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Guillaume Havard[†], Mariusz Urbański[‡] and Michel Zinsmeister[§]

Abstract. In this paper we deal with the following family of exponential maps $(f_\lambda : z \mapsto \lambda(e^z - 1))_{\lambda\in[1,+\infty)}$. Denoting $d(\lambda)$ the hyperbolic dimension of f_λ. It is proved in [Ur,Zd[1]] that the function $\lambda \mapsto d(\lambda)$ is real analytic in $(1, +\infty)$, and in [Ur,Zd[2]] that it is continuous in $[1, +\infty)$. In this paper we prove that this map is C^1 on $[1, +\infty)$, with $d'(1^+) = 0$. Moreover we prove that depending on the value of $d(1)$

$$\begin{cases} d'(1+\varepsilon) & \sim & -\varepsilon^{2d(1)-2} & \text{if } d(1) < \frac{3}{2}, \\ |d'(1+\varepsilon)| & \lesssim & -\varepsilon\log\varepsilon & \text{if } d(1) = \frac{3}{2}, \\ |d'(1+\varepsilon)| & \lesssim & \varepsilon & \text{if } d(1) > \frac{3}{2}. \end{cases}$$

In particular, if $d(1) < \frac{3}{2}$, then there exists $\lambda_0 > 1$ such that $d(\lambda) < d(1)$ for any $\lambda \in (1, \lambda_0)$.

Hausdorff dimension, Julia set, Exponential family, Parabolic points, Thermodynamic Formalism, Conformal Measures

Contents

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1. Introduction

1.1. **An overview of the problem.** In this paper we deal with maps of the form $f_\lambda : z \mapsto \lambda(e^z - 1)$, for $\lambda \geq 1$. As long as λ is strictly greater than 1, 0 is a repelling fixed point and there exists an attracting fixed point $q_\lambda < 0$. Those two points collapse to 0 for $\lambda = 1$, and 0 becomes parabolic. We are interested in J_λ, the set of points that do not escape to ∞ under iterations of f_λ. The Hausdorff dimension of this set, that we denote $d(\lambda)$, is an element of $(1, 2)$, and is called the Hyperbolic Dimension of the map f_λ. While for any λ the Julia set of f_λ has Hausdorff dimension constant equal to 2, cf. [McMu[1]], the Hyperbolic Dimension varies with λ. Moreover, any invariant probability measure gives full mass to J_λ, and $d(\lambda)$ is, in the hyperbolic case, equal to the first zero of the pressure of the map $t \mapsto -t \log |f'_\lambda|$[1], cf. [Ma,Ur[1]].

Variations of $\lambda \mapsto d(\lambda)$ with respect to λ, is an interesting feature that reflects changes in geometry after perturbation of a dynamical system. The philosophy is that d behaves smoothly, and even real analytically, if we perturb a conformal hyperbolic dynamical system, in a real analytic way.

This philosophy was proposed in 1981 Rio de Janeiro's conference by Sullivan [Su]. The same year Ruelle [Ru] proved that it was true for a class of Hyperbolic Conformal Repellers. His strategy, used since then in other contexts, see [Ur,Zd[1]] for the exponential family and [Ma,Ur[2]] for meromoprhic functions, was the following : prove a Bowen's formula that identifies the dimension as the zero of a pressure function, prove that this pressure is the logarithm of a simple and isolated eigenvalue of a Perron-Frobenius(-Ruelle) operator, then use some results about perturbation theory of operators.

When approaching the boundary of an Hyperbolic components one can not expect any smoothness. Nevertheless there still exists some paths along thus we still have continuity of the Hausdorff dimension. This was first proved by Bodart and Zinsmeister in [Bo,Zi] for the quadratic family, $z \mapsto$

[1]This result is known as *Bowen's formula*.

$z^2 + c$, for $c \in \mathbb{R}$ approaching $\frac{1}{4}$ from the left. Then it has been proved for other parameters c, [Ri], or other rational maps, [McMu[4]], [Bu,Le], or in other situations see [McMu[3]] for Kleinian Groups, [Ur,Zd[2]] for the exponential family. The strategy for such results is to control conformal measures, or Patterson-Sullivan measures, in order to prove that they converge towards the "good" conformal/Patterson-Sullivan-measure. This usually boils down in proving that any limiting measure is non-atomic. Note that this strategy may also be used to proved discontinuity of the Hausdorff dimension, or more precisely to prove convergence towards something bigger than the Hausdorff dimension of the "limit set", [Do,Se,Zi], [Ur,Zi[1]] and [Ur,Zi[2]].

The problem of the derivative of the Hausdorff dimension is, to our knowledge, investigated in two other papers than the present one. In [Ha,Zi[1]] for the quadratic family it is proved that $d'(c)$, the derivative of $d(c) := \mathrm{Hdim}(J_c)$, diverges towards $+\infty$ as c converges towards $\frac{1}{4}$ from the left. In [Ja[1]], still for the quadratic family, but this time for c converging from the right towards $-\frac{3}{4}$, and under the realistic hypothesis that $d(-\frac{3}{4}) < \frac{4}{3}$, it is proved that $d'(c)$ converges towards $-\infty$. In order to control the derivative the starting point in all those papers is first to get an exact formula for the derivative. This is done using thermodynamic formalism by differentiating the Bowen's formula. Then some uniform estimates of distorsion in a neighborhood of the fixed point are used in order to control measures of fondamental annuli. Conclusions then comes from a precise analysis of a certain integral. This is that last point that explains why such a study has not been yet done in a more general setting. In the present paper, as well as in [Ha,Zi[1]] and [Ja[1]], some very particular properties of the case studied are used to conclude.

1.2. **Main result.** When one notes that if τ_λ denotes the translation by $-\lambda$, then we have $f_\lambda \circ \tau_\lambda = \tau_\lambda \circ g_\lambda$, with $g_\lambda(z) = \alpha(\lambda)e^z$ and $\alpha(\lambda) = \lambda e^{-\lambda}$, this philosophy (real analyticity of d) is in [Ur,Zd[1]] proved to be the case. More precisely, it is proved there that $d : \lambda \mapsto d(\lambda)$ is real-analytic on $(1, +\infty)$, and in [Ur,Zd[2]], that it is continuous on $[1, +\infty)$. In this paper we study the asymptotic behavior of the function $\lambda \mapsto d'(\lambda)$, and we prove the following.

Theorem 1.1. *There exist $\lambda_0 > 1$ and $K > 1$ such that $\forall \lambda \in (1, \lambda_0)$*

$$\begin{cases} \frac{-1}{K}(\lambda-1)^{2d(1)-2} \leq d'(\lambda) \leq -K(\lambda-1)^{2d(1)-2} & \text{if } d(1) < \frac{3}{2}, \\ |d'(\lambda)| \leq K(\lambda-1)\log\frac{1}{\lambda-1} & \text{if } d(1) = \frac{3}{2}, \\ |d'(\lambda)| \leq K(\lambda-1) & \text{if } d(1) > \frac{3}{2}. \end{cases}$$

In particular the function $\lambda \mapsto d(\lambda)$ is C^1 on $[1, +\infty)$, with $d'(1) = 0$.

Remark : As already mentioned, conjugating f_λ by the translation τ_λ, we get the family $g_\lambda := \tau_\lambda \circ f_\lambda \circ \tau_\lambda^{-1}$, with $g_\lambda(z) = \lambda e^{-\lambda} e^z$. Changing variable to $\varepsilon := \lambda e^{\lambda-1} - 1$, we get the family $g_\varepsilon : z \mapsto (1+\varepsilon)e^{-1}e^z$ with

$\varepsilon \sim (\lambda - 1)$. Let $D(\varepsilon)$ be the hyperbolic dimension of g_ε, then

$$\begin{cases} D'(\varepsilon) & \sim & \varepsilon^{2D(0)-3} & \text{if } D(0) < \frac{3}{2}, \\ |D'(\varepsilon)| & \lesssim & \log\frac{1}{\varepsilon} & \text{if } D(0) = \frac{3}{2}, \\ |D'(\varepsilon)| & \lesssim & K & \text{if } D(0) > \frac{3}{2}. \end{cases}$$

Note in particular that, in case $D(0) < \frac{3}{2}$, we get exactly the same asymptotic as the one in [Ha,Zi[1]] for the family $c \mapsto z^2 + c$, with $c < \frac{1}{4}$. For this last family we were able to prove that $d(\frac{1}{4}) < \frac{3}{2}$, see [Ha,Se,Zi]. Inequality that we do not know for the exponential family.

Note also that if $d(1) < \frac{3}{2}$ then we have a control on the sign of the derivative in a right neighborhood of 1. It asserts that $d(1^+)$ is a local maximum of the Hyperbolic Dimension.

Remark : There is to our knowledge no algorithm to compute accurately Hausdorff dimension of parabolic Julia sets. In [Ha,Se,Zi] an estimate of the Hausdorff dimension of the cauliflower (the Julia set of $z \mapsto z^2 + \frac{1}{4}$) is given using by calculating, with a computer, the first terms of a sum, then by estimating its tail. This method uses strongly particular properties of the map. More generally, one could build an infinite iterated function system whose limit set would have Hausdorff dimension equal to the hyperbolic dimension of the Julia set. Then, using results from [He,Ur], one could approximate this Hausdorff dimension by finite subsystems keeping track of the error. Finally, there are algorithms to calculate Hausdorff dimension of finite IFSs with any desired accuracy [McMu[2]], [Je,Po]. However, to realize such program would be a tedious extremely time consuming task.

The proof of the main result will follow exactly the same lines as the one of [Ha,Se,Zi], but will make an extensive use of the Thermodynamic Formalism for Meromorphic Functions, as developed by, Urbański, Urbański and Kotus, Urbański and Zdunik, and Urbański and Mayer. The reader will find in [Ma,Ur[2]] all proofs of results we need in this paper, as well as a complete bibliography on the subject.

1.3. **Organization of the paper.** In the first part we use Chapter 8 of [Ma,Ur[2]] to get a formula for $d'(\lambda)$, for any $\lambda \in (1, +\infty)$. This mainly consists of conjugating the dynamics and differentiating the pressure.

In the second part we collect some estimates of the distortion around the fixed point 0. They are crucial since the formula obtained in the first part of this paper involves two integrals with respect to an invariant measure that has unbounded Radon-Nikodym derivative with respect to the Hausdorff measure, in any neighborhood of 0.

In the third part we use those estimates to control the integrals and to prove the main result.

In the first appendix we prove the estimates used in the second part of this paper in a more general setting than needed in this paper. Namely, we allow the repelling fixed point to converge towards a parabolic fixed point with several petals. The second appendix is devoted to the study of partial sums of some sequences that will be needed several times in the paper.

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2. A formula for the derivative of the function $\lambda \mapsto d(\lambda)$

Before giving and proving the formula of the derivative, this is done below in Proposition 2.1, we introduce some notations and recall some results concerning the thermodynamical formalism for that family of exponential maps.

2.1. Thermodynamic formalism.

Let P be be the cylinder $\{z \in \mathbb{C} \,|\, -\pi < \mathcal{I}m\; z < \pi\}$. As it is done in [Ur,Zd[1]] we associate to f_λ the map $F_\lambda : P \to P$ defined by

$$F_\lambda \circ \pi = \pi \circ f_\lambda,$$

with π being the natural projection on the cylinder $P = \mathbb{C}/\sim$, with $z_1 \sim z_2$ if and only if $(z_1 - z_2) = 2ik\pi$, for some $k \in \mathbb{Z}$. In particular for any $z \in P$ we have $f_\lambda(z) = F_\lambda(z)$, and $F_\lambda(z) = F_\lambda(z')$ if and only if there exists $k \in \mathbb{Z}$ such that $f_\lambda(z) - f_\lambda(z') = 2ik\pi$. This tells us that for any $z \in P$, we have $F_\lambda^{-1}(z) = \{z_k \in P \,|\, f_\lambda(z_k) = z + 2ik\pi,\, k \in \mathbb{Z}\}$. We also see that $J(F_\lambda) = \pi(J(f_\lambda)) = J(f_\lambda) \cap P$.

Let us now introduce some notation and collect some results, where we mainly refer to [Ma,Ur[2]], see also [Ur,Zd[1]], [Ur,Zd[2]], [Ur].

- For any $\lambda \geq 1$ we define $\mathcal{L}_{\lambda,t}$, the Perron-Frobenius operator associated with the potential $-t\log|F_\lambda'|$. It acts on $\mathcal{H}_\alpha^\lambda$, the set of bounded α-Hölder functions defined on $J(F_\lambda)$, in the following way, let $g \in \mathcal{H}_\alpha^\lambda$, and $z \in J(F_\lambda)$

$$\begin{aligned}\mathcal{L}_{\lambda,t}(g)(z) &= \sum_{F_\lambda(y)=z} \frac{1}{|(F_\lambda)'(y)|^t} g(y)\\ &= \sum_{k\in\mathbb{Z}} \frac{1}{|z+\lambda+2ik\pi|^t} g(z_k), \quad \text{with } z_k \in P \text{ such that } f_\lambda(z_k) = z + 2ik\pi\cdot\end{aligned}$$

- The only $d(\lambda)$-conformal measure supported on J_λ is denoted m_λ[2].

- The only equilibrium measure for the potential $-d(\lambda)\log|F_\lambda'|$ and the dynamical system (J_λ, F_λ) is denoted μ_λ.

