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Течії Стокса в тривимірних областях

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Stokes flows in 3D containers

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Дана робота складається з двох частин. Спочатку ми обговорюємо аналітичний підхід до розв'язання задачі Стокса про стаціонарну течію в'язкої нестисливої рідини в тривимірних порожнинах. Цей підхід базується на методі суперпозиції. В другій частині ми описуємо особливості ламінарного перемішування рідини в тривимірних течіях.

Ключові слова: течія Стокса, вихори Моффата, перемішування, періодичні лінії.

This study consists of two parts. First we consider an analytical approach for solving the problem of steady Stokes flow in some 3D containers with arbitrary velocities prescribed over the surfaces. The approach is based on the superposition method. First we discuss the Stokes problem solution in a finite cylinder. This is the simplest problem because the flow domain is restricted with only two families of coordinate surfaces and the edge (rim) is a smooth line. Then we discuss the analytical solution of the Stokes problem in more complicated domains, such as a circular cone, a rectangular trihedral corner and a 3D rectangular cavity. The Moffatt eddies in such domains are described. In the second part of the study we consider the laminar mixing process in the Stokes flow in a 3D container. We show that in 3D flows a much richer variety of mixing regimes is observed than in 2D flow configurations. The mixing processes in a 3D flow, containing periodic lines, possess essentially two-dimensional characteristics. In the flows, where only isolated periodic points exist, the liquid elements stretch or compress in all three directions.

Key Words: Stokes flow, Moffatt eddies, mixing, periodic lines.

Статтю представив академік НАН України Перестюк М.О.

Stokes flows. Analytical approach

In 1996, V.V. Meleshko published a paper which addressed a general analytical approach for solving the problem of two-dimensional steady Stokes flow in a rectangular cavity [1]. The flow inside the cavity was caused by a nonzero velocity distribution over two opposite sides of the cavity. The analytical approach called the superposition method was proposed for solving this problem. The principal idea of the method consists in representing the velocity field in a cavity as a sum of several velocity fields. Each of these velocity fields is represented in the form of Fourier series with arbitrary unknown coefficients. The terms of these series satisfy the governing equations and the superposition provides sufficient arbitrariness for fulfilling the boundary condition. In [1] the 2D rectangular cavity was treated as the intersection of two infinite strips. So, the solution of the Stokes problem was represented as the superposition of the well-known solutions for these infinite strips. The satisfaction of the boundary conditions led the author to an infinite system of linear algebraic equations, which was regular and consequently could be solved by the method of reduction. In the case of non-smooth distribution of the velocity the convergence of the series was not sufficient to provide us with enough accuracy. But it was shown that knowledge of asymptotic behaviour of the unknowns could considerably improve the convergence of the series and consequently the accuracy of the solution. It was shown that the first term of the asymptotics was responsible for the discontinuity at the corner of the cavity.

Later we decided that the method of superposition suggested in [1] could be applied to more complicated 3D problems. The simplest of them is the Stokes problem in a finite circular cylinder. The solution of this problem was published in [2]. In this case the cylinder was treated as intersection of the infinite cylinder and the infinite layer. So, the solution of the Stokes problem was represented as the superposition of the solutions for these two geometries. For the layer part the velocity field was expressed in the form of Dini and Fourier-Bessel series. For the cylinder part the velocity was represented in the form of Fourier series. The asymptotic behaviour of the unknowns was established by the Mellin transform technique developed by Gomilko and Meleshko and it was shown that the main term of the asymptotics was responsible for the discontinuity of the velocity at the rim. Knowledge of the main asymptotic term allowed the authors to separate and present in closed form the part of solution containing the discontinuity at the rim. It was established that the asymptotic behaviour of the velocity field near the rim coincides with the well-known Goodier-Taylor solution in a 2D wedge with constant tangential velocity applied at its sides. That is why the Stokes problem in a finite cylinder can be considered as the simpliest 3D problem. Smoothness of the cylinder's rim makes the velocity singularity near the rim to actually have 2D nature. However the streamlines near the calm rim were completely different form the well-known 2D Moffatt eddies. Because of the third coordinate the streamlines in the plane of symmetry had the form of spirals. A small displacement form the symmetry plane made the streamline really three-dimensional. It moved away from the symmetry plane at one corner and approaches the plane at another corner.

The Stokes problem solution becomes much more complicated in domains with 3D singular points at the surface. For example, a cuboid has eight trihedral corners. The unknowns of the solution depend on two indexes and therefore the technique for improvement of the series convergence based on the asymptotic analysis cannot be applied.

The simplest example of the 3D singular point at the surface is the vertex of a cone. This problem was considered in [3]. The flow inside a circular cone was induced by a non-zero velocity prescribed at the boundary within a ring at some the distance from the vertex. Therefore the problem was a 3D analogous of the 2D corner flow considered by Moffatt [4]. Since this canonical domain is bounded with a coordinate surface in the spherical coordinate system, it is not necessary to apply the superposition method. The solution of the problem was represented in the form of a Fourier series on the trigonometric system $\cos(m\varphi)$, where φ was the angular coordinate. The solution was constructed for each term by use of the Mellin transform. The contribution of each term of the Fourier expansion to the local velocity field near the vertex was studied. The following two conclusions were drawn: 1) at any m, providing α is less than the critical value, there exists a sequence of eddies near the vertex; 2) when the Fourier expansion of the velocity prescribed at the boundary includes several terms, the term at $m=1$ will dominate near the vertex. If this term is missing in the Fourier expansion, the term at $m=2$ will dominate. If both the terms are missing, the axisymmetric term ($m=0$) will play a leading role.

