subset in the space of marked finitely generated groups.

1. Introduction

In this expository note, we are interested in groups whose actions on some particular kind of spaces always have (global) fixed points.

Definition 1.1. Let G be a (discrete) group. We say that G has:
- Serre's *Property (FH)*, if any isometric G-action on an affine Hilbert space has a fixed point [HV89, Chap 4];
- Serre's *Property (FA)*, if any G-action on a simplicial tree (by automorphisms and without inversion) has a fixed point [Ser77, Chap I.6];
- *Property (F$\mathbb{R}$A)*, if if any isometric G-action on a complete $\mathbb{R}$-tree has a fixed point [HV89, Chap 6.b].

These definitions extend to topological groups: one has then to require the actions to be continuous.

Such properties give information about the structure of the group G. Serre proved that a countable group has Property (FA) if and only if (i) it is finitely generated, (ii) it has no infinite cyclic quotient, and (iii) it is not an amalgam [Ser77, Thm I.15]. Among locally compact, second countable groups, Guichardet and Delorme proved that Property (FH) is equivalent to Kazhdan's Property (T) [Gui77, Del77]. Kazhdan groups are known to be compactly generated and to have a compact abelianization; see e.g. Chapter 1 in [HV89]. It is known that Property (FH) implies Property (F$\mathbb{R}$A) [1], which itself obviously implies Property (FA).

We are interested in the behavior of these properties in Grigorchuk's space of marked (finitely generated) groups (see Section 2 for definition). One main aim of this note is to provide a simple proof of the following result, which implies that any finitely generated Kazhdan group is a quotient of a finitely presented Kazhdan group:

Theorem 1.2 (Shalom [Sha00])**.** *Property (FH) defines an open subset in the space of marked groups.*

Rather than just prove this result, our purpose is to indicate a general scheme, which gives a common strategy for proving Shalom's result and Theorems 1.3 and 1.4 below.

Theorem 1.3 (Korevaar-Schoen [KS97], Shalom [Sha00])**.** *A finitely generated group G has Property (FH) if and only if every isometric G-action on a Hilbert space almost has fixed points.*

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1. This was noticed by several people; see [HV89, Chap 6.b].

Theorem 1.4 (Culler-Morgan)**.** *Property (F$\mathbb{R}$A) defines an open subset in the space of marked groups.*

In fact, one deduces the latter Theorem from [CM87, Thm 4.5] by applying it to a free group $\mathbb{F}_n$. On the other hand, it is an open problem whether Property (FA) defines an open subset in the space of marked groups.

The general (simple) idea for the scheme is, starting from actions without (global) fixed points, to pass to a "limit" of the spaces to get uniform actions (that is, not almost having fixed points; see Section 2). It follows in fact the same strategy as in [Gro03, Sect 3.8.B] [2], and Theorems 1.2 – 1.4 are parcular cases of Gromov's results.

Theorem 1.5 (Gromov)**.** *Let $(G_k, S_k)_{k\in\mathbb{N}}$ be a sequence of marked groups converging to (G, S). If each group G_k acts without fixed point on a (non-empty) complete metric space (X_k, d_k), then G acts uniformly on some ultralimit of the spaces (X_k, d_k).*

The definition of ultralimits will be given in Section 3. Note that we allow to rescale the spaces before taking the limit (see Theorem 3.12 for a more precise statement). The idea to take "limits" of metric spaces is not new, even for such purposes. Asymptotic cones, introduced by Gromov in [Gro81] and defined rigorously in [Gro93, DW84] are a major particular case of ultralimits which is very useful in the study of metric spaces and groups, see e.g. [Dru] and references therein. Ultralimits appear explicitely in [KL97, BH99], for instance. Finally, let us mention that Korevaar and Schoen [KS97] introduced limits of CAT(0) spaces (with another process) and proved in this context results of the same spirit as Theorem 1.5.

Section 2 gives the necessary preliminaries. In Section 3, we recall what ultralimits are and prove Theorem 1.5. Finally, Section 3 is devoted to applications to Properties (FH) and (F$\mathbb{R}$A).

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2. Preliminaries

2.1. **Terminology about group actions.** In this note, every metric space is assumed to be non-empty and all groups considered are discrete ones. Let (X, d) be a metric space and let G be a group acting on it by isometries. The action is said to *almost have fixed points* if, for all $\varepsilon > 0$ and for all finite subset $F \subseteq G$, there exists $x \in X$ such that $d(g \cdot x, x) < \varepsilon$ for all $g \in F$; it is said to be *uniform* otherwise.

An action with a global fixed point almost has fixed points, but the converse strongly does not hold. Indeed, the following examples show that an action with almost fixed points can be *metrically proper*, that is such that for any $x \in X$ and $R > 0$, the set $\{g \in G : d(x, gx) \leq R\}$ is finite.

