Continuous Large Deviation Multifractal Spectrum: Definition and Estimation

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CONTINUOUS LARGE DEVIATION MULTIFRACTAL SPECTRUM: DEFINITION AND ESTIMATION

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The large deviation multifractal spectrum gives important statistical informations on irregular measures. However it is difficult to estimate. In this paper, we propose two new definitions of the large deviation spectrum better adapted to the design of various estimators. They rely on the computation of the Lebesgue measure of the reunion of all intervals of same size whose coarse grain Hölder exponent is equal to a Hölder exponent. In particular, we introduce the continuous large deviation spectrum for which we construct different estimators. We finally show some numerical results obtained on both deterministic and random synthetical signals.

keywords

Multifractal analysis, large deviation spectrum, kernel density estimation.

1 Introduction

Multifractal Analysis (MA) is concerned with the analysis, characterization and classification of irregular measures, capacities or functions[?,?,?]. Here, we will restrict ourselves to the study of probability measures μ defined on the unit interval $[0,1)$. In this framework, it is natural to consider $\mathcal{P} := (\mathcal{P}_n)_{n\geq 1}$ the sequence of partitions $\mathcal{P}_n$ of the unit interval $[0,1)$, each $\mathcal{P}_n$ made of dyadic intervals, i.e $\mathcal{P}_n := (I_n^k)_{0\leq k\leq 2^n-1}$ with $I_n^k := [k2^{-n},(k+1)2^{-n})$. On this dyadic net, the *large deviation spectrum* is defined as[?,?]

$$f_g(\alpha) := \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log_2 N_n^\varepsilon(\alpha), \tag{1}$$

where $N_n^\varepsilon(\alpha) := \#\{I_n^k \subset \mathcal{P}_n : |\alpha_n^k - \alpha| \leq \varepsilon\}$ and $\alpha_n^k := -1/n \log_2 \mu(I_n^k)$. Loosely speaking, (1) reflects the exponential decreasing rate of $N_n^\varepsilon(\alpha)$, which is the number of dyadic intervals having a *coarse grain Hölder exponent* α_n^k close to a *Hölder exponent* α up to a precision ε, when the resolution n tends to ∞. More precisely, it is equal, up to 1, to a Large Deviation (LD) rate function [?,?,?]. It is closely linked to two other well-known spectrum definitions, namely: the *Hausdorff spectrum*, defined as the *Hausdorff dimension* of sets of points having the same Hölder exponent, and the *Legendre spectrum*, based on the *Legendre transform* of the so-called *Rényi exponents* $\tau(q)$ of the q^{th} moments of the measure. In short, the large deviation spectrum gives a *statistical* description of the measure, whereas the Hausdorff spectrum f_h gives a *geometrical* description and the Legendre spectrum f_l is naturally linked to thermodynamics. Though in some cases it can be proved that $f_h = f_g = f_l$ – we say that the *multifractal formalism* holds – , in general we only have $f_h \leq f_g \leq f_l$ and $f_g^{**} = f_l$, where f_g^{**} is the concave hull of f_g [?,?,?].

The problem of getting accurate estimates of the Hausdorff spectrum f_h is not solved yet. Since f_g gives important informations related to the statistical properties of irregular measures, it seems worth trying to design estimators for it. When the weak version of the multifractal formalism holds, we have $f_g = f_l$ and thus f_g is given by the estimation of f_l, which is easy and robust. Indeed, it only involves average quantities, and only one limit is needed for its estimation [?,?]. However, in a general setting, its use implies a sever loss of information since the Legendre spectrum f_l is always a concave function. For instance, compound multiplicative processes have a non-concave large deviation spectrum [?,?] and the detection of this specific large deviation property is of interest for applications. As a consequence, it seems necessary to keep the large deviation approach together with an attempt to make its estimation easier and more robust.

Large deviation spectrum estimators have been considered by a number of authors. Histogram methods [?,?] yield satisfactory results, when well adapted, on strictly multiplicative cascades, but they fail to estimate a good approximation in more complex situations. In [?], an estimator of f_g has been proposed which is based on a classical tool in non-parametric estimation: the kernel method [?]. Such a method has been used extensively with good results in density estimation and rather precise theorems are known that assess the quality of the results. The difficulty here is that f_g is not a probability density function, but a double logarithmic normalization of a density. However, it has proved to be a valuable tool in the MA framework as it gave good results on both synthetic and real signals, such as fully developed turbulence flows and LAN traffic [?,?].