-The pressure of the potential $-t\log|F_\lambda'|$ is denoted $P(\lambda,t)$, and is defined by

$$P(\lambda,t) = \sup\{h_\mu - t\chi_\mu\},$$

[2] We refer to section 3 of this paper for a definition and more details about conformal measures.

where the supremum is taken over all invariant probability measures μ supported on $J(F_\lambda)$, such that $\chi_\mu < +\infty$, where h_μ denotes the metric entropy of the measure μ, and $\chi_\mu = \int \log |F'_\lambda| d\mu$ is its Lyapunov exponent.

We will derive our formula for $d'(\lambda)$ starting from Bowen's formula that asserts that for any $\lambda > 1$, $d(\lambda)$ is the only real number so that $P(\lambda, d(\lambda)) = 0$ (see [Ur,Zd[1]]). We want to differentiate this formula with respect to λ, and in order to do so we need to appropriately conjugate the dynamics of F_λ.

- Let $\lambda_0 > 1$ be fixed. For any $\lambda > 1$, we denote h_λ the conjugating map from J_{λ_0} to J_λ such that $F_\lambda \circ h_\lambda = h_\lambda \circ F_{\lambda_0}$.

- We then set : $\varphi_{\lambda,t} := -t \log |F'_\lambda \circ h_\lambda|$. It is a potential which is defined on J_{λ_0}. We then use Corollary 8.10 in [Ma,Ur[2]] that tells us that $(\lambda, t) \mapsto P_0(\varphi_{\lambda,t})$ is real analytic for λ close enough to λ_0[3]. Bowen's formula then implies that $\frac{\partial}{\partial \lambda} P_0(\varphi_{\lambda,d(\lambda)}) = 0$. It is this calculation that leads to the desired formula.

2.2. **The formula and its proof.** In this section we prove the following formula

Proposition 2.1. *For any $\lambda \in (1, +\infty)$ we have*

$$d'(\lambda) = -\frac{d(\lambda)}{\chi_{\mu_\lambda}} \left(1 - \frac{1}{\lambda}\right) \int_{J_\lambda} \sum_{k=1}^{+\infty} \mathcal{R}e \left(\frac{1}{(F^k_\lambda)'}\right) d\mu_\lambda\cdot, \tag{2.1}$$

where μ_λ is the only equilibrium measure for the potential $-d(\lambda) \log |F'_\lambda|$.

Let $\lambda_0 > 1$ be fixed and let h_λ denote the conjugating map : $F_\lambda \circ h_\lambda = h_\lambda \circ F_{\lambda_0}$. Since μ_λ is the equilibrium measure for the potential $-d(\lambda) \log |F'_\lambda|$, we deduce that the potential $\varphi_{\lambda,d(\lambda)}$ has a unique equilibrium measure which is $\tilde{\mu_\lambda} := h_{\lambda *}(\mu_\lambda)$. We shall now use Theorem 6.14 in [Ma,Ur[2]] which asserts that given a tame function φ and a weakly tame function ψ we have

$$\frac{\partial}{\partial t} P_0(\varphi + t\psi)_{|\,t=0} = \int \psi d\mu_\varphi, \tag{2.2}$$

with μ_φ the equilibrium measure for the potential φ. We refer to chapter 4 of [Ma,Ur[2]] for definition of tame and loosely tame functions. By Lemma 8.9 in [Ma,Ur[2]], we know that for $R > 0$ small enough, there exists $\beta > 0$ such that $\forall \lambda \in (\lambda_0 - R, \lambda_0 + R)$ $\varphi_{\lambda,t}$ is β-tame. We then deduce from (2.2) that

$$0 = \frac{\partial}{\partial \lambda} P_0(\varphi_{\lambda,d(\lambda)}) = \int_{J_{\lambda_0}} \frac{\partial}{\partial \lambda} \left(\varphi_{\lambda,d(\lambda)}\right) d\tilde{\mu_\lambda}. \tag{2.3}$$

We thus have to compute $\frac{\partial}{\partial \lambda} \varphi_{\lambda,d(\lambda)}$. Note that

$$\varphi_{\lambda,d(\lambda)} = -d(\lambda) \log |F'_\lambda \circ h_\lambda| = -d(\lambda)(\log \lambda + \mathcal{R}e\ h_\lambda).$$

[3]We denote here P_0 the pressure with respect to the dynamical system $(J_{\lambda_0}, F_{\lambda_0})$.

Differentiating with respect to λ we get

$$\frac{\partial}{\partial\lambda}\varphi_{\lambda,d(\lambda)} = -d'(\lambda)\log|F'_\lambda \circ h_\lambda| - d(\lambda)\left(\frac{1}{\lambda} + \mathcal{R}e\,\frac{\partial}{\partial\lambda}h_\lambda\right)\cdot \tag{2.4}$$

Lemma 2.2. *For any $\lambda \in (1,+\infty)$ and any $z \in J_{\lambda_0}$ we have*

$$\frac{\partial}{\partial\lambda}h_\lambda(z) = \left(1 - \frac{1}{\lambda}\right)\sum_{k=1}^{+\infty}\frac{1}{(F_\lambda^k)'(h_\lambda(z))} - \frac{1}{\lambda}\cdot \tag{2.5}$$

In order to prove this formula we use two results from [Ur], Lemma 13.2 and Proposition 13.4, that we give in the following Lemma

Lemma 2.3. *For any $\lambda_0 \in (1,+\infty)$ one can find $R > 0$, $K > 0$, and $\alpha > 0$ such that*

$$\forall \lambda \in B(\lambda_0, R) \quad \forall n \in \mathbb{N} \quad \forall z \in J_\lambda \qquad |(F_\lambda^n)'(z)| \geq K(1+\alpha)^n\cdot \tag{2.6}$$

$$\forall \lambda \in B(\lambda_0, R) \quad \forall z \in J_\lambda \qquad \left|\frac{\partial}{\partial\lambda}h_\lambda(z)\right| < K\cdot \tag{2.7}$$

We can now prove Lemma 2.2.

Proof. In order to simplify notation, we write $\dot{h_\lambda}$ instead of $\frac{\partial}{\partial\lambda}h_\lambda$, and we drop z. We start with the conjugating formula : $h_\lambda \circ F_{\lambda_0} = F_\lambda \circ h_\lambda = \lambda(e^{h_\lambda} - 1)$, that we differentiate with respect to λ. We thus get,

$$\dot{h_\lambda} \circ F_{\lambda_0} = \dot{F_\lambda} \circ h_\lambda + \dot{h_\lambda} F'_\lambda \circ h_\lambda.$$

So that we have

$$\dot{h_\lambda} = \frac{\dot{h_\lambda} \circ F_{\lambda_0}}{F'_\lambda \circ h_\lambda} - \frac{\dot{F_\lambda} \circ h_\lambda}{F'_\lambda \circ h_\lambda}\cdot$$

Iterating this formula we end up for $n \in \mathbb{N}$ with

$$\dot{h_\lambda}(z) = \frac{\dot{h_\lambda} \circ F_{\lambda_0}^n}{(F_\lambda^n)' \circ h_\lambda} - \sum_{k=1}^{n}\frac{\dot{F_\lambda} \circ F_\lambda^{k-1} \circ h_\lambda}{(F_\lambda^k)' \circ h_\lambda}\cdot$$

Using Lemma 2.3 we deduce that

$$\frac{\dot{h_\lambda}(F_{\lambda_0}^n(z))}{(F_\lambda^n)'(h_\lambda(z))} \quad \text{is converging towards } 0\cdot$$

On the other hand, since $\dot{F_\lambda}(z) = e^z - 1 = \frac{1}{\lambda}F'_\lambda(z) - 1$, for any k we have

$$\frac{\dot{F_\lambda} \circ F_\lambda^{k-1}}{(F_\lambda^k)'} = \frac{1}{\lambda}\frac{1}{(F_\lambda^{k-1})'} - \frac{1}{(F_\lambda^k)'}\cdot$$

This leads to

$$\sum_{k=1}^{n}\frac{\dot{F_\lambda} \circ F_\lambda^{k-1}}{(F_\lambda^k)'} = \frac{1}{\lambda} - \frac{1}{(F_\lambda^n)'} + \left(\frac{1}{\lambda} - 1\right)\sum_{k=1}^{n-1}\frac{1}{(F_\lambda^k)'}\cdot$$

Using (2.6) in Lemma 2.3 we get that the series on the left above is converging towards

$$\sum_{k=1}^{+\infty} \frac{1}{(F_\lambda^k)'},$$

which finishes the proof. ■

Using (2.4) and Lemma 2.2 in (2.3) we get

(2.8)

$$-d'(\lambda) \int_{J_{\lambda_0}} \log |F_\lambda'| d\tilde{\mu}_\lambda - d(\lambda) \left(1 - \frac{1}{\lambda}\right) \int_{J_{\lambda_0}} \mathcal{R}e \sum_{k \geq 1} \frac{1}{(F_\lambda^k)' \circ h_\lambda} = 0\cdot$$

For any function g continuous on J_λ we have $\tilde{\mu}_\lambda(g \circ h_\lambda) = \mu_\lambda(g)$. We deduce from (2.8) that Proposition 2.1 is true.

3. Local dynamic and uniform estimates

In this section we introduce some notations and collect estimates proved in the appendix in a more general setting[4]. We then use these estimates in order to control uniformly conformal measures (m_λ) and equilibrium measures (μ_λ).

3.1. **Notation.** We know that $J_\lambda \cap P \subset \{z \in \mathbb{C} \,|\, -\frac{\pi}{2} < \mathcal{I}m\ z < \frac{\pi}{2}\}$.

Given $0 < \theta < \frac{\pi}{2}$ we denote S_θ the sector $\{re^{i\alpha} \,|\, r > 0,\ -\theta < \alpha < \theta\}$.

For $r_0 << 1$ we fix $0 < \theta < \frac{\pi}{2}$ to be such that $J_1 \cap B(0, r) \subset S_\theta$. Then we choose $\varepsilon_0 > 0$ small enough so that for any $0 \leq \lambda = 1 + \varepsilon \leq \lambda_0 = 1 + \varepsilon_0$ we have $f_\lambda^{-1}(S_\theta) \subset S_\theta$ and $J_\lambda \cap B(0, r) \subset S_\theta$. We then set $\gamma_0 = \{r_0 e^{it} \mid t \in]-\theta, \theta[\}$, $\gamma_1(\lambda) = f_\lambda^{-1}(\gamma_0)$. Joining $r_0 e^{i\theta}$ with $f_\lambda^{-1}(r_0 e^{i\theta})$ by a line, and doing the same with $r_0 e^{-i\theta}$ and its image by f_λ^{-1}, we get a cell $C_0(\lambda)$. It is a simply connected domain. A compactness argument tells us that if $1 \leq \lambda \leq \lambda_0$, then there exists a simply connected domain $V \subset S_\theta$ such that the closure of $\cup_\lambda C_0(\lambda)$ is a subset of V. In particular, Kœbe distorsion Theorem gives us a constant $K > 1$, only depending on r_0 and λ_0, such that for any univalent function h on V and any point x and y in $\overline{\cup_\lambda C_0(\lambda)}$ we have $\frac{1}{K} \leq \frac{|h'(x)|}{|h'(y)|} \leq K$. We will use later on this fact with inverse branches of f_λ^n. They are well defined on V since the post-singular set of the f_λ's, i.e. the orbit $-\lambda$ under f_λ, is a subset of $(-\infty, 0)$.

We then define for each integer n the set $C_n(\lambda) := f_\lambda^{-n}(C_0(\lambda))$, with f_λ^{-n} being the n^{th} iterates of the inverse branch of f_λ defined on $B(0, r_0)$ that fixes 0. In the following we are working with respect to measures concentrated on J_λ of dimension strictly greater to 1. One checks easily in that context that with respect to such measure $(C_n(\lambda))_{n \in \mathbb{N} \cup \{0\}}$ is a partition of $B(0, r_0)$. Moreover the set $C_0(\lambda)$ is mapped univalently by f_λ^{-n} to $C_n(\lambda)$.

[4]We deal in the appendix with a family of germ of holomorphic in a neighborhood of a repelling fixed point which degenerates into a parabolic fixed point with p petals.

Let N_ε be an integer[5] and defined the sequence $(a_{n,\varepsilon})_{n\in\mathbb{N}}$ as $a_{n,\varepsilon} = \frac{1}{n}$, if $n \leq N_\varepsilon$, and $a_{n,\varepsilon} = \varepsilon(1+\varepsilon)^{-n}$, if $n \geq N_\varepsilon$. Note that $a_{n,\varepsilon} \to 0$. In order to simplify notations, we let $a_n := a_{n,\varepsilon}$. We consider now the one parameter familly of sequences, $(a_n(\alpha))_{n\in\mathbb{N}}$, defined for $n \in \mathbb{N}$ by $a_n(\alpha) := a_n^\alpha$. We are also interested in partial sums of $\sum a_n(\alpha)$. For $k \leq n$ we let $S_{k,n}(\alpha) := \sum_{l=k}^n a_l(\alpha)$. The sequence $(a_n(\alpha))$ will describe, for different values of α, the distorsion around 0, the conformal measure of partition sets of a neighborhood of 0, and the partial sums $S_{k,n}(\alpha)$ will play a role in controlling the invariant measure of the same partition sets, as well as evaluating the integral which is crucial in order to get our main result. Those estimates are easy and we use them in this section but we postponed their proofs to the appendix.

