Multifractal analysis of the divergence of Fourier series: the extreme cases

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MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF FOURIER SERIES: THE EXTREME CASES

FRÉDÉRIC BAYART, YANICK HEURTEAUX

Abstract. We study the size, in terms of the Hausdorff dimension, of the subsets of $\mathbb{T}$ such that the Fourier series of a generic function in $L^1(\mathbb{T})$, $L^p(\mathbb{T})$ or in $\mathcal{C}(\mathbb{T})$ may behave badly. Genericity is related to the Baire category theorem or to the notion of prevalence. This paper is a continuation of [2].

1. Introduction

This paper, which can be seen as a continuation of [2], deals with the divergence of Fourier series of functions in $L^p(\mathbb{T})$, $p \geq 1$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, or in $\mathcal{C}(\mathbb{T})$, from the multifractal point of view. More precisely, let f be in $L^p(\mathbb{T})$, or in $\mathcal{C}(\mathbb{T})$, and let $(S_n f)_{n\geq 0}$ the sequence of partial sums of its Fourier series. We are interested in the size of the sets of the real numbers x such that $(S_n f(x))_{n\geq 0}$ diverges with a prescribed growth.
We will measure the size of subsets of $\mathbb{T}$ using the Hausdorff dimension. Let us recall the relevant definitions (we refer to [5] and to [8] for more on this subject). If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function satisfying $\phi(0) = 0$ (ϕ is called a *dimension function* or a *gauge function*), the ϕ-Hausdorff outer measure of a set $E \subset \mathbb{R}^d$ is

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon\to 0} \inf_{r\in R_\varepsilon(E)} \sum_{B\in r} \phi(|B|),$$

where $R_\varepsilon(E)$ is the set of (countable) coverings of E with balls B of diameter $|B| \leq \varepsilon$. When $\phi_s(x) = x^s$, we write for short $\mathcal{H}^s$ instead of $\mathcal{H}^{\phi_s}$. The Hausdorff dimension of a set E is defined by

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0; \mathcal{H}^s(E) > 0\} = \inf\{s > 0;\ \mathcal{H}^s(E) = 0\}.$$

The first result studying the Hausdorff dimension of the divergence sets of Fourier series is due to J-M. Aubry [1].

Theorem 1.1. *Let $f \in L^p(\mathbb{T})$, $1 < p < +\infty$. If $\beta \geq 0$, define*

$$\mathcal{E}(\beta, f) = \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} n^{-\beta}|S_n f(x)| > 0\right\}.$$

Then $\dim_{\mathcal{H}}\big(\mathcal{E}(\beta, f)\big) \leq 1 - \beta p$. Conversely, given a set E such that $\dim_{\mathcal{H}}(E) < 1 - \beta p$, there exists a function $f \in L^p(\mathbb{T})$ such that, for any $x \in E$, $\limsup_{n\to+\infty} n^{-\beta}|S_n f(x)| = +\infty$.

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This result motivated us to introduce in [2] the notion of divergence index. For a given function $f \in L^p(\mathbb{T})$ and a given point $x_0 \in \mathbb{T}$, we can define $\beta(x_0)$ as the infimum of the nonnegative real numbers β such that $|S_n f(x_0)| = O(n^\beta)$. The real number $\beta(x_0)$ will be called the *divergence index* of the Fourier series of f at point x_0. It is well-known that, for any function $f \in L^p(\mathbb{T})$ $(1 \leq p < +\infty)$ and any point $x_0 \in \mathbb{T}$, $0 \leq \beta(x_0) \leq 1/p$ (see [11]). Moreover, when $p > 1$, Carleson's theorem implies that $\beta(x_0) = 0$ almost surely. In [2], we gave precise estimates on the size of the level sets of the function β. These are defined as

$$\begin{aligned} E(\beta, f) &= \{x \in \mathbb{T};\ \beta(x) = \beta\} \\ &= \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{\log|S_n f(x)|}{\log n} = \beta \right\}. \end{aligned}$$

Theorem 1.2 ([2]). *Let $1 < p < +\infty$. For quasi-all functions $f \in L^p(\mathbb{T})$, for any $\beta \in [0, 1/p]$, $\dim_{\mathcal{H}}\big(E(\beta, f)\big) = 1 - \beta p$.*

The terminology "quasi-all" used here is relative to the Baire category theorem. It means that this property is true for a residual set of functions in $L^p(\mathbb{T})$.

In the case of continuous functions, the situation breaks down dramatically. If $(D_n)_{n\geq 0}$ denotes the Dirichlet kernel, we can first observe that, when $f \in \mathcal{C}(\mathbb{T})$,

$$\|S_n f\|_\infty \leq \|D_n\|_1 \|f\|_\infty \leq C\|f\|_\infty \log n.$$

This motivated us in [2] to introduce the following level sets:

$$\begin{aligned} \mathcal{F}(\beta, f) &= \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} (\log n)^{-\beta} |S_n f(x)| > 0 \right\} \\ F(\beta, f) &= \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{\log|S_n f(x)|}{\log\log n} = \beta \right\}. \end{aligned}$$

Whereas, on $L^p(\mathbb{T})$, $1 < p < +\infty$, the divergence index takes its biggest value ($\beta(x) = 1/p$) on small sets, this is far from being the case on $\mathcal{C}(\mathbb{T})$, as the following very surprizing result indicates.

Theorem 1.3 ([2]). *For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, for any $\beta \in [0, 1]$, $F(\beta, f)$ is non-empty and has Hausdorff dimension 1.*

However, several questions were left open in [2].

Question 1: what happens on $L^1(\mathbb{T})$? In view of the differences between $L^p(\mathbb{T})$, $p \in (1, +\infty)$, and $\mathcal{C}(\mathbb{T})$, it seems *a priori* not clear what situation should be expected on $L^1(\mathbb{T})$. Moreover, Carleson's theorem is false on $L^1(\mathbb{T})$ and Kolmogorov Theorem ensures that there exist functions in $L^1(\mathbb{T})$ with everywhere divergent Fourier series.
The proof of Theorem 1.2 proceeds in two steps. In a first time, we build a residual set of functions in $L^p(\mathbb{T})$ such that, if f lies in this residual set and if $0 \leq \beta \leq 1/p$, $\dim_{\mathcal{H}}(E(\beta, f)) \geq 1 - \beta p$. In a second time, we use Theorem 1.1 to conclude that necessarily $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$. The first step works as well in $L^1(\mathbb{T})$ and the trouble comes from Aubry's result, which uses the Carleson Hunt maximal inequality. In Section 2, we

succeed to overcome this difficulty by proving a (very weak!) version of Carleson's maximal inequality in $L^1(\mathbb{T})$ which is sufficient to prove the analogue of Theorem 1.1. Thus, we will show that

Theorem 1.4. *For quasi-all functions $f \in L^1(\mathbb{T})$, for any $\beta \in [0,1]$,*

$$\dim_{\mathcal{H}}\big(E(\beta,f)\big) = 1-\beta.$$

Question 2: what about the size of the set of multifractal functions? Theorem 1.2 and Theorem 1.4 say that, in $L^p(\mathbb{T})$ ($p \geq 1$), the set of multifractal functions is big in a topological sense. One can ask if it remains big for other points of view. We deal here with an infinite-dimensional version of the notion of "almost-everywhere". This notion, called *prevalence*, has been introduced by J. Christensen in [4] and has been widely studied since then. In multifractal analysis, some properties which are true on a dense G_δ-set are also prevalent (see for instance [7] or [6]), whereas some are not (see for instance [7] or [10]). This motivated us to examine Theorem 1.2 and Theorem 1.4 under this point of view.