In section 2, we recall the histogram method and the principle of the kernel estimator of the large deviation spectrum f_g. In section 3, we propose two new definitions of the large deviation spectrum better adapted to the design of various estimators. They rely on the computation of the Lebesgue measure of the reunion of all intervals of same size whose coarse grain Hölder exponent is equal to a Hölder exponent. In particular, we introduce the *continuous large deviation spectrum* f_g^c for which we construct different estimators in section 4. We finally show in section **??** some numerical results obtained on both deterministic and random synthetical signals.

2 Large Deviation Multifractal Spectrum Estimation

2.1 Histogram method

Roughly speaking, all histogram algorithms involve the following steps:

1. for each n, compute all coarse grain Hölder exponents α_n^k;
2. compute the minimum and maximum coarse grain Hölder exponents $\alpha_n^{\min} = \min_k \alpha_n^k$ and $\alpha_n^{\max} = \max_k \alpha_n^k$;
3. divide $[\alpha_n^{\min}, \alpha_n^{\max}]$ into N boxes;
4. compute the number N_n^i of intervals I_n^k whose α_n^k falls in the i-th box, for $i = 0, \cdots, N-1$;
5. Estimate f^i by a linear regression on $(\log n, \log N_n^i)$.

There are two types of limitations with those methods:

- The choice of the averaging number (step 1): if we deal with, say, a trinomial measure, decimating (that is, summing the measure of two adjacent intervals) at each resolution n produces strong oscillations with induce errors in the linear regression estimation (step 5).
- The choice of the number of "boxes" on the α axis (step 3): this choice is ad-hoc, although it influences a lot the result. In particular, it does not take into account the precision ε limit of (1).

2.2 Kernel method

The starting point of this estimation method is to remark that $N_n^\varepsilon(\alpha)$ may be re-written as the following convolution

$$N_n^\varepsilon(\alpha) := \sum_{k=0}^{2^n-1} 1\!\!1_{[-\varepsilon,+\varepsilon]}(\alpha - \alpha_n^k) = 2^n \int_{\alpha-\epsilon}^{\alpha+\epsilon} p_n(\beta)d\beta = 2^n(p_n * 1\!\!1_{[-\varepsilon,+\varepsilon]})(\alpha), \quad (2)$$

where $1\!\!1_{[-\varepsilon,+\varepsilon]}$ is the indicator function of the centered interval $[-\varepsilon, +\varepsilon]$ and $p_n(\alpha)$ is the *empirical probability density function* of the coarse grain Hölder exponents

$$p_n(\alpha) = 2^{-n} \sum_{k=0}^{2^n-1} \delta(\alpha - \alpha_n^k). \quad (3)$$

It is easily shown that the indicator function $1\!\!1_{[-\varepsilon,+\varepsilon]}$ can be replaced by any compactly supported positive kernel K such as $\int_{\mathcal{D}} K = 1$, where $\mathcal{D}$ is the support of K. Let us note $K_\varepsilon(\alpha) = \frac{1}{2\varepsilon} K(\frac{\alpha}{\varepsilon})$ and $p_n^\varepsilon(\alpha) = 2\varepsilon(p_n * K_\varepsilon)(\alpha)$. We obtain

$$f_g(\alpha) := 1 + \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log_2 p_n^\varepsilon(\alpha), \quad (4)$$

One of the key problem in non-parametric density estimation is the choice of the optimal bandwidth h such that the empirical density convoluted by the kernel $p_n * K_h$ is “as close as possible” to the density p. In this setting, h becomes a function of n denoted $h_n^\star$. Adapting classical techniques allows us to reduce the two limits to one limit together with an optimal choice of precision $\varepsilon_n^\star$. However, the first thing to check is under which conditions it is indeed possible to get rid of this limit on ε [?]. Another crucial topic is the spatial adaptivity of h_n. Indeed, if the use of smooth kernels allows us to obtain more regular estimates, it can also hide changes of concavity. Data adaptive local bandwidth[?] allows us to obtain accurate estimators, the precision ε being adapted to the spatial variable α. Note that no linear regression needs to be done, since the spectrum is estimated at only one resolution. Of course, in the general case, one needs to verify that the spectra estimated at each resolution n are consistent. Such a verification should be performed carefully[?].