Example 2.1 (on the hyperbolic plane). Let $\mathbb{H}^2$ be the Poincaré upper half-plane and define $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ by $\varphi(z) = z + 1$. This gives a $\mathbb{Z}$-action by isometries on $\mathbb{H}^2$ which is metrically proper and almost has fixed points.

2. This was pointed out to me by Yves de Cornulier, after this paper was accepted for publication and posted on the arXiv. There also exist good unpublished notes [Cor] on this topic.

Example 2.2 (on a Hilbert space). Let S be the shift operator on $\ell^2\mathbb{Z}$ (defined by $(S\xi)(n) = \xi(n-1)$) and $\delta_0 \in \ell^2\mathbb{Z}$ the Dirac mass at 0. The affine map $\xi \mapsto S\xi + \delta_0$ defines a $\mathbb{Z}$-action by isometries on $\ell^2\mathbb{Z}$ which is metrically proper and almost has fixed points.

Remark 2.3. In the case of finitely generated groups, the definition can be made easier: if S is a finite generating set of G, a G-action by isometries on X almost has fixed points if and only if, for all $\varepsilon > 0$, there exists $x \in X$ such that $\max\{d(x, s\cdot x) : s \in S\} < \varepsilon$.

2.2. **The space of marked groups.** Let us recall that a *marked group on n generators* is a pair (G, S) where G is a group and $S \in G^n$ generates G. A marked group (G, S) defines canonically a quotient $\phi : \mathbb{F}_n \to G$, and vice-versa. Moreover, for such a quotient, we may consider the normal subgroup $N = \ker(\phi) \lhd \mathbb{F}_n$. Two marked groups, or two quotients, are said to be equivalent if they define the same normal subgroup of $\mathbb{F}_n$. Abusing terminology, we denote by $\mathcal{G}_n$ the set of (equivalence classes of) marked groups on n generators, or the set of (equivalence classes of) quotients of $\mathbb{F}_n$, or the set of normal subgroups of $\mathbb{F}_n$.

We now describe Grigorchuk's topology on $\mathcal{G}_n$ [Gri84], which corresponds to an earlier construction by Chabauty [Chab50]; for introductory expositions, see [Cham00, CG05, Pau04]. Denote by B_r the ball of radius r in $\mathbb{F}_n$ (centered at the trivial element). Given normal subgroups $N \neq N' \lhd \mathbb{F}_n$, we set

$$d(N, N') := \exp\left(-\max\left\{r \in \mathbb{N} : N' \cap B_r = N \cap B_r\right\}\right) .$$

This turns $\mathcal{G}_n$ into a compact, ultrametric, separable space. The map

$$\big(G, (s_1, \dots s_n)\big) \mapsto \big(G, (s_1, \dots s_n, 1_G)\big)$$

defines an isometric embedding $\mathcal{G}_n \to \mathcal{G}_{n+1}$ for all $n \in \mathbb{N}$. We denote by $\mathcal{G}$ the direct limit of this directed system of topological spaces and call $\mathcal{G}$ the *space of (finitely generated) marked groups*. Note that $\mathcal{G}_n$ is open and closed in $\mathcal{G}$ for all n. Given $N \lhd \mathbb{F}_n$ and $N_k \lhd \mathbb{F}_n$ for $k \in \mathbb{N}$, one has $\lim N_k = N$ if and only if, for all $g \in \mathbb{F}_n$: ($g \in N \iff g \in N_k$ for k sufficiently large) and ($g \notin N \iff g \notin N_k$ for k sufficiently large).

3. From actions without fixed point to uniform actions

3.1. **Being far from almost fixed points.** Let (G, S) be a marked (finitely generated) group. Let it act by isometries on a metric space (X, d) and set $\delta(x) = \max\{d(x, sx) : s \in S\}$ for any $x \in X$. A point $x \in X$ is fixed by G if and only if $\delta(x) = 0$, and the action almost has fixed points if and only if $\inf\{\delta(x) : x \in X\} = 0$.

Remark 3.1. As G is finitely generated, a G-action on an $\mathbb{R}$-tree T almost has fixed points if and only if it has a fixed point. Indeed, if the action on T has no fixed point, there exists $g \in G$ which induces a hyperbolic isometry of T — this can be deduced from [Tig79, Cor 2.3]; see also [MS84, Prop II.2.15] or [Bes02, Exercise 2.8]. Thus, the G-action on T is uniform.

This property is specific to $\mathbb{R}$-trees, as illustrated in Examples 2.1 and 2.2. For general metric spaces, this Section will explain how to produce uniform actions from actions without fixed points.

Lemma 3.2. *We have* $|\delta(x) - \delta(y)| \leq 2d(x, y)$ *for all* $x, y \in X$.

