The p-median polytope of restricted Y-graphs

Mourad Baïou, Francisco Barahona

THE p-MEDIAN POLYTOPE OF RESTRICTED Y-GRAPHS

MOURAD BAÏOU AND FRANCISCO BARAHONA

Abstract. We further study the effect of odd cycle inequalities in the description of the polytopes associated with the p-median and uncapacitated facility location problems. We show that the obvious integer linear programming formulation together with the odd cycle inequalities completely describe these polytopes for the class of *restricted Y-graphs*. This extends our results for the class of *Y-free graphs*. We also obtain a characterization of both polytopes for a bidirected path.

1. Introduction

Let $G = (V, A)$ be a directed graph, not necessarily connected, where each arc $(u, v) \in A$ has an associated cost $c(u, v)$. The *p-median problem* (pMP) consists of selecting p nodes, usually called *centers*, and then assign each nonselected node to a selected node. The goal is to select p nodes that minimize the sum of the costs yield by the assignment of the nonselected nodes. This problem has several applications such as location of bank accounts [6], placement of web proxies in a computer network [12], semistructured data bases [11, 10]. When the number of centers is not specified and each opened center induces a given cost, this is called the *uncapacitated facility location problem* (UFLP).

The facets of p-median polytope have been studied in [1] and [8]. The facets of the uncapacitated facility location polytope have been studied in [9], [7], [4], [5], [3]. In [2] we studied the effect of odd cycle inequalities in the description of the polytopes associated with the pMP and the UFLP for the class of Y-free graphs. In this paper we further study these inequalities, namely we show that the obvious integer linear programming formulation together with the odd cycle inequalities completely describe these polytopes for the class of *restricted Y-graphs*.

Let $G = (V, A)$ be a directed graph. We are going to use variables y associated with the nodes in V, and variables x associated with the arcs in A. For a directed cycle

$$C = v_1, (v_1, v_2), v_2, (v_2, v_3), \dots, v_{k-1}, (v_{k-1}, v_k), v_k, (v_k, v_1), v_1,$$

we denote by $A(C)$ the set of arcs in C. We say that C is *odd* if k is odd. We plan to study the following linear system:

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$$\sum_{v \in V} y(v) = p, \tag{1}$$

$$\sum_{v:(u,v) \in A} x(u,v) = 1 - y(u) \quad \forall u \in V, \tag{2}$$

$$x(u,v) \le y(v) \quad \forall (u,v) \in A, \tag{3}$$

$$0 \le y(v) \le 1 \quad \forall v \in V, \tag{4}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A, \tag{5}$$

$$\sum_{a \in A(C)} x(a) \le \frac{|A(C)| - 1}{2} \quad \text{for each odd directed cycle } C. \tag{6}$$

Inequalities (1)-(5) give a linear programming relaxation of the pMP, by adding inequalities (6) we obtain a stronger relaxation. Analogously (2)-(5) give a linear programming relaxation of the UFLP and adding inequalities (6) yields a stronger relaxation.

Denote by $P_p(G)$ the polytope defined by (1)-(5), let $PC_p(G)$ be the polytope defined by (1)-(6), and let $pMP(G)$ be the convex hull of $P_p(G) \cap \{0,1\}^{|V|+|A|}$. In general we have

$$pMP(G) \subseteq PC_p(G) \subseteq P_p(G).$$

Also let $P(G)$ be the polytope defined by (2)-(5), let $PC(G)$ be the polytope defined by (2)-(6), and let $UFLP(G)$ be the convex hull of $P(G) \cap \{0,1\}^{|V|+|A|}$. We have

$$UFLP(G) \subseteq PC(G) \subseteq P(G).$$

For a undirected graph $G = (V,E)$ we denote by $\overleftrightarrow{G} = (V,A)$ the directed graph obtained from G by replacing each edge $uv \in E$ by two arcs (u,v) and (v,u). A directed graph $G = (V,A)$ is called *1-directed* if $(u,v) \in A \Rightarrow (v,u) \notin A$. A directed graph $G = (V,A)$, not necessarily connected, is called *Y-free* if it is 1-directed and it does not contain as induced subgraph the graph of Figure 1. In [2] we proved the following.

Theorem 1. *If G is a Y-free graph then $pMP(G) = PC_p(G)$ and $UFLP(G) = PC(G)$.*

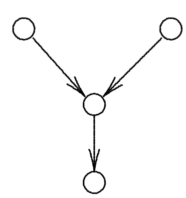

FIGURE 1. The graph Y.

In Figure 2 we show three graphs, that are not Y-free, and for each of them a fractional extreme point of $P_p(G)$. The numbers near the nodes correspond to the variables y and the numbers near the arcs correspond to the variables x. In this paper we study graphs for which these three configurations are forbidden.

A directed graph $G = (V,A)$, not necessarily connected, is called a *Y-graph* if it is 1-directed and it does not contain as induced subgraphs the graphs H_1, H_2 and H_3 of Figure 2. For a directed graph $G = (V,A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u,v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of

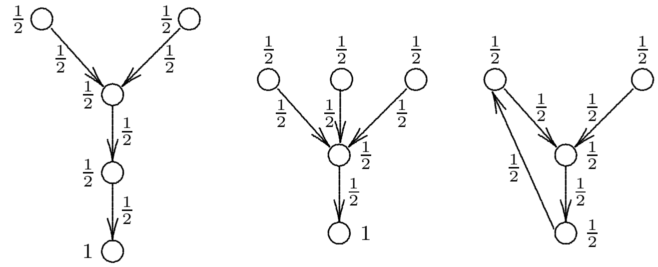

FIGURE 2

arcs (u, v), with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ_G^+ and δ_G^-. A node u with $\delta^+(u) = \emptyset$ is called a *pendent* node.