2.3 Desired Properties

Ideally, the properties that any estimator of f_g should possess are the following ones:

- P1) If a implicit dependency between precision ε and resolution n is introduced, it should be designed carefully;
- P2) The precision ε should be spatially adaptive;
- P3) The resolution range $n_{\min} \leq n \leq n_{\max}$ on which the spectrum estimation results are stable and which represents the multifractal behavior of the measure should be well determined.

Let us comment the first property. As said in subsection 2.1, histogram methods introduce an implicit dependency between the resolution n and the precision ε. Even though, from an estimation point of view, it is obviously desirable to pass from the two limits, on n and then on ε, to one limit on n, it is clear the dependency between precision and resolution should be designed carefully. The kernel estimator take this requirement into account. The second property is needed in the case of spatially non-stationary measures such as lumping or sum of measures, which results in changes in the concavity of f_g. The last property is necessary because of finite size effects.

3 New definitions of the Large Deviation Multifractal Spectrum

Instead of trying to improve kernel estimators according to these properties, we propose here two new definitions of the large deviation multifractal spectrum which are both better adapted than f_g to the construction of various estimators, and share its large deviation properties. In addition, these new spectra coincide with f_g on some classical multifractal signals such as deterministic or random multiplicative cascades and other stochastic processes[?,?]. In particular, we used these theoretical spectra as benchmarks for the estimators constructed on these new definitions, see section **??**.

3.1 A first approach

A first candidate is defined as follows. For every interval I of size $|I| = \eta$, let $\alpha_\eta(I) := \log \mu(I)/ \log \eta$ be the coarse grain Hölder of I (this notation will be used throughout this paper). Define first the set $E_\eta(\alpha)$ as the reunion of all intervals of same size η whose coarse grain Hölder exponent is equal to a Hölder exponent α

$$E_\eta(\alpha) := \cup\{I \in [0,1) : |I| = \eta,\ \alpha_\eta(I) = \alpha\}. \tag{5}$$

Let $p^c_\eta(\alpha)$ be its Lebesgue measure

$$p^c_\eta(\alpha) := |E_\eta(\alpha)|. \tag{6}$$

We define

$$f^c(\alpha) := 1 - \underline{\lim}_{\eta\to 0} \frac{\log p^c_\eta(\alpha)}{\log \eta}. \tag{7}$$

This new definition, close to (1), allows us to use all available values $\mu(I)$ for all intervals I of the unit interval $[0, 1)$ and to get rid of the double limit problem. It gives good results only for random measures, where the values $\mu(I)$ for all I of a given size follow a smooth distribution (see subsection **??**). Results are especially bad on such “geometrical” measures as the binomial one: in this case the exact calculations of $p^c_\eta(\alpha)$ is extremely tedious and the experimental results are far from the theoretic spectrum.

3.2 Continuous Large Deviation Spectrum

To avoid these difficulties, let us re-introduce the precision ε, as in the discrete spectrum definition of (4). One way is to compute the following Lebesgue measure

$$|E^\varepsilon_\eta(\alpha)| = |\cup\{I \in [0,1) : |I| = \eta,\ |\alpha_\eta(I) - \alpha| \le \varepsilon\}|. \tag{8}$$

Unfortunately, this quantity is hard to evaluate and prefer to use an integral on horizontal sections as follows

$$p^{c,\varepsilon}_\eta(\alpha) := \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} p^c_\eta(\beta)d\beta = 2\varepsilon(p^c_\eta * \mathbb{1}_{[-\varepsilon,+\varepsilon]})(\alpha). \tag{9}$$

Then the *continuous large deviation multifractal spectrum* stands as

$$f^c_g(\alpha) := 1 - \lim_{\varepsilon\to 0} \underline{\lim}_{\eta\to 0} \frac{\log p^{c,\varepsilon}_\eta(\alpha)}{\log \eta}. \tag{10}$$

Of course, as in 2.2, the indicator function in (9) can be replaced by any admissible kernel K.