Proof. For any $s \in S$, the triangle inequality gives

$$d(x, sx) \leq d(x, y) + d(y, sy) + d(sy, sx) \leq d(y, sy) + 2d(x, y) ,$$

which implies $\delta(x) \leq \delta(y) + 2d(x, y)$. We then deduce similarly $\delta(y) \leq \delta(x) + 2d(x, y)$. ☐

We now introduce one key ingredient to produce uniform actions, which has been directly inspired from Lemma 6.3 in [Sha00]; see also Proposition 4.1.1 in [KS97] and (the proof of) Lemma 2 in [Laf06]. It asserts, that we may find points which are, roughly speaking, far from almost fixed points.

Lemma 3.3. *Assume the space X is complete and the action has no fixed point. Then, for all $n \in \mathbb{N}^*$, there exists $x_n \in X$ such that:*

$$\text{for all } y \in X, \quad d(y, x_n) \leq n\delta(x_n) \implies \delta(y) \geq \frac{\delta(x_n)}{2} \ .$$

Proof. Let us assume by contradiction that there exists some $n \in \mathbb{N}^*$ such that, for all $x \in X$, there exists $y = y(x) \in X$ which satisfies both $d(y, x) \leq n\delta(x)$ and $\delta(y) < \delta(x)/2$. Let us now take some $z_0 \in X$ (recall that X is non-empty). Then, we define inductively a sequence of points z_k such that $d(z_{k+1}, z_k) \leq n\delta(z_k)$ and $\delta(z_{k+1}) < \delta(z_k)/2$ for all $k \in \mathbb{N}$. Consequently, we have $\delta(z_k) < \delta(z_0)/2^k$, whence $d(z_{k+1}, z_k) \leq n\delta(z_0)/2^k$. Since X is complete, this shows that z_k converges to some point z as k tends to ∞. Thanks to Lemma 3.2, we obtain $\delta(z) = \lim_{k\to\infty} \delta(z_k) = 0$. Hence, z is a fixed point of the G-action, a contradiction. □

Note that the hypotheses on X made in Lemma 3.3 cannot be dropped. Indeed, to obtain counter-examples, consider $\mathbb{Z}$ acting by rotations on $\mathbb{C}$, respectively $\mathbb{C} \setminus \{0\}$.

3.2. **Ultrafilters and ultralimits of metric spaces.** Bourbaki defines ultrafilters to be maximal filters [Bou71]. However, we think slightly differently to ultrafilters in this note.

Definition 3.4. An *ultrafilter* on some (non-empty) set E is a finitely-additive, $\{0, 1\}$-valued measure on $\mathcal{P}(E)$, that is a function $\omega : \mathcal{P}(E) \to \{0, 1\}$ which satisfies: (i) $\omega(A \cup B) = \omega(A) + \omega(B)$ whenever $A \cap B = \emptyset$, and (ii) $\omega(E) = 1$

Note that (i) and (ii) imply $\omega(\emptyset) = 0$. In this note, we shall only need to consider ultrafilters on $\mathbb{N}$. The following well-known Lemma establishes the equivalence with Bourbaki's definition. Its proof is given for completeness.

Lemma 3.5. *A function $\omega : \mathcal{P}(E) \to \{0, 1\}$ is an ultrafilter if and only if $\mathcal{F} = \{A \in \mathcal{P}(E) : \omega(A) = 1\}$ satisfies:*

1. *$\emptyset \notin \mathcal{F}$ and $E \in \mathcal{F}$;*
2. *if $A \in \mathcal{F}$ and $A \subseteq B \subseteq E$, then $B \in \mathcal{F}$;*
3. *if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;*
4. *for any $A \subseteq E$, one has $A \in \mathcal{F}$ or $E \setminus A \in \mathcal{F}$.*

Remark 3.6. Properties (1)–(3) are precisely the axioms of *filters* in [Bou71].

Proof. Suppose first ω is an ultrafilter. Then (1), (2) and (4) are obvious. If $A, B \in \mathcal{F}$, then (2) gives $\omega(A \cup B) = 1$ and (i) implies $\omega((A \cup B) \setminus B) = 0$, $\omega((A \cup B) \setminus A) = 0$ and finally

$$\omega(A \cap B) = \omega(A \cup B) - \omega((A \cup B) \setminus B) - \omega((A \cup B) \setminus A) = 1 \ .$$

Hence (3) is proved.

