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Abstract. We give an example of supersonic solutions to a one-dimensional steady state Euler-Poisson system arising in the modeling of plasmas and semiconductors. The existence of the supersonic solutions which correspond to large current density is proved by Schauder's fixed point theorem. We show also the uniqueness of solutions in the supersonic region.

Keywords. Euler-Poisson equations, steady state flow, supersonic solutions, existence, uniqueness.

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1 Introduction

The Euler-Poisson system plays an important role in the mathematical modeling and numerical simulation for plasmas and semiconductors [2, 7, 8]. In the steady state isentropic case the existence and uniqueness of smooth solutions are obtained in the subsonic region for a one-dimensional flow [3] or potential flows [4]. See also [1] for the subsonic solutions to a one-dimensional non-isentropic model. In [5, 6], the stationary transonic solutions are studied by an artificial viscosity approximation. The existence of the transonic solutions is proved by passing to the limit in the approximate Euler-Poisson system as the viscosity coefficient goes to zero. However, the existence of the purely supersonic solutions has not been discussed yet.

In this paper, we give an example of the supersonic solutions in a one-dimensional steady state Euler-Poisson system :

$$\partial_x j = 0, \tag{1.1}$$

$$\partial_x \left(\frac{j^2}{n} + p(n) \right) = n\partial_x \phi - j/\tau, \tag{1.2}$$

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$$-\partial_{xx}\phi = b - n. \tag{1.3}$$

Equation (1.1) implies that j is a constant. Here, n, j and ϕ are the electron density, the current density and the electric potential, respectively. The parameter $\tau > 0$ stands for the momentum relaxation time depending on n and j in general. For simplicity, we assume that τ is a constant. The given function $b = b(x)$ is the doping profile for the semiconductors. The pressure function $p = p(n)$ is assumed to be smooth and strictly increasing for $n > 0$. As in [3], we consider equations (1.1)-(1.3) in the interval $(0, 1)$ subject to the following Dirichlet boundary conditions :

$$n(0) = n_0, \;\; n(1) = n_1, \;\; \phi(0) = \phi_0, \;\; \phi(1) = \phi_1, \tag{1.4}$$

where $n_0 > 0$, $n_1 > 0$ and ϕ_0, $\phi_1 \in I\!R$ are given data. If $n > 0$ is a smooth function, after eliminating ϕ in (1.2)-(1.3), we obtain a Dirichlet problem for n :

$$-\partial_{xx}F_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n-b) = 0 \quad \text{in } (0,1), \tag{1.5}$$

$$n(0) = n_0, \;\; n(1) = n_1, \tag{1.6}$$

where

$$F_j(n) = \frac{1}{2n^2} + \frac{h(n)}{j^2} \;\text{ with } h(n) = \int_1^n \frac{p'(y)}{y}\,dy.$$

Once n is solved, from (1.2) ϕ is given explicitly by :

$$\phi(x) = \phi_0 + j^2(F_j(n(x)) - F_j(n_0)) + \int_0^x \frac{j}{\tau n(y)}dy. \tag{1.7}$$

Then ϕ_1 is linked with j by the following relation

$$\phi_1 = \phi_0 + j^2(F_j(n_1) - F_j(n_0)) + \int_0^1 \frac{j}{\tau n(y)}dy. \tag{1.8}$$

It is easy to see that (n, ϕ) with $n > 0$ is a smooth solution of (1.2)-(1.4) if and only if (n, ϕ) is a smooth solution of (1.5)-(1.7). Therefore, we may first solve n to the Dirichlet problem (1.5)-(1.6) and then determine ϕ by (1.7).

Now the equation (1.5) is elliptic if and only if $F_j'(n) \neq 0$. Since p is strictly increasing, there is a unique $n_c(j)$ such that $F_j'(n_c(j)) = 0$, or equivalently

$$\sqrt{p'(n_c(j))} = \frac{|j|}{n_c(j)}.$$

Here the quantities $c = \sqrt{p'(n)}$ and j/n stand for the speed of sound and the electron velocity, respectively. If $n \longrightarrow n^2p'(n)$ is strictly increasing, we obtain the following alternative :

$$\text{subsonic flow} \Longleftrightarrow F'(n) > 0 \Longleftrightarrow n > n_c(j) \Longrightarrow \text{(1.5) is elliptic,} \tag{1.9}$$

$$\text{supersonic flow} \Longleftrightarrow F'(n) < 0 \Longleftrightarrow n < n_c(j) \Longrightarrow (1.5) \text{ is elliptic.} \quad (1.10)$$

Note that the linear term n/j^2 in (1.5) has not a good sign. Nevertheless, it is small as j is large and then can be controlled by the $L^2(0,1)$ norm of $\partial_x n$ by Poincaré's inequality. Similar argument holds for the term $\partial_x(1/j\tau n)$. This is the main feature of the problem to yield the existence and uniqueness of solutions.