Definition 1.5. Let E be a complete metric vector space. A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that, for any $x \in E$, $\mu(x + A) = 0$. If this property holds, the measure μ is said to be *transverse* to A. A subset of E is called *Haar-null* if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a *prevalent* set.

The following results enumerate important properties of prevalence and show that this notion supplies a natural generalization of "almost every" in infinite-dimensional spaces:

- If A is Haar-null, then $x + A$ is Haar-null for every $x \in E$.
- If $\dim(E) < +\infty$, A is Haar-null if and only if it is negligible with respect to the Lebesgue measure.
- Prevalent sets are dense.
- The intersection of a countable collection of prevalent sets is prevalent.
- If $\dim(E) = +\infty$, compacts subsets of E are Haar-null.

In Section 3, we will prove the following result.

Theorem 1.6. *Let $1 \leq p < +\infty$. The set of functions $f \in L^p(\mathbb{T})$ such that, for any $\beta \in [0, 1/p]$, $\dim_{\mathcal{H}}(E(\beta,f)) = 1 - \beta p$, is prevalent.*

Thus, almost every function in $L^p(\mathbb{T})$ is multifractal with respect to the summation of its Fourier series.

Question 3: can we say more on $\mathcal{C}(\mathbb{T})$? Theorem 1.3 implies that there exists a residual subset $A \subset \mathcal{C}(\mathbb{T})$ such that, if $f \in A$ and if $\beta < 1$, one can find a set $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{(\log n)^\beta} = +\infty \text{ for any } x \in E. \tag{1}$$

On the other hand, we know that, for any fixed $f \in \mathcal{C}(\mathbb{T})$, $\|S_n f\|_\infty$ is negligible compared to $\log n$ and that, conversely, given any sequence $(\delta_n)_{n\geq 2}$ of positive real numbers going to zero, we can find $f \in \mathcal{C}(\mathbb{T})$ such that

$$\limsup_{n\to+\infty} \frac{|S_n f(0)|}{\delta_n \log n} = +\infty.$$

These statements can be found for example in [11]. It seems then natural to ask whereas this property can be ensured in a set with Hausdorff dimension equal to 1 ((1) meaning that this is true when $\delta_n = (\log n)^{\beta-1}$, $0 < \beta < 1$). This is indeed true.

Theorem 1.7. *Let $(\delta_n)_{n\geq 2}$ be a sequence of positive real numbers going to zero. For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, there exists $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that, for any $x \in E$,*

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The same result also holds in a prevalent subset of $\mathcal{C}(\mathbb{T})$.

Theorem 1.8. *Let $(\delta_n)_{n\geq 2}$ be a sequence of positive real numbers going to zero. For almost every function $f \in \mathcal{C}(\mathbb{T})$, there exists $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that, for any $x \in E$,*

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The proof of Theorems 1.7 and 1.8 are proposed in Section 4.

2. Multifractal analysis of the divergence of Fourier series in $L^1(\mathbb{T})$

We first recall some basic facts on Fourier series and Fourier transforms in L^p. Let $\xi \in \mathbb{R}$ and $e_\xi : t \mapsto e^{2\pi i \xi t}$. The Fourier transform of $f \in L^1(\mathbb{R})$ is the continuous function

$$\hat{f} : \xi \mapsto \int_{\mathbb{R}} f(x)\bar{e_\xi}(x)dx.$$

The operator makes also sense in the space $L^p(\mathbb{R})$ when $1 \leq p < +\infty$. In that case, $\hat{f} \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$. In $L^p(\mathbb{R})$ we can define the band-limiting operator S_n by

$$\widehat{S_n f} = \mathbf{1}_{[-n,n]}\hat{f}.$$

It is well known that, on $L^p(\mathbb{R})$, the projections $(S_n)_{n\geq 0}$ are uniformly bounded; this is the Riesz theorem. This is not the case on $L^1(\mathbb{R})$. However, there exists some absolute constant $C > 0$ such that, for any $n \geq 2$ and any $f \in L^1(\mathbb{R})$,

$$\|S_n f\|_1 \leq C \log n \|f\|_1.$$

A function $g \in L^1(\mathbb{T})$ is identified to a 1-periodic function on $\mathbb{R}$. Its Fourier transform is the tempered distribution

$$\hat{g} = \sum_{k\in\mathbb{Z}} \langle g, e_k\rangle \delta_k,$$

where $\langle g, e_k\rangle = \int_{\mathbb{T}} g(t)\bar{e_k}(t)dt$ are the Fourier coefficients of g and δ_k denotes the Dirac mass at point k. If $g \in L^1(\mathbb{T})$, the band limiting operator corresponds to taking the partial sum of the Fourier series,

$$S_n g : t \mapsto \sum_{k=-n}^{n} \langle g, e_k\rangle e_k(t).$$

We can also write $S_n g = D_n * g$ where

$$D_n(t) = \sum_{k=-n}^{n} e_k(t) = \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}$$

is the Dirichlet kernel and the Riesz theorem always occurs in this context.
Let us also recall the definition of $\sigma_n g$, the n-th Féjer sum of g, namely

$$\sigma_n(g) = \frac{1}{n}\big(S_0 g + \cdots + S_{n-1} g\big).$$

We write $\mathcal{E}_n(\mathbb{T}) := S_n(L^1(\mathbb{T}))$ the set of trigonometric polynomials of degree less than n and $\mathcal{E}_n(\mathbb{R}) := S_n(L^1(\mathbb{R}))$. The classical Nikolsky inequality (see for example [9]) says that if $P \in \mathcal{E}_n(\mathbb{T})$ or $P \in \mathcal{E}_n(\mathbb{R})$ and $1 \le p \le q \le \infty$, then

$$\|P\|_q \le n^{\frac{1}{p}-\frac{1}{q}} \|P\|_p.$$

Our first lemma will be helpful to control a function which is locally a Dirichlet kernel.

Lemma 2.1. *There exists a constant $A > 0$ such that, for any $N \ge 2$, for any measurable function $n : \mathbb{T} \to \{1, \dots, N\}$, for any $t \in \mathbb{T}$, then*

$$\int_{\mathbb{T}} |D_{n(x)}(x-t)| dx \le A \log N.$$

Proof. It is obvious from the above expression of D_n that, if $k \le N$ and if $u \in [-1/2, 1/2]$,

$$|D_k(u)| \le \left\{ \begin{array}{l} CN \\ \frac{C}{|u|} \end{array} \right.$$

for some absolute constant $C > 0$. We then split the integral into two parts:

$$\int_{|x-t| \le 1/N} |D_{n(x)}(x-t)| dx \le 2CN \frac{1}{N}$$

and

$$\int_{1/N < |x-t| \le 1/2} |D_{n(x)}(x-t)| dx \le C \int_{1/N < |x-t| \le 1/2} \frac{dx}{|x-t|} \le 2C \log N.$$

□

Writing $S_{n(x)} f(x) = (f \star D_{n(x)})(x)$ and using Fubini's theorem, it is straightforward to deduce the following inequality on partial sums of Fourier series of L^1-functions.

Lemma 2.2. *There exists a constant $A > 0$ such that, for any $N \ge 2$, for any measurable function $n : \mathbb{T} \to \{1, \dots, N\}$, for any $f \in L^1(\mathbb{T})$, then*

$$\int_{\mathbb{T}} |S_{n(x)} f(x)| dx \le A \log N \, \|f\|_1.$$

We are now ready to prove the following weak version of the maximal inequality of Carleson and Hunt, on $L^1(\mathbb{T})$.