4 Discretization and estimation

4.1 Discretization of (6)

In an experimental setting, we consider samples of a signal, that is measures $\mu(I^k_n)$ of dyadic intervals. Let us consider the special case of intervals I of the form

$I_\eta(x) = [x, x+\eta)$ where $x \in [0, 1-\eta)$. Let us define μ_n as the approximation of measure μ at resolution n: it is uniform on each dyadic interval and is equal to $\mu(I_n^k)$. It admits a piece-wise constant density $d\mu_n(x)$ and we can write

$$\mu_n(I_\eta(x)) = \int_x^{x+\eta} d\mu_n(x)dx, \tag{11}$$

where $d\mu_n(x) = 2^n \mu_n(I_n^k)$ for all $x \in I_n^k$. The continuous integral (11) can be discretized and we obtain for all $x \in [0, 1-\eta)$ and for all (n, η) such that $\eta \geq \eta_n = 2^{-n}$

$$\mu_n(I_\eta(x)) = (x_n^{k+1} - x)2^n \mu(I_n^k) + \sum_{m=k+1}^{l-1} \mu(I_n^m) + (x + \eta - x_n^l)2^n \mu(I_n^l), \tag{12}$$

where $k = \lfloor x\, 2^n \rfloor$, $l = \lfloor (x+\eta)\, 2^n \rfloor$ and $x_n^k = k\, 2^{-n}$. Note that when $\eta = \eta_n$, (12) reduces to the following convex linear formula

$$\mu_n([x, x + 2^{-n})) = (k + 1 - x2^n)\mu(I_n^k) + (x2^n - k)\mu(I_n^{k+1}). \tag{13}$$

Moreover, if $x = k\, 2^n$ then $\mu_n(I_{\eta_n}(x)) = \mu(I_n^k)$. The coarse grain Hölder exponent of measure μ_n on the interval $I_\eta(x)$ is $\alpha_{n,\eta}(x) := \log \mu_n(I_\eta(x)) / \log \eta$, which converges to $\alpha_\eta(x) := \log \mu(I_\eta(x)) / \log \eta$ when n tends to ∞. Using the continuity of the coarse grain Hölder exponent $\alpha_\eta(x)$, for all α, $\alpha_{min} \leq \alpha \leq \alpha_{max}$, it is possible to find at least one x such that $\alpha_\eta(x) = \alpha$ as in (6).

4.2 Solutions of $\alpha_\eta(x) = \alpha$ when $\eta = \eta_n$

Each dyadic interval I_n^k contains one and only one solution iff:

- either $\alpha = \alpha_n^k$ and $x = x_n^k$;
- or $\alpha_n^k > \alpha \geq \alpha_n^{k+1}$ or $\alpha_n^k < \alpha \leq \alpha_n^{k+1}$. In this case we must solve the following equation
$$(k + 1 - x2^n)\mu(I_n^k) + (x2^n - k)\mu(I_n^{k+1}) = 2^{-n\alpha},$$
which yields
$$x = x_n^k + 2^{-n} \frac{2^{-n\alpha} - \mu(I_n^k)}{\mu(I_n^{k+1}) - \mu(I_n^k)}, \tag{14}$$
Replacing the measure on the dyadics with its coarse grain Hölder exponent
$$x = x_n^k + 2^{-n} \frac{2^{-n\alpha} - 2^{-n\alpha_n^k}}{2^{-n\alpha_n^{k+1}} - 2^{-n\alpha_n^k}}. \tag{15}$$

Each dyadic interval contains zero or one solution in $[\alpha_{\min}, \alpha_{\max}]$, denoted x_l. We sort these solutions by increasing order $(x_l)_{0 \leq l \leq L-1}$, $L \leq 2^{-n}$ and compute

$$\widehat{p_\eta^c}(\alpha) = \left| \bigcup_{0 \leq l \leq L-1} [x_l, x_l + \eta) \right| = \sum_{0 \leq l \leq L-1} \min\{\eta, x_{l+1} - x_l\}. \tag{16}$$

Finally, the continuous *coarse grain spectrum* can be obtained with

$$\widehat{p_\eta^{c,\varepsilon}}(\alpha) = 2\varepsilon(K_\varepsilon * \widehat{p_\eta^c})(\alpha), \tag{17}$$

and

$$\widehat{f_{g,\eta}^{c,\varepsilon}}(\alpha) = 1 - \frac{\log \widehat{p_\eta^{c,\varepsilon}}(\alpha)}{\log \eta}. \tag{18}$$

Note that the following coarse grain spectrum may also give good results (see **??**)

$$\widehat{f_\eta^c}(\alpha) = 1 - \frac{\log \widehat{p_\eta^c}(\alpha)}{\log \eta}. \tag{19}$$