Conversly, we now assume (1)–(4). Then (ii) is obvious. Properties (1) et (3) imply: (5) for any $C, D \subseteq E$ such that $C \cap D = \emptyset$, one has $C \notin \mathcal{F}$ or $D \notin \mathcal{F}$. To prove (i), let us now take $A, B \subseteq E$ such that $A \cap B = \emptyset$. The case $\omega(A) = 1 = \omega(B)$ is impossible by (5). In case $\omega(A) = 1$ and $\omega(B) = 0$ (or $\omega(A) = 0$ and $\omega(B) = 1$), property (2) implies $\omega(A \cup B) = 1$. Finally, if $\omega(A) = 0 = \omega(B)$, (4) gives

$E \setminus A \in \mathcal{F}$ and $E \setminus B \in \mathcal{F}$. We deduce then $(E \setminus A) \cap (E \setminus B) \in \mathcal{F}$ by (3) and $\omega(A \cup B) = 0$ by (5). Hence (i) is proved. □

Example 3.7. Given any $a \in E$, there is an ultrafilter δ_a defined by $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ otherwise. Such an ultrafilter is called *principal*.

Definition 3.8. Let ω be an ultrafilter on $\mathbb{N}$ and let X be a metric space. A sequence (x_n) in X is said to converge to $x \in X$ relative to ω if, for any neighborhood V of x, the set $\{n \in \mathbb{N} : x_n \in V\}$ has ω-measure 1.

The limit of a sequence, provided it exists, is unique, and we write $\lim_{n\to\omega} x_n = x$, or $\lim_\omega x_n = x$. From this point of view, the interesting ultrafilters are the non-principal ones: for instance, any sequence (x_n) converges to x_k, relative to the principal ultrafilter δ_k. The existence of non-principal ultrafilters follows from Zorn's Lemma (see e.g [Bou71], or Exercise I.5.48 in [BH99]). We shall use the following well-known result implicitly in the text. It ensures that any bounded sequence of real (or complex) numbers is ω-convergent.

Proposition 3.9 ([Bou71])**.** *If X is a compact metric space and ω is an ultrafilter on $\mathbb{N}$, then any sequence (x_n) in X is ω-convergent.*

We now define ultralimits of metric spaces, essentially as in [KL97] or [BH99], (except that we allow to "rescale" the spaces before to take the limit, as is done in the construction of asymptotic cones, for instance). Let us consider sequences $(X_k, d_k, *_k)_{k\in\mathbb{N}}$ of pointed metric spaces and $r = (r_k)$ of positive numbers. Set

$$\mathcal{B}_r = \left\{ x \in \prod_{k\in\mathbb{N}} X_k : \text{ the sequence } (r_k d_k(x_k, *_k))_k \text{ is bounded} \right\} .$$

If some group G acts by isometries on the spaces X_k, its diagonal action may not stabilize $\mathcal{B}_r$. A necessary and sufficient condition is that the sequence $(r_k d_k(g \cdot *_k, *_k))$ is bounded for any generator g of G. If this condition is fulfilled, we say that G *acts diagonally* on $\mathcal{B}_r$. For any ultrafilter ω on $\mathbb{N}$, we may endow $\mathcal{B}_r$ with the pseudo-distance $d_{\omega,r}(x, y) = \lim_\omega r_k d_k(x_k, y_k)$. If G acts diagonally on $\mathcal{B}_r$, the diagonal action is isometric.

Definition 3.10. Let ω be some non-principal ultrafilter on $\mathbb{N}$. The *ultralimit* (relative to scaling factors (r_k) and to ω) of the sequence $(X_k, d_k, *_k)$ is the pointed metric space $(X_{\omega,r}, d_{\omega,r}, *_{\omega,r})$, where $X_{\omega,r}$ is the separation of $(\mathcal{B}_r, d_{\omega,r})$ and $*_{\omega,r}$ denotes either the point $(*_k)_k \in \mathcal{B}_r$, or its image in $X_{\omega,r}$.

Note that if G acts diagonally on $\mathcal{B}_r$, the diagonal action induces an isometric action on every ultralimit $X_{\omega,r}$, which we call again diagonal. If the sequence $(X_k, d_k)_k$ is constant and if $r_k \to 0$, one gets the notion of asymptotic cone, due to Gromov [Gro81, Gro93], and van den Dries and Wilkie [DW84].

Proposition 3.11 (Gromov [Gro03])**.** *Let G be a finitely generated group acting by isometries on complete metric spaces (X_k, d_k) for $k \in \mathbb{N}$. If these actions have no fixed point, then there exist scaling factors $r_k > 0$ and base points $*_k \in X_k$ such that:*

(1) *the group G acts diagonally on $\mathcal{B}_r$;*
(2) *for any non-principal ultrafilter ω on $\mathbb{N}$, the diagonal action of G on $X_{\omega,r}$ is uniform.*

Korevaar and Schoen [KS97] used the same idea to rescale spaces, and then take a limit, to produce uniform actions from actions without fixed points on CAT(0) spaces. On the other hand their construction of limits uses properties of CAT(0) spaces.