Corollary 2.3. *Let $\alpha > 0$. There exists $C := C_\alpha > 0$ such that, for any $f \in L^1(\mathbb{T})$,*

$$\int_{\mathbb{T}} \sup_{n \ge 2} \frac{|S_n f(x)|}{(\log n)^{1+\alpha}} dx \le C \, \|f\|_1.$$

Proof. Using the monotone convergence theorem, we first observe that it is sufficient to prove that, for any $N\geq 2$,

$$\int_{\mathbb{T}}\sup_{2\leq n\leq N}\frac{|S_nf(x)|}{(\log n)^{1+\alpha}}dx\leq C\,\|f\|_1 \tag{2}$$

where, of course, C does not depend on N. Now, we take a measurable function $n:\mathbb{T}\to\mathbb{N}\backslash\{0,1\}$ not necessarily bounded, and observe that (2) will be proved if we are able to show that

$$\int_{\mathbb{T}}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx\leq C\,\|f\|_1$$

for some constant C independent of the function n. If $k\geq 0$, let

$$A_k=\{x\in\mathbb{T};\ 2^{2^k}\leq n(x)<2^{2^{k+1}}\}.$$

Lemma 2.2 ensures that

$$\begin{aligned}\int_{\mathbb{T}}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx &= \sum_{k\geq 0}\int_{A_k}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx\\ &\leq \sum_{k\geq 0}\frac{1}{(2^k\log 2)^{1+\alpha}}\int_{A_k}|S_{n(x)}f(x)|dx\\ &\leq \sum_{k\geq 0}C\frac{2^{k+1}\log 2}{2^{k(1+\alpha)}(\log 2)^{1+\alpha}}\,\|f\|_1\\ &= C_\alpha\,\|f\|_1.\end{aligned}$$

□

The following lemma is inspired by Aubry's paper. It means that, as soon as a trigonometric polynomial is large at some point $a\in\mathbb{T}$, it is also large in small intervals around a, with a rather good control of the L^p-norm.

Lemma 2.4. *Let $p\geq 1$ and $\varepsilon>0$. There exists $\delta>0$ such that, if n is large enough, if $P\in\mathcal{E}_n(\mathbb{T})$ and if $a\in\mathbb{T}$ is such that $|P(a)|\geq\|P\|_p$, then, for any interval I with center a and with length $|I|\leq\frac{1}{n}$,*

$$\|P\|_{L^p(I)}\geq\delta|P(a)|\times|I|^{1/p}\times\begin{cases}\dfrac{1}{(\log n)^{(1+\varepsilon)/p}} & \textit{provided } p>1\\[2mm] \dfrac{1}{(\log n)^{1+\varepsilon}\log(1/|I|)} & \textit{provided } p=1.\end{cases}$$

Remarks:
- Such a point a does exist because P is continuous.
- In fact, we will only need the lemma in the case $p=1$, but we give the general case for completeness.

Proof of Lemma 2.4. Without loss of generality, we may assume that $a=0$. The idea is to localize P around 0, and to use Nikolsky inequality to estimate the L^p-norm knowing the L^∞-norm. Let $\gamma\in(0,1)$ such that $\gamma(1+\varepsilon)>1$. We introduce a function w with support in $[-1,1]$ satisfying $0\leq w\leq 1$, $w(0)=1$ and for which there exist two strictly positive constants D and E such that

$$\forall\xi\in\mathbb{R},\quad |\hat{w}(\xi)|\leq De^{-E|\xi|^\gamma}.$$

It is a classical result in Fourier analysis that such a function does exist (see e.g. [1, Lemma 6]). We then set $w_I(x)=w(x/|I|)$. We decompose Pw_I as f_1+f_2 with $f_1=S_NPw_I$ and $N=[|I|^{-1}(\log n)^{1+\varepsilon}]$, the integer part of $|I|^{-1}(\log n)^{1+\varepsilon}$. On the one hand, if $p>1$ we get

$$\begin{aligned}\|f_1\|_\infty &\leq N^{1/p}\|f_1\|_p \text{ (Nikolsky inequality)}\\ &\leq C_p|I|^{-1/p}(\log n)^{(1+\varepsilon)/p}\|Pw_I\|_p \text{ (Riesz theorem)}\\ &\leq C_p|I|^{-1/p}(\log n)^{(1+\varepsilon)/p}\|P\|_{L^p(I)}.\end{aligned}$$

When $p=1$, we have to add the norm of the Riesz projection, and we get

$$\|f_1\|_\infty\leq C_1|I|^{-1}(\log n)^{1+\varepsilon}\log(1/|I|)\|P\|_{L^1(I)}.$$

On the other hand, we may write

$$\begin{aligned}\hat{f_2}(\xi) &= \mathbf{1}_{\{|\xi|>N\}}(\xi)(\hat{P}\star\hat{w}_I)(\xi)\\ &= \sum_{j=-n}^{n}\mathbf{1}_{\{|\xi|>N\}}(\xi)\hat{P}(j)\hat{w}_I(\xi-j).\end{aligned}$$

Now, if n is large enough and $j\leq n$, we have

$$\begin{aligned}\int_{|\xi|>N}|\hat{w}_I(\xi-j)|d\xi &\leq \int_{|\xi|>\frac12|I|^{-1}(\log n)^{1+\varepsilon}}|\hat{w}_I(\xi)|d\xi\\ &= \int_{|\xi|>\frac12(\log n)^{1+\varepsilon}}|\hat{w}(\xi)|d\xi.\end{aligned}$$

Observe that

$$\int_A^{+\infty}e^{-E\xi^\gamma}d\xi=\frac{1}{\gamma}\int_{A^\gamma}^{+\infty}e^{-Et}t^{1/\gamma-1}dt\leq Ce^{-(E/2)A^\gamma}.$$

It follows easily that

$$\int_{|\xi|>N}|\hat{w}_I(\xi-j)|d\xi\leq Cn^{-2}$$

provided n is large enough. This implies

$$\begin{aligned}\|f_2\|_\infty\leq\|\hat{f_2}\|_1 &\leq Cn^{-2}\sum_{j=-n}^{n}|\hat{P}(j)|\\ &\leq Cn^{-2}(2n+1)\|P\|_1\\ &\leq Cn^{-2}(2n+1)\|P\|_p\\ &\leq \frac12\|P\|_p\end{aligned}$$

provided n is large enough. If we recall that $|P(0)|\geq\|P\|_p$, we get

$$\|f_1\|_\infty\geq|P(0)|-\|f_2\|_\infty\geq\frac12|P(0)|$$

and the result follows from the above estimates of $\|f_1\|_\infty$. $\square$

We can now conclude by proving the following proposition (Proposition 2.5) and its corollary on the Hausdorff dimension of $E(\beta,f)$ (Corollary 2.6). Recall that it is all that we need to obtain Theorem 1.4 since the construction done in [2] is always true when $p=1$ and shows that there exists a residual set of functions $f\in L^1(\mathbb{T})$ with $\dim_{\mathcal{H}}(E(\beta,f))\geq 1-\beta$ for any $\beta\in[0,1]$.