Remark 2. *Remark that for any arc (v, w) with w not a pendent node, $|\delta^-(v)| \leq 1$. Thus in such graphs the only nodes v different from a pendent node that may have $|\delta^-(v)| = 2$ have the property: if $(v, u) \in A$ then u is a pendent node. Call such a node a Y-node and denote by Y_G the set of Y-nodes in G.*

A simple cycle C is an ordered sequence

$$v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p,$$

where

- v_i, $0 \leq i \leq p-1$, are distinct nodes,
- a_i, $0 \leq i \leq p-1$, are distinct arcs,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i, for $0 \leq i \leq p-1$, and
- $v_0 = v_p$.

By setting $a_p = a_0$, we associate with C three more sets as below.

- We denote by $\hat{C}$ the set of nodes v_i, such that v_i is the head of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.
- We denote by $\dot{C}$ the set of nodes v_i, such that v_i is the tail of a_{i-1} and also the tail of a_i, $1 \leq i \leq p$.
- We denote by $\tilde{C}$ the set of nodes v_i, such that either v_i is the head of a_{i-1} and also the tail of a_i, or v_i is the tail of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it will be called *even*. A cycle C with $C = \tilde{C}$ is a *directed* cycle. A cycle C is called a Y*-cycle* if all the nodes in $\hat{C}$ are Y-nodes. Remark that when $\hat{C} = \emptyset$ then C is a directed cycle and also a Y-cycle.

A polyhedron is called *integral* if all its extreme points are integral. A *restricted* Y*-graph* is a Y-graph that does not contain an odd Y-cycle C with $\hat{C} \neq \emptyset$. A restricted Y-graph may have directed odd cycles, and it may contain odd cycles such that some of the nodes in $\hat{C}$ are pendent nodes. In this paper we prove that Theorem 1 also holds for restricted Y-graphs. We obtain as a corollary that if G is a 1-directed graph with no odd Y-cycle, then $P_p(G)$ is integral if and only if G does not contain any of the graphs H_1, H_2 and H_3 as induced subgraphs. Also we obtain as a corollary that for an undirected graph G the polytope $P_p(\overleftrightarrow{G})$ is integral if and only if G is a path. Other than the classes mentioned here, we do not know of any other classes of graphs for which the polytopes of the pMP or the UFLP have been characterized.

For simplicity, in what follows we use z to denote the vector (x, y), i. e. $z(u) = y(u)$ and $z(u, v) = x(u, v)$.

This paper is organized as follows. In Section 2 we give some polyhedral preliminaries. In Section 3 we prove our main result. Section 4 is devoted to bidirected paths.

2. Some basic polyhedral facts

Consider a polyhedron P defined by

$$P = \{x \in \mathbb{R}^n \,|\, Ax \leq b\}.$$

Denote by $A^= x \leq b^=$ a maximal subsystem of $Ax \leq b$ such that $A^= x = b^=$ for all $x \in P$. Then the dimension of P is

$$n - \text{rank}(A^=).$$

A face F of P is obtained by setting into equation some of the inequalities defining P. Clearly F is a polyhedron. An extreme point of P is a face of dimension 0.

Lemma 3. *Let P be a polyhedron defined by*

$$P = \{x \in \mathbb{R}^n \,|\, Ax \leq b\},$$

whose extreme points are all $0-1$ vectors. Let P' be defined by

$$P' = \{x \in P \,|\, cx = d\}.$$

If $\hat{x}$ is an extreme point of P' then all its components are in

$$\{0, 1, \alpha, 1 - \alpha\},$$

for some number $\alpha \in [0, 1]$.

Proof. Let $A^= x \leq b^=$ be a maximal subsystem of $Ax \leq b$ such that $A^= \hat{x} = b^=$. If $\text{rank}(A^=) = n$ then $\hat{x}$ is an extreme point of P and it is a $0-1$ vector. If $\text{rank}(A^=) = n-1$ then

$$F = \{x \in P \,|\, A^= x = b^=\}$$

is a face of P of dimension 1. Therefore $\hat{x}$ is a convex combination of two extreme points of P. □

3. Characterization of $pMP(G)$ and $UFLP(G)$ when G is a restricted Y-graph

In this section we show that if G is a restricted Y-graph then $pMP(G)$ is defined by (1)-(6), and $UFLP(G)$ is defined by (2)-(6). First we need several lemmas.

Lemma 4. *Let G be a restricted Y-graph. Let C be an even Y-cycle, then there is no intersection between the arc set of C and the arc set of any odd directed cycle.*

Proof. Let $C = v_0, a_0, \dots, a_{p-1}, v_p = v_0$ and let $C' = v'_0, a'_0, \dots, a'_{k-1}, v'_k = v'_0$ be an odd directed cycle. If C' intersects C and C is a directed cycle, then we would have the configuration H_1 or H_3. So we should assume that C is not a directed even cycle.

Suppose that $A(C) \cap A(C') \neq \emptyset$. We must have $|\dot{C} \cap V(C')| \geq 1$, otherwise the configuration H_1, H_2 or H_3 is present.

If $|\dot{C} \cap V(C')| = 1$, then we can assume that $\dot{C} \cap V(C') = \{v_0\}$, $A(C) \cap A(C') = \{a_0, \dots, a_r\}$, $v_0 = v'_0$, and $a_0 = a'_0, \dots, a_r = a'_r$. Let C'' be the cycle whose arc set is $A(C) \triangle A(C')$ (the symmetric difference between $A(C)$ and $A(C')$). Then the parity of $|A(C'')|$ is different from the parity of $|A(C)|$. Because G is a Y-graph we have that none of $v_1, \dots, v_{r+1}$ is in $\hat{C}$, thus $\hat{C} = \hat{C}''$. Thus C'' is an odd Y-cycle, we have a contradiction.

Now we use induction, so we assume that $A(C) \cap A(D) = \emptyset$ when D is an odd directed cycle and $|\dot{C} \cap V(D)| \leq m$.