4.3 Estimation using vertical sections

Let us remark that it is not necessary to evaluate the horizontal sections to estimate $p_\eta^{c,\varepsilon}(\alpha)$ in (10). Indeed, if we define

i) $\alpha_\eta^-(x) := \inf\{\alpha_\eta(I) \ : x \in I,\ |I| = \eta\}$;

ii) $\alpha_\eta^+(x) := \sup\{\alpha_\eta(I) \ : x \in I,\ |I| = \eta\}$;

iii) $g_\eta^\varepsilon(x,\alpha) := \frac{1}{2\varepsilon}\left(\min\{\alpha+\varepsilon, \alpha_\eta^+(x))\} - \max\{\alpha-\epsilon, \alpha_\eta^-(x)\}\right)$,

we get

$$g_\eta^\varepsilon(x,\alpha) = \frac{1}{2\varepsilon}\begin{cases} \alpha_\eta^+(x) - \alpha_\eta^-(x) & \text{if} \quad \alpha-\varepsilon \le \alpha_\eta^-(x) \le \alpha_\eta^+(x) \le \alpha+\varepsilon \\ \alpha_\eta^+(x) - \alpha + \varepsilon & \text{if} \quad \alpha_\eta^-(x) \le \alpha-\varepsilon \le \alpha_\eta^+(x) \le \alpha+\varepsilon \\ \alpha+\varepsilon - \alpha_\eta^-(x) & \text{if} \quad \alpha-\varepsilon \le \alpha_\eta^-(x) \le \alpha+\varepsilon \le \alpha_\eta^+(x) \\ 2\varepsilon & \text{if} \quad \alpha_\eta^-(x) \le \alpha-\varepsilon \le \alpha+\varepsilon \le \alpha_\eta^+(x) \end{cases}.$$

Note that $2\varepsilon g_\eta^\varepsilon(x,\alpha)$ does not differ much from the η-oscillation of $\alpha_\eta(I)$ which is

$$\mathrm{osc}_\eta(\alpha_\eta, I) := \alpha_\eta^+(x) - \alpha_\eta^-(x).$$

The two-variables function g_η^ε can be re-written as

$$g_\eta^\varepsilon(x,\alpha) = \frac{1}{2\varepsilon}|[\alpha-\varepsilon, \alpha+\varepsilon] \cap [\alpha_\eta^-(x), \alpha_\eta^+(x)]|. \tag{20}$$

Let us show that $p_\eta^{c,\varepsilon}(\alpha)$ can be obtained by the integration of $g_\eta^\varepsilon(x,\alpha)$ on the support of measure μ, denoted suppμ. Indeed

$$\begin{aligned} 2\varepsilon \int_{\mathrm{supp}\mu} g_\eta^\varepsilon(x,\alpha)dx &= 2\varepsilon \int_{\mathrm{supp}\mu} \frac{1}{2\varepsilon}\left(\int_{\alpha_\eta^-(x)}^{\alpha_\eta^+(x)} \mathbb{1}_{[-1,1]}\left(\frac{\beta-\alpha}{\varepsilon}\right)d\beta\right)dx \\ &= \int_{\mathrm{supp}\mu}\int_{-\infty}^{+\infty} \mathbb{1}_{[\alpha_\eta^-(x),\alpha_\eta^+(x)]}(\beta)\mathbb{1}_{[\alpha-\varepsilon,\alpha+\varepsilon]}(\beta)d\beta dx \\ &= \int_{-\infty}^{+\infty} \mathbb{1}_{[\alpha-\varepsilon,\alpha+\varepsilon]}(\beta)\left(\int_{\mathrm{supp}\mu} \mathbb{1}_{[\alpha_\eta^-(x),\alpha_\eta^+(x)]}(\beta)dx\right)d\beta \end{aligned}$$

Since

$$\beta \in [\alpha_\eta^-(x), \alpha_\eta^+(x)] \iff \exists I,\, \alpha_\eta(I) = \beta : x \in I$$
$$\iff x \in E_\eta(\beta),$$

then $\mathbb{1}_{[\alpha_\eta^-(x),\alpha_\eta^+(x)]} = \mathbb{1}_{E_\eta(\beta)}(x)$ for every (x, β). Therefore

$$\int_{\text{supp}\mu} \mathbb{1}_{[\alpha_\eta^-(x),\alpha_\eta^+(x)]}(\beta)dx = |E_\eta(\beta)| = p_\eta(\beta).$$

Finally

$$2\varepsilon \int_{\text{supp}\mu} g_\eta^\varepsilon(x,\alpha)dx = \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} p_\eta(\beta)d\beta = p_\eta^{c,\varepsilon}(\alpha). \tag{21}$$

Note that this technique of changing an integral from horizontal sections to vertical sections is used in fractal dimension theory [?,?]. In particular, the quantity $p_\eta^c(\alpha)$ is closely related to the η-variation V_η and to the irregularity function $S_\eta^{(\infty,1)}$.