3.2. **Uniform estimates of the distorsion.** In this section we give uniform estimates depending on λ for the local dynamics next to the repelling-parabolic fixed point 0. We recall that the family we are studying is given for $\lambda := 1+\varepsilon \geq 1$ by $f_\lambda(z) = \lambda(e^z - 1)$. In particular, in a neighborhood of 0, the local dynamic is given by the following Taylor expansion

$$F_\lambda(z) = f_\lambda(z) = \lambda z + z^2 + z^3 g_\lambda(z)\cdot$$

With $g_\lambda(z)$ uniformly bounded, independently of λ, as soon as a neighborhood of zero has been fixed. Note in particular that for $\varepsilon = 0$, the point 0 is a parabolic fixed point with one petal.

We apply the general results of the first appendix of this paper to this special family f_λ. In the remaining of the paper we set $\lambda = 1+\varepsilon$ and we denote the relevant quantities by indexing them equally well either by ε or λ. Moreover, in the remainder of this section F_λ^{-n} will be the inverse branch of F_λ^{-n} that fixes 0. From Proposition 5.7 we deduce that

Proposition 3.1. *Let $0 < r_0$, $1 < \lambda_0$ being fixed. Then there exists $K > 1$ such that $\forall\lambda \in (1, \lambda_0)$, $\forall z \in C_0(\lambda)$, and $\forall n \in \mathbb{N}$*

$$\frac{1}{K} a_n(2) \leq |(F_\lambda^{-n})'(z)| \leq K a_n(2)\cdot$$

The following result is technical but will be crucial in order to control the sign of the derivative $d'(\lambda)$.

Lemma 3.2. *Let $0 < r_0$, $1 < \lambda_0$ being fixed. There exists an integer N such that $\forall n > N$, $\forall k \in \mathbb{N} \cap [1, n-N]$, $\forall\lambda \in (1, \lambda_0)$ and $\forall z \in C_n(\lambda)$*

$$\frac{\sqrt{3}}{2}|(F_\lambda^k)'(z)| \leq \mathcal{R}e\ (F_\lambda^k)'(z)\cdot$$

Proof. Let $z \in C_n$ and $\theta_k(z) = \arg(F_\lambda^k)'(z)$. The Lemma boils down to proving that $|\theta_k(z)| \leq \frac{\pi}{6}$.
One computes that $(F_\lambda^k)'(z) = \prod_{j=0}^{k-1} F_\lambda'(F_\lambda^j(z)) = \lambda^k \exp(\sum_{j=0}^{k-1} F_\lambda^j(z))$. So

[5] In our study we have $N_\varepsilon \sim \frac{1}{\varepsilon} = \frac{1}{\lambda-1}$.

that we have $\theta_k(z) = \sum_{j=0}^{n-1} \mathcal{I}m\ (F_\lambda^j(z))$. Since $F_\lambda^j(z)$ belongs to C_{n-j} we may use Corollary 5.8 which asserts that $|\mathcal{I}m\ (Z)| \lesssim \frac{1}{(n-j)^2}$ for any $Z \in C_{n-j}$. We thus have

$$|\theta_k(z)| \lesssim \sum_{j=1}^{n-k} \frac{1}{(n-j)^2} \leq \sum_{j=N}^{+\infty} \frac{1}{j^2}.$$

This is less than $\frac{\pi}{6}$ if N is big enough and we are done. ∎

We end this section with two more estimates of the distorsion. The first one needs the following observation on the localization of $J(f_\lambda)$.

Lemma 3.3. *For every $R > 0$ there exists $\Delta > 0$ such that for all $\lambda > 1$,*

$$J(f_\lambda) \setminus \bigcup_{n=-\infty}^{+\infty} B(2\pi ni, R) \subset \{z \in \mathbb{C} : \mathcal{R}e\ z \geq \Delta\}.$$

Proof. First notice that

$$f_\lambda(\{z \in \mathbb{C} : \mathcal{R}e\ z < 0\}) = B(-\lambda, \lambda) \subset \{z \in \mathbb{C} : \mathcal{R}e\ z < 1-\lambda\} \subset \{z \in \mathbb{C} : \mathcal{R}e\ z < 0\}.$$

Thus

$$\{z \in \mathbb{C} : \mathcal{R}e\ z < 0\} \subset \mathcal{F}(f_\lambda) := \text{Fatou set of } f_\lambda. \tag{3.1}$$

Now write $z = x + iy$. Then

$$\mathcal{R}e\ (f_\lambda(z)) = \mathcal{R}e\ (\lambda(e^x \cos y + ie^x \sin y - 1)) = \lambda(e^x \cos y - 1).$$

Note that there exists $\Delta_1 > 0$ so small that if $0 < x < \Delta_1$ and $x + iy \notin \bigcup_{n=-\infty}^{+\infty} B(2\pi ni, R)$, then $\text{dist}(y, \{2\pi ni : n \in \mathbb{Z}\}) > R/2$, and consequently, $\cos y < \cos(R/2)$. Hence, $\mathcal{R}e\ (f_\lambda(z)) < \lambda(e^{\Delta_1} \cos(R/2) - 1)$. Take now $0 < \Delta \leq \Delta_1$ so small that $e^\Delta \cos(R/2) < 1$. So $\mathcal{R}e\ (f_\lambda(z)) < 0$ and, by (3.1), $f_\lambda(z) \in \mathcal{F}(f_\lambda)$. Therefore we have proved that

$$\{z \in \mathbb{C} \setminus \bigcup_{n=-\infty}^{+\infty} B(2\pi ni, R) : \mathcal{R}e\ z < \Delta\} \ \subset \mathcal{F}(f_\lambda).$$

We are done. ∎

Now notice that if $\mathcal{R}e\ z \geq \Delta$, then

$$|f_\lambda'(z)| = \lambda e^{\mathcal{R}e\ z} \geq \lambda e^\Delta > 1.$$

Combining this and Lemma 3.3, we obtain the following.

Lemma 3.4. *For every $R > 0$ there exists $\gamma > 1$ such that for every $z \in J(F_\lambda) \setminus B(0, R)$,*

$$|F_\lambda'(z)| \geq \gamma.$$

Using Proposition 5.7, Lemma 3.4 and the same reasoning as for the proof of Lemma 3.6 in [Ha,Zi[1]] we prove the following result

Lemma 3.5. *There exist $0 < r_0$, $1 < \lambda_0$ and $1 < K$ such that $\forall \lambda \in (1, \lambda_0)$ and $\forall z \in J_\lambda$,*

$$F_\lambda^n(z) \notin B(0, r_0) \Rightarrow Kn^2 \leq |(F_\lambda^n)'(z)|.$$

3.3. **Conformal measures.** Let us recall that a probability measure m_λ is called *conformal* if its strong Jacobian is equal to $|F'_\lambda|^{d(\lambda)}$. This means that for any measurable set A on which f_λ is 1-1 we have

$$m_\lambda(F_\lambda(A)) = \int_A |F'_\lambda|^{d(\lambda)} dm_\lambda. \tag{3.2}$$

Those measures are usually a powerful tool to study Hausdorff dimension of Julia sets. In fact their definition is dynamical but they very often carry a geometrically significant information about the Julia set. In many of cases they coincide (up to a multiplicative constant) with Hausdorff or packing measures on the Julia set.

Using Proposition 5.7 and the notation introduced below we get the following.

Proposition 3.6. *Let $0 < r_0$, $1 < \lambda_0$ being fixed. Then there exists $K > 1$ such that $\forall \lambda \in (1, \lambda_0)$, and $\forall n \in \mathbb{N}$*

$$\frac{1}{K} a_n(2d(\lambda)) \leq m_\lambda(C_n(\lambda)) \leq K a_n(2d(\lambda))$$

Proof. This is not difficult when one observes that for each λ the function F^n_λ is univalent on $C_n(\lambda)$. In particular using the definition of a conformal measure we deduce that :

$$m_\lambda(C_0(\lambda)) = m_\lambda(F^n_\lambda(C_n(\lambda))) = \int_{C_n(\lambda)} |(F^n_\lambda)'|^{d(\lambda)} dm_\lambda. \tag{3.3}$$

We then use estimates of Proposition 3.1, since $|(F^n_\lambda)'|$ on $C_n(\lambda)$ is comparable with $|(F^{-n}_\lambda)'|^{-1}$ on $C_0(\lambda)$. We deduce that there exists a constant $K > 0$ such that for any $z \in C_0(\lambda)$

$$\frac{1}{K} |(F^n_\lambda)'(z)|^{-d(\lambda)} \leq m_\lambda(C_n(\lambda)) \leq K |(F^n_\lambda)'(z)|^{-d(\lambda)}. \tag{3.4}$$

We can now conclude the proof by using again Proposition 5.7. ∎

Remark : Let m_∞ be any accumulation point of the family of probability measures $(m_\lambda)_{\lambda>1}$. Let (λ_n) be a sequence of real numbers converging from above towards 1 such that the sequence (m_{λ_n}) converges weakly to m_∞, and $(d(\lambda_n))$ converges to some $d \geq 0$. For any $r > 0$ small enough one may find $N(r)$ such that

$$\forall n \geq N(r) \qquad B(0,r) \cap J_{\lambda_n} \subset \{0\} \cup \bigcup_{k \leq N(r)} C_k(\lambda_n)$$

And in particular we conclude if $r > 0$ is such that $m_\infty(\{|z| = r\}) = 0$, that we have

$$m_\infty(B(0,r)) = \lim_{n\to\infty} m_{\lambda_n}(B(0,r)) \leq \lim_{n\to\infty} \sum_{k>N(r)} m_{\lambda_n}(C_k(\lambda_n)) \leq \frac{K}{N(r)^{2d(\lambda_n)-1}}.$$

So that we conclude that m_∞ has no atom at 0. And it is one of the main point in order to conclude that $d(\lambda) \to d(1)$ when $\lambda \to 1$, see [Ur,Zd^2].

We end this section about conformal measures with a technical Lemma. It will be used in the next section concerning invariant measures.

Before stating and proving this result we recall that $P = \{z \in \mathbb{C} \,|\, -pi < \mathcal{I}m\ z \leq pi\}$, and for any $M > 0$, and any $r > 0$, we introduce the following notation : $P_M := \{z \in P \,|\, \mathcal{R}e\ z \leq M\}$, $\mathcal{B}_r := P \setminus B(0,r)$ and $\mathcal{B}_{r,M} := P_M \cap \mathcal{B}_r$.

Lemma 3.7. *There exists* $0 < \alpha < \beta$ *such that* $\forall M \geq 2$, $\forall \lambda \in [1, \lambda_0]$, *with* $\lambda_0 < \frac{\pi}{3}$, $\forall r \in]0, \frac{\pi}{3} - \lambda_0]$ *and* $\forall A \subset B(0,r)$ *measurable, we have*

$$\alpha m_\lambda(A) \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq \beta m_\lambda(A). \tag{3.5}$$

and

$$m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_{r,M}) \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq 54\beta m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_{r,M}). \tag{3.6}$$

Proof. Let B_k be the connected component of $F_\lambda^{-1}(B(0,r))$ such that $f_\lambda(B_k) = B(2ik\pi, r)$. For any $z \in B_k$ we have :

$$|F'_\lambda(z)| = |f'_\lambda(z)| = |f_\lambda(z) + \lambda| = |\lambda + 2ik\pi + ae^{i\theta}|, \tag{3.7}$$

with $a < r$ and $0 \leq \theta < 2\pi$. With our assumptions this leads, for $|k| \geq 1$, to

$$2|k|\pi - \frac{\pi}{3} \leq |F'_\lambda(z)| = |f'_\lambda(z)| \leq 2|k|\pi + \frac{\pi}{3}\cdot \tag{3.8}$$

Since $|f'_\lambda(z)| = \lambda \exp(\mathcal{R}e\ z)$, we also get, for any $|k| \geq 1$, that

$$\forall z \in B_k \quad \log 5 \leq \mathcal{R}e\ z\cdot$$

As a consequence we see that $F_\lambda^{-1}(B(0,r)) \cap \mathcal{B}_r = \cup_{|k|\geq 1} B_k$.

The measure m_λ being conformal we have

$$m_\lambda(A) = m_\lambda(F_\lambda(A_k)) = \int_{A_k} |F'_\lambda|^{d(\lambda)} dm\lambda,$$

with $A_k := F_\lambda^{-1}(A) \cap B_k$. From (3.8) we deduce that

$$\frac{m_\lambda(A)}{(2|k|\pi + \frac{\pi}{3})^{d(\lambda)}} \leq m_\lambda(A_k) \leq \frac{m_\lambda(A)}{(2|k|\pi - \frac{\pi}{3})^{d(\lambda)}}, \tag{3.9}$$

so that

$$2m_\lambda(A) \sum_{k\geq 1} \frac{1}{(2k\pi + \frac{\pi}{3})^{d(\lambda)}} \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq 2m_\lambda(A) \sum_{k\geq 1} \frac{1}{(2k\pi - \frac{\pi}{3})^{d(\lambda)}}\cdot \tag{3.10}$$

The function $\lambda \mapsto d(\lambda)$ being continuous on $[1, \lambda_0]$ one may consider its minimum δ which is strictly greater than 1. With $\alpha = 2\sum_{k\geq 1} \frac{1}{(2k\pi + \frac{\pi}{3})^2}$ and $\beta = 2\sum_{k\geq 1} \frac{1}{(2k\pi - \frac{\pi}{3})\delta}$ we have :

$$\alpha m_\lambda(A) \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq \beta m_\lambda(A).$$

This is (3.5).