2 Existence of solutions

Assume $b \in L^\infty(0,1)$. In view of (1.9), the subsonic solutions to (1.2)-(1.4) correspond to the small value of j. They have been considered in [3]. We study here the supersonic solutions which correspond to the case (1.10). To this end, let M_1 and M_2 be any two constants satisfying

$$0 < M_1 < \min(n_0, n_1), \quad \max(n_0, n_1) < M_2. \quad (2.1)$$

Choosing j such that $n_c(j) > M_2$, then (1.10) and (2.1) imply that the boundary data n_0 and n_1 are in the supersonic region. Since the maximum principle can not be applied to (1.5) in the supersonic region, the solutions of (1.5)-(1.6) may not be supersonic flow. To seek for a supersonic solution, we define a smooth and strictly decreasing function $\tilde{F}_j$ on $I\!R^+$ such that

$$\tilde{F}_j(+\infty) = 0, \quad \tilde{F}_j(n) = F_j(n) \ \text{ for all } n \le M_2.$$

Then we study the following problem instead of (1.5)-(1.6) :

$$-\partial_{xx}\tilde{F}_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n-b) = 0 \quad \text{in } (0,1), \quad (2.2)$$

$$n(0) = n_0, \ \ n(1) = n_1. \quad (2.3)$$

Our strategy is to prove the existence of a smooth solution n to (2.2)-(2.3) such that $0 < n \le M_2$. Then n is a supersonic solution of (1.5)-(1.6) by the definition of $\tilde{F}_j$.

Since $\tilde{F}_j$ is smooth and strictly decreasing from $I\!R^+$ to $I\!R^+$, we may make a change of variable $v = \tilde{F}_j(n)$ for $n > 0$. Let G_j be the inverse of $\tilde{F}_j$, which is also smooth and strictly decreasing from $I\!R^+$ to $I\!R^+$. Then the problem (2.2)-(2.3) is equivalent to

$$-\partial_{xx}v - \frac{1}{j}\partial_x\left(\frac{1}{\tau G_j(v)}\right) + \frac{1}{j^2}(G_j(v) - b) = 0 \quad \text{in } (0,1), \quad (2.4)$$

$$v(0) = v_{0j} = F_j(n_0), \ \ v(1) = v_{1j} = F_j(n_1). \quad (2.5)$$

To study the problem (2.4)-(2.5), we will apply Schauder's fixed point theorem. For this purpose, let's define a closed convex set

$$S = \{v \in C([0,1]);\ F_j(M_2) \le v \le F_j(M_1)\},$$

and a map T by $v = T(\sigma)$ for $\sigma \in S$, where v solves the linear problem :

$$-\partial_{xx}v + \frac{1}{j\tau}\alpha_j(\sigma)\partial_x v + \frac{1}{j^2}\beta_j(x,\sigma) = 0 \quad \text{in } (0,1), \tag{2.6}$$

$$v(0) = v_{0j}, \quad v(1) = v_{1j}, \tag{2.7}$$

with

$$\alpha_j(\sigma) = \frac{G_j'(\sigma)}{G_j^2(\sigma)} = \frac{1}{G_j^2(\sigma)\tilde{F}_j'(G_j(\sigma))}, \qquad \beta_j(x,\sigma) = G_j(\sigma) - b(x).$$

We observe that $\sigma \in S$ implies that

$$F_j(M_2) \le \sigma \le F_j(M_1).$$

From $\tilde{F}_j(\sigma) = F_j(\sigma)$ for $\sigma \le M_2$, we have

$$M_1 \le G_j(\sigma) \le M_2.$$

Therefore, from the definition of F_j, there is a $j_1 > 0$ depending only on M_1 and M_2 such that α_j and β_j are two bounded functions with bounds depending on M_1 and M_2 but independent of j and σ for any $j \in I\!R$ satisfying $|j| \ge j_1$.