4.4 Using a kernel

Estimation results can be improved with the help of a kernel K as defined in section 2.2. Let $p > 1$. We generalize (20) as follows

$$g_\eta^{K_\varepsilon}(x,\alpha) := \left[\int_{\alpha_\eta^-(x)}^{\alpha_\eta^+(x)} [K_\varepsilon(\beta-\alpha)]^p \, d\beta\right]^{1/p}. \tag{22}$$

- If $p = 1$

  $$g_\eta^{K_\varepsilon}(x,\alpha) = \frac{1}{2\varepsilon}\int_{\alpha_\eta^-(x)}^{\alpha_\eta^+(x)} K\left(\frac{\beta-\alpha}{\varepsilon}\right) d\beta.$$

  If moreover $K = \mathbb{1}_{[-1,1]}$, we find

  $$g_\eta^{K_\varepsilon}(x,\alpha) = \frac{1}{2\varepsilon}|[\alpha-\varepsilon, \alpha+\varepsilon] \cap [\alpha_\eta^-(x), \alpha_\eta^+(x)]|.$$

- If $p = \infty$

  $$g_\eta^{K_\varepsilon}(x,\alpha) = \sup_{\beta \in [\alpha_\eta^-(x),\alpha_\eta^+(x)]} K_\varepsilon(\beta - \alpha).$$

  As K is a positive and non-decreasing function on the positive real line

  $$g_\eta^{K_\varepsilon}(x,\alpha) = \begin{cases} K_\varepsilon(\alpha_\eta^+(x) - \alpha) & \text{if} \quad \alpha \geq \frac{1}{2}(\alpha_\eta^+(x) - \alpha_\eta^-(x)) \\ \\ K_\varepsilon(\alpha_\eta^-(x) - \alpha) & \text{if} \quad \alpha \leq \frac{1}{2}(\alpha_\eta^+(x) - \alpha_\eta^-(x)) \end{cases}.$$

  Moreover, if $K = \mathbb{1}_{[-1,1]}$, we have

  $$g_\eta^{K_\varepsilon}(x,\alpha) = 1 \iff [\alpha_\eta^-(x), \alpha_\eta^+(x)] \cap [\alpha-\varepsilon, \alpha+\varepsilon] \neq \emptyset$$
  $$\iff x \in E_\eta^\varepsilon(\alpha),$$

  whence $p_\eta^\varepsilon(\alpha) = |E_\eta^\varepsilon(\alpha)|$.

When K is any admissible kernel and $p = 1$, the quantity $p_\eta^{c,\varepsilon}$ can be obtained with horizontal sections as in section 4.3, which gives

$$\begin{aligned} p_\eta^{c,\varepsilon}(\alpha) &= \int_{\text{supp}\mu} \int_{\alpha_\eta^-(x)}^{\alpha_\eta^+(x)} K_\varepsilon(\beta - \alpha) d\beta dx \\ &= 2\varepsilon \int_{-\infty}^{+\infty} p_\eta^c(\beta) K_\varepsilon(\beta - \alpha) d\beta = 2\varepsilon (p_\eta^c * K_\varepsilon)(\alpha). \end{aligned} \quad (23)$$

Note that if $p \neq 1$, integrals can not be inverted.