Proposition 2.5. *Let $f\in L^1(\mathbb{T})$ and $\tau:(0,+\infty)\to(0,+\infty)$ be an increasing function. Define*

$$E(\tau,f):=\left\{x\in\mathbb{T};\ \limsup_{n\to+\infty}\frac{|S_nf(x)|}{\tau(n)}=+\infty\right\}.$$

If $\nu>3$ and if ϕ is a dimension function satisfying $c_1 s\leq\phi(s)\leq c_2\frac{s\tau(s^{-1})}{\log(s^{-1})^{\nu}}$, then

$$\mathcal{H}^{\phi}(E(\tau,f))=0.$$

Proof. Let $M>0$ and $\varepsilon=\nu-3$. Define

$$E_M(\tau,f)=\left\{x\in\mathbb{T};\ \limsup_{n\to+\infty}\frac{|S_nf(x)|}{\tau(n)}>M\right\}.$$

If $x\in E_M(\tau,f)$, one can find n_x as large as we want such that $|S_{n_x}f(x)|\geq M\tau(n_x)$. Set $I_x=\left[x-\frac{1}{2n_x},x+\frac{1}{2n_x}\right]$ and observe that $\|S_{n_x}f\|_1\leq C(\log n_x)$. The hypothesis on the function τ implies that, if n_x is large enough, $\|S_{n_x}f\|_1\leq|S_{n_x}f(x)|$. We can then apply Lemma 2.4 and we get

$$\|S_{n_x}f\|_{L^1(I_x)}\geq\delta\frac{M\tau(n_x)}{n_x(\log n_x)^{2+\varepsilon/2}}.$$

$(I_x)_{x\in E_M(\tau,f)}$ is a covering of $E_M(\tau,f)$. We can extract a Vitali's covering, namely a countable family of disjoint intervals $(I_i)_{i\in\mathbb{N}}$, of length $1/n_i$, such that $E_M(\tau,f)\subset\bigcup_{i\in\mathbb{N}}5B_i$. Then, Corollary 2.3 implies

$$\begin{aligned}C\|f\|_1&\geq\int_{\mathbb{T}}\sup_{n\geq 2}\frac{|S_nf(x)|}{(\log n)^{1+\varepsilon/2}}dx\\&\geq\sum_i\int_{I_i}\frac{|S_{n_i}f(x)|}{(\log n_i)^{1+\varepsilon/2}}dx\\&\geq\delta M\sum_i\frac{|I_i|\tau(1/|I_i|)}{\big(\log(1/|I_i|)\big)^{3+\varepsilon}}.\end{aligned}$$

This yields $\sum_i\phi(5|I_i|)\leq\frac{C\|f\|_1}{\delta M}$ (we recall that τ is increasing), with C another absolute constant and $M>0$ as large as we want. Hence, $\mathcal{H}^{\phi}(E_M(\tau,f))\leq\frac{C\|f\|_1}{\delta M}$ (the length of the intervals of the covering can be arbitrarily small). This in turn implies $\mathcal{H}^{\phi}(E(\tau,f))=0$, since $E(\tau,f)=\bigcap_{M>0}E_M(\tau,f)$. □

By applying the previous proposition to $\tau(s)=s^{\beta}$ and $\phi(s)=s^{1-\beta}/\log(s^{-1})^4$, we get:

Corollary 2.6. *For any $f\in L^1(\mathbb{T})$ and any $\beta\in[0,1]$, $\dim_{\mathcal{H}}(E(\beta,f))\leq 1-\beta$.*

3. Prevalence of multifractal behaviour

3.1. **Strategy.** In all this part, p is a fixed real number such that $1 \leq p < +\infty$. To prove that a set $A \subset E$ is Haar-null, the Lebesgue measure on the unit ball of a finite-dimensional subspace V can often play the role of the transverse measure. Precisely, if there exists a finite-dimensional subspace V of E such that, for any $x \in E$, $V \cap (x + A)$ has full Lebesgue-measure, then A is prevalent. Such a finite-dimensional subspace V is called a *probe* for A. Of course, it is the same to prove that for any $x \in E$, $(x + V) \cap A$ has full Lebesgue-measure.

We shall use this property to prove prevalence. More precisely, we shall first prove that, for a fixed $\beta \in [0, 1/p]$, the set of functions f in $L^p(\mathbb{T})$ satisfying $\dim_{\mathcal{H}} \big(E(\beta, f)\big) = 1 - \beta p$ is prevalent. Then we will conclude because a countable intersection of prevalent sets is prevalent.

3.2. **The construction of saturating functions with disjoint spectra.** In this subsection, $\alpha > 1$ is fixed. For $j \geq 1$, we define $J = [j/\alpha] + 1$, which is smaller than $j - 2$ if j is large enough, say $j \geq j_\alpha$. For $0 \leq K \leq 2^J - 1$, we define the dyadic intervals

$$I_{K,j} := \left[\frac{K}{2^J} - \frac{1}{2^j}; \frac{K}{2^J} + \frac{1}{2^j}\right].$$

We also define

$$\mathbf{I}_j := \bigcup_{K=0}^{2^J-1} I_{K,j} \quad \text{and} \quad \mathbf{I}'_j := \bigcup_{K=0}^{2^J-1} 2I_{K,j}.$$

The condition $j \geq j_\alpha$ ensures that the $2I_{K,j}$ do not overlap. We finally introduce D_α the set of real numbers in $[0, 1]$ which are α-approximable by dyadics. Namely, $x \in [0, 1]$ belongs to D_α if there exist two sequences of integers $(k_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ such that

$$\left| x - \frac{k_n}{2^{j_n}} \right| \leq \frac{1}{2^{\alpha j_n}}.$$

It is easy to check that D_α is contained in $\limsup_{j\to+\infty} \mathbf{I}_j$. Indeed, let $x \in D_\alpha$. One may find J as large as we want and K such that $|x - K/2^J| \leq 1/2^{\alpha J}$. Let j be an integer such that $J - 1 = [j/\alpha]$ (such an integer exists because $\alpha \geq 1$). We get

$$\left| x - \frac{K}{2^J} \right| \leq \frac{1}{2^j}.$$

Finally, $x \in \mathbf{I}_j$. Furthermore, it is well-known that $\dim_{\mathcal{H}}(D_\alpha) = 1/\alpha$ and even that $\mathcal{H}^{1/\alpha}(D_\alpha) = +\infty$ (see for instance [3] and the mass transference principle). It follows that

$$\dim_{\mathcal{H}} \left(\limsup_{j\to+\infty} \mathbf{I}_j \right) \geq \frac{1}{\alpha}.$$

We are going to build finite families of functions which behave badly on each $\mathbf{I}_j$, and which have disjoint spectra. The starting point is a modification of the basic construction of [2].

Lemma 3.1. *Let $j \geq j_\alpha$ and $J = [j/\alpha] + 1$. There exists a trigonometric polynomial P_j with spectrum contained in $(0, 2^{j+1} - 1]$ such that*

- $\|P_j\|_p \leq 1$

- *$|P_j(x)| \geq C2^{-(J-j)/p}$ for any $x \in \mathbf{I}_j$*

where the constant C is independant of j.

Proof. Let χ_j be a continuous piecewise linear function equal to 1 on $\mathbf{I}_j$, equal to 0 outside $\mathbf{I}'_j$ and satisfying $0 \leq \chi_j \leq 1$ and $\|\chi'_j\|_\infty \leq 2^j$. P_j is defined by

$$P_j := 2^{-(J-j+2)/p} e_{2^j} \sigma_{2^j} \chi_j.$$

The L^p-norm of P_j is clearly less than or equal to 1 (observe that the measure of $\mathbf{I}'_J$ is 2^{J-j+2}). Applying Lemma 1.7 of [2] to $1-\chi_j$, we find that $\sigma_{2^j}\chi_j(x) \geq 1/4$ for any $x \in \mathbf{I}_j$. This gives the second assertion of the lemma. □

We now collapse these polynomials to get as many saturating functions as necessary, with disjoint spectra.