Consider now a directed odd cycle C' with $|\dot{C} \cap V(C')| = m + 1 \geq 2$. Consider a maximal directed path contained in $A(C) \cap A(C')$ going from u to v. So $u \in \dot{C} \cap V(C')$ and $v \in \big(\tilde{C} \cap V(C')\big)$, otherwise we have the configuration H_1 or H_3. Since $|\dot{C} \cap V(C')| \geq 2$ there is a node $w \in \dot{C} \cap V(C')$, $w \neq u$, such that the directed path in C' from v to w does not contain the node u. Denote by C'^v_w this path. We can assume that $V(C'^v_w) \cap V(C) = \{v, w\}$. Consider the path in C from v to w that does not contain u, denote it by C^v_w. The junction of C'^v_w and C^v_w is a cycle C''. It contains at least one Y-node, because C^v_w contains at least one Y-node. Also notice that all the nodes in $\hat{C}''$ are Y-nodes, since these nodes are in $V(C)$. Thus C'' is a Y-cycle with $\hat{C}'' \neq \emptyset$. Since G is a restricted Y-graph C'' is even. Notice that now $w \notin \dot{C}''$, $v \in \dot{C}''$ and $u \notin C''$. Thus $|\dot{C}'' \cap V(C')| \leq m$. This implies $A(C'') \cap A(C') = \emptyset$, which is impossible. □

Now we study a vector z assuming that it is a fractional extreme point of $PC(G)$ or $PC_p(G)$, we plan to arrive to a contradiction. We denote by $G_z = (V_z, A_z)$ the graph induced by the arcs $(u,v) \in A$ such that $0 < z(u,v) < 1$. Below we state several properties of G_z.

Lemma 5. *We may assume that $|\delta^-_{G_z}(v)| = 1$ for every pendent node v in G_z.*

Proof. If v is a pendent node in G_z and $\delta^-_{G_z}(v) = \{(u_1, v), \dots, (u_k, v)\}$, we can split v into k pendent nodes $\{v_1, \dots, v_k\}$ and replace every arc (u_i, v) with (u_i, v_i). Then we define z' such that $z'(u_i, v_i) = z(u_i, v)$, $z'(v_i) = 1$, for all i, and $z'(u) = z(u)$, $z'(u, w) = z(u, w)$ for all other nodes and arcs. Let G' be this new graph. We have that the constraints that are tight for z are also tight for z', so z' is a fractional extreme point of $PC_{p+k-1}(G')$. □

The lemma above implies that we can assume that every cycle is a Y-cycle.

Lemma 6. *G_z does not contain an even cycle.*

Proof. Let $C = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$ be an even cycle in G_z, that is $|C| + |\hat{C}|$ is even. If $v \in \hat{C}$ then v is not a pendent node in G_z, since $|\delta^-_{G_z}(v)| > 1$. Hence v must be a Y-node in G_z.

Also, for every node $v \in \tilde{C}$, $|\delta^-_{G_z}(v)| = 1$, otherwise the configuration H_1 is present. Thus the unique arc directed into v belongs to C.

Assume $v_0 \in \dot{C}$. Assign labels to the nodes and arcs of C as follows:

- $l(v_0) \leftarrow 0$; $l(a_0) \leftarrow 1$.

- For $i = 1$ to $p - 1$ do the following:
  - If v_i is the head of a_{i-1} and is the tail of a_i, then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow -l(v_i)$.
  - If v_i is the head of a_{i-1} and is the head of a_i, then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
  - If v_i is the tail of a_{i-1} and is the head of a_i, then $l(v_i) \leftarrow -l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
  - If v_i is the tail of a_{i-1} and is the tail of a_i, then $l(v_i) \leftarrow 0$, $l(a_i) \leftarrow -l(a_{i-1})$.

Now define z^* as follows. For every arc a_i of C, $i = 0, \dots, p-1$, let $z^*(a_i) = z(a_i) + l(a_i)\epsilon$. For every node v_i, $i = 0, \dots, p-1$, let $z^*(v_i) = z(v_i) + l(v_i)\epsilon$. Also for every node $u \in \hat{C}$, pick an arc $(u, v) \in \delta^+_{G_z}(u)$ and let $z^*(u, v) = z(u, v) - l(u)\epsilon$. Finally let $z^*(u) = z(u)$, $z^*(u, v) = z(u, v)$ for all other nodes and arcs of G.

It should be clear that z^* satisfies constraints (2) for every node $v \neq v_0$, and that every constraint (3) that is tight for z is also tight for z^*. In order to show that constraint (2) with respect to v_0 is satisfied by z^*, and that equation (1) is also satisfied by z^*, we have to discuss some properties of the labeling. Let $v_{j(0)}, v_{j(1)}, \dots, v_{j(k)}$ be the ordered sequence of nodes in $\dot{C}$, with $v_{j(0)} = v_{j(k)}$. A path in C

$$v_{j(i)}, a_{j(i)}, \dots, a_{j(i+1)-1}, v_{j(i+1)}$$

from $v_{j(i)}$ to $v_{j(i+1)}$ will be called a *segment* and denoted by S_i. A segment is *odd* (resp. *even*) if it contains and *odd* (resp. *even*) number of arcs. Let

$$l(S_i) = \sum_{v \in S_i \cap V} l(v).$$

Let r be the number of even segments and t the number of odd segments. We have that $r + t = |\dot{C}|$, and since the parity of $|C|$ is equal to the parity of t, we have that $t + |\dot{C}|$ is even. Therefore $r = |\dot{C}| - t$ is also even. The labeling has the following properties:

a) If the segment is odd then $l(a_{j(i)}) = -l(a_{j(i+1)-1})$.
b) If the segment is even then $l(a_{j(i)}) = l(a_{j(i+1)-1})$.
c) If S_i is odd then $l(S_i) = 0$.
d) If S_i is even then $l(S_i) = l(a_{j(i)})$.
e) Let $S_1, \dots, S_r$ be the ordered sequence of even segments in C. Then $l(S_i) = -l(S_{i+1})$, for $i = 1, \dots, r - 1$.