5 Estimation results

5.1 Deterministic measures

We first tested our estimators on the binomial measure, which is the paradigm of multifractal measures. The binomial measure μ of parameter p_0, $0. < p_0 < 1.$, is a probability measure which is defined conveniently via a recursive construction. Start by splitting $I_0^0 := [0, 1)$ into two subintervals I_1^0 and I_1^1 of equal length and assign the masses p_0 and $p_1 := 1 - p_0$ to them. With the two subintervals, proceed in the same manner: at step two, the four subintervals I_2^0, I_2^1, I_2^2 and I_2^3 have masses p_0^2, $p_0 p_1$, $p_0 p_1$ and p_1^2 and so on. This defines a sequence of measures $(\mu_n)_{n \geq 1}$ on partitions $\mathcal{P}_n$ with $\mu_n(I_n^k) := {p_0}^{n\varphi_{n,0}^k} {p_1}^{n(1-\varphi_{n,0}^k)}$, where $\varphi_{n,0}^k$ is the proportion of 0 in the binary decomposition of k at the n^{th} order, which converges weakly toward μ. Figure **??** displays the four first steps of the recursive construction of the binomial measure (a), its theoretical spectrum and estimation results (b) and related compound multiplicative cascades (lumping (c) and sum (d) of two binomials). The vertical bias in (b, c, d) on the bounds is due to the symmetry of the kernel. Horizontal bias in (c, d) are caused by the finite size of the samples. Indeed, to synthesize a pre-multifractal measure of resolution n corresponding to the lumping or the sum of two measures, we need to synthesize two pre-multifractal measures of resolution $n - 1$. In the case of a lumping of two binomial measures μ, resp. ν, of parameter p_0, resp. q_0, we know [?] that the theoretical minimum, resp. maximum, Hölder exponent are $\alpha^{\min} = \min\{\alpha_\mu^{\min}, \alpha_\nu^{\min}\} = -\log_2 p_0$, resp. $\alpha^{\max} = \max\{\alpha_\mu^{\max}, \alpha_\nu^{\max}\} = -\log_2 p_1$ with $p_0 < q_0$. The estimated minimum coarse grain Hölder exponent is $\hat{\alpha}_n^{\min} = -(1 - 1/n)\log_2 p_0$ and the maximum is $\hat{\alpha}_n^{\max} = -(1 - 1/n)\log_2 p_1$. In the case of the sum, the theoretical minimum, resp. maximum, Hölder exponent is $\alpha^{\min} = \min\{\alpha_\mu^{\min}, \alpha_\nu^{\min}\} = -\log_2 p_0$, resp. $\alpha^{\max} = \min\{\alpha_\mu^{\max}, \alpha_\nu^{\max}\} = -\log_2 q_1$. Here, $\hat{\alpha}_n^{\min} = -1/n \log_2(p_0^n + q_0^n)$ and $\hat{\alpha}_n^{\max} = -1/n \log_2(p_1^n + q_1^n)$.

5.2 Random measure

We also tested our estimator on a random uniform measure W, as defined in [?]. The iterative construction of this random measure W is the same as the one of the binomial measure, except that, at each step of the construction, the parameter p_0 is replaced by a random variable P uniformly chosen on the unit interval $(0, 1)$ and parameter p_1 by $1 - P$. Figure **??** shows one realization of this measure (a) and both theoretical spectrum (its analytic expression can be found in[?]) and estimation result. Note that the estimator based on (7) already yields good results as the distribution of the coarse grain Hölder exponents is smooth enough. In this case, introducing a kernel hardly improves the results.

6 Conclusion

In this paper, we have defined new multifractal spectra which are related the large deviation properties of the analyzed measure but are easier to estimate than f_g. Results obtained on synthetical signals, such as deterministic (subsection **??**) or random cascades (subsection **??**), and on real signals show small dependencies w.r.t. the precision ε and the form of kernel K. Current work is being done to define methods where the precision ε is adapted to the spatial distribution of the Hölder exponents α, and to determine the scale range $\eta_{\min} \leq \eta \leq \eta_{\max}$ on which the multifractal spectrum can be estimated. All described methods have been implemented and tested in FRACLAB, a fractal analysis toolbox for signal processing[a].

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[a]You can download this free of charge toolbox from our W^3 pages at the following URL: `http://www.inria.fr/fractales/`