Note that (3.8) tells us that for any $z \in B_1$ we have $\mathcal{R}e\ z \leq \log(2\pi + \frac{\pi}{3}) < 2 \leq M$. This implies that $B_1 \subset \mathcal{B}_{r,M}$. In particular we have

$$m_\lambda(A_1) \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_{r,M})\cdot$$

We then deduce from (3.9) that

$$\frac{m_\lambda(A)}{(2\pi + \frac{\pi}{3})^2} \leq m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_{r,M})\cdot$$

Together with (3.5) we conclude that

$$m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq (2\pi + \frac{\pi}{3})^2 \beta m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_{r,M})\cdot$$

Since $(2\pi + \frac{\pi}{3})^2 \leq 54$ we conclude that the left hand side inequality of (3.6) holds. The right hand side being obvious the proof is finished. ∎

3.4. **Invariant measures.** Let us first recall that $\mu_\lambda = \rho_\lambda m_\lambda$ is the unique F_λ-invariant probability measure equivalent with m_λ. This measure is also the unique equilibrium state for the potential $-d(\lambda)\log|F_\lambda'|$ i.e.

$$h_{\mu_\lambda} - d(\lambda)\int \log|F_\lambda'|d\mu_\lambda = \sup\{h_\mu - d(\lambda)\int \log|F_\lambda'|d\mu\},$$

where supremum is taken over all F_λ-invariant ergodic probability measures such that $\int \log|F_\lambda'|d\mu < +\infty$. The function ρ_λ is obtained in [Ma,Ur^2] as the limit of the sequence $\mathcal{L}_\lambda^n(1)$. The main results of this section is

Proposition 3.8. *Let $0 < r_0$, $1 < \lambda_0$ being fixed. Then there exists $K > 1$ such that $\forall \lambda \in (1, \lambda_0)$, and $\forall n \in \mathbb{N}$*

i- $\frac{1}{K} a_n(2d(\lambda) - 1) \leq \mu_\lambda(C_n(\lambda)) \leq K a_n(2d(\lambda) - 1)$ *if* $n \leq N_\varepsilon$.

ii- $\frac{1}{K}\frac{a_n(2d(\lambda))}{\varepsilon} \leq \mu_\lambda(C_n(\lambda)) \leq K\frac{a_n(2d(\lambda))}{\varepsilon}$ *if* $n \geq N_\varepsilon$.

Proof. Let $\mathcal{B}_r := P \setminus B(0,r)$. We know that $\mu_\lambda(\mathcal{B}_r) > 0$ so that the first return time $N_{\lambda,r}(z) := \inf\{n \geq 1 \,|\, F_\lambda^n(z) \in \mathcal{B}_r\}$ is finite μ_λ-almost-surely. Let $\mathcal{B}_{\lambda,n} := \{N_{\lambda,r} = n\}$. We recall that the sets (C_n) are introduced at the beginning of this section. Note that for r small enough we have $\mathcal{B}_{\lambda,n} \cap B(0,r) = C_{n-1}(\lambda)$. Since μ_λ is F_λ-invariant its restriction to $\mathcal{B}_r$ is invariant for the first return map in $\mathcal{B}_r$, that we denote T_λ. Moreover, μ_λ can be built from this T_λ-invariant measure and this leads, for any measurable set A, to the formula

$$\mu_\lambda(A) = \sum_{n\geq 1}\sum_{k=0}^{n-1} \mu_\lambda(F_\lambda^{-k}(A) \cap \mathcal{B}_{\lambda,n} \cap \mathcal{B}_r).$$

We are interested in the sets C_l for which we get

$$\mu_\lambda(C_l) = \sum_{n\geq 1}\sum_{k=0}^{n-1} \mu_\lambda(F_\lambda^{-k}(C_l) \cap \mathcal{B}_{\lambda,n} \cap \mathcal{B}_r).$$

Note now that the set $F_\lambda^{-k}(C_l) \cap \mathcal{B}_{\lambda,n} \cap \mathcal{B}_r$ is empty unless $n > l+1$ and $k = n-l-1$. In this case we have $F_\lambda^{-(n-l)}(C_l) \cap \mathcal{B}_{\lambda,n} \cap \mathcal{B}_r = F_\lambda^{-1}(C_{n-2}) \cap \mathcal{B}_r$. We thus conclude that

$$\mu_\lambda(C_l) = \sum_{n \geq l} \mu_\lambda(F_\lambda^{-1}(C_n) \cap \mathcal{B}_r)\cdot$$

In Corollary 3.10, that we admit for the moment, we show that there exists $K_1 > 0$, independent of λ, such that for any $A \subset B(0,r)$ we have

$$\frac{1}{K_1} m_\lambda(A) \leq \mu_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq K_1 m_\lambda(A)\cdot$$

So,

$$\frac{1}{K_1} \sum_{n \geq l} m_\lambda(C_n) \leq \mu_\lambda(C_l) \leq K_1 \sum_{n \geq l} m_\lambda(C_n).$$

From Proposition 3.6 we deduce that there exists $K_2 > 0$ such that

$$\frac{1}{K_2} \sum_{n \geq l} \sum_{n \geq l} a_n(2d(\lambda)) \leq \mu_\lambda(C_l) \leq K_2 \sum_{n \geq l} a_n(2d(\lambda)).$$

With the notations used in the appendix this is exactly

$$\frac{1}{K_2} S_{l,+\infty}(2d(\lambda)) \leq \mu_\lambda(C_l) \leq K_2 S_{l,+\infty}(2d(\lambda))\cdot$$

We then use Corollary 5.10 to finish the proof. ∎

Lemma 3.9. *There exists $K > 0$ such that for all $\lambda = 1 + \varepsilon$, with $\varepsilon > 0$ small enough, all $r > 0$ small enough, and all $M > 0$ big enough we have,*

$$\frac{1}{K} \leq \rho_\lambda \leq K \quad \text{on } \mathcal{B}_{r,M}, \quad \text{and} \quad \rho_\lambda \leq K \quad \text{on } \mathcal{B}_r\cdot$$

From this Lemma and Lemma 3.7 we easily conclude this.

Corollary 3.10. *There exists $K > 0$ such that for all $\lambda = 1 + \varepsilon$, with $\varepsilon > 0$ small enough, all $r > 0$ small enough, and for any measurable set $A \subset B(0,r)$ we have*

$$\frac{1}{K} m_\lambda(A) \leq \mu_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq K m_\lambda(A)\cdot$$

Proof. Let $r > 0$ and $\varepsilon > 0$ be small enough so that the assertions of Lemmas 3.7 and 3.9 hold. Let $K > 0$ coming from Lemma 3.9 be larger than $\max\{\beta, \alpha^{-1}\}$, both α and β coming from Lemma 3.7. By Lemma 3.7 we know that for any $A \subset B(0,r)$ we have

$$\frac{1}{K} m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq m_\lambda(A) \leq K m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r)\cdot$$

From the right hand side inequality in Lemma 3.9 we know that

$$\mu_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq K m_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r)\cdot$$

These two inequalities give us

$$\mu_\lambda(F_\lambda^{-1}(A) \cap \mathcal{B}_r) \leq K^2 m_\lambda(A)\cdot$$

For the other inequality we first note that Lemma 3.7 also asserts that

$$\frac{1}{K}m_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M}) \le m_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_r) \le Km_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M})\cdot$$

Since Lemma 3.9 implies that

$$\frac{1}{K}m_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M}) \le \mu_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M}) \le Km_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M}),$$

we conclude that

$$m_\lambda(A) \le Km_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_r) \le K^2 m_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M}) \le K^3\mu_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_{r,M})\cdot$$

We easily deduce that

$$m_\lambda(A) \le K^3\mu_\lambda(F_\lambda^{-1}(A)\cap\mathcal{B}_r)\cdot$$

This is the left hand side inequality of the Corollary and its proof is finished.

∎

Proof. Before starting the proof of Lemma 3.9 we sketch the strategy. We first use a result of Urbański and Zdunik, Lemma 3.4 in [Ur,Zd[1]], that asserts that as long as we stay far away from the post-singular set, iterates of $\mathcal{L}_\lambda$ are uniformly bounded from above by a constant that does not depend on λ. This gives us that ρ_λ is bounded from above in some $\mathcal{B}_r$. And this allows us to prove that for r and ε small enough, and for M big enough we have

$$\frac{1}{2} \le \mu_\lambda(\mathcal{B}_{r,M}) \le 1.$$

In order to control ρ_λ on $\mathcal{B}_{r,M}$ we use Kœbe's distortion Theorem on $\mathcal{B}_{r,M}$ and prove that the measures m_λ have the bounded distortion property on $\mathcal{B}_{r,M}$, with a constant which only depends on r and M. This implies, see [Ma] (compare [Ha] Propositions 1.2.7 and 1.2.8), that there exists an F_λ-invariant measure ν_λ which gives mass 1 to $\mathcal{B}_{r,M}$ and which is equivalent with m_λ. Its Radon-Nikodym derivative is such that $\frac{1}{K} \le \frac{d\nu_\lambda}{dm_\lambda} \le K$ on $\mathcal{B}_{r,M}$, with some $K>0$ independent of λ. Since m_λ is ergodic and conservative, there is, up to a multiplicative constant, only one possible invariant measure equivalent to it. This means that $\mu_\lambda = \alpha_\lambda\nu_\lambda$. Integrating on $\mathcal{B}_{r,M}$ we conclude that $\alpha_\lambda = \mu_\lambda(\mathcal{B}_{r,M})$. This leads to $\frac{1}{2K} \le \rho_\lambda \le K$.

We now go into further details. Note that the singular set of F_λ is the one point $-\lambda$ which sequence of iterates converges towards 0 from the left. In particular $\mathcal{B}_{r,M}$ is a simply connected domain on which inverse branches of F_λ are well defined. Since J_λ is a subset of $\{-\frac{\pi}{2} \le \mathcal{I}m\ z \le \frac{\pi}{2}\}$ one may find an open simply connected domain $U_{r,M}$ such that : $\overline{U_{r,M}} \subset \mathcal{B}_{\frac{r}{2},2M}$ and $J_\lambda \cap \mathcal{B}_{r,M} \subset U_{r,M}$. We have thus an annulus $\mathcal{B}_{\frac{r}{2},2M}\setminus U_{r,M}$ and an associate Kœbe constant $\sqrt{K_{r,M}}$. We conclude that for any λ and any $n\in\mathbb{N}$ we have

$$\forall x\in U_{r,M}\quad \forall y\in U_{r,M}\qquad \frac{1}{K_{r,M}} \le \frac{\mathcal{L}_\lambda^n(1)(x)}{\mathcal{L}_\lambda^n(1)(y)} \le K_{r,M}\cdot \tag{3.11}$$

Since for a measurable set A we have $m_\lambda(F_\lambda^{-n}(A)) = \int_A \mathcal{L}_\lambda^n(1)dm_\lambda$, we conclude, if $A \subset U_{r,M}$, that

$$\frac{1}{K_{r,M}}\frac{m_\lambda(A)}{m_\lambda(U_{r,M})} \leq \frac{m_\lambda(F_\lambda^{-n}(A))}{m_\lambda(F_\lambda^{-n}(U_{r,M}))} \leq K_{r,M}\frac{m_\lambda(A)}{m_\lambda(U_{r,M})}\cdot$$

This is precisely the bounded distortion property for m_λ on $U_{r,M}$ as it is used in [Ha]. Since $(J_\lambda, F_\lambda, m_\lambda)$ is ergodic and conservative there is, up to a multiplicative constant, only one invariant measure equivalent with m_λ. Let ν_λ be the one that gives mass 1 to $\mathcal{B}_{r,M}$. It follows from Propositions 1.2.7 and 1.2.8 in [Ha] that

$$m_\lambda\text{-almost surely on } \mathcal{B}_{r,M} \qquad \frac{1}{K_{r,M}} \leq \frac{d\nu_\lambda}{dm_\lambda} \leq K_{r,M}\cdot$$

The measures μ_λ and ν_λ only differ by a multiplicative constant which can be computed by integrating the function 1 over $\mathcal{B}_{r,M}$. We deduce that $\mu_\lambda = \mu_\lambda(\mathcal{B}_{r,M})\nu_\lambda$ and we conclude that

$$m_\lambda\text{-almost surely on } \mathcal{B}_{r,M} \qquad \frac{\mu_\lambda(\mathcal{B}_{r,M})}{K_{r,M}} \leq \rho_\lambda \leq K_{r,M}\mu_\lambda(\mathcal{B}_{r,M}). \tag{3.12}$$

Using inequalities (3.11) one may now adapt the reasoning of Lemma 3.4 in [Ur,Zd[1]] to our situation. Let M be large enough and r small enough so that : $\frac{\log M - 1}{2} \geq r$ and for all $\lambda \in [1, \lambda_0]$ if $\mathcal{R}e\ z > M$ then $\mathcal{L}_\lambda(1)(z) \leq 1$. The purpose of the first requirement is the following

$$\forall z \in P \qquad (\mathcal{R}e\ z > M \quad \text{and} \quad F_\lambda(y) = z) \Rightarrow |y| > r \quad (\text{i.e. } y \in \mathcal{B}_r)\cdot \tag{3.13}$$

We prove by induction that H_n is true for all n with

$$H_n \Leftrightarrow ||\mathcal{L}_\lambda^n(1)\chi_{B_r}||_\infty \leq \frac{K_{r,M}}{m_\lambda(\mathcal{B}_{r,M})}\cdot$$