Properties a) and b) imply that $l(a_{p-1}) = -l(a_0)$. It follows that constraints (2) are satisfied by z^*.

Properties c), d) and e) imply that equation (1) is satisfied by z^*.

It follows from the remarks above and Lemma 4 that any constraint among (1)-(6) that was tight for z remains tight for z^*. This contradicts the fact that z is an extreme point of $PC_p(G)$ or $PC(G)$. □

Lemma 7. *The graph G_z must contain at least one Y-node.*

Proof. Suppose that z is an extreme point of $PC_p(G)$. Suppose that G_z is Y-free. Let G' be the graph obtained by adding to G_z all arcs (u, v) with $z(u, v) = 1$, and all nodes u with $z(u) = 1$. It is easy to see that G' is also Y-free. Let $z^{G'}$ be the restriction of z to G'. Theorem 1 implies that $PC_p(G')$ is an integral polytope. Clearly $z^{G'} \in PC_p(G')$. Since $z^{G'}$ is fractional, we have that $z^{G'} = 1/2z_1 + 1/2z_2$ with $z_1, z_2 \in PC_p(G')$, $z_1 \neq z_2$. Let $\bar{z}_1$

(resp. $\bar{z}_2$) be the vector obtained by adding zeros to z_1 (resp. z_2) for all $(u,v) \in G \backslash G'$. We have to see that these two new vectors belong to $PC_p(G)$. For that we study constraints (6), it is easy to see that the other constraints are satisfied. Suppose that we add a zero component associated with the arc (u,v). Consider the odd cycle C with

$$A(C) = \{(w_i, w_{i+1}) \,|\, i = 1, \ldots, 2l\} \cup \{(u,v)\},$$

where $u = w_{2l+1}$ and $v = w_1$. We have that

$$z_1(w_{2i-1}, w_{2i}) + z_1(w_{2i}, w_{2i+1}) \leq 1, \text{ for } i = 1, \ldots, l.$$

This implies $\sum_{a \in A(C)} \bar{z}_1(a) \leq l$. The same is true for $\bar{z}_2$. Therefore $\bar{z}_1$ and $\bar{z}_2$ are in $PC_p(G)$.

Since $z = 1/2\bar{z}_1 + 1/2\bar{z}_2$ we have a contradiction. The same proof holds when z is an extreme point of $PC(G)$. □

Lemma 8. *There is a Y-node t in G such that:*

- *The arcs (u_1,t), (u_2,t), (t,w) are in A.*
- *V can be partitioned into W_1 and W_2 so that $\{u_1,t,w\} \subseteq W_1$ and $u_2 \in W_2$.*
- *The unique arc in G_z between W_1 and W_2 is (u_2,t).*
- *Any arc in G between W_1 and W_2 does not belong to an odd directed cycle. In other words, all directed odd cycles have all their arcs either in $A(W_1)$ or in $A(W_2)$. See Figure 3.*

Proof. Any Y-node in G_z is also a Y-node in G. Let t be a Y-node in G_z, Lemma 7 shows that such a node exists. The node t is incident to a pendent node w and there are exactly two arcs (u_1,t) and (u_2,t) directed into t. Let $G_1 = (S_1, A_1), \ldots, G_r = (S_r, A_r)$ be the connected components of G_z. Set $S_{r+1} = V \setminus V_z$. Let G_1 be the connected component that contains t. It follows from lemmas 5 and 6 and from the definition of a restricted Y-graph that t does not belong to any cycle in G_z. Hence if we remove t from G_z then we disconnect G_1 into two connected components. Let S'_1 and S'_2 be the node sets of these two components, containing u_1 and u_2 respectively. Define $S_0 = S'_1 \cup \{t,w\}$ and redefine $S_1 = S'_2$. The unique arc of G_z that may have one endnode in S_i and the other endnode in S_j, $i \neq j$, is (u_2,t). Define two sets W_1 and W_2 as follows.

Step 0. $W_1 \leftarrow S_0$, $W_2 \leftarrow S_1$.
Step 1. If there is a set $S_i \not\subset W_1$ such that there is an arc in G of an odd directed cycle having one endnode in W_1 and the other endnode in S_i, then set $W_1 \leftarrow W_1 \cup S_i$.
Step 2. If there is a set $S_i \not\subset W_2$ such that there is an arc in G of an odd directed cycle having one endnode in W_2 and the other endnode in S_i, then set $W_2 \leftarrow W_2 \cup S_i$. Go to Step 1.
Step 3. For every set S_i not included in $W_1 \cup W_2$ include S_i in W_2.

By definition we have $W_1 \cup W_2 = \cup_{i=0}^{r+1} S_i$. We have to see that $W_1 \cap W_2 = \emptyset$. For that suppose $S_j \in W_1 \cap W_2$. Notice that a directed cycle cannot go through w because we would have the configuration H_1 or H_3. Then there is a cycle C in G containing a node in S_j, the node u_1, the node u_2 and t. Because G is a restricted Y-graph this cycle should be even. The cycle C contains arcs of odd directed cycles, this contradicts Lemma 4. □

Based on this last lemma, we define the graphs G^1 and G^2 as follows. Let $A(W_1)$ and $A(W_2)$ be the set of arcs in G having their both endnodes in W_1 and W_2, respectively.