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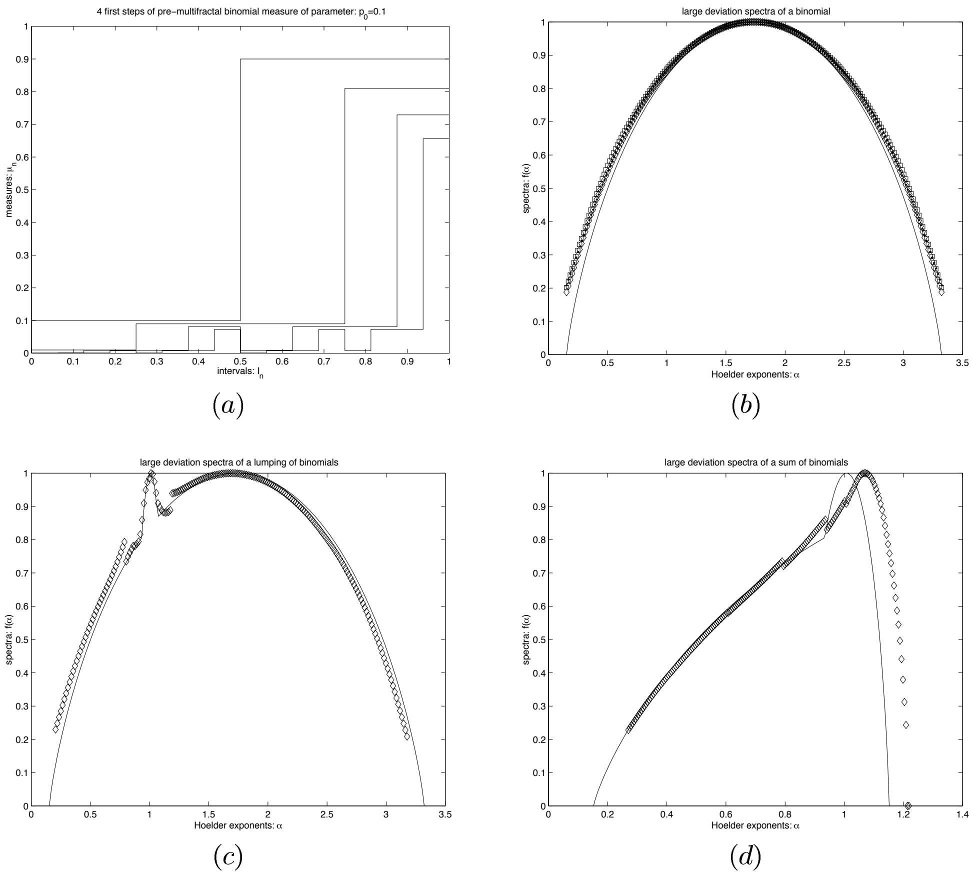
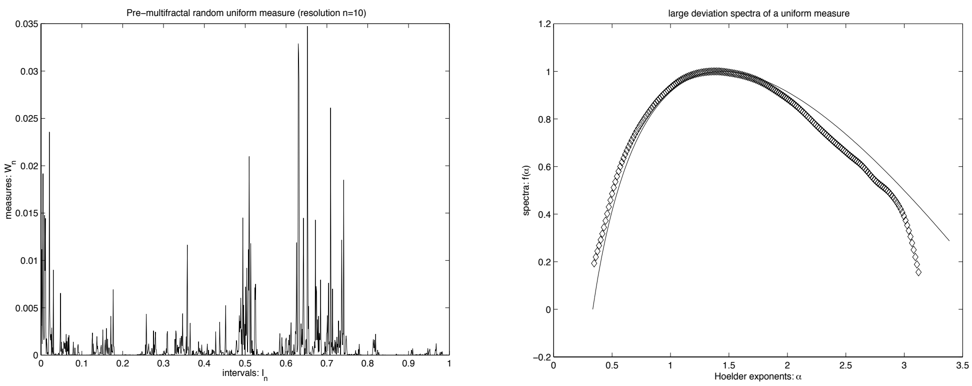


Figure 1: (a): four first steps of the recursive construction of the binomial measure of parameter $p_0 = .1$; (b): theoretical spectrum and estimation results obtained for different decimated resolutions ($n_{\min} = 5$ ($\diamond$) to $n_{\max} = 10$ ($\star$)) with a Gaussian kernel for a binomial measure (parameter $p_0 = .1$ and $n = 10$); (c): theoretical spectrum and estimation result for the lumping of two binomial measures; (d): theoretical spectrum and estimation result for the sum of two binomial measures (first measure μ of parameter $p_0 = .1$, second measure ν of parameter $q_0 = .45$ and compound measure of resolution $n = 16$ in both (c) and (d)).



(a) (b)

Figure 2: (a): one realization of a uniform random measure W (resolution $n = 10$); (b): theoretical spectrum and estimation result for this realization.