Lemma 3.2. *Let $s \geq 1$. There exist functions $g_1, \dots, g_s$ in $L^p(\mathbb{T})$ and sequences of integers $(n_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$, $(m_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$ satisfying*

- *$1 \leq m_{j,r} < n_{j,r} \leq C2^j$ for any j and any r;*
- *for any $j \geq j_\alpha$, any $x \in \mathbf{I}_j$, any $r \in \{1, \dots, s\}$,*

$$\left|S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x)\right| \geq \frac{C}{j^2} 2^{(j-J)/p}$$

- *for any $r \in \{1, \dots, s\}$, the spectrum of g_r is included in $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}] =: G_r$*
- *if $r_1 \neq r_2$, $G_{r_1} \cap G_{r_2} = \emptyset$.*

Proof. For $r \in \{1, \dots, s\}$, we set

$$g_r := \sum_{j \geq j_\alpha} \frac{1}{j^2} e_{(s+r)2^{j+1}} P_j.$$

Define

$$\begin{aligned} m_{j,r} &:= (s+r)2^{j+1} \\ n_{j,r} &:= (s+r)2^{j+1} + (2^{j+1}-1) \end{aligned}$$

so that each g_r belongs to L^p with spectrum included in $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}]$. Moreover, the intervals $(m_{j,r}, n_{j,r}]$ are disjoint, so that

$$\left|S_{n_{j,r}} g_r - S_{m_{j,r}} g_r\right| = \frac{1}{j^2} |P_j|.$$

Let us also remark that, for any $j \geq j_\alpha$ and any $r < s$, $n_{j,r} < m_{j,r+1}$ and $n_{j,s} < m_{j+1,1}$ so that the spectra $G_1, \cdots, G_s$ are disjoint. This ends up the proof. □

It is easy to show that, if $x \in \limsup_j \mathbf{I}_j$, $r \in \{1, \dots, s\}$ and $\beta < \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$, then

$$\limsup_{n \to +\infty} \frac{|S_n g_r(x)|}{n^\beta} = +\infty.$$

In some sense, the functions g_r have the worst possible behaviour on $\mathbf{I}_j$ if we keep in mind that they have to belong to $L^p(\mathbb{T})$. We now show that this property remains true almost everywhere (in the sense of the lebesgue measure) on any affine subspace $f + \operatorname{span}(g_1, \dots, g_s)$ provided s is large enough. This is the main step towards the proof of Theorem 1.6.

3.3. **Prevalence of divergence for a fixed divergence index.** We keep the notations of the previous subsection.

Proposition 3.3. *Let $0 < \beta < \frac{1}{p}\left(1-\frac{1}{\alpha}\right)$. There exists $s \geq 1$ such that, for every $f \in L^p(\mathbb{T})$, for almost every $c=(c_1,\dots,c_s)$ in $\mathbb{R}^s$, the function $g = f + c_1g_1 + \cdots + c_sg_s$ satisfies for every $x \in D_\alpha$*

$$\limsup_{n\to+\infty} \frac{|S_ng(x)|}{n^\beta} = +\infty.$$

Proof. We set $\varepsilon = \frac{1}{p}\left(1-\frac{1}{\alpha}\right) - \beta$. Let $s > 4/\varepsilon$ and let f be an arbitrary function in $L^p(\mathbb{T})$. For such a value of s, we will prove the conclusion of the proposition for every $x \in \limsup_j \mathbf{I}_j$ (recall that $D_\alpha \subset \limsup_j \mathbf{I}_j$).

Let $M > 0$ and let us introduce

$$S_M := \left\{ g \in L^p(\mathbb{T});\ \exists x \in \limsup_{j\to+\infty} \mathbf{I}_j \text{ s.t. } \forall n \geq 1,\ |S_ng(x)| \leq Mn^\beta \right\}.$$

It is enough to show that for every $R > 0$, the set of $c \in \mathbb{R}^s$ satisfying $\|c\|_\infty \leq R$ and such that $f + c_1g_1 + \cdots + c_sg_s$ belongs to S_M has Lebesgue measure 0. In the sequel, we will fix such values of M and R.

If $j \geq 1$, we split each interval $I_{K,j}$ into 2^j subintervals. Each of them has size 2^{-2j+1}, and we get 2^{J+j} intervals $O_{l,j}$ with $\bigcup_{l=1}^{2^{J+j}} O_{l,j} = \mathbf{I}_j$. For $j \geq 1$, $l \in \{1,\dots,2^{J+j}\}$, we set

$$S_M^{(l,j)} := \left\{ g \in L^p(\mathbb{T});\ \exists x \in O_{l,j} \text{ s.t. } \forall n \geq 1,\ |S_ng(x)| \leq Mn^\beta \right\}.$$

Clearly,

$$S_M \subset \limsup_{j\to+\infty} \bigcup_{l=1}^{2^{J+j}} S_M^{(l,j)}$$

and we shall first control the size of the $c \in \mathbb{R}^s$ with $\|c\|_\infty \leq R$ such that

$$f + c_1g_1 + \cdots + c_sg_s \in S_M^{(l,j)}.$$

We denote by λ_s the Lebesgue measure on $\mathbb{R}^s$ and we fix $j \geq j_\alpha$, l in $\{1,\dots,2^{J+j}\}$ and c, c^0 in $\mathbb{R}^s$ such that $\|c\|_\infty \leq R$, $\|c^0\|_\infty \leq R$ and

$$\begin{cases} f + c_1g_1 + \cdots + c_sg_s & \in & S_M^{(l,j)} \\ f + c_1^0g_1 + \cdots + c_s^0g_s & \in & S_M^{(l,j)}. \end{cases}$$

Let $r \in \{1,\dots,s\}$ and let us apply the definition of $S_M^{(l,j)}$ with $n = n_{j,r}$ and $n = m_{j,r}$. The spectra $(G_l)_{l\neq r}$ being disjoint from G_r, we can find $x \in O_{l,j}$ such that

$$\left|S_{n_{j,r}}f(x) - S_{m_{j,r}}f(x) + c_r\left(S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)\right)\right| \leq Mn_{j,r}^\beta + Mm_{j,r}^\beta \leq 2CM2^{\beta j}.$$

In the same way, we can find $y \in O_{l,j}$ such that

$$\left|S_{n_{j,r}}f(y) - S_{m_{j,r}}f(y) + c_r^0\left(S_{n_{j,r}}g_r(y) - S_{m_{j,r}}g_r(y)\right)\right| \leq 2CM2^{\beta j}.$$

Using the triangle inequality, we get

$$\begin{aligned}\left|c_r\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-c_r^0\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right| &\leq \\ 4CM2^{\beta j}+|S_{n_{j,r}}f(x)-S_{n_{j,r}}f(y)|+|S_{m_{j,r}}f(x)-S_{m_{j,r}}f(y)|.\end{aligned} \tag{3}$$

Now, by combining the norm of the Riesz projection, Nikolsky's inequality and Bernstein's inequality, we know that

$$\|(S_nf)'\|_\infty\leq C(\log n)n^{1+1/p}\|f\|_p$$

(the factor $\log n$ disappears when $p>1$). This yields

$$\begin{aligned}|S_{n_{j,r}}f(x)-S_{n_{j,r}}f(y)| &\leq C\log(n_{j,r})n_{j,r}^{1+1/p}|x-y|\|f\|_p\\ &\leq Cj2^{j(1+1/p)}2^{-2j+1}\|f\|_p\\ &\ll 2^{\beta j}.\end{aligned}$$