Proof. Let S be a finite generating set of G and set $\delta_k(x) = \max\{d(x, sx) : s \in S\}$ for all $x \in X_k$. By Lemma 3.3, we obtain points $*_k \in X_k$ for all $k \in \mathbb{N}$ such that:

$$\text{for all } y_k \in X_k, \quad d(y_k, *_k) \leq k\delta_k(*_k) \implies \delta_k(y_k) \geq \frac{\delta_k(*_k)}{2} .$$

We now set $r_k = \delta_k(*_k)^{-1}$, which are well-defined since the actions have no fixed point. Thus, we have $r_k d_k(*_k, s \cdot *_k) \leq 1$ for all k, so that (1) is satisfied.

To prove (2), we consider some non-principal ultrafilter ω on $\mathbb{N}$. For any $y = (y_k) \in \mathcal{B}_r$, the sequence $(d(y_k, *_k)/\delta_k(*_k))_k$ is bounded. Hence, for k sufficiently large, one has $d(y_k, *_k) \leq k\delta_k(*_k)$, which implies $\delta_k(y_k) \geq \delta_k(*_k)/2$.

For all $s \in S$, set now $A_s = \{k \in \mathbb{N} : r_k d_k(y_k, sy_k) \geq 1/2\}$. The former argument implies $k \in \bigcup_{s\in S} A_s$ for k large enough, whence $\omega\big(\bigcup_{s\in S} A_s\big) = 1$. Since S is finite, there exists $s \in S$ such that $\omega(A_s) = 1$, which shows that $d_{\omega,r}(y, sy) = \lim_\omega r_k d_k(y_k, sy_k) \geq 1/2$. □

Let us now make Theorem 1.5 precise.

Theorem 3.12 (Gromov [Gro03]). *Let $(G_k, S_k)_{k\in\mathbb{N}}$ be a sequence of marked groups converging to (G, S) in the space $\mathcal{G}_n$. If each group G_k acts without fixed point on a complete metric space (X_k, d_k), then there exists scaling factors $r_k > 0$ and base points $*_k \in X_k$ such that:*

1. *the free group $\mathbb{F}_n$ acts diagonally on $\mathcal{B}_r$;*
2. *for any non-principal ultrafilter ω on $\mathbb{N}$, the diagonal action of $\mathbb{F}_n$ on the ultralimit $X_{\omega,r}$ is uniform;*
3. *the diagonal action factors through the epimorphism $\mathbb{F}_n \to G$ associated to (G, S).*

Proof. Let N_k and N be the normal subgroups of F_n associated with (G_k, S_k) and (G, S) respectively. The G_k-actions on the spaces X_k give $\mathbb{F}_n$-actions which are trivial on N_k. Proposition 3.11 gives then scaling factors $r_k > 0$ and points $*_k \in X_k$ such that conditions (1) and (2) are fulfilled.

To prove (3), it suffices to see that the diagonal action is trivial on N. Let us take $g \in N$. As $N_k \to N$, we have $g \in N_k$ for k sufficiently large. Take now $y = (y_k) \in \mathcal{B}_{\omega,r}$. Since $g \cdot y_k = y_k$ for k large enough, we get $\lim_\omega r_k d_k(y_k, gy_k) = 0$, whence $d_{\omega,r}(y, gy) = 0$. The subgroup N acts trivially on $X_{\omega,r}$, as desired. □

4. Applications to fixed point properties

In this Section, we apply Theorem 1.5 (or Theorem 3.12) to obtain results about fixed point properties on groups. Another ingredient is to identify classes of metric spaces which are stable by ultralimits. Let us record two easy observations:

Remark 4.1. When a group acts isometrically on a metric space, the action can be extended to the completion. Moreover, if the action is uniform, then so is the extension.

Remark 4.2. The subsets $\mathcal{G}_n$ being an open cover of $\mathcal{G}$, a property defines an open set in $\mathcal{G}$ if and only if it defines an open set in every $\mathcal{G}_n$.

4.1. **Fixed points in (affine) Hilbert spaces.**

We consider (affine) Hilbert spaces over $\mathbb{R}$, by forgetting the complex structure and replacing the inner-product by its real part, if necessary. For such spaces, it is well-known that any isometry is an affine map. In Example 2.2, we exhibited an isometric action on a Hilbert space without fixed point, which almost has fixed points. On the other hand, Theorem 1.5 allows to unify the proof of Theorems 1.2 and 1.3, that we now recall.

Theorem 4.3 (Shalom [Sha00]). *Property (FH) defines an open subset in $\mathcal{G}$.*

Theorem 4.4 (Korevaar-Schoen [KS97], Shalom [Sha00]). *A finitely generated group G has Property (FH) if and only if every isometric G-action on a Hilbert space almost has fixed points.*

Korevaar and Schoen also prove the following: if Γ is a finitely generated group with Property (FH) and if X is a geodesically complete CAT(0) space with curvature bounded from below, then any non-uniform isometric action of Γ on X has a fixed point. To do this, they show that some limits (in their sense) of geodesically complete CAT(0) space with curvature bounded from below are Hilbert spaces (compare with Lemma 4.5 below).