The Stokes flow in a trihedral rectangular corner was studied in [5]. The flow was induced by a non-zero velocity distribution over one of the corner's walls. We considered two possible motions of the wall: uniform translation or rotation about arbitrary point. Such a situation occurs for example in a cubic cavity where the motion of a lid generates the fluid motion. As mentioned above, the unknown coefficients depend now on two indexes and consequently the technique for investigating the asymptotic behaviour of the unknowns, used in 2D problems, is useless in 3D case. Within the limits of the linear approximation of the problem, this difficulty may be readily overcome provided the local behaviour of the flow field near the points of discontinuity is known. Indeed, since the superposition principle is valid, the solution may be presented as a superposition of the known solution responsible for the velocity discontinuity and some new solution satisfying continuous boundary conditions. The solution of the Stokes problem in a trihedral corner was constructed with the method of superposition. For this purpose three spherical coordinate systems with a common origin O at the vertex were introduce. These systems were such that each of the corner walls lay in the equatorial plane of the corresponding coordinate system. In order to obtain sufficient functional arbitrariness to fulfil all the boundary conditions at the walls we represent the velocity field as the superposition of three vector fields. Each term of this superposition was a solution of the Stokes equations in the corresponding spherical coordinate system. The velocity field was represented as a local similarity solution $U(r,\theta,\varphi)=r^n\cdot u(\theta,\varphi)$. This representation reduces dimensionality and allows us to formulate the boundary value problem for the corresponding spherical triangle. Now the problem becomes 2D since it is formulated at a spherical surface. Satisfaction of the boundary conditions leads to the infinite system of linear equations. But now the unknown coefficients depend on one index. So, the asymptotic analysis by means of Mellin transform developed by Gomilko and Meleshko can now be applied. The analysis of the asymptotic behaviour of

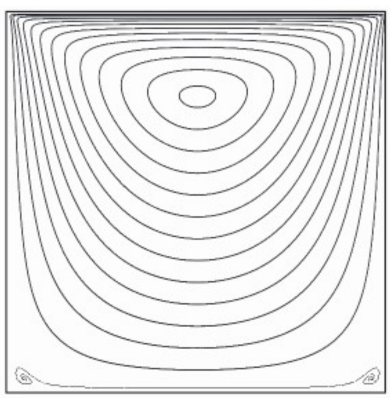

Fig. 1. Streamlines in the symmetry plane. The top wall is moving

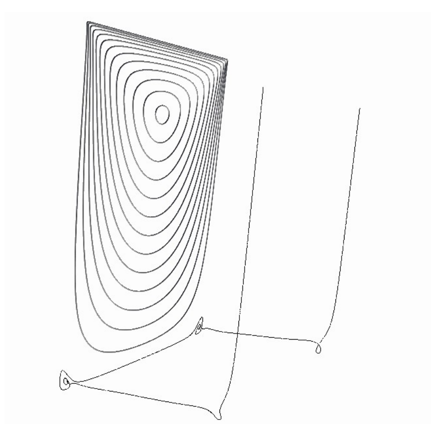

Fig. 2. Streamlines. The top wall is moving

the unknowns provided a way of refining the technique to obtain high accuracy everywhere in the corner. The velocity field was shown to behave near the edges, where the discontinuity of the tangential velocities was preassigned, in accordance with the Goodier-Taylor solution. The numerical analysis of the flow topology near the edge formed by two fixed walls conformed the existence of eddies. The flow driven by the wall rotation about the vertex was shown to be strictly 2D. In this case the streamlines lay on a spherical surface. The flow became essentially 3D when the wall rotated about a centre displaced from the vertex. The centrelines of the corner eddies contained also saddle points.

In 2005 paper [6] was published by Albensoeder and Kuhlmann. They investigated the flow in a lid-driven cubic cavity and concluded that the singularities in the boundary conditions cannot be fully removed by solution [5]. They said that if we compose the superposition of solutions [5] in every of the eight corners then the singularity at the edge connecting two neighbouring vertices is given twice. If we take the solution [5] in every corner with weight 1/2, then the singularity at the edge is now formulated correctly, but at the vertices of the trihedral corners we take only a half of the singularity into account. In the present paper we show haw to construct the singular solution to extract all the singularities correctly.

The velocities prescribed at the surface of the cuboid are discontinuous at the eight edges and at the eight vertices. So, along the edges between the moving and the stationary walls the flow fields are singular. Since the local asymptotic solutions near both the edges and the vertices are known, we may eliminate these singularities. The flow fields may be presented as

$$\mathbf{U}=\mathbf{u}+\mathbf{u}_s,\qquad P=p+p_s\,. \tag{1}$$

Here $\mathbf{u}_s$ and p_s represent the singular solution of the Stokes problem which satisfies the discontinuous boundary condition. Then the flow fields **u** and *p* are continuous. The singular solution of the Stokes problem may be constructed as a superposition of the local flow fields in the trihedral rectangular corners [5]. But in this case the singularity at any edge is taken care of twice [6]. Therefore we have to subtract one singular solution valid at the dihedral corner formed