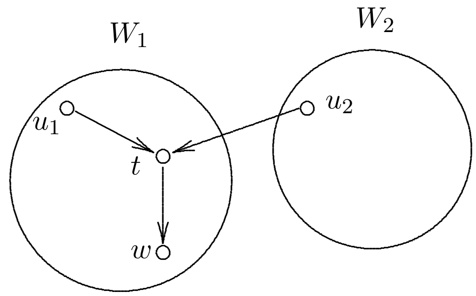

FIGURE 3

Let $G^1 = (W_1, A(W_1))$ and $G^2 = (W_2 \cup \{t', v', w'\}, A(W_2) \cup \{(u_2, t'), (t', v'), (v', w')\})$, see Figure 4. Now we can prove the following.

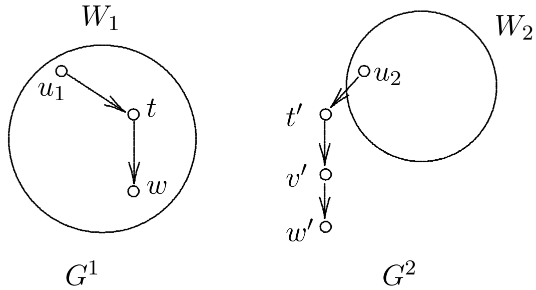

FIGURE 4

Theorem 9. *If G is a restricted Y-graph then $PC(G)$ is an integral polytope.*

Proof. The proof is by induction on the number of Y-nodes. If the graph is Y-free the result follows from Theorem 1. Then we assume that the result is true for any restricted Y-graph G' with $|Y_{G'}| < |Y_G|$.

Let z_1 be the restriction of z to G^1. Clearly $z_1 \in PC(G^1)$. Define $z_2 \in PC(G^2)$ as follows: $z_2(u_2, t') = z(u_2, t)$, $z_2(t') = z(t)$, $z_2(t', v') = 1 - z(t')$, $z_2(v') = 1 - z(t')$, $z_2(v', w') = z(t')$, $z_2(w') = 1$ and $z_2(u) = z(u)$, $z_2(u, v) = z(u, v)$ for all other nodes and arcs of G^2. We have that $z_2 \in PC(G^2)$.

Notice that G^1 and G^2 are both restricted Y-graphs. Also $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Since z_1 and z_2 are both fractional, by the induction hypothesis they are not extreme points of $PC(G^1)$ and $PC(G^2)$, respectively. Thus there must exist a 0-1 vector $z'_1 \in PC(G^1)$ with $z'_1(t) = 0$ and such that the same constraints that are tight for z_1 are also tight for z'_1. Also there must exist a 0-1 vector $z'_2 \in PC(G^2)$ with $z'_2(t') = 0$ such that the same constraints that are tight for z_2 are also tight for z'_2. Combine z'_1 and z'_2 to define a solution $z' \in PC(G)$ as follows.

$z'(u) = z'_1(u)$,	for every node of G^1,
$z'(u, v) = z'_1(u, v)$	for every arc of G^1,
$z'(u_2, t) = 0$,	
$z'(v) = z'_2(v)$,	for all node $v \in W_2$,
$z'(u, v) = z'_2(u, v)$,	for all arc$(u, v) \in A(W_2)$,
$z'(u, v) = z(u, v) = 0$,	for all arc having one endnode in W_1 and the other in W_2.

Clearly that any constraint among (2)-(5), that is tight for z is also tight for z'. By Lemma 8, any odd directed cycle is included either in $G(W_1)$ or in $G(W_2)$, thus any constraint (6) that is tight for z is also tight for z'. This contradicts the fact that z is an extreme point of $PC(G)$. □

Now we plan to prove that if G is a restricted Y-graph then $PC_p(G)$ is integral. The proof is by induction on the number of Y-nodes. If the graph is Y-free the result follows from Theorem 1. Then we assume that it is true for any restricted Y-graph G' with $|Y_{G'}| < |Y_G|$. We keep working with the graphs G^1 and G^2 defined before. Now let z be a fractional extreme point of $PC_p(G)$. We need the following lemmas.

Lemma 10. *The values of z are in $\{0, 1, \alpha, 1-\alpha\}$, for some number $\alpha \in [0, 1]$.*

Proof. Since $PC(G)$ is an integral polytope and $PC_p(G)$ is obtained from $PC(G)$ by adding exactly one equation, the result follows from Lemma 3. □

Lemma 11. $z(u_1, t) = z(u_2, t) = z(t) = \frac{1}{2}$.

Proof. Suppose $z(u_1, t) < z(t)$. Consider the graph G' obtained from G by removing the arc (u_1, t) and adding the arc (u_1, w). Let z' be defined as $z'(u_1, w) = z(u_1, t)$ and $z'(u) = z(u)$, $z'(u, v) = z(u, v)$ for all other nodes and arcs. We have that $z' \in PC_p(G')$. Since G' is a restricted Y-graph with $|Y_{G'}| < |Y_G|$, then z' is not an extreme point of $PC_p(G')$. Hence, there exists a vector $z^* \in PC_p(G')$, $z^* \neq z'$, such that all constraints that are tight for z' are also tight for z^*. Define $\bar{z}(u_1, t) = z^*(u_1, w)$ and $\bar{z}(u) = z^*(u)$, $\bar{z}(u, v) = z^*(u, v)$ for all other nodes and arcs. Then $\bar{z} \neq z$ and since the arc (u_1, t) does not belong to any odd directed cycle in G then all constraints that are tight for z are also tight for $\bar{z}$. This is impossible since z is an extreme point of $PC_p(G)$. (Notice that we do not need that $\bar{z} \in PC_p(G)$). The same may be done if $z(u_2, t) < z(t)$.

Thus we may assume that $z(u_1, t) = z(u_2, t) = z(t)$. Now we have to prove that $z(t) = \frac{1}{2}$.

Consider the graph G' defined from G as follows. Remove (u_2, t) and add (u_2, t'), (t', v') and (v', w). Here t' and v' are new nodes, see Figure 5.