Let us now mention a result by Mok in the same vein [Mok95]: if M is a compact riemannian manifold and if $G = \pi_1(M)$, then G has property (T) if and only if, for any irreducible unitary representation π of G, there is no non-zero E_π-valued harmonic 1-form (where E_π is the locally constant Hilbert bundle on M induced from π). As this text was almost finished, we saw in the Appendix of [Kle] a "weak version of some results in [FM05]", which implies Theorem 4.4. The proof in [Kle] uses ultralimits of Hilbert spaces in a very similar way as in this note (with less details).

For proofs of Theorems 4.3 and 4.4, we use the following easy observation.

Lemma 4.5. *Any ultralimit of affine Hilbert spaces is an affine Hilbert space.*

Proof. Let us consider a sequence $(\mathcal{H}_k, d_k, *_k)_{k\in\mathbb{N}}$ of (pointed) affine Hilbert spaces, scaling factors $(r_k)_{k\in\mathbb{N}}$ and some non-principal ultrafilter ω on $\mathbb{N}$. We denote by $\mathcal{H}_k^0$ the (Hilbert) vector space under $\mathcal{H}_k$. Let $(\mathcal{H}^0_{\omega,r}, d_{\omega,r}, 0_{\omega,r})$ be the ultralimit of the sequence $(\mathcal{H}_k^0, ||.||_k, 0)$, relative to (r_k) and to ω. Then, $\mathcal{H}^0_{\omega,r}$ is a vector space with respect to operations $(u_k) + (v_k) := (u_k + v_k)$ and $\lambda \cdot (v_k) := (\lambda \cdot v_k)$, and the formula $\langle u | v \rangle := \lim_\omega \langle r_k u_k | r_k v_k \rangle$ defines a bilinear form on $\mathcal{H}^0_{\omega,r}$. Moreover, for all $u, v \in \mathcal{H}^0_{\omega,r}$, we have:

$$d_{\omega,r}(u,v)^2 = \lim_{k\to\omega} d_k(r_k u_k, r_k v_k)^2 = \lim_{k\to\omega} \langle r_k(u_k - v_k) | r_k(u_k - v_k) \rangle = \langle u - v | u - v \rangle \ ,$$

so that $\mathcal{H}^0_{\omega,r}$ is an inner-product space. Since any ultralimit of metric spaces is complete [3], $\mathcal{H}^0_{\omega,r}$ is a Hilbert space.

Finally, we consider the ultralimit $(\mathcal{H}_{\omega,r}, d_{\omega,r}, *_{\omega,r})$ of the sequence $(\mathcal{H}_k, d_k, *_k)$, relative to (r_k) and to ω. The action $(u_k) + (x_k) := (u_k + x_k)$ turns it into an affine space over $\mathcal{H}^0_{\omega,r}$. Hence it is an affine Hilbert space. ☐

Proof of Theorems 4.3 and 4.4. Let (G_k) be a sequence in $\mathcal{G}$ which converges to some $G \in \mathcal{G}$. As the subspaces $\mathcal{G}_n$ form an open cover of $\mathcal{G}$, we find $n \in \mathbb{N}^*$ such that $G, G_k \in \mathcal{G}_n$. Assuming that every G_k admits an isometric action without fixed point on some affine Hilbert space $\mathcal{H}_k$, Theorem 1.5 gives then an ultralimit $\mathcal{H} = \mathcal{H}_{\omega,r}$ on which G acts uniformly. Moreover, $\mathcal{H}$ is an affine Hilbert space by Lemma 4.5.

This proves Theorem 4.3. Moreover, if we specialize to the case $G_k = G_0$ and $\mathcal{H}_k = \mathcal{H}_0$ for all k, it also proves the non-trivial part of Theorem 4.4. ☐

4.2. **Fixed points in (complete) $\mathbb{R}$-trees.** Let us recall that $\mathbb{R}$-trees have been invented by Tits [Tit77] [4]. Let us also recall (see e.g. Lemmata 1.2.6 and 2.4.3 in [Chi01]) that a metric space (X, d) is an $\mathbb{R}$-*tree* if and only if the following conditions both hold:

3. See e.g. Lemma 2.4.2 in [KL97] or Lemma I.5.53 in [BH99]
4. Unlike the definition we follow, Tits required $\mathbb{R}$-trees to be complete.