For $v \in H^1(0,1)$ and $z \in H_0^1(0,1)$, let

$$a(v,z) = \int_0^1 \left(\partial_x v\partial_x z + \frac{1}{j\tau}\alpha_j(\sigma)z\partial_x v\right)dx, \quad l(z) = -\frac{1}{j^2}\int_0^1 \beta_j(x,\sigma)z dx.$$

It is clear that $l(\cdot)$ is linear and continuous on $H_0^1(0,1)$, and $a(\cdot,\cdot)$ is bilinear and continuous on $H_0^1(0,1) \times H_0^1(0,1)$. Moreover, by Poincaré's inequality,

$$\begin{aligned} a(z,z) &= \int_0^1 \left((\partial_x z)^2 + \frac{1}{j\tau}\alpha_j(\sigma)z\partial_x z\right)dx \\ &\ge \|\partial_x z\|^2_{L^2(0,1)} - \frac{1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\|z\|_{L^2(0,1)}\|\partial_x z\|_{L^2(0,1)} \\ &\ge \left(1 - \frac{C_1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\right)\|\partial_x z\|^2_{L^2(0,1)}, \quad \forall\, z \in H_0^1(0,1), \end{aligned}$$

where $C_1 > 0$ is the constant in Poincaré's inequality. Then there exists a $j_2 \ge \frac{2C_1}{\tau}\|\alpha_j\|_{L^\infty(0,1)}$ depending only on M_1 and M_2 such that

$$a(z,z) \ge \frac{1}{2}\|\partial_x z\|^2_{L^2(0,1)}, \quad \forall\, |j| \ge j_2,\ \forall\, z \in H_0^1(0,1). \tag{2.8}$$

Therefore, $a(\cdot,\cdot)$ is coercive. By Lax-Milgram's theorem, there exists a unique solution $v \in H^1(0,1)$ to the variational problem $a(v,z) = l(z)$, $\forall\, z \in H_0^1(0,1)$ and (2.7). This shows that the map T is well defined.

We prove now that $T(S)$ is a compact set of $C([0,1])$. Indeed, let $\overline{v}_j = (1-x)v_{0j} + xv_{1j}$. Then $v - \overline{v}_j \in H_0^1(0,1)$. From the continuity of $l(\cdot)$ and $a(\cdot,\cdot)$, the coercivity estimate (2.8) and

$$a(v - \overline{v}_j, v - \overline{v}_j) = l(v - \overline{v}_j) - a(\overline{v}_j, v - \overline{v}_j),$$

it is easy to obtain

$$\|\partial_x(v-\overline{v}_j)\|_{L^2(0,1)} \le \frac{2C_1}{j^2}\|\beta_j\|_{L^\infty(0,1)} + \frac{2C_1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\|\partial_x\overline{v}_j\|_{L^2(0,1)}. \tag{2.9}$$

Recall that α_j and β_j are bounded independent of σ. We conclude from Poincaré's inequality and the compact imbedding from $H^1(0,1)$ into $C([0,1])$ that $T(S)$ is a compact set of $C([0,1])$. Moreover, there are constants $C_2 > 0$ and $j_3 \ge j_2$ which depend only on M_1 and M_2 such that

$$|v(x) - \overline{v}_j(x)| \le \frac{C_2}{|j|}, \quad \forall\, |j| \ge j_3, \quad \forall\, x \in [0,1].$$

Since

$$F_j(\max(n_0,n_1)) \le \overline{v}_j(x) \le F_j(\min(n_0,n_1)), \quad \forall\, x \in [0,1],$$

it follows that

$$F_j(\max(n_0,n_1)) - \frac{C_2}{|j|} \le v(x) \le F_j(\min(n_0,n_1)) + \frac{C_2}{|j|}, \quad \forall\, |j| \ge j_3, \quad \forall\, x \in [0,1].$$

The function $n \longrightarrow F_j(n)$ being strictly decreasing for $n \le M_2$, from (2.1) there is a $j_4 \ge j_3$ depending only on M_1 and M_2 such that

$$F_j(M_2) \le v(x) \le F_j(M_1), \quad \forall\, |j| \ge j_4, \quad \forall\, x \in [0,1]. \tag{2.10}$$

Hence, $v \in S$ and then T is a self map from S to S. Finally, the continuity of T follows from a standard argument. More precisely, for $\sigma_1, \sigma_2 \in S$, we can prove that there is a constant $C_3 > 0$ depending only on M_1 and M_2 such that

$$\left(1 - \frac{C_3}{|j|\tau}\right)\|T(\sigma_1) - T(\sigma_2)\|_{C([0,1])} \le \frac{C_3}{|j|\tau}\|\sigma_1 - \sigma_2\|_{C([0,1])}.$$

Thus, T is continuous for $|j| > j_5 = \max(j_4, C_3/\tau)$. We conclude from Schauder's fixed point theorem the existence of a solution $v \in H^1(0,1) \cap S$ of $v = T(v)$.