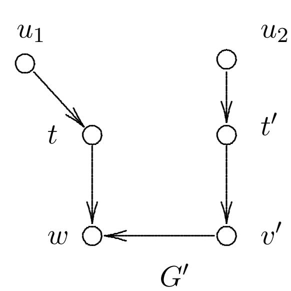

FIGURE 5

Define z' to be $z'(u_2, t') = z(u_2, t)$, $z'(t') = z(u_2, t)$, $z'(t', v') = 1 - z(u_2, t)$, $z'(v') = 1 - z(u_2, t)$, $z'(v', w) = z(u_2, t)$ and $z'(u) = z(u)$, $z'(u, v) = z(u, v)$ for all other nodes and arcs. We have that $z' \in PC_{p+1}(G')$ and G' is a restricted Y-graph with $|Y_{G'}| < |Y_G|$. Hence z' is not an extreme point of $PC_{p+1}(G')$. Thus there must exist a 0-1 vectors $z'_1, \ldots, z'_r$ in $PC_{p+1}(G')$ such that z' is a convex combination of $z'_1, \ldots, z'_r$, and all constraints that are tight for z' are also tight for $z'_1, \ldots, z'_r$. Thus

$$z' = \sum_{i=1}^{r} \lambda_i z'_i, \tag{7}$$

$$\sum_{i=1}^{r} \lambda_i = 1, \tag{8}$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, r. \tag{9}$$

If there exists a vector z'_k with $z'_k(t) = z'_k(t')$, then we can define from z'_k a 0-1 vector $z'' \in PC_p(G)$ such that the same constraints tight for z are also tight for z''. Thus we may suppose that for all z'_i, $i = 1 \ldots, r$, we have $z'_i(t) \neq z'_i(t')$. Let $z'_i(t) = 1$, $z'_i(t') = 0$, for $i = 1, \ldots, r_1$, and $z'_i(t) = 0$, $z'_i(t') = 1$, for $i = r_1 + 1, \ldots, r$. Then

$$z'(t) = \sum_{i=1}^{r_1} \lambda_i, \tag{10}$$

$$z'(t') = \sum_{i=r_1+1}^{r} \lambda_i, \tag{11}$$

and since by definition $z'(t) = z'(t')$ and $\sum_{i=1}^{r} \lambda_i = 1$, the result is obtained. □

Lemma 12. *If z is a fractional extreme point of $PC_p(G)$ then each component of z is in $\{0, 1, \frac{1}{2}\}$.*

Proof. Immediate from Lemma 10 and Lemma 11. □

Now we can state the final result of this section.

Theorem 13. *The polytope $PC_p(G)$ is integral.*

Proof. Define $p_1 = \sum_{v \in W_1} z(v)$ and $p_2 = \sum_{v \in W_2} z(v)$, so $p = p_1 + p_2$. We distinguish two cases:

Case 1. The numbers p_1 and p_2 are integers.

Consider the graphs G^1 and G^2 of Figure 4, as defined above. Let z_1 be the restriction of z to G^1. Clearly $z_1 \in PC_{p_1}(G^1)$. Define z_2 as follows. $z_2(u_2, t') = z(u_2, t) = \frac{1}{2}$, $z_2(t') = \frac{1}{2}$, $z_2(t', v') = \frac{1}{2}$, $z_2(v') = \frac{1}{2}$, $z_2(v', w') = \frac{1}{2}$, $z_2(w') = 1$ and $z_2(u) = z(u)$, $z_2(u, v) = z(u, v)$ for all other nodes and arcs of G^2. We have that $z_2 \in PC_{p_2+2}(G^2)$.

G^1 and G^2 are both restricted Y-graphs and $|Y_{G^1}| < |Y_G|$, $|Y_{G^2}| < |Y_G|$. Since z_1 and z_2 are both fractional, by the induction hypothesis they are not extreme points of $PC_{p_1}(G^1)$ and $PC_{p_2+2}(G^2)$, respectively. Thus there must exist a 0-1 vector $z'_1 \in PC_{p_1}(G^1)$ with $z'_1(t) = 0$ so that the same constraints that are tight for z_1 are also tight for z'_1. Also there must exist a 0-1 vector $z'_2 \in PC_{p_2+2}(G^2)$ with $z'_2(t') = 0$ such that the same constraints that are tight for z_2 are also tight for z'_2. Combine z'_1 and z'_2 to define a solution $z' \in PC_p(G)$ as follows.

$$\begin{array}{ll} z'(u) = z'_1(u), & \text{for every node } u \text{ of } G^1, \\ z'(u,v) = z'_1(u,v), & \text{for every arc } (u,v) \text{ of } G^1, \\ z'(u_2,t) = 0, & \\ z'(v) = z'_2(v), & \text{for every node } v \in W_2, \\ z'(u,v) = z'_2(u,v), & \text{for every arc}(u,v) \in A(W_2), \\ z'(u,v) = z(u,v) = 0, & \text{for every arc } (u,v) \text{ having one endnode in } W_1 \\ & \text{and the other in } W_2. \end{array}$$

Remark that $\sum_{v \in V} z'(v) = p$. Also any constraint among (2)-(5), that is tight for z is also tight for z'. By Lemma 8, any odd directed cycle is included either in $A(W_1)$ or in $A(W_2)$, thus any constraint (6) that is tight for z remains tight for z'. Then the same constraints of $PC_p(G)$ that are tight for z are also tight for z'. This contradicts the fact that z is an extreme point of $PC_p(G)$.

Case 2. The values of p_1 and p_2 are not integers.