The same is true for $|S_{m_{j,r}}f(x)-S_{m_{j,r}}f(y)|$ and we get

$$\left|c_r\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-c_r^0\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right|\leq\kappa 2^{\beta j} \tag{4}$$

for some constant κ depending on M and $\|f\|_p$ but not on j.
In the same way,

$$\|(S_ng_r)'\|_\infty\leq C(\log n)n^{1+1/p}\|g_r\|_p\leq C(\log n)n^{1+1/p}.$$

It follows that

$$\begin{aligned}\left|c_r^0\left(\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right)\right| &\leq CRj2^{j(1+1/p)}2^{-2j+1}\\ &\ll 2^{\beta j}.\end{aligned}$$

Combining with (4) we obtain a new constant κ depending on M, $\|f\|_p$ and R but not on j such that

$$\left|(c_r-c_r^0)\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)\right|\leq\kappa 2^{\beta j}. \tag{5}$$

Dividing (5) by $|S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)|$ (which is not equal to zero), we find

$$\begin{aligned}|c_r-c_r^0| &\leq \kappa 2^{\beta j}|S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)|^{-1}\\ &\leq \frac{\kappa}{C}2^{\beta j}j^2 2^{-(j-J)/p}\\ &\leq \frac{\kappa 2^{1/p}}{C}j^2 2^{-\varepsilon j}\\ &\leq 2^{-\varepsilon j/2}\end{aligned}$$

provided j is large enough. Thus, the set of $c\in\mathbb{R}^s$ with $\|c\|_\infty\leq R$ and such that $f+c_1g_1+\dots+c_sg_s\in S_M^{(l,j)}$ is contained in a ball (for the l^∞-norm) of radius $2^{-\varepsilon j/2}$. Taking the s-dimensional Lebesgue measure, this yields

$$\lambda_s\left(\left\{c\in\mathbb{R}^s;\ \|c\|_\infty\leq R \text{ and } f+c_1g_1+\dots+c_sg_s\in S_M^{(l,j)}\right\}\right)\leq 2^s2^{-\varepsilon sj/2}.$$

This in turn gives

$$\lambda_s\left(\left\{c\in\mathbb{R}^s;\ \|c\|_\infty\le R \text{ and } f+c_1g_1+\cdots+c_sg_s\in\bigcup_{l=1}^{2^{J+j}}S_M^{(l,j)}\right\}\right)\le 2^s2^{2j-\varepsilon sj/2}.$$

Thus, since $\varepsilon s/2>2$, this last quantity is the general term of a convergent series. Remember that

$$S_M\subset\limsup_{j\to+\infty}\bigcup_{l=1}^{2^{J+j}}S_M^{(l,j)}.$$

The conclusion of Proposition 3.3 follows from Borel Cantelli's lemma. □

Corollary 3.4. *Let $\alpha>1$. For almost every function f in $L^p(\mathbb{T})$, for every $x\in D_\alpha$,*

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\ge\frac1p\left(1-\frac1\alpha\right).$$

Proof. This follows immediately from Proposition 3.3, taking a sequence (β_n) increasing to $\frac1p\left(1-\frac1\alpha\right)$ and using the fact that a countable intersection of prevalent sets remains prevalent. □

3.4. **The general case.** We are now able to complete the proof of Theorem 1.6, that is to prove that almost every function $f\in L^p(\mathbb{T})$ in the sense of prevalence has a multifractal behaviour with respect to the summation of its Fourier series. Indeed, let $(\alpha_k)_{k\ge0}$ be a dense sequence in $(1,+\infty)$. By Corollary 3.4, for almost every function $f\in L^p(\mathbb{T})$, for every $k\in\mathbb{N}$ and every $x\in D_{\alpha_k}$,

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\ge\frac1p\left(1-\frac1{\alpha_k}\right).$$

Now, let $\alpha>1$ and consider a subsequence $(\alpha_{\phi(k)})_{k\ge0}$ which increases to α. Then $D_\alpha\subset\bigcap_{k\ge0}D_{\alpha_{\phi(k)}}$ and for any $x\in D_\alpha$,

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\ge\frac1p\left(1-\frac1\alpha\right).$$

The conclusion follows now exactly the argument of [2]. For the sake of completeness, we give a complete account. Define

$$\begin{aligned}D_\alpha^1&=\left\{x\in D_\alpha;\ \limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}=\frac1p\left(1-\frac1\alpha\right)\right\}\\ D_\alpha^2&=\left\{x\in D_\alpha;\ \limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}>\frac1p\left(1-\frac1\alpha\right)\right\},\end{aligned}$$

so that $\mathcal{H}^{1/\alpha}(D_\alpha^1\cup D_\alpha^2)=\mathcal{H}^{1/\alpha}(D_\alpha)=+\infty$. It suffices to prove that $\mathcal{H}^{1/\alpha}(D_\alpha^2)=0$. Let $(\beta_n)_{n\ge0}$ be a sequence of real numbers such that

$$\beta_n>\frac1p\left(1-\frac1\alpha\right)\quad\text{and}\quad\lim_{n\to+\infty}\beta_n=\frac1p\left(1-\frac1\alpha\right).$$

Let us observe that

$$D_\alpha^2\subset\bigcup_{n\ge0}\mathcal{E}(\beta_n,f).$$

Moreover, Theorem 1.1 for $p>1$ and Corollary 2.6 for $p=1$ imply that $\mathcal{H}^{1/\alpha}(\mathcal{E}(\beta_n,f))=0$ for all n. Hence, $\mathcal{H}^{1/\alpha}(D_\alpha^2)=0$ and $\mathcal{H}^{1/\alpha}(D_\alpha^1)=+\infty$, which proves that

$$\dim_{\mathcal{H}}\left(E\left(\frac{1}{p}\left(1-\frac{1}{\alpha}\right),f\right)\right)\geq\frac{1}{\alpha}.$$

By Theorem 1.1 and Corollary 2.6 again, this inequality is necessarily an equality. Finally, such a function f satisfies the conclusion of Theorem 1.6, setting $1-\beta p=1/\alpha$.

4. Rapid divergence on big sets for Fourier series of continuous functions

This section is devoted to the proof of Theorem 1.7 and Theorem 1.8. We need to construct functions in $\mathcal{C}(\mathbb{T})$ for which the Fourier series behave badly on a set with Hausdorff dimension 1. We will construct these functions by blocks. For $k\geq 1$ and $\omega>1$, we set

$$J_k^\omega:=\bigcup_{j=0}^{k-1}\left[\frac{j}{k}-\frac{1}{2\omega k},\frac{j}{k}+\frac{1}{2\omega k}\right]$$

which will be seen as a subset of $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The construction makes use of holomorphic functions, so that we will also see $\mathbb{T}$ as the boundary of the unit disk $\mathbb{D}$ and J_k^ω as a part of $\partial\mathbb{D}$.