This shows the existence of a solution $v \in H^1(0,1) \cap S$ to the problem (2.4)-(2.5), and then the existence of a solution $n = G_j(v) \in H^1(0,1)$ to the problem (2.2)-(2.3). Since $v = \tilde{F}_j(n) = F_j(n)$ for $n \le M_2$, from (2.10) we obtain

$$M_1 \le n(x) \le M_2, \quad \forall\, |j| \ge j_5, \quad \forall\, x \in [0,1]. \tag{2.11}$$

Therefore, $n \in H^1(0,1)$ is a supersonic solution to the problem (1.5)-(1.6). Thus, we have proved

Theorem 1 *Let $n_0 > 0$ and $n_1 > 0$. Let M_1, M_2 be two constants satisfying (2.1) and $b \in L^\infty(0,1)$. Then there exists a $j_e > 0$ depending only on M_1 and M_2 such that for any current density j satisfying $|j| \ge j_e$, the problem (1.2)-(1.4) admits a solution $(n,\phi) \in H^1(0,1) \times H^1(0,1)$. This solution is located in the supersonic region and satisfies (2.11).*

3 Uniqueness of solutions

There doesn't exist a general result on the uniqueness of solutions when the boundary data are located in the supersonic region. Indeed, for large j the formation of shocks cannot be avoided and the transonic solutions should be investigated. We refer to [5, 6] for the analysis of the transonic solutions. Here we give a uniqueness result in the supersonic region for large j. This result can be stated as follows.

Theorem 2 *Let M_1 and M_2 be two constants with $0 < M_1 < M_2$. Let $(n^{(1)}, \phi^{(1)})$ and $(n^{(2)}, \phi^{(2)})$ be two supersonic solutions of (1.2)-(1.3) in $H^1(0,1) \times H^1(0,1)$ with $M_1 \leq n^{(1)}, n^{(2)} \leq M_2$. Then there exists a $j_u > 0$ depending only on M_1 and M_2 such that for any current density $j \in I\!R$ satisfying $|j| \geq j_u$, we have $(n^{(1)}, \phi^{(1)}) = (n^{(2)}, \phi^{(2)})$.*

Proof. In view of (1.7), it suffices to show that $n^{(1)} = n^{(2)}$. Let $w = n^{(2)} - n^{(1)}$. By subtracting the equation (1.5) satisfied by $n^{(1)}$ and $n^{(2)}$ we obtain :

$$\partial_{xx}(A_j(x)w) + \frac{1}{j\tau}\partial_x(B(x)w) + \frac{1}{j^2}w = 0 \quad \text{in } (0,1), \tag{3.1}$$

where

$$A_j(x) = -\int_0^1 \frac{\partial F_j}{\partial n}\left(n^{(1)}(x) + s\left(n^{(2)}(x) - n^{(1)}(x)\right)\right) ds,$$

$$\frac{1}{M_2^2} \leq B(x) = \frac{1}{n^{(1)}n^{(2)}} \leq \frac{1}{M_1^2} \quad \text{in } (0,1).$$

From

$$F_j'(n) = -\frac{1}{n^3} + \frac{h'(n)}{j^2},$$

it is easy to check that there are constants $C_4 > 0$ and $j_6 > 0$ which depend only on M_1 and M_2 such that

$$A_j(x) \geq C_4, \quad \forall\, |j| \geq j_6, \ \forall\, x \in [0,1].$$

Multiplying (3.1) by $A_j w \in H_0^1(0,1)$ and integrating over $(0,1)$ give :

$$\int_0^1 [\partial_x(A_j(x)w)]^2\, dx = \int_0^1 \left(-\frac{1}{j\tau}B(x)w\partial_x(A_j(x)w) + \frac{1}{j^2}A_j(x)w^2\right) dx.$$

It follows from Poincaré's inequality that :

$$\|\partial_x(A_j w)\|^2_{L^2(0,1)} \leq \frac{1}{C_4}\left(\frac{C_0}{M_1^2|j|\tau} + \frac{C_0^2}{j^2}\right)\|\partial_x(A_j w)\|^2_{L^2(0,1)}.$$

This shows that $A_j w = 0$ and then $w = 0$ provided that $|j| \geq j_7$ for some large $j_7 > 0$ depending only on M_1 and M_2.

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