(1) it is *geodesic*, that is, for any $x, y \in X$, there exists a map $c : [0, \ell] \to X$ such that $c(0) = x$, $c(\ell) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, \ell]$;

(2) it is 0-*hyperbolic*, that is, for all $x, y, z, t \in X$:

$$d(x, y) + d(z, t) \le \max\{d(x, z) + d(y, t), d(y, z) + d(x, t)\} \ .$$

A map c as in point (1) is called a *geodesic* from x to y. In what follows, we shall need the fact that any ultralimit of $\mathbb{R}$-trees is an $\mathbb{R}$-tree. One may argue by saying that the ultralimit is a quotient of some subtree of a Λ-tree (where Λ is the ultrapower of $\mathbb{R}$ with respect to ω), in which we identify points at infinitesimal distance. However, we prefer a more pedestrian way.

Lemma 4.6. (1) *Any ultralimit of geodesic spaces is a geodesic space;*

(2) *Any ultralimit of* 0*-hyperbolic spaces is a* 0*-hyperbolic space.*

This Lemma is easy. For instance, part (1) is an exercise in [BH99]. We nevertheless give a proof for completeness.

Proof. Let us consider some non-principal ultrafilter ω, pointed metric spaces $(X_k, d_k, *_k)$, and scaling factors r_k, for $k \in \mathbb{N}$.

(1) Assume the spaces X_k are geodesic and take $x \neq y \in X_{\omega,r}$, represented by elements $(x_k), (y_k)$ in $\mathcal{B}_r$. We set $\ell_k = d_k(x_k, y_k)$ and consider geodesics $c_k : [0, \ell_k] \to X_k$ from x_k to y_k (note that we may assume $x_k \neq y_k$ for all k). Setting $\ell = \lim_\omega r_k \ell_k = d_{\omega,r}(x, y)$, we define a map

$$c \ : \ [0, \ell] \to \mathcal{B}_r \ ; \ t \mapsto \left(c_k \left(t \frac{\ell_k}{\ell} \right) \right)_{k \in \mathbb{N}}$$

Then $d(c(t), c(t')) = \lim_\omega r_k d(c_k(t\ell_k/\ell), c_k(t'\ell_k/\ell)) = \lim_\omega r_k(\ell_k/\ell) \cdot |t - t'| = |t - t'|$ for all $t, t' \in [0, \ell]$. Hence c is a geodesic from x to y.

(2) Assume the spaces X_k are 0-hyperbolic and take $x, y, z, t \in X_{\omega,r}$, which are represented by elements $(x_k), (y_k), (z_k), (t_k)$ in $\mathcal{B}_r$. Fixing $\varepsilon > 0$, we have

$$\omega \left(\left\{ k \in \mathbb{N} : \begin{array}{ccc} r_k d_k(x_k, z_k) \le d_{\omega,r}(x, z) + \frac{\varepsilon}{2} & , & r_k d_k(y_k, t_k) \le d_{\omega,r}(y, t) + \frac{\varepsilon}{2} \\ r_k d_k(y_k, z_k) \le d_{\omega,r}(y, z) + \frac{\varepsilon}{2} & , & r_k d_k(x_k, t_k) \le d_{\omega,r}(x, t) + \frac{\varepsilon}{2} \end{array} \right\} \right) = 1 \ .$$

We now use 0-hyperbolicity of the spaces X_k, which shows that the set

$$\left\{ k \in \mathbb{N} : \ r_k d_k(x_k, y_k) + r_k d_k(z_k, t_k) \le \max \left\{ d_{\omega,r}(x, z) + d_{\omega,r}(y, t) + \varepsilon, d_{\omega,r}(y, z) + d_{\omega,r}(x, t) + \varepsilon \right\} \right\}$$

has ω-measure 1. Hence, we obtain

$$d_{\omega,r}(x, y) + d_{\omega,r}(z, t) \le \max \left\{ d_{\omega,r}(x, z) + d_{\omega,r}(y, t), d_{\omega,r}(y, z) + d_{\omega,r}(x, t) \right\} + \varepsilon \ .$$

As ε is arbitrary, this shows that the ultralimit $X_{\omega,r}$ is 0-hyperbolic. □

We now recall and prove Theorem 1.4 of the Introduction.

Theorem 4.7 (Culler-Morgan [CM87])**.** *Property (F$\mathbb{R}$A) defines an open subset in $\mathcal{G}$.*

Remark 4.8. Let G be a *finitely generated* group. Then, G has property (F$\mathbb{R}$A) if and only if every G-action on an $\mathbb{R}$-tree has a fixed point.

Proof. The completion of an $\mathbb{R}$-tree is an $\mathbb{R}$-tree [Im77] — see also [MS84, Cor II.1.10] or [Chi01, Thm 2.4.14]. Hence, we are done by Remarks 3.1 and 4.1. □

Proof of Theorem 4.7. Let (G_k) be a sequence in $\mathcal{G}_n$ which converges to some point $G \in \mathcal{G}_n$, and such that any G_k acts without fixed point on some complete $\mathbb{R}$-tree T_k. By remark 4.2, it suffices to show that G acts without fixed point on some complete $\mathbb{R}$-tree.