Notice that H_0 is obvious and assume that H_n is true. Since

$$\mathcal{L}_\lambda(1)(z) \leq \sum_{k \geq \mathcal{R}e\ z} \frac{2}{k^{d(\lambda)}},$$

and since $d(\lambda)$ is converging towards $d(\lambda_0)$, one deduces that $\mathcal{L}_\lambda(1)(z)$ is, uniformly in λ, converging towards 0 as $\mathcal{R}e\ z \to \infty$. We deduce that $||\mathcal{L}_\lambda(1)\chi_{\mathcal{B}_r}||_\infty$ is achieved for some $z_1 \in \mathcal{B}_r$. An easy induction leads, for all integers $n \geq 0$, to the existence of some $z_n \in \mathcal{B}_r$ such that

$$||\mathcal{L}_\lambda^n(1)\chi_{B_r}||_\infty = \mathcal{L}_\lambda^n(1)(z_n)\cdot$$

Consider z_{n+1} and assume that it lies in $\mathcal{B}_{r,M}$. Then we have

$$1 = \int \mathcal{L}_\lambda^{n+1}(1)dm_\lambda \geq \int \mathcal{L}_\lambda^{n+1}(1)\chi_{\mathcal{B}_{r,M}}dm_\lambda \geq \frac{\mathcal{L}_\lambda^{n+1}(1)(z_{n+1})}{K_{r,M}}m_\lambda(\mathcal{B}_{r,M})\cdot$$

The last inequality is an application of (3.11) and we conclude that H_{n+1} is true. But z_{n+1} might be with a real part greater than M. In this case we have

$$\mathcal{L}_\lambda^{n+1}(1)(z_{n+1}) = \mathcal{L}_\lambda(\mathcal{L}_\lambda^n(1))(z_{n+1}) \leq \mathcal{L}_\lambda^n(1)(z_n)\mathcal{L}_\lambda(1)(z_{n+1}) \leq \mathcal{L}_\lambda^n(z_n)\cdot$$

Those inequalities are implied by our assumptions on M and r that ensure us first, that any pre-image of z_{n+1} is in $\mathcal{B}_r$, and second, that $\mathcal{L}_\lambda(1)(z_{n+1}) \leq 1$. We may now apply our inductive assumption to conclude that H_{n+1} is true so that this hypothesis is true for any integer n. Let α_{r,M,λ_0} be defined as the infimum of the set $(m_\lambda(\mathcal{B}_{r,M}))$ where $\lambda \in [1, \lambda_0]$. Since $\lambda \mapsto m_\lambda(\mathcal{B}_{r,M})$ is continuous on $[1, \lambda_0]$, this infimum is achieved and is strictly greater than 0. Fix r small and choose $M(r)$ such that all assumptions are fulfilled and set $C_{r,\lambda_0} = \frac{K_{r,M(r)}}{\alpha_{r,M(r),\lambda_0}}$. We deduce from our analysis that $\lim_{n\to\infty} \mathcal{L}_\lambda^n(1) = \rho_\lambda \leq C_{r,\lambda_0}$ on $\mathcal{B}_r$. We have thus proved the left hand side inequality of Lemma 3.9. In order to finish the proof of this Lemma we need to prove that $\frac{1}{K} \leq \rho_\lambda \leq K$ on $\mathcal{B}_{r,M}$. By (3.12) this will be done if one can prove that $\mu_\lambda(\mathcal{B}_{r,M}) \geq \frac{1}{2}$ for suitable r and M.

Since we know that $\rho_\lambda \leq C_{r,\lambda_0}$ on $\mathcal{B}_r$, we may already use the left-hand side inequalities of Proposition 3.8. In particular for any n we have

$$\mu_\lambda(C_n) \leq \frac{C_{r,\lambda_0}}{n^{2\delta-1}} \qquad \text{with } 1 < \delta = \inf\{d(\lambda)\}, \text{ well defined by continuity.}$$

Let now N be big enough so that

$$\sum_{n\geq N} \frac{1}{n^{2\delta-1}} \leq \frac{1}{4C_{r,\lambda_0}}\cdot$$

Chose r' small enough so that for any $\lambda \in [1, \lambda_0]$ we have

$$B(0, r') \subset \cup_{n\geq N} C_n(\lambda)\cdot$$

Such a choice is possible because of Proposition 5.7. We then easily conclude that $\mu_\lambda(B(0, r')) \leq \frac{1}{4}$. As a consequence, one may assume, without loss of generality, that we have started our analysis with $r > 0$ small enough so that $\mu_\lambda(B(0, r)) \leq \frac{1}{4}$.

By Lemma 4.1 in [Ur,Zd[2]], we know that the sequence of measures (m_λ) is tight. In particular, if M is chosen large enough, then for any $\lambda \in [1, \lambda_0]$ we have $m_\lambda(P_M^c) \leq \frac{1}{4C_{r,\lambda_0}}$. From where we deduce that $\mu_\lambda(P_M^c) \leq \frac{1}{4}$.

Note now that $\mu_\lambda(\mathcal{B}_{r,M}) = 1 - \mu_\lambda(B(0, r)) - \mu_\lambda(P_M^c) \geq \frac{1}{2}$. As already mentioned this inequality finishes the proof of the Lemma. ∎

4. Controlling the integrals

In this section we mainly reproduce the reasoning of [Ha,Zi[1]]. Nevertheless there are some differences we would like to emphasize : the main being that we do not know whether the dimension of $J(F_1)$ is less than $\frac{3}{2}$ or not. Note also that the Markov partition used in [Ha,Zi[1]] is replaced in the present article by the backward images of the fundamental domain C_0. Finally, note that we work directly on J_λ without conjugating the dynamics.

Before we start the proofs and in order to simplify some expressions and calculations, we introduce the following notation. Let

$$\Psi_n = \sum_{k=1}^{n} \frac{1}{(F_\lambda^k)'},$$

$$\Phi_n = \sum_{k=1}^{n} \frac{1}{|(F_\lambda^k)'|},$$

$$\Psi = \sum_{k=1}^{\infty} \frac{1}{(F_\lambda^k)'},$$

and

$$\Phi = \sum_{k=1}^{+\infty} \frac{1}{|(F_\lambda^k)'|}$$

so that formula (2.1) may be written

$$d'(\lambda) = -\frac{d(\lambda)}{\chi_{\mu_\lambda}} \left(1 - \frac{1}{\lambda}\right) \int_{J_\lambda} \mathcal{R}e \ (\Psi)\, d\mu_\lambda \cdot$$

We will need the following equation which is an easy computation

$$\Psi = \frac{1}{(F_\lambda^n)'} \Psi \circ F_\lambda^n + \Psi_n, \qquad \Phi = \frac{1}{|(F_\lambda^n)'|} \Phi \circ F_\lambda^n + \Phi_n \cdot \tag{4.1}$$

4.1. **Lyapunov exponents.** In this paragraph we prove that the Lyapunov exponents do not play any role in our estimates of the derivative. In order to do this we only need to check that they are uniformly bounded above and separated away from zero. More precisely we prove the following.

Proposition 4.1. *There exist $r_0 > 0$, $\lambda_0 > 1$ and $K > 1$ such that $\forall \lambda \in (\lambda, \lambda_0)$ we have*

$$\frac{1}{K} \le \chi_{\mu_\lambda} := \int_{J_\lambda} \log |F_\lambda'| d\mu_\lambda \le K \cdot$$

Proof. First note that $\forall \lambda \ge 1$ and $\forall z \in J_\lambda$ we have $|F_\lambda'(z)| \ge 1$. In particular we have

$$\int_{C_0} \log |F_\lambda'| d\mu_\lambda \le \chi_{\mu_\lambda} \cdot$$

There is $K_1 > 0$ such that $\mathcal{R}e \ z \ge K_1$ for any $z \in C_0(\lambda)$ and any $\lambda \in (1, \lambda_0)$, and by Proposition 5.5 there is K_2 such that $\mu_\lambda(C_0) \ge K_2$. Since $\log |F_\lambda'(z)| = \log \lambda + \mathcal{R}e \ z$ we deduce that

$$0 < K_1 K_2 \le \int_{C_0} \log |F_\lambda'| d\mu_\lambda \le \chi_{\mu_\lambda} \cdot$$

This is the first part of the proof.

For the other part note first that continuity of $\lambda \mapsto d(\lambda)$ and the fact that $d(1) > 1$ imply that there exist $\alpha > 1$ and $\beta > 0$ such that $\alpha + \beta \leq d(\lambda)$ for any $\lambda \in (1, \lambda_0)$. This implies in particular that $\forall \lambda \in (1, \lambda_0)$ and $\forall z \in J_\lambda$

$$\frac{1}{|(F_\lambda)'(z)|^{d(\lambda)}} \leq \frac{1}{|(F_\lambda)'(z)|^{\alpha+\beta}}.$$

Consider now the following partition of the strip $P : (A_n)_{n\in\mathbb{N}}$, with $A_n := \{z \in P \,|\, n-1 < \mathcal{R}e\ z \leq n\}$. We have

$$\chi_{\mu_\lambda} = \sum_{n=1}^{+\infty} \int_{A_n} \log |F_\lambda'| d\mu_\lambda \leq \log \lambda_0 + \sum_{n=1}^{+\infty} \int_{A_n} \mathcal{R}e\ z d\mu_\lambda(z)$$
$$\leq \log \lambda_0 + \sum_{n=1}^{+\infty} n\mu_\lambda(A_n).$$

Lemma 3.9 implies that there exists $K_3 > 0$ such that $\mu_\lambda(A_n) \leq K_3 m_\lambda(A_n)$ for $n \geq 2$. Note now that

$$m_\lambda(A_n) = \int_{J_\lambda} \chi_{A_n} dm_\lambda = \int_{J_\lambda} \mathcal{L}_\lambda(\chi_{A_n}) dm_\lambda.$$

For any $z \in J_\lambda$ and any $k \in \mathbb{Z}$ we let z_k be the preimage of z for F_λ such that $f_\lambda(z_k) = z + 2ik\pi$. We thus have

$$\mathcal{L}_\lambda(\chi_{A_n})(z) = \sum_{k\in\mathbb{Z}} \frac{1}{|F_\lambda'(z_k)|^{d(\lambda)}} \chi_{A_n}(z_k).$$

With α and β defined above, this gives that

$$\mathcal{L}_\lambda(\chi_{A_n})(z) \leq \sum_{k\in\mathbb{Z}} \frac{1}{|F_\lambda'(z_k)|^{\alpha+\beta}} \chi_{A_n}(z_k).$$

Since $|F_\lambda'(z_k)| = \lambda e^{\mathcal{R}e\ z_k} = |z + \lambda + 2ik\pi|$, we have

$$\frac{1}{|F_\lambda'(z_k)|^{\alpha+\beta}} \chi_{A_n}(z_k) \leq \frac{1}{|z + \lambda + 2ik\pi|^\alpha} \lambda^{-\beta} e^{-\beta(n-1)},$$

so that

$$\mathcal{L}_\lambda(\chi_{A_n})(z) \leq \lambda^{-\beta} e^{-\beta n} \sum_{k\in\mathbb{Z}} \frac{1}{|z + \lambda + 2ik\pi|^\alpha}.$$

As we have $\alpha > 1$, there is $K_4 > 0$, independent of λ and z, such that

$$\sum_{k\in\mathbb{Z}} \frac{1}{|z + \lambda + 2ik\pi|^\alpha} < K_4 \lambda^\beta.$$

This tells us that

$$\mathcal{L}_\lambda(\chi_{A_n})(z) \leq K_4 e^{-\beta n}.$$

Integrating with respect to m_λ, and summing over $n \geq 2$, we get

$$\chi_{\mu_\lambda} \leq \log \lambda_0 + K_3 m_\lambda(A_1) + K_3 K_4 \sum_{n\geq 2} e^{-\beta n} \leq K_5,$$

With $K_5 := \log \lambda_0 + K_3 + K_3 K_4 \frac{e^{-2\beta}}{1-e^{-\beta}}$. This is clearly independent of λ and we are done ∎

Note that with some more work one can indeed prove that χ_{μ_λ} converges towards χ_{μ_1} as λ converges towards 1 from above.

4.2. **Controlling the integral away from** 0**.** Let N be an integer[6] and set $M_N = \bigcup_{n \geq N+1} C_n$ and $B_N = J_\lambda \setminus M_N$. Note that both set M_N and B_N depends on λ.

Proposition 4.2. *There exists $k(N) > 0$ such that $\forall \lambda \in [1, \lambda_0]$ we have* $\displaystyle\int_{B_N} \Phi d\mu_\lambda \leq k(N)$.

