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Complex dynamics in simple mechanical systems: Similarities to neuronal bursting

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Abstract. We present an overview of our studies on some simple mechanical systems including the 'simple' nonlinear pendulum and its variants. We show that these systems exhibit numerous types of regular bursting oscillations which are seen in biological neurons. In particular, we discuss bow-tie shaped bursts which we found in a driven pendulum with linear velocity damping, under constant torque and dynamic feedback. Similar bursts of identical bow-tie shape have been reported by us previously in a system of two resistively coupled Josephson junctions in a certain parameter regime under certain conditions. We discuss the bifurcation mechanism producing some of these bursts.

Keywords. Bursting oscillations; pendulum; bifurcations; mechanical systems.

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1. Introduction

Seemingly unrelated phenomena and systems can at times exhibit nearly identical dynamical behaviour. While these might normally come as a surprise, often one finds on further study that the basic governing equations underlying the dynamics of the two varied systems are the same. Many other times, however, this may not be the case and it is not so straightforward to understand how the differing systems produce the same behaviour. In all cases, we believe that understanding the governing mechanism underlying a common observed dynamical behaviour seen even in an unrelated system is useful and can help in providing insights on a system under study.

Here we discuss examples of some latter situations. We show that the nonlinear oscillations of certain simple mechanical systems, in particular bursting oscillations, closely resemble those exhibited by certain biological cells including neurons.

2. The forced pendulum with damping and dynamic feedback

Consider a damped pendulum with a constant torque I and forced externally with frequency ω. Further, consider a dynamic feedback which modulates the frequency of forcing. The equation of motion describing the evolution of the angle variable θ in time t is [1]

$$\ddot{\theta} = I - \sin(\theta) - \alpha\dot{\theta} + \sin(\omega t - \theta), \tag{1}$$

where α denotes the damping parameter. For forcing frequencies much smaller than the natural frequency of the unforced system, the system exhibits bursting oscillations which have an intriguing bow-tie shape [1] (shown in figure 1). Bursts such as these are observed in isolated-CA3 pyramidal neurons [2].

Equation (1) has two fixed points $(\theta^*, \dot{\theta}^*)$ which exist only for $\omega = 0$ and $I \leq 2$ which are located at $P = (\arcsin(I/2), 0)$ and $Q = (\pi - \arcsin(I/2), 0)$. For $I > 2$, the system has no fixed points.

The characteristic equation at P is:

$$\lambda(\lambda^2 + \alpha\lambda + 2h) = 0, \tag{2}$$

where $h = \sqrt{1 - (I/2)^2}$. The eigenvalues are:

0 and $\lambda_{+,-} = (-\alpha \pm \sqrt{\alpha^2 - 8h})/2$.

The system is critically damped at $\alpha = 2\sqrt{2h}$. For $\alpha < 2\sqrt{2h}$, the two eigenvalues form complex conjugate pairs and P is a stable focus, while for $\alpha > 2\sqrt{2h}$, it is a stable node.

At Q, the characteristic equation is given by:

$$\lambda(\lambda^2 + \alpha\lambda - 2h) = 0. \tag{3}$$

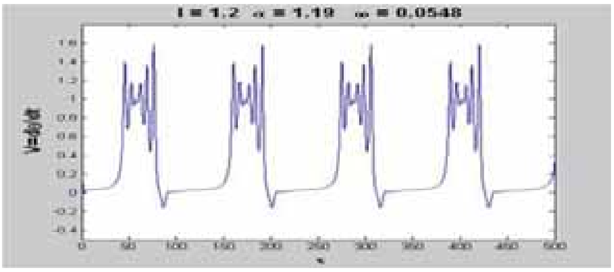

Figure 1. Bow-tie shaped bursting oscillations produced in a certain parameter regime by the system described by eq. (1). The bursts appear only for forcing frequencies which are much smaller than the natural frequency of the unforced system. Parameter values: $I = 1.2, \alpha = 1.19$ and $\omega = 0.0548$.