Theorem 1.5 gives an ultralimit $T = T_{\omega,r}$ of the $\mathbb{R}$-trees T_k on which G acts uniformly, and T is an $\mathbb{R}$-tree by Lemma 4.6. Hence, Remark 4.8 concludes the proof. [5] □

Remark 4.9. The last proof does not work for simplicial trees: we used the fact that the class of $\mathbb{R}$-trees is closed under ultralimits.

In fact, as the referee pointed out, using Theorem 1.5 to prove Theorem 4.7 is a little awkward: as Remark 3.1 tells us immediately that the G_k-actions on the $\mathbb{R}$-trees T_k are uniform, it is unnecessary to make a clever choice of base points as in Proposition 3.11 and Theorem 3.12.

References

[Bes02] M. Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.

[BH99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Bou71] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 1 à 4.* Hermann, Paris, 1971.

[CG05] C. Champetier and V. Guirardel. Limit groups as limits of free groups. *Israel J. Math.*, 146:1–75, 2005.

[Chab50] C. Chabauty. Limite d'ensembles et géométrie des nombres. *Bull. Soc. Math. France*, 78:143–151, 1950.

[Cham00] C. Champetier. L'espace des groupes de type fini. *Topology*, 39(4):657–680, 2000.

[Chi01] I. Chiswell. *Introduction to Λ-trees.* World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[Cor] Y. de Cornulier. Fixed points and almost fixed points, after Gromov and V. Lafforgue. Unpublished notes.

[CM87] M. Culler and J. W. Morgan Group actions on $\mathbb{R}$-trees. *Proc. London Math. Soc*, 55(3):571–604, 1987.

[Del77] P. Delorme. 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations. *Bull. Soc. Math. France*, 105(3):281–336, 1977.

[DW84] L. van den Dries and A. J. Wilkie. Gromov's theorem on groups of polynomial growth and elementary logic. *J. Algebra*, 89(2):349–374, 1984.

[Dru] C. Druţu. Quasi-isometry rigidity of groups. `arXiv:math/0612256v2 [math.GT]`.

[FM05] D. Fisher and G. Margulis. Almost isometric actions, property (T), and local rigidity. *Invent. Math.*, 162(1):19–80, 2005.

[Gri84] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[Gro81] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gro03] M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[Gui77] A. Guichardet. Étude de la 1-cohomologie et de la topologie du dual pour les groupes de Lie à radical abélien. *Math. Ann.*, 228(3):215–232, 1977.

[HV89] P. de la Harpe and A. Valette. *La propriété (T) de Kazhdan pour les groupes localement compacts.* Société Mathématique de France, Paris, 1989. Avec un appendice de Marc Burger, Astérisque, No. 175.

[Im77] W. Imrich. On metric properties of tree-like spaces. In *Contributions to graph theory and its applications (Internat. Colloq., Oberhof, 1977)*, pages 129–156. Tech. Hochschule Ilmenau, Ilmenau, 1977.

5. In fact, use of Remark 4.8 is superfluous, as it is known that every ultralimit of metric spaces is complete: see e.g. Lemma 2.4.2 in [KL97] or Lemma I.5.53 in [BH99]

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.

[Kle] B. Kleiner. A new proof of Gromov's Theorem on groups of polynomial growth. `arXiv:0710.4593v3 [math.GR]`.

[KS97] N. J. Korevaar and R. M. Schoen. Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom.*, 5(2):333–387, 1997.

[Laf06] V. Lafforgue. Une remarque sur les fonctions conditionnellement de type négatif. *C. R. Math. Acad. Sci. Paris*, 342(7):497–499, 2006.

[Mok95] N. Mok. Harmonic forms with values in locally constant Hilbert bundles. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, number Special Issue, pages 433–453, 1995.

[MS84] J. W. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[Pau04] F. Paulin. Sur la théorie élémentaire des groupes libres (d'après Sela). *Astérisque*, (294):ix, 363–402, 2004.

[Ser77] J.-P. Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977. Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[Sha00] Y. Shalom. Rigidity of commensurators and irreducible lattices. *Invent. Math.*, 141(1):1–54, 2000.

[Tig79] J.-P. Tignol. Remarques sur le groupe des automorphismes d'un arbre *Ann. Soc. Sci. Bruxelles Sér. I*, 93:196–202, 1979.

[Tit77] J. Tits. A "Theorem of Lie-Kolchin" for trees. In *Contributions to Algebra: a collection of papers dedicated to Ellis Kolchin*, pages 377–388 Academic Press, 1977.

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