Proof. Let $D_0 = B_N$ and for any $n \in \mathbb{N}$ let $D_n = C_{N+n}$. Following [Ha,Zi[1]] let U_n be the set of points which arrive or come back to B_N after exactly n iterates, which means that $U_n = F_\lambda^{-1}(D_{n-1})$. Note that $U_n \cap M_n = D_n$. Given $N_0 \in \mathbb{N}$ we set $A_n = F_\lambda^{-N_0}(U_n) \cap B_N$. Since (U_n) is a partition of J_λ, (A_n) is a partition of B_N and we have

$$\int_{B_N} \Phi d\mu_\lambda = \sum_{k=1}^{+\infty} \int_{A_k} \Phi d\mu_\lambda.$$

Using relation 4.1 with $n = N_0 + k$ we get

$$\int_{A_k} \Phi d\mu_\lambda = \int_{A_k} \left(\frac{1}{|(F_\lambda^{N_0+k})'|} \Phi \circ F_\lambda^{N_0+k} + \Phi_{N_0+k} \right) d\mu_\lambda.$$

Using the fact that $F_\lambda^{N_0+k}(A_k) \subset B_N$, Lemma 3.5 and Lemma 3.4 we deduce that

$$\int_{A_k} \Phi d\mu_\lambda \leq \frac{\kappa(N)}{(N_0+k)^2} \int_{A_k} \Phi \circ F_\lambda^{N_0+k} d\mu_\lambda + (N_0+k)\mu_\lambda(A_k)$$

The fact that $F_\lambda^{N_0+k}(A_k) \subset B_N$ also implies that $\chi_{A_k} \leq \chi_{B_N} \circ F_\lambda^{N_0+k}$, from the invariance of μ_λ we thus get

$\displaystyle\int_{A_k} \Phi \circ F_\lambda^{N_0+k} d\mu_\lambda \leq \int \chi_{B_N} \circ F_\lambda^{N_0+k} \Phi \circ F_\lambda^{N_0+k} d\mu_\lambda \leq \int_{B_N} \Phi d\mu_\lambda$ which leads to

$$\int_{A_k} \Phi d\mu_\lambda \leq \frac{\kappa(N)}{(N_0+k)^2} \int_{B_N} \Phi d\mu_\lambda + (N_0+k)\mu_\lambda(A_k).$$

In order to estimate $\mu_\lambda(A_k)$, we first use Lemma 3.9 to conclude that $\mu_\lambda(A_k) \leq K m_\lambda(A_k) \leq K m_\lambda(F_\lambda^{-N_0}(U_k))$, for some constant K independent of k, N_0 and λ. Since $U_k = F_\lambda^{-1}(D_k)$, we get $\mu_\lambda(A_k) \leq K m_\lambda(F_\lambda^{-(N_0+1)}(D_k))$. Moreover

$$m_\lambda(F_\lambda^{-(N_0+1)}(D_k)) = \int \chi_{D_k} \circ F_\lambda^{N_0+1} dm_\lambda = \int_{D_k} \mathcal{L}_\lambda^{N_0+1}(1) dm_\lambda,$$

since there exists $K_1(N_0)$ independent of λ and k such that $\mathcal{L}_\lambda^{N_0+1}(1) \leq$

[6]This integer will be chosen later big enough to ensure that for any $z_n \in C_n$ we have $\sum_{n \geq N} \arg(z_n) \leq \frac{\pi}{6}$.

$K_1(N_0)$, using Lemma 3.6 and the fact that $D_k = C_{N+k}$, we get
$m_\lambda(F_\lambda^{-(N_0+1)}(D_k) \leq K_1(N_0) m_\lambda(C_{N+k}) \leq \frac{K_2}{(N+k)^{2d(\lambda)}}$.
Using the fact that $(N_0+k) \leq N_0(N+k)$, we thus conclude that
$\int_{A_k} \Phi d\mu_\lambda \leq \frac{\kappa(N)}{(N_0+k)^2} \int_{B_N} \Phi d\mu_\lambda + \frac{KK_2N_0}{(N+k)^{2d(\lambda)-1}}$.
Summing over k we end up with
$\int_{B_N} \Phi d\mu_\lambda \leq \frac{\kappa(N)}{N_0-1} \int_{B_N} \Phi d\mu_\lambda + \frac{KK_2N_0}{(N-1)^{2d(\lambda)-2}}$.
The integer N being fixed, one may now choose N_0 big enough so that $\frac{\kappa(N)}{N_0-1} \leq frac12$, so that
$\int_{B_N} \Phi d\mu_\lambda \leq \frac{2KK_2N_0}{(N-1)^{2d(\lambda)-2}}$. This last constant depends only on N and we are done. ∎

4.3. **Controlling the integral in a neighborhood of 0.** In this paragraph we deal with the remaining part of $\int \mathcal{R}e\ (\Psi) d\mu_\lambda$. If we note $M_N = J_\lambda \setminus B_N$ we prove

Proposition 4.3. *There exists $K > 0$ and $N \in \mathbb{N}$ such that for $\forall \lambda \in (1, \lambda_0)$*

$$\frac{1}{K}(\lambda-1)^{2d(\lambda)-3} \leq \int_{M_N} \mathcal{R}e\ (\Psi) d\mu_\lambda \leq K(\lambda-1)^{2d(\lambda)-3}, \quad \text{if } d(\lambda) < \tfrac{3}{2},$$

$$-\frac{1}{K}\log(\lambda-1) \leq \int_{M_N} \mathcal{R}e\ (\Psi) d\mu_\lambda \leq -K\log(\lambda-1), \quad \text{if } d(\lambda) = \tfrac{3}{2},$$

$$\left| \int_{M_N} \mathcal{R}e\ (\Psi) d\mu_\lambda \right| \leq K, \quad \text{if } d(\lambda) > \tfrac{3}{2}.$$

Proof. We split this integral into several pieces. First we note using 4.1 that

$$\int_{M_N} \mathcal{R}e\,(\Psi) d\mu_\lambda = \sum_{n=N+1}^{+\infty} \left[\int_{C_n} \mathcal{R}e\ \left(\frac{1}{(F_\lambda^{n-N})'} \Psi \circ F_\lambda^{n-N} \right) d\mu_\lambda + \int_{C_n} \mathcal{R}e\ \left(\Psi_{n-N}\right) d\mu_\lambda \right].$$

We first deal with the left hand side of the sum that we bound integrating the modulus of the function.

$$\left| \int_{C_n} \mathcal{R}e\ \left(\frac{1}{(F_\lambda^{n-N})'} \Psi \circ F_\lambda^{n-N} \right) d\mu_\lambda \right| \leq \int_{C_n} \frac{1}{|(F_\lambda^{n-N})'|} \Phi \circ F_\lambda^{n-N} d\mu_\lambda.$$

We use Lemma 3.5 and the fact that for $z \in C_n$, we have $F_\lambda^{n-N}(z) \in C_N \subset B_N$ to conclude that

$$\left| \int_{C_n} \mathcal{R}e\ \left(\frac{1}{(F_\lambda^{n-N})'} \Psi \circ F_\lambda^{n-N} \right) d\mu_\lambda \right| \leq \frac{K}{(n-N)^2} \int_{B_N} \Phi d\mu_\lambda.$$

Summing over $n \geq N$ we get

$$\left| \sum_{n=N+1}^{+\infty} \left[\int_{C_n} \mathcal{R}e\ \left(\frac{1}{(F_\lambda^{n-N})'} \Psi \circ F_\lambda^{n-N} \right) d\mu_\lambda \right] \right| \leq K \int_{B_N} \Phi d\mu_\lambda \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

By Proposition 4.2 we conclude that there exists $K(N) > 0$ such that

$$\left|\sum_{n=N}^{+\infty}\left[\int_{C_n}\mathcal{R}e\ \left(\frac{1}{(F_\lambda^{n-N})'}\Psi\circ F_\lambda^{n-N}\right)d\mu_\lambda\right]\right|\leq K(N)\cdot \tag{4.2}$$

We now deal with the right hand side. We have

$$\int_{C_n}\mathcal{R}e\ (\Psi_{n-N})d\mu_\lambda=\sum_{k=1}^{n-N}\int_{C_n}\mathcal{R}e\ \left(\frac{1}{(F_\lambda^k)'}\right)d\mu_\lambda\cdot$$

Choose N big enough so that conclusions of Lemma 3.2 hold. For any $z\in C_n$ and any $k\leq n-N$ we have

$$\frac{\sqrt{3}}{2}|F_\lambda^k(z)|\leq\mathcal{R}e\ (F_\lambda^k)'(z),$$

so that

$$\int_{C_n}\mathcal{R}e\ (\psi_{n-N})d\mu_\lambda\sim\sum_{k=1}^{n-N}\int_{C_n}\mathcal{R}e\ \left(\frac{1}{|F_\lambda^k|'}\right)d\mu_\lambda\cdot$$

Note now that for any $z\in C_n$, we have by the Chain Rule that

$$(F_\lambda^k)'(z)=\frac{(F_\lambda^n)'(z)}{(F_\lambda^{n-k})'(F_\lambda^k(z))},$$

with $F_\lambda^k(z)\in C_{n-k}$. We deduce, using Proposition 3.1, that

$$\frac{1}{|(F_\lambda^k)'(z)|}\sim\frac{a_n(2)}{a_{n-k}(2)}\cdot$$

Estimates of $\mu_\lambda(C_n)$ are given by Proposition 3.8 and we conclude that

$$\int_{C_n}\mathcal{R}e\ (\psi_{n-N})d\mu_\lambda\sim\begin{cases}a_n(2d(\lambda)-1)a_n(2)\sum_{k=1}^{n-N}a_{n-k}(-2) & \text{if } n\leq N_\varepsilon,\\ \frac{1}{\varepsilon}a_n(2d(\lambda))a_n(2)\sum_{k=1}^{n-N}a_{n-k}(-2) & \text{if } n\geq N_\varepsilon.\end{cases}$$

Since $a_n(\alpha)a_n(\beta)=a_n(\alpha+\beta)$, and with $S_{k,n}(\alpha)=\sum_k^n a_j(\alpha)$, this can also be written

$$\int_{C_n}\mathcal{R}e\ (\psi_{n-N})d\mu_\lambda\sim\begin{cases}a_n(2d(\lambda)+1)S_{N,n-1}(-2) & \text{if } n\leq N_\varepsilon,\\ \frac{1}{\varepsilon}a_n(2d(\lambda)+2)S_{N,n-1}(-2) & \text{if } n\geq N_\varepsilon.\end{cases}$$

Use now Corollary 5.11 we have $S_{N,n-1}(-2)\sim(a_n(-3)-a_N(-3))$ if $n\leq N_\varepsilon$ and $S_{N,n-1}(-2)\sim\frac{a_n(-2)}{\varepsilon}$ if $n\geq N_\varepsilon$ and we get

$$\int_{C_n}\mathcal{R}e\ (\psi_{n-N})d\mu_\lambda\sim\begin{cases}a_n(2d(\lambda)+1)(a_n(-3)-a_N(-3)) & \text{if } n\leq N_\varepsilon,\\ \frac{1}{\varepsilon^2}a_n(2d(\lambda)+2)a_n(-2) & \text{if } n\geq N_\varepsilon.\end{cases}$$

Since $a_n(\alpha)a_n(\beta)=a_n(\alpha+\beta)$ we get

$$\int_{C_n}\mathcal{R}e\ (\psi_{n-N})d\mu_\lambda\sim\begin{cases}a_n(2d(\lambda)-2)-a_N(-3)a_n(2d(\lambda)+1) & \text{if } n\leq N_\varepsilon,\\ \frac{1}{\varepsilon^2}a_n(2d(\lambda)) & \text{if } n\geq N_\varepsilon.\end{cases}$$

Summing over $n \geq N$ this gives us $\sum_{n\geq N} \int_{C_n} \mathcal{R}e\ (\psi_{n-N}) d\mu_\lambda$ is comparable with

$$\max\left((S_{N,N_\varepsilon}(2d(\lambda)-2) - a_N(-3) S_{N,N_\varepsilon}(2d(\lambda)+1)), \frac{1}{\varepsilon^2} S_{N_\varepsilon,+\infty}(2d(\lambda)) \right)$$

We then deduce from Corollary 5.10 and Corollary 5.11 that $S_{N,N_\varepsilon}(2d(\lambda)+1) \sim a_N(2d(\lambda)) \sim 1$, and also that $S_{N_\varepsilon,+\infty}(2d(\lambda)) \sim \frac{a_{N\varepsilon}(2d(\lambda))}{\varepsilon} \sim \varepsilon^{2d(\lambda)-1}$. Estimates of $S_{N,N_\varepsilon}(2d(\lambda)-2)$ depend on the comparison of $d(\lambda)$ with $\frac{3}{2}$. More precisely, if $d(\lambda) > \frac{3}{2}$ then Corollary 5.11 tells us that $S_{N,N_\varepsilon}(2d(\lambda)-2) \sim 1$, if $d(\lambda) = \frac{3}{2}$ then it tells us that $S_{N,N_\varepsilon}(2d(\lambda)-2) \sim \log N_\varepsilon$, and if $d(\lambda) < \frac{3}{2}$ then $S_{N,N_\varepsilon}(2d(\lambda)-2) \sim \varepsilon^{2d(\lambda)-3}$. Summarizing all those estimates we get

$$\sum_{n\geq N} \int_{C_n} \mathcal{R}e\ (\psi_{n-N}) d\mu_\lambda \sim \begin{cases} 1 & \text{if } d(\lambda) > \frac{3}{2} \\ \log N_\varepsilon & \text{if } d(\lambda) = \frac{3}{2} \\ \varepsilon^{2d(\lambda)-3} & \text{if } d(\lambda) < \frac{3}{2} \end{cases}$$

4.4. **Proof of the main result.** We are now in position to prove the main result of this paper that we recall here.

Theorem 4.4. *There exists $\lambda_0 > 1$, and $K > 1$ such that*

$$\begin{cases} \frac{-1}{K}(\lambda-1)^{2d(1)-2} \leq d'(\lambda) \leq -K(\lambda-1)^{2d(1)-2} & \text{if } d(1) < \frac{3}{2}, \\ |d'(\lambda)| \leq K(\lambda-1)\log\frac{1}{\lambda-1} & \text{if } d(1) = \frac{3}{2}, \\ |d'(\lambda)| \leq K(\lambda-1) & \text{if } d(1) > \frac{3}{2}. \end{cases}$$

In particular the function $\lambda \mapsto d(\lambda)$ is C^1 on $[1, +\infty)$, with $d'(1) = 0$.