The eigenvalues are:

$0, \lambda_{+,-} = (-\alpha \pm \sqrt{\alpha^2 + 8h})/2.$

λ_- is always stable while λ_+ is not; thus Q is a saddle point.

The bursts occur for $\omega \neq 0$ for which the system has no fixed points. Therefore, to understand the mechanism of the bursting oscillations, the $\theta - \dot{\theta}$ phase space was studied at different instants of time [1]. The vector field at any time instant is like that of a damped pendulum under constant torque. The flow lines of the vector field as well as the θ and $\dot{\theta}$ nullclines (in green and red, respectively) at different instants of time are depicted in figure 2.

As time progresses, driven by the external force, points

$$P = \left(\theta^* = \sin^{-1}\left[\frac{I}{2}\sec\left(\frac{\omega t}{2}\right)\right] + \left(\frac{\omega t}{2}\right),\ \dot{\theta}^* = 0\right)$$

and

$$Q = \left(\theta^* = \pi - \sin^{-1}\left[\frac{I}{2}\sec\left(\frac{\omega t}{2}\right)\right] + \left(\frac{\omega t}{2}\right),\ \dot{\theta}^* = 0\right) \quad (4)$$

approach each other, coalesce and disappear and then reappear after some time in a periodic manner. The merging of the focus at P and the saddle at Q occurs at $I = 2$, when the V-nullcline is tangent to the x-axis ($V = 0$). It was shown in [1] that both the vanishing of the fixed points and their emergence again occur through a saddle-focus bifurcation.

The resting period between two bursts was estimated analytically and found to be [1]

$$T_{\text{rest}} = \frac{4}{\omega}\cos^{-1}\left(\frac{I}{2}\right). \quad (5)$$

Identical bow-tie shaped bursting oscillations were reported by us in a different study [3] in a very different system – that of two resistive capacitive shunted junction (RCSJ) models of the Josephson junction coupled together resistively in a certain parameter regime, with one junction kept in an oscillatory mode and the other in an excitable mode. In the latter system, however, there was no external forcing and the oscillatory junction was the driver for the excitable junction. It is interesting that the bifurcation mechanisms for bursting in these two systems differ slightly. While in the damped-forced pendulum with dynamic feedback both the initiation and termination of a bow-tie burst are via a saddle–focus bifurcation, in the case of the coupled Josephson junctions, bursting is initiated by a saddle node on an invariant circle (SNIC) bifurcation and also ends via the same mechanism, but the modulation of amplitudes in the spiking region of the burst occurs via the saddle-focus bifurcation [3].

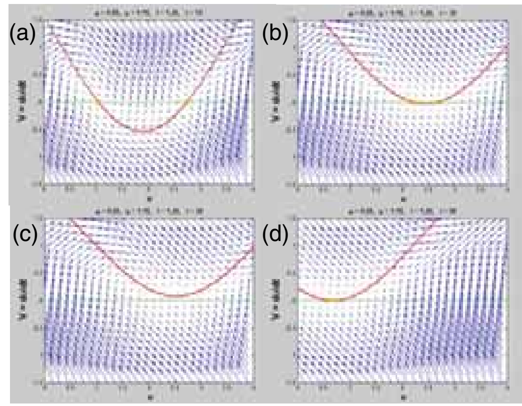

Figure 2. Evolution of the vector field with time t of the system described in eq. (1) in the θ–$\dot{\theta}$ plane at different times: (**a**) $t = 13$, (**b**) $t = 35$, (**c**) $t = 38$ and (**d**) $t = 90$. The θ-nullcline is in red while the V-nullcline is in green. The focus and the saddle at the two fixed points P and Q merge in a saddle–focus bifurcation at the end of a burst and reappear at the initiation of the next burst (parameter values: $I = 1.25, \alpha = 1.19$ and $\omega = 0.05$).