Lemma 4.1. *There exist three absolute constants $C_1,C_2,C_3>0$ such that, for any $k\geq 3$, for any $\omega\geq\log k$, one can find a function f which is holomorphic in a neighbourhood of $\mathbb{D}$ and which satisfies :*

$$\begin{array}{lrrcl}
(6) & \forall z\in\overline{\mathbb{D}}, & \Re e f(z) & \geq & \dfrac{C_1}{\omega k}\\
(7) & \forall z\in J_k^\omega, & |f(z)| & \geq & C_2\omega\\
(8) & \forall z\in\mathbb{T}, & |f(z)| & \leq & C_3\omega\\
(9) & \forall z\in\mathbb{T}, & \left|\dfrac{f'(z)}{f(z)}\right| & \leq & \omega k.
\end{array}$$

Proof. We set:

$$\begin{aligned}
\varepsilon &= \frac{1}{\omega k}\\
z_j &= e^{\frac{2\pi ij}{k}},\ j=0,\dots,k-1\\
f(z) &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}
\end{aligned}$$

and we claim that f is the function we are looking for. Indeed, for any $z\in\overline{\mathbb{D}}$ and any $j\in\{0,\dots,k-1\}$,

$$\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right)=\frac{1+\varepsilon}{|1+\varepsilon-\overline{z_j}z|^2}\Re e\big(1+\varepsilon-z_j\overline{z}\big)\geq\frac{1+\varepsilon}{(2+\varepsilon)^2}\times\varepsilon\geq C_1\varepsilon, \tag{10}$$

which proves (6). To prove (7), we may assume that $z=e^{2\pi i\theta}$ with $\theta\in\left[\frac{-\varepsilon}{2};\frac{\varepsilon}{2}\right]$. Then

$$\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_0}z}\right)=\frac{1+\varepsilon}{|1+\varepsilon-z|^2}\Re e\big(1+\varepsilon-z\big)\geq\frac{C_2}{\varepsilon}.$$

Moreover, (10) says that for any j, $\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right)\geq 0$. It follows that

$$\Re e f(z)\geq \frac{C_2}{k\varepsilon}=C_2\omega.$$

Conversely, we want to control $\sup_{z\in\mathbb{T}}|f(z)|$. Pick any $z=e^{2\pi i\theta}\in\mathbb{T}$. By symmetry, we may and shall assume that $|\theta|\leq\frac{1}{2k}$. Then we get

$$\left|\frac{1+\varepsilon}{1+\varepsilon-\overline{z_0}z}\right|\leq\frac{C}{\varepsilon}$$

for some constant $C>0$. Now, for any $j\in\{1,\dots,k/4\}$, we can write

$$\begin{aligned}|1+\varepsilon-\overline{z_j}z| &\geq |\Im m(\overline{z_j}z)|\\ &\geq \sin\left(\frac{2\pi j}{k}-2\pi\theta\right)\\ &\geq \frac{2}{\pi}\times 2\pi\left(\frac{j}{k}-\theta\right)\\ &\geq \frac{4}{k}\left(j-\frac{1}{2}\right).\end{aligned}$$

Taking the sum,

$$\left|\sum_{j=1}^{k/4}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq\frac{k(1+\varepsilon)}{4}\sum_{j=1}^{k/4}\frac{1}{j-1/2}\leq Ck\log k$$

(the constant C may change from line to line). In the same way, we have

$$\left|\sum_{j=3k/4}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq Ck\log k.$$

If $j\in[k/4,3k/4]$, we also have $|1+\varepsilon-\overline{z_j}z|\geq C$, so that

$$\left|\sum_{j=k/4}^{3k/4}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq Ck.$$

Putting this together, we get

$$|f(z)|=\left|\frac{1}{k}\sum_{j=0}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq C\left(\frac{1}{k\varepsilon}+\log k+1\right)\leq C_3\omega$$

(this is the place where we need that $\omega\geq\log k$). Finally, it remains to prove (9). We observe that

$$\frac{f'(z)}{f(z)}=\frac{\sum_{j=0}^{k-1}\frac{\overline{z_j}}{(1+\varepsilon-\overline{z_j}z)^2}}{\sum_{j=0}^{k-1}\frac{1}{1+\varepsilon-\overline{z_j}z}}.$$

We deduce that

$$\begin{aligned}\left|\frac{f'(z)}{f(z)}\right| &\leq \frac{\sum_{j=0}^{k-1}\frac{1}{|1+\varepsilon-\overline{z_j}z|^2}}{\sum_{j=0}^{k-1}\frac{\Re e(1+\varepsilon-z_j\bar{z})}{|1+\varepsilon-\overline{z_j}z|^2}}\\ &\leq \frac{\sum_{j=0}^{k-1}\frac{1}{|1+\varepsilon-\overline{z_j}z|^2}}{\sum_{j=0}^{k-1}\frac{\varepsilon}{|1+\varepsilon-\overline{z_j}z|^2}}\\ &\leq \frac{1}{\varepsilon}=\omega k.\end{aligned}$$

□

The crucial step is given by the following lemma.

Lemma 4.2. *Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive real numbers decreasing to zero. Then, if n is large enough, one can find an integer k_n, a real number $\omega_n>1$ and a trigonometric polynomial P_n with spectrum in $[1,2n-1]$ such that*

- $\|P_n\|_\infty\leq 1$;
- *For any $x\in J_{k_n}^{\omega_n}$, $|S_nP_n(x)|\geq\varepsilon_n\log(n)$.*

Moreover, we can choose k_n and ω_n such that (k_n) goes to $+\infty$ and $\omega_n=o(k_n^\alpha)$ for any $\alpha>0$.

Proof. It is clear that the conclusion of the lemma is more difficult to obtain when the sequence (ε_n) is large. Thus, we may assume that

$$\varepsilon_n\geq\frac{\log\log n}{4\pi\log n}.$$

In particular, $\varepsilon_n\log n$ goes to infinity. We define k_n and ω_n by

- ω_n is equal to $\exp\big(4\pi(\log n)\varepsilon_n\big)$
- k_n is the biggest integer k satisfying

$$2\pi k\omega_n\leq n.$$

Observe that $\omega_n\geq\log n$ and $\omega_n=o(n^\alpha)$ for all $\alpha>0$. Then, the inequalities

$$2\pi k_n\omega_n\leq n\leq 2\pi(k_n+1)\omega_n$$

ensure that

$$k_n\leq n\leq Ck_nn^{1/2}$$

if n is large enough. It follows that (k_n) goes to $+\infty$, that $\omega_n\geq\log k_n$ and that $\omega_n=o(k_n^\alpha)$ for any $\alpha>0$.
Let f_n be the holomorphic function given by Lemma 4.1 for the values $k=k_n$ and $\omega=\omega_n$. We take $h_n(z)=\log(f_n(z))$, which defines a holomorphic function in a neighbourhood of $\overline{\mathbb{D}}$ (remember (6)). Moreover, $|\Im m(h_n(z))|\leq\pi/2$ for any $z\in\overline{\mathbb{D}}$ and $h_n(0)=0$. Now, we look at the function h_n on the boundary of the unit disk $\mathbb{D}$, that is we introduce the

function $g_n(x)=h_n(e^{2i\pi x})$ defined on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The properties satisfied by f_n translate into

$$\begin{array}{rrcl}\forall x\in J_{k_n}^{\omega_n}, & |g_n(x)| & \geq & \log\omega_n+\log C_2\\ \forall x\in\mathbb{T}, & |g_n(x)| & \leq & \log\omega_n+\log C_3\\ \forall x\in\mathbb{T}, & |g_n'(x)| & \leq & 2\pi k_n\omega_n\leq n.\end{array}$$