Proof. Let us recall that we have

$$d'(\lambda) = -\frac{d(\lambda)}{\chi_{\mu_\lambda}}\left(1 - \frac{1}{\lambda}\right) \int_{J_\lambda} \mathcal{R}e\ \Psi d\mu_\lambda.$$

We first use [Ur,Zd[2]], where it is proved that $\lambda \mapsto d(\lambda)$ is continuous on $[1, +\infty)$, and Proposition 4.1 to conclude that there exists $\lambda_1 > 1$ and $K_1 > 1$ such that $\forall \lambda \in (1, \lambda_1)$ we have

$$\frac{1}{K_1}(\lambda-1) \leq \frac{d(\lambda)}{\chi_{\mu_\lambda}}\left(1-\frac{1}{\lambda}\right) \leq K_1(\lambda-1).$$

Note that given any integer N we have

$$\int_{J_\lambda} \mathcal{R}e\ \Psi d\mu_\lambda = \int_{B_N} \mathcal{R}e\ \Psi d\mu_\lambda + \int_{M_N} \mathcal{R}e\ \Psi d\mu_\lambda,$$

so that

$$(4.3) \quad |d'(\lambda)| \leq 2K_1(\lambda-1)\max\left(\left|\int_{B_N} \mathcal{R}e\ \Psi d\mu_\lambda\right|, \left|\int_{M_N} \mathcal{R}e\ \Psi d\mu_\lambda\right| \right).$$

We may thus use Proposition 4.2 and Proposition 4.3 to conclude that $d'(\lambda)$ is converging towards 0 when λ is converging towards 0 from above. In particular there is $\lambda_2 > 1$ such that $\forall \lambda \in [1, \lambda_2)$,

$$-\frac{1}{2} \leq d'(\lambda) \leq \frac{1}{2}\cdot$$

We deduce that

$$-\frac{1}{2}(\lambda - 1) \leq d(\lambda) - d(1) \leq \frac{1}{2}(\lambda - 1),$$

so that

$$\begin{aligned}(\lambda-1)^{\lambda-1}(\lambda-1)^{2d(1)-3} \leq (\lambda-1)^{2d(\lambda)-3} &= (\lambda-1)^{2d(1)-3}(\lambda-1)^{2(d(\lambda)-d(1))} \\ &\leq (\lambda-1)^{-(\lambda-1)}(\lambda-1)^{2d(1)-3}\end{aligned}$$

Since $\lambda \mapsto (\lambda-1)^{\lambda-1}$ is continuous on $[1, \lambda_2]$ there exists $K_3 > 1$ such that

$$\frac{1}{K_3}(\lambda-1)^{2d(1)-3} \leq (\lambda-1)^{2d(\lambda)-3} \leq K_3(\lambda-1)^{2d(1)-3}\cdot$$

Using again Proposition 4.2 and Proposition 4.3, and the fact we just proved that allows us to replace $d(\lambda)$ with $d(1)$, we conclude the proof of the main result in case $d(1) < \frac{3}{2}$.

In case $d(1) = \frac{3}{2}$, propositions 4.2 and 4.3 tells us that the maximum in (4.3) is dominated by $-\log(\lambda - 1)$. In case $d(1) > \frac{3}{2}$, the same proposition leads to the fact that this maximum is bounded. ∎

5. Appendices

5.1. Estimates close to a repelling/parabolic fixed point. In this appendix we show how to get estimates in case of a degeneracy towards a multi-petal parabolic fixed point. It is a two steps proof : first we deal with the real axis then we extend estimates obtained in the real line to the complex plane using Kœbe's distortion Theorem.

Consider the following family of germs of holomorphic functions defined in a neighborhood of 0 that we denote by $\mathcal{U}$:

$$f_\varepsilon(z) = (1+\varepsilon)z + z^{p+1} + z^{p+2}g_\varepsilon(z)\cdot$$

Assume that there is an inverse branch f_ε^{-1} well defined on $\mathcal{U}$ that leaves a sector $S_\theta := \{re^{i\alpha} \mid -\theta \leq \alpha \leq \theta\}$ invariant, for some $0 < \theta < \frac{\pi}{2}$. Let $\mathcal{U}_\theta := \mathcal{U} \cap S_\theta$. Assume also that $\forall z \in \mathcal{U}$ we have $|zg_\varepsilon(z)| < \frac{1}{2}$. Let $I = \mathcal{U} \cap \mathbb{R}^+$ and assume that $f_\varepsilon^{-1}(I) \subset I$ and that f_ε is not decreasing on I.

This appendix is organized as follow : in the first two paragraphs we study those germs giving in the second paragraph uniform estimates for $|(f_\varepsilon^{-n})'|$.

5.1.1. *The mean value Theorem and its consequences.* We start with the following easy fact.

Lemma 5.1. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a decreasing map with antiderivative F on $\mathbb{R}$ and let $(u_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers. Suppose that there exist $n > 1$ such that for all $k \leq n$ we have*

i- $K_1 \leq (u_k - u_{k+1})f(u_k)$, *then* $K_1 k \leq F(u_0) - F(u_k)\cdot$
ii- $(u_k - u_{k+1})f(u_{k+1}) \leq K_2$, *then* $F(u_0) - F(u_k) \leq K_2 k\cdot$

Proof. One only needs to check that our assumptions imply

$$(u_k - u_{k+1})f(u_k) \leq \int_{u_{k+1}}^{u_k} f(t)dt \leq (u_k - u_{k+1})f(u_{k+1})\cdot$$

∎

In particular we point out the following two particular cases :

Corollary 5.2. *Let (x_n) be a decreasing sequence of positive real numbers. Assume that there exist $0 < K_1 < K_2$ and $n \in \mathbb{N}$ such that $\forall k \leq n$,*

$$K_1 x_n^{p+1} \leq (x_n - x_{n+1}) \leq K_2 x_{n+1}^{p+1}.$$

Then there exist $\tilde{K_1}$ and $\tilde{K_2}$[7] *such that for $\forall k \leq n$*

$$\tilde{K_1} \leq k^{\frac{1}{p}} x_k \leq \tilde{K_2}\cdot$$

Corollary 5.3. *Let (u_n) be a decreasing sequence of real numbers. Assume that there are $\alpha > 0$, $\beta > 0$, $p > 0$ and $n \in \mathbb{N}$ such that $\forall k \leq n$*

$$(u_k - u_{k+1}) \leq \alpha + \beta e^{p u_{k+1}}.$$

Then $\forall k \leq n$ we have

$$\frac{\alpha^{\frac{1}{p}}}{(\alpha + \beta e^{p u_0})^{\frac{1}{p}}} e^{-\alpha k} \leq e^{u_k - u_0}.$$

Let us provide a short argument of how these corollaries can be deduced from the Lemma 5.1.

Proof. For Corollary 5.2 we use the Lemma with the function $f : x \mapsto x^{-(p+1)}$ so that one may take $F : x \mapsto -\frac{1}{p}x^{-p}$. We deduce that we have :

$$K_1 n \leq \frac{1}{p}\left(\frac{1}{x_n^p} - \frac{1}{x_0^p}\right) \leq K_2 n\cdot$$

Elementary computations then lead to the desired inequalities.

For Corollary 5.3 we now consider the function $f : x \mapsto (1 + \frac{\beta}{\alpha}e^{px})^{-1}$. One first checks that $F : x \mapsto x - \frac{1}{p}\log f(x)$ is an antiderivative of f. Our assumptions on (u_n) may now be written as

$$(u_k - u_{k+1})f(u_k) \leq \alpha\cdot$$

[7] One can take for instance $\tilde{K_2} = (pK_1)^{-\frac{1}{p}}$ and $\tilde{K_1} = (pK_2 + \frac{1}{x_0^p})^{-\frac{1}{p}}$.

Using the Lemma 5.1 we deduce that $F(u_0) - F(u_k) \leq \alpha k$. This can be written in the form

$$u_0 - u_k + \frac{1}{p} \log \left(\frac{\alpha + \beta e^{pu_k}}{\alpha + \beta e^{pu_0}} \right) \leq \alpha k \cdot$$

Applying exponents to both sides of this last inequality, we deduce that

$$\left(\frac{\alpha + \beta e^{pu_k}}{\alpha + \beta e^{pu_0}} \right)^{\frac{1}{p}} e^{-\alpha k} \leq e^{u_k - u_0}.$$

From this we get our estimates. ∎

5.1.2. *Uniform estimates along the real axis.* We now come back to our dynamical setting. Let $x_0 \in I$ be a fixed element. Assume for convenience that $x_0 < 1$. Define for any $n \geq 0$, $f_\varepsilon(x_{n+1}(\varepsilon)) = x_n(\varepsilon)$, where $x_0(\varepsilon) = x_0$. For each $\varepsilon > 0$ sufficiently small, we define N_ε as $N_\varepsilon = \sup\{n \in \mathbb{N} \,|\, x_n^p \geq \varepsilon\}$, and for $\varepsilon = 0$ as $N_0 = +\infty$. Note that for any $\varepsilon > 0$ small enough, the sequence $(x_n(\varepsilon))$ is strictly decreasing towards 0. So that N_ε is a well defined integer. Our main results in this paragraph is the following.

Proposition 5.4. *There exists $K > 1$ such that for all $\varepsilon > 0$ small enough,*

$$K^{-1} \leq \varepsilon N_\varepsilon \leq K, \quad \varepsilon > 0, \tag{5.1}$$

$$K^{-1} \leq x_n n^{\frac{1}{p}} \leq K, \quad \forall n < N_\varepsilon, \tag{5.2}$$

$$K^{-1} \leq x_n \varepsilon^{-\frac{1}{p}} (1+\varepsilon)^n \leq K, \quad \forall n \geq N_\varepsilon. \tag{5.3}$$

This result may be interpreted in the following way : N_ε is a "parabolic time". During that time, the fixed point 0 acts on the orbit of x_0, (x_n), as if it was a parabolic fixed point with p petals. For n greater than N_ε the orbit of x_0 is close enough to 0 and realize that it is indeed an attracting fixed point for f_ε^{-1}.

In the following Lemma we obtain estimates which are true for all $n \in \mathbb{N}$ and part of proposition 5.4.

Lemma 5.5. *There exists $K > 1$ such that for all $\varepsilon > 0$ small enough,*

$$K^{-1} \varepsilon^{\frac{1}{p}} (1+\varepsilon)^{-n} \leq x_n \leq K \frac{1}{n^{\frac{1}{p}}}, \quad \forall n \in \mathbb{N} \cdot \tag{5.4}$$

Proof. All our estimates will result from the following very definition of (x_n).

$$x_n = (1+\varepsilon) x_{n+1} + x_{n+1}^{p+1} (1 + x_{n+1} g_\varepsilon(x_{n+1})). \tag{5.5}$$

Assuming that $\varepsilon < 1$, we easily deduce from this equality that for any n we have

$$1 \leq \frac{x_n}{x_{n+1}} \leq (2 + 2x_0^p) \leq 4 \tag{5.6}$$

From 5.5 we deduce that

$$\frac{x_n - x_{n+1}}{x_n^{p+1}} = \frac{\varepsilon x_{n+1}}{x_n^{p+1}} + \left(\frac{x_{n+1}}{x_n}\right)^{p+1} (1 + x_{n+1} g_\varepsilon(x_{n+1})), \tag{5.7}$$

which, with (5.6) leads to

$$K_1 := \frac{1}{4^{p+2}} \le \frac{1}{2}\left(\frac{x_{n+1}}{x_n}\right)^{p+1} \le \frac{x_n - x_{n+1}}{x_n^{p+1}}.$$

Using now Corollary 5.2 we conclude that $\forall n$, $x_n n^{\frac{1}{p}} \le \left(p\frac{1}{4^{p+2}}\right)^{\frac{-1}{p}} \le 64$. This is precisely the right hand side (5.4).

The left hand side of 5.4 is obtained when one notes that (5.5) also implies that

$$\log x_n - \log x_{n+1} = \log(1 + \varepsilon + x_{n+1}^p + x_{n+1}^{p+1} g_\varepsilon(x_{n+1})) \le \log(1+\varepsilon) + 2x_{n+1}^p.$$

We may thus apply Corollary 5.3 with the sequence $u_n := \log x_n$, $\alpha = \log(1+\varepsilon)$, and $\beta = 2$, and deduce that

$$\frac{\alpha^{\frac{1}{p}}}{(\alpha + \beta e^{pu_0})^{\frac{1}{p}}} e^{-\alpha n} \le e^{u_n - u_0}.$$

Assuming that ε is small enough so that $\frac{\varepsilon}{3} \le \alpha = \log(1+\varepsilon) \le e^{pu_0}$ we get

$$\frac{1}{9} \varepsilon^{\frac{1}{p}} (1+\varepsilon)^{-n} \le e^{u_n} = x_n.$$

This ends the proof of lemma 5.5. ∎ We are now in position to give a proof of Proposition 5.4, but first note that the right hand side of (5.2) and the left hand side of (5.3) are given by Lemma 5.5.

Proof. In order to get estimate (5.2), we check that the assumptions on g_ε, the definition of N_ε and relation (5.7) leads for all $n < N_\varepsilon$ to

$$\frac{x_n - x_{n+1}}{x_{n+1}^{p+1}} \le \frac{5}{2}.$$

Corollary 5.2 then tells us that $\forall n < N_\varepsilon$ we have

$$\tilde{K_1} \le x_n n^{\frac{1}{p}},$$

with, for instance, $\tilde{K_1} = (\frac{5p}{2} + \frac{1}{x_0^p})^{-\frac{1}{p}}$.