3. Other mechanical systems

There is another mechanical system which may be briefly mentioned – that of a pendulum in which the length $l(t)$ is varied in time:

$$\frac{d^2\theta}{dt^2} + \frac{2}{l}\frac{dl}{dt}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0. \quad (6)$$

We showed in [4] that under certain conditions, such a system produces symmetric bursting oscillations. This is depicted in figure 3.

In a different study [5] we have shown that a slightly different variant of the mechanical system described in

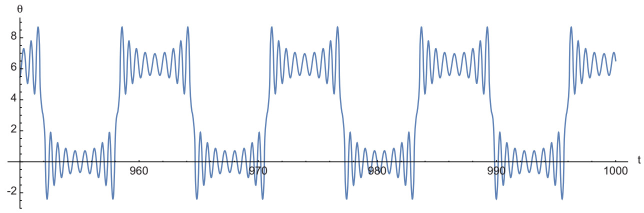

Figure 3. An example of symmetric bursting behaviour is seen in a pendulum with slowly varying lengths under certain conditions (Ref. [4]). The vertical axis stands for the angle variable θ and the horizontal axis represents the time t.

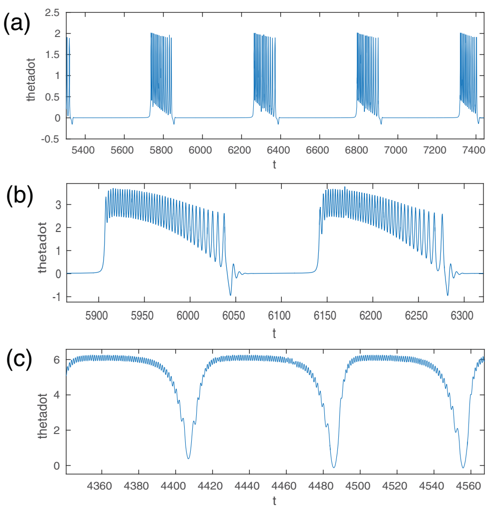

Figure 4. Two different kinds of bursting oscillations (**a**) and (**b**) and mixed mode oscillations (**c**) are seen in the model in Ref. [5]. The vertical axis (theta dot) represents the membrane voltage of the model neuron.

eq. (1) exhibits different burst types which are seen in different kinds of neurons and pancreatic beta cells. A few examples of the numerous types of bursting and mixed mode oscillations exhibited by that model are shown in figure 4. It is interesting that a single model reproduces a host of different bursts involving different bifurcations to start and terminate each burst.

Bursts have been studied extensively and classified by Rinzel [6], Bertram [7], Ermentrout [8], Hoppensteadt [9], Izhikevich [10] and several others. Both spontaneous and evoked bursts have been studied in vivo as well as in vitro in a variety of cells – sensory cells, gustatory cells, hair cells of the inner ear, pancreatic beta cells, neurons in different regions of the brain, etc. and many models have been constructed to reproduce their dynamical behaviour. Yet we are far from getting a reasonably good understanding of their significance for living systems. Nevertheless, newer models involving seemingly unrelated systems could contribute to securing a clearer view of these intriguing phenomena.

4. Conclusions

Simple nonlinear mechanical systems can exhibit complex bursting oscillatory behaviour akin to those seen in different biological cells, in particular sensory cells, neurons and pancreatic beta cells. The focus of this overview was on the intriguing bow-tie burst produced by a damped pendulum with external force and a dynamical feedback in a certain parameter regime, though a few glimpses were provided of other bursts and complex oscillations in some other mechanical systems studied in Refs [4, 5].

Although the significance of the differing shapes of bursts in living systems is not known, it is believed that bursts have some significant roles to play in the normal functioning of the respective system they are produced in, for instance in cognitive tasks such as feature extraction or for associative memory in the nervous system. Understanding the dynamical mechanism producing different types of bursts even in unrelated mechanical systems could help in providing better insights into the way living systems operate.

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