Thus from Lemma 12, $\sum_{v \in W_1} z(v) = p_1 = \alpha + \frac{1}{2}$ and $\sum_{v \in W_2} z(v) = p_2 = \beta - \frac{1}{2}$, where α and β are integers and $\alpha + \beta = p$. Define G^1 and G^2 from G as follows. $G^1 = \big(W_1 \cup \{u'_1\}, (A(W_1) \setminus \{(u_1,t)\}) \cup \{(u_1,u'_1),(u'_1,t)\}\big)$ and $G^2 = \big(W_2 \cup \{t',w'\}, A(W_2) \cup \{(u_2,t'),(t',w')\}\big)$, see Figure 6.

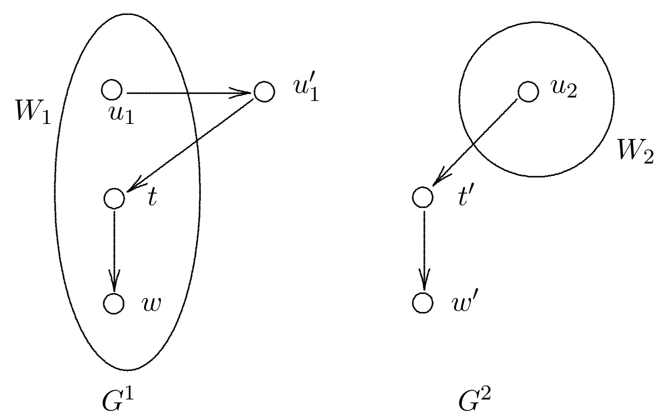

FIGURE 6

Define z^1 to be:

$$\begin{aligned} &z^1(u_1,u'_1) = z^1(u'_1) = z^1(u'_1,t) = \frac{1}{2}, \\ &z^1(u) = z(u) \quad \text{for all other nodes of } G^1, \\ &z^1(u,v) = z(u,v) \quad \text{for all other arcs of } G^1. \end{aligned}$$

Let z^2 be defined by:

$$\begin{aligned} &z^2(u_2,t') = z^2(t') = z^2(t',w') = \frac{1}{2}, \\ &z^2(w') = 1, \\ &z^2(u) = z(u) \quad \text{for all other nodes of } G^2, \\ &z^2(u,v) = z(u,v) \quad \text{for all other arcs of } G^2. \end{aligned}$$

Notice that $z^1 \in PC_{\alpha+1}(G^1)$ and $z^2 \in PC_{\beta+1}(G^2)$. Notice also that G^1 and G^2 are restricted Y-graphs with $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Thus there must exist a 0-1 vector $\bar{z}^1 \in PC_{\alpha+1}(G^1)$ such that the same constraints that are tight for z^1 are also tight for $\bar{z}^1$, and such that $\bar{z}^1(u_1, u_1') = 0$. Also there must exist a 0-1 vector $\bar{z}^2 \in PC_{\beta+1}(G^2)$ such that the same constraints that are tight for z^2 are also tight for $\bar{z}^2$ and such that $\bar{z}^2(t') = 0$. Now from $\bar{z}^1$ and $\bar{z}^2$ define $\bar{z} \in PC_p(G)$ as follows.

$$
\begin{array}{ll}
\bar{z}(u_2, t) = 0, & \\
\bar{z}(u) = \bar{z}^1(u), & \text{for all } u \in W_1 \setminus \{t\}, \\
\bar{z}(u,v) = \bar{z}^1(u,v), & \text{for all } (u,v) \in A(W_1) \setminus \{(u_1,t),(t,w)\}, \\
\bar{z}(t) = 0, & \\
\bar{z}(u_1,t) = 0, & \\
\bar{z}(t,w) = 1, & \\
\bar{z}(u) = \bar{z}^2(u), & \text{for all } u \in W_2, \\
\bar{z}(u,v) = \bar{z}^2(u,v), & \text{for all } (u,v) \in A(W_2), \\
\bar{z}(u,v) = z(u,v), & \text{for all other arcs.}
\end{array}
$$

It is easy to see that $\bar{z} \in PC_p(G)$ and the same constraints that are tight for z are also tight for $\bar{z}$. We have a contradiction because z is an extreme point. □

4. Bidirected paths

A bidirected path is a graph $G = (V, A)$ where $V = \{u_1, \ldots, u_n\}$, and

$$A = \{(u_i, u_{i+1}), (u_{i+1}, u_i),\ i = 1, \ldots, n-1\}.$$

In this section we show that $pMP(G) = P_p(G)$ and $UFLP(G) = P(G)$ when G is a bidirected path. For that we first consider a graph $H = (V, A)$ where

$$V = \{u_1, \ldots, u_n\} \bigcup_{i=1}^{n} V_i,$$

with $V_i = \{v_1^i, \ldots, v_{p(i)}^i\}$, for $i = 1, \ldots, n$.

The set of arcs A is composed by two arc-subsets. The arcs (u_i, v_j^i) for $i = 1, \ldots, n$ and $j = 1, \ldots, p(i)$. And the arc-subset between any two consecutive nodes u_i and u_{i+1}, for $i = 1, \ldots, n-1$, that consists of one of the following possibilities:

- $(u_i, u_{i+1}) \in A$ and $(u_{i+1}, u_i) \notin A$, or
- $(u_i, u_{i+1}) \notin A$ and $(u_{i+1}, u_i) \in A$, or
- $(u_i, u_{i+1}) \in A$ and $(u_{i+1}, u_i) \in A$, or
- there is no arc between u_i and u_{i+1}.

Notice that H may not be connected. Call such a graph an *extended path.* For H an extended path, denote by $Pair(H)$ the set of pair of nodes $\{u_i, u_{i+1}\}$ such that both arcs (u_i, u_{i+1}) and (u_{i+1}, u_i) belong to the set of arcs of H.