We are now in position to give estimates for N_ε. They easily come out from the following inequalities we have already proved:

(5.8)

$$\frac{\tilde{K_1}}{K_0^{2p}} \frac{1}{N_\varepsilon^{\frac{1}{p}}} \le \frac{\tilde{K_1}}{K_0^{2p}} \frac{1}{(N_\varepsilon - 1)^{\frac{1}{p}}} \le \frac{x_{N_\varepsilon - 1}}{K_0^{2p}} \le \frac{x_{N_\varepsilon}}{K_0} \le x_{N_\varepsilon + 1} \le \varepsilon^{\frac{1}{p}} \le x_{N_\varepsilon} \le \frac{2}{N_\varepsilon}.$$

From there we deduce that

$$\tilde{K_3} \le \varepsilon^{\frac{1}{p}} N_\varepsilon \le 2, \tag{5.9}$$

with $\tilde{K_3} = \frac{\tilde{K_1}}{K_0^{2p}}$.

Now we only need to take care of (5.3). We start by noticing that for all n we have $(1+\varepsilon)x_{n+1} \leq x_n$. For any $n \geq N_\varepsilon$ we thus have $(1+\varepsilon)^{n-N_\varepsilon} x_n \leq x_{N_\varepsilon}$. This leads to

$$x_n \leq (1+\varepsilon)^{-n} x_{N_\varepsilon} (1+\varepsilon)^{N_\varepsilon}. \tag{5.10}$$

By definition of N_ε and relation (5.6) we have

$$x_{N_\varepsilon} K_0 x_{N_\varepsilon+1} \leq K_0 \varepsilon^{\frac{1}{p}}.$$

By relation (5.9) we also have

$$(1+\varepsilon)^{N_\varepsilon} \leq (1+\frac{2^p}{N_\varepsilon})^{N_\varepsilon} \leq e^{2^p}.$$

From this and (5.10) we deduce that

$$x_n \leq K_0 e^{2^p} \varepsilon^{\frac{1}{p}} (1+\varepsilon)^{-n}. \tag{5.11}$$

Taking $K = K_0^{2p} e^{2^p}$ finishes the proof of the Proposition.

∎

Let now $\alpha(p) := \frac{p+1}{p}$. The following corollary is useful

Corollary 5.6. *There exists $K > 1$ such that for all $\varepsilon > 0$ small enough we have*

$$\begin{array}{llcl} i\text{-} & K^{-1} \leq & (x_n - x_{n+1}) n^{\alpha(p)} & \leq K \quad \forall n \leq N_\varepsilon \\ ii\text{-} & K^{-1} \leq & (x_n - x_{n+1}) \varepsilon^{-\alpha(p)} (1+\varepsilon)^{(p+1)n} & \leq K \quad \forall n > N_\varepsilon \end{array}$$

Proof. From relation (5.5) we deduce that $\forall n \in \mathbb{N}$,

$$x_n - x_{n+1} = \varepsilon x_{n+1} + x_{n+1}^{p+1}(1 + x_{n+1} g_\varepsilon(x_{n+1})), \tag{5.12}$$

so that $\varepsilon x_{n+1} \leq x_n - x_{n+1}$, and Lemma 5.5 tells us that the left hand side inequality of ii- is true. Moreover, for $n \leq N_\varepsilon$ we have $\varepsilon \leq x_{n+1}^p$, and (5.12) leads to $x_n - x_{n+1} \leq 3x_{n+1}^{p+1}$. With Lemma 5.5, we get the right-hand side inequality of i-.

Let $n < N_\varepsilon$. Then, from (5.2) and (5.6), we deduce that

$$\frac{1}{2KK_0^{p+1}} \frac{1}{n^{\alpha(p)}} \leq \frac{x_n^{p+1}}{K_0^{p+1}} \leq x_{n+1}^{p+1} \leq (x_n - x_{n+1}).$$

This is the left-hand side inequality of i-.

In order to finish fix $n \geq N_\varepsilon$. Then, by relation (5.5) and Proposition 5.4, we get that

$$x_n - x_{n+1} \leq 3x_{n+1}^{p+1} \leq 3K^{p+1} \varepsilon^{\alpha(p)} (1+\varepsilon)^{-(p+1)n}.$$

The proof of the Corollary is now complete. ∎

5.1.3. *Extension to the complex plane.* As already mentioned, this extension is done via Kœbe's distortion Theorem. It asserts that given two simply connected domains in $\mathbb{C}$, $V \subset V'$, such that the boundary of V is at a positive distance from the boundary of V', there exists a constant $K > 0$, which depends only on the modulus of the annulus $V' \setminus V$, and such that *for any univalent function f defined in V'* we have for all $x, y \in V$ we have,

$$\frac{1}{K} \leq \frac{|f'(x)|}{|f'(y)|} \leq K\cdot$$

Proposition 5.7. *Let V be a domain such that $\bar{V} \subset \mathcal{U}_\theta$. Then there exists $K > 0$ such that $\forall \varepsilon$ small enough, $\forall n \in \mathbb{N}$ and $\forall z \in V$ we have*

i- $\frac{1}{K} \leq n^{\alpha(p)}|(f_\varepsilon^{-n})'(z)| \leq K$ *if* $n < N_\varepsilon$.

ii- $\frac{1}{K}(1+\varepsilon)^{-(p+1)n} \leq |(f_\varepsilon^{-n})'(z)| \leq K\varepsilon^{\alpha(p)}(1+\varepsilon)^{-(p+1)n}$ *if* $n \geq N_\varepsilon$.

Proof. Enlarging V if necessary one may assume that there is $x_0 \in V \cap \mathbb{R}^+ \cap \mathcal{U}_\theta$ such that for all ε small enough $x_1(\varepsilon) := f_\varepsilon^{-1}(x_0)$ is also in V. Kœbe's distortion Theorem implies that for all n, all ε and all $z \in V$ we have

$$\frac{1}{K}\frac{(x_n(\varepsilon) - x_{n+1}(\varepsilon))}{x_0 - x_1(\varepsilon)} \leq |(f_\varepsilon^{-n})'(z)| \leq K\frac{(x_n(\varepsilon) - x_{n+1}(\varepsilon))}{x_0 - x_1(\varepsilon)}\cdot$$

Applying Corollary 5.6, and noticing that $x_0 - x_1(\varepsilon) > a > 0$ with some real a independent of ε, lead to the desired inequalities. ∎

The following result gives uniform estimates on how closely the orbits are tangent to the real axis.

Corollary 5.8. *There exists $K > 0$ such that $\forall \varepsilon$ small enough, $\forall n \in \mathbb{N}$ and $\forall z_0 \in V$, we have*

$$|\mathcal{I}m\ (f_\varepsilon^{-n}(z_0))| \leq K\frac{1}{n^{\alpha(p)}}\cdot$$

In particular, the series $\sum_{n=0}^{\infty} \mathcal{I}m\ (f_\varepsilon^{-n}(z_0))$ converges.

Proof. Note that $|\mathcal{I}m\ (z_n)| = |\mathcal{I}m\ (z_n - x_n)| \leq |z_n - x_n|$. Kœbe's distortion theorem leads to $|z_n - x_n| \leq K\frac{1}{|(f_\varepsilon^n)'(z_n)|}$ and Proposition 5.7 gives the result. ∎

5.2. **Estimates of some partial sums.** In this appendix we single out the behaviour of the partial sums we need to evaluate at several steps in the proof of our main result. It seemed to us that postponing those estimates to an appendix will clarify the exposition. We are thus in this paragraph dealing with a sequence of real numbers defined by : $a_n = 1/n$ for $n \leq N_\varepsilon$ and $a_n = \varepsilon(1+\varepsilon)^{-n}$ for $n > N_\varepsilon$, where N_ε is comparable with $1/\varepsilon$. We are indeed interested in the sequences $(a_n(\alpha))_{n\in\mathbb{N}}$, with $\alpha \in \mathbb{R}$ and $a_n(\alpha) = a_n^\alpha$, and partial sums $S_{k,n}(\alpha) = \sum_{j=k}^{n} a_j(\alpha)$.

The first Lemma, whose proof is straightforward and left to the reader asserts, the following.

Lemma 5.9. *For any $k < n$ in $\mathbb{N}$ we have*

$$S_{k,n}(\alpha) \sim \begin{cases} \frac{1}{1-\alpha}(n^{1-\alpha} - k^{1-\alpha}) & \text{if } n \le N_\varepsilon \text{ and } \alpha \ne 1 \\ \log \frac{n}{k} & \text{if } n \le N_\varepsilon \text{ and } \alpha = 1 \end{cases}$$

$$S_{k,n}(\alpha) \sim \frac{1}{\alpha\varepsilon}(a_k(\alpha) - a_n(\alpha)), \text{ if } k > N_\varepsilon \text{ and } \alpha \ne 0.$$

As its consequence, we get the following.

Corollary 5.10. *If $\alpha > 0$ then*

i-	$S_{n,+\infty}(\alpha)$	$\sim$	$\frac{a_n(\alpha)}{\varepsilon}$	*if $n > N_\varepsilon$,*
ii-	$S_{n,+\infty}(\alpha)$	$\sim$	$a_n(\alpha - 1)$	*if $n \le N_\varepsilon$ and $\alpha > 1$,*
iii-	$S_{n,+\infty}(\alpha)$	$\sim$	$\log \frac{N_\varepsilon}{n} + K$	*if $n \le N_\varepsilon$, $\alpha = 1$, for some $K > 0$.*
iv-	$S_{n,+\infty}(\alpha)$	$\sim$	$N_\varepsilon^{1-\alpha}$	*if $n \le N_\varepsilon$ and $\alpha < 1$,*

Proof. Since $\alpha > 0$, we see that the sequence $(1+\varepsilon)^{-\alpha n}$ converges to 0, and Lemma 5.9 implies that i- is true. Note that we have

$$\max(S_{n,N_\varepsilon}(\alpha), S_{N_\varepsilon,+\infty}(\alpha)) \le S_{n,+\infty} \le 2\max(S_{n,N_\varepsilon}(\alpha), S_{N_\varepsilon,+\infty}(\alpha)).$$

Using i- that we have just proved, the fact that we have $a_{N_\varepsilon} \sim a_{N_\varepsilon+1}$, and the fact that $N_\varepsilon \sim \varepsilon^{-1}$, we conclude that

$$S_{N_\varepsilon,+\infty} \sim \frac{a_{N_\varepsilon}}{\varepsilon} \sim \varepsilon^{\alpha-1} \sim N_\varepsilon^{1-\alpha}.$$

Let us now estimate S_{n,N_ε} by considering three cases. We start with the case when $\alpha = 1$. Indeed, Lemma 5.9 implies that $S_{n,N_\varepsilon} \sim \log(\frac{N_\varepsilon}{n})$. This gives us iii-.

Assume now that $\alpha > 1$. Then $S_{N_\varepsilon,+\infty} \sim N_\varepsilon^{1-\alpha} \le n^{1-\alpha} = a_n(\alpha-1)$. Moreover, in virtue of Lemma 5.9, we have $S_{n,N_\varepsilon} \sim n^{1-\alpha} - N_\varepsilon^{1-\alpha}$. Thus

$$S_{n,N_\varepsilon} \sim a_n(\alpha - 1)\left(1 - \left(\frac{n}{N_\varepsilon}\right)^{\alpha-1}\right).$$

In particular $S_{n,N_\varepsilon} \lesssim a_n(\alpha-1)$. So, we can conclude that $S_{n,+\infty}(\alpha) \lesssim a_n(\alpha-1)$. If $\frac{n}{N_\varepsilon} \le \frac{1}{2}$, we have $(1 - \frac{n}{N_\varepsilon})^{\alpha-1} \ge (1-\frac{1}{2})^{\alpha-1}$. And we also have $S_{n,+\infty}(\alpha) \gtrsim a_n(\alpha-1)$; so, we are done. On the other hand, if $\frac{n}{N_\varepsilon} \ge \frac{1}{2}$, then

$$S_{n,+\infty}(\alpha) \ge S_{N_\varepsilon,+\infty}(\alpha) \sim N_\varepsilon^{1-\alpha} \sim n^{1-\alpha} = a_n(\alpha-1)\cdot$$

This ends the proof of ii-.

Assume finally that $0 < \alpha < 1$. Then Lemma 5.9 tells us that

$$S_{n,N_\varepsilon}(\alpha) \sim (N_\varepsilon^{1-\alpha} - n^{1-\alpha}) \le N_\varepsilon^{1-\alpha} \sim S_{N_\varepsilon,\infty}.$$

We thus conclude that $\max(S_{n,N_\varepsilon}(\alpha), S_{N_\varepsilon,+\infty}(\alpha)) \sim S_{N_\varepsilon,+\infty}$. This proves iv- and ends the proof of the Corollary. ∎

We can also prove the following result with the same kind of arguments. So we omit them.

Corollary 5.11. *Let N be a fixed integer such that $2N < N_\varepsilon \sim \frac{1}{\varepsilon}$. Then we have the following estimates of $S_{N,n}(\alpha)$ for $N \leq n$:*

$$S_{N,n}(\alpha) \sim \begin{cases} \left.\begin{array}{ll} a_N(\alpha-1) - a_n(\alpha-1) & \textit{for } 1 < \alpha \\ \log \frac{n}{N} & \textit{for } \alpha = 1 \\ a_n(\alpha-1) - a_N(\alpha-1) & \textit{for } \alpha < 1 \end{array}\right\} & \textit{for } n \leq N_\varepsilon \\ \left.\begin{array}{ll} 1 & \textit{for } 1 < \alpha \\ \log N_\varepsilon & \textit{for } 1 = \alpha \\ N_\varepsilon^{1-\alpha} & \textit{for } 0 \leq \alpha < 1 \\ \frac{a_n(\alpha)}{\varepsilon} & \textit{for } \alpha < 0 \end{array}\right\} & \textit{for } n \geq N_\varepsilon \end{cases}$$

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