We apply Lemma 1.7 of [2], which is a precised version of Féjer's theorem, to the function $\theta_x(t)=g_n(t)-g_n(x)$ for $x\in\mathbb{T}$. Since $\|\theta_x\|_\infty\leq 2\log\omega_n+2\log C_3$, $\|\theta_x'\|_\infty\leq n$ and $\theta_x(x)=0$, we get

$$|\sigma_n\theta_x(x)|\leq\frac{1}{2}\log\omega_n+C_4$$

for some absolute constant C_4. If $x\in J_{k_n}^{\omega_n}$ we deduce that

$$|\sigma_n g_n(x)|\geq\frac{1}{2}\log\omega_n-C_5.$$

Finally we set

$$P_n=\frac{2}{\pi}e_n\,\sigma_n(\Im m g_n)=\frac{2}{\pi}e_n\Im m(\sigma_n g_n),$$

so that $\|P_n\|_\infty\leq 1$. Now, remember that g_n is the restriction to the circle of an holomorphic function h_n satisfying $h_n(0)=0$. We can then write $\sigma_n g_n=\sum_{j=1}^{n-1}a_je_j$, so that $2i\Im m\;\sigma_n g_n=-\sum_{j=1}^{n-1}\overline{a_j}e_{-j}+\sum_{j=1}^{n-1}a_je_j$. Thus, the spectrum of P_n is contained in $[1,2n-1]$. Moreover, for any $x\in J_{k_n}^{\omega_n}$, we get

$$\begin{aligned}|S_nP_n(x)| &= \frac{1}{\pi}\left|\sum_{j=1}^{n-1}\overline{a_j}e_{-j+n}\right|\\ &= \frac{1}{\pi}|\sigma_n g_n(x)|\\ &\geq \frac{1}{2\pi}\log\omega_n-C_6\\ &= 2\varepsilon_n\log n-C_6\\ &\geq \varepsilon_n\log n\end{aligned}$$

if n is large enough. □

We are now ready to construct the dense G_δ-set of functions required in Theorem 1.7.

Proof of Theorem 1.7. Let $(\delta_n)_{n\geq 2}$ be a sequence going to 0. We first consider an auxiliary sequence $(\delta_n')_{n\geq 1}$ such that

$$\lim_{n\to+\infty}\delta_n'=0,\quad \lim_{n\to+\infty}\frac{\delta_n'}{\delta_n}=+\infty\quad\text{and}\quad\lim_{n\to+\infty}\delta_n'\log n=+\infty\ .$$

Let $(g_n)_{n\geq 1}$ be a dense sequence in $\mathcal{C}(\mathbb{T})$, such that the spectrum of g_n is contained in $[-n,n]$. We set $\eta_n=\max(\delta_k';\ n\leq k)$. The sequence $(\eta_n)_{n\geq 1}$ decreases to zero. Moreover, we fix a sequence $(\varepsilon_n)_{n\geq 1}$, going to zero, such that ε_n/η_n tends to infinity. Lemma 4.2 gives us an integer N, a sequence $(P_j)_{j\geq N}$ of trigonometric polynomials with spectrum contained in $[1,2j-1]$, a sequence $(k_j)_{j\geq N}$ of integers going to $+\infty$ and a sequence $(\omega_j)_{j\geq N}$ satisfying $\omega_j>1$, such that

$$|S_jP_j(x)|\geq\varepsilon_j\log j$$

for any $x\in J_{k_j}^{\omega_j}$. Moreover, we can choose ω_j such that $\omega_j=o(k_j^\alpha)$ for any $\alpha>0$.
Let us define for $j\geq N$

$$h_j:=g_j+\frac{\eta_j}{\varepsilon_j}e_jP_j.$$

The sequence $(h_j)_{j\geq N}$ remains dense in $\mathcal{C}(\mathbb{T})$. Let us also observe that the spectra of g_j and $\frac{\eta_j}{\varepsilon_j}e_jP_j$ are disjoint. It follows that if $x\in J_{k_j}^{\omega_j}$,

$$|S_{2j}h_j(x)-S_jh_j(x)|=\left|\frac{\eta_j}{\varepsilon_j}S_jP_j(x)\right|\geq\eta_j\log j.$$

Thus, for any $x\in J_{k_j}^{\omega_j}$, one may find $n\in\{j,2j\}$ such that

$$|S_nh_j(x)|\geq\frac{1}{2}\eta_j\log j\geq\frac{1}{2}\delta_n'(\log n-\log 2).$$

Let $r_j>0$ be small enough so that

$$|S_nh(x)|\geq|S_nh_j(x)|-1$$

for any $h\in B(h_j,r_j)$ and any $n\in\{j,2j\}$ (the open balls are related to the norm $\|\ \|_\infty$). Then, we claim that the following dense G_δ-set of $\mathcal{C}(\mathbb{T})$ fulfills all the requirements:

$$G:=\bigcap_{p\geq N}\bigcup_{j\geq p}B(h_j,r_j).$$

Indeed, pick any h in G and any increasing sequence (j_p) such that h belongs to $B(h_{j_p},r_{j_p})$. Setting $\rho_p=\omega_{j_p}$ and $s_p=k_{j_p}$, it is not hard to show that

$$E:=\limsup_{p\to+\infty}E_p,\ \text{ with } E_p=J_{s_p}^{\rho_p}$$

has Hausdorff dimension 1. Indeed, remember that for any $\alpha>0$, $\omega_j=o(k_j^\alpha)$. It follows for any $\alpha>0$ and for p large enough, E_p contains

$$F_p=\bigcup_{j=0}^{s_p-1}\left[\frac{j}{s_p}-\frac{1}{2s_p^{1+\alpha}};\frac{j}{s_p}+\frac{1}{2s_p^{1+\alpha}}\right],$$

Now, it is well-known that $\limsup_p F_p$ has Hausdorff dimension equal to $1/(1+\alpha)$ (this follows for instance from the mass transference principle of [3]). Finally, $\dim_{\mathcal{H}}(E)\geq\frac{1}{1+\alpha}$. Moreover, for any $x\in E$, the work done before and the fact that $\delta_n'\log n$ goes to $+\infty$ show that

$$|S_nh(x)|\geq\frac{1}{2}\delta_n'(\log n-\log 2)-1\geq\frac{1}{4}\delta_n'\log n$$

for infinitely many values of n. We then get

$$\frac{|S_nh(x)|}{\delta_n\log n}\geq\frac{\delta_n'}{4\delta_n}$$

for infinitely many values of n. This achieves the proof of Theorem 1.7. □

We can finally construct the prevalent set of functions required in Theorem 1.8.

Proof of Theorem 1.8. Let $(\delta_n)_{n\geq 2}$ be a sequence going to 0 and denote by A the set of continuous functions $f \in \mathcal{C}(\mathbb{T})$ such that

$$\dim_{\mathcal{H}}\left(\left\{x\in\mathbb{T}\ ;\ \limsup_{n\to+\infty}\frac{|S_nf(x)|}{\delta_n\log n}=+\infty\right\}\right)<1.$$

We have to prove that A is Haar-null in $\mathcal{C}(\mathbb{T})$.

Let f_0 be a fixed function in the complementary of A (such a function does exist by Theorem 1.7) and let g be an arbitrary function in $\mathcal{C}(\mathbb{T})$. Suppose that t_1 and t_2 are two real numbers such that

$$t_1f_0\in(g+A)\quad\text{ and }\quad t_2f_0\in(g+A).$$

We can then find $f_1\in A$ and $f_2\in A$ such that $(t_1-t_2)f_0=f_1-f_2$. It is clear that $f_1-f_2\in A$ (A is a vector subspace of $\mathcal{C}(\mathbb{T})$). It follows that $t_1=t_2$, so that

$$\#\left(\mathrm{span}\,(f_0)\cap(g+A)\right)\leq 1.$$

In particular, the Lebesgue-measure in $\mathrm{span}\,(f_0)$ is transverse to A and A is Haar-null in $\mathcal{C}(\mathbb{T})$. □

Remark: We have just only proved that a proper subspace in a complete metric vector space is Haar-null. This property is probably well-known.

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