Theorem 14. *If H is an extended path then $pMP(H) = P_p(H)$ and $UFLP(H) = P(H)$.*

Proof. The proof is by induction on $|Pair(H)|$. If $|Pair(H)| = 0$ then H is a restricted Y-graph with no odd directed cycle. Hence from Theorem 13 we have that $pMP(H)$ is defined by inequalities (1)-(5). Suppose that the theorem is true for every extended path H' with $|Pair(H')| \leq m$. Let H be an extended path with $|Pair(H)| = m+1$ and assume that z is a fractional extreme point of $P_p(H)$.

Define $H_z = (V_z, A_z)$ to be the graph induced by the arcs $(u, v) \in A$, with $0 < z(u, v) < 1$. Remark that H_z is also an extended path.

Claim 1. $|Pair(H_z)| \geq 1$.

Proof. Suppose $|Pair(H_z)| = 0$. Then H_z is a restricted Y-graph with no odd directed cycle. Let H' be the graph obtained by adding to H_z all arcs (u, v) with $z(u, v) = 1$, and all nodes u with $z(u) = 1$. It is easy to see that H' is also a restricted Y-graph with no odd directed cycle. Then $P_p(H')$ is an integral polytope. Let $z^{H'}$ be the restriction of z to H'. Clearly $z^{H'} \in P_p(H')$. Since $z^{H'}$ is fractional, we have that $z^{H'} = 1/2z_1 + 1/2z_2$ with $z_1, z_2 \in P_p(H')$, $z_1 \neq z_2$.

Let $\bar{z}_1$ (resp. $\bar{z}_2$) be the vector obtained by adding zeros to z_1 (resp. z_2) for each $(u, v) \in H \setminus H'$. It is easy to see that $\bar{z}_1$ and $\bar{z}_2$ belong to $P_p(H)$. Since $z = 1/2\bar{z}_1 + 1/2\bar{z}_2$, we have a contradiction. □

From the claim above, we may assume that there are at least two arcs (u_i, u_{i+1}) and (u_{i+1}, u_i) with $0 < z(u_i, u_{i+1}) < 1$ and $0 < z(u_{i+1}, u_i) < 1$. Define H' from H by removing the arc (u_i, u_{i+1}) and adding the arc (u_i, w) where w is new a pendent node. Define z' to be $z'(u_i, w) = z(u_i, u_{i+1})$, $z'(w) = 1$ and $z'(u) = z(u)$, $z'(u, v) = z(u, v)$ for all other nodes and arcs of H'. Notice that H' is an extended path with $|pair(H')| = m$ and that $z' \in P_{p+1}(H')$. Then by the induction hypothesis $P_{p+1}(H')$ must be integral. Hence there must exist a 0-1 solution $\bar{z} \in P_{p+1}(H')$ with $\bar{z}(u_{i+1}, u_i) = 1$, so that the same constraints that are tight for z' are also tight for $\bar{z}$.

From $\bar{z}$ define $z^* \in P_p(H)$ as follows: $z^*(u_i, u_{i+1}) = \bar{z}(u_i, w)$ and $z^*(u) = \bar{z}(u)$, $z^*(u, v) = \bar{z}(u, v)$ for all other nodes and arcs. All constraints that are tight for z are also tight for z^*. To see this, it suffices to remark that $z^*(u_{i+1}) = \bar{z}(u_{i+1}) = 0$ and $z^*(u_i, u_{i+1}) = \bar{z}(u_i, w) = 0$. This contradicts the fact that z is an extreme point of $P_p(H)$.

The proof for $UFLP(H)$ is similar. □

We also have the following two corollaries.

Corollary 15. *Let G be a 1-directed graph with no odd Y-cycle. Then $P_p(G)$ is integral for all p if and only if G does not contain any of the graphs H_1, H_2 and H_3 as induced subgraph.*

Proof. Let $G = (V, E)$ be a 1-directed graph with no odd Y-cycle. If G does not contain none of the graphs H_1, H_2 and H_3 as induced subgraph, then G is a restricted Y-graph with no odd directed cycle. Thus $PC_p(G) = P_p(G)$ and the integrality of $P_p(G)$ follows from Theorem 13. Now suppose that G contains $H_1 = (V_1, E_1)$ as induced subgraph. Call v^* the unique pendent node in H_1. Define z^* as follows:

$$z^*(v) = \begin{cases} \frac{1}{2} & \text{if } v \in V_1 \setminus \{v^*\} \\ 1 & \text{otherwise} \end{cases}, \quad z^*(u, v) = \begin{cases} \frac{1}{2} & \text{if } (u, v) \in E_1 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to verify that z^* is an extreme point of $P_{|V|-2}(G)$. By the same manner one can construct fractional extreme point of $P_p(G)$ when G contains H_2 or H_3 as induced subgraph. □

Corollary 16. *Let G be a undirected graph. Then $P_p(\overleftrightarrow{G})$ is integral for all p if and only if G is a path.*

Proof. If G is a path the result follows from Theorem 14. Suppose G is not a path. Thus G contains a node w of degree at least 3. Let w_1, w_2 and w_3 three nodes adjacent to w. The solution z^* defined below is an extreme fractional point of $P_{|V|-2}(\overleftrightarrow{G})$.

$$z^*(v) = \begin{cases} \frac{1}{2} & \text{if } v \in \{w, w_1, w_2, w_3\} \\ 1 & \text{otherwise} \end{cases} \quad , \quad z^*(u,v) = \begin{cases} \frac{1}{2} & \text{if } (u,v) \in \{(w,w_1),(w_1,w), \\ & \qquad (w_2,w),(w_3,w)\} \\ 0 & \text{otherwise} \end{cases}$$

□

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(M. Baïou) CUST and Laboratoire LIMOS, Université Clermont II, Campus des cézeaux BP 125, 63173 Aubière cedex, France.

E-mail address, M. Baïou: `baiou@custsv.univ-bpclermont.fr`

(F. Barahona) IBM T. J. Watson research Center, Yorktown Heights, NY 10589, USA

E-mail address, F. Barahona: `barahon@us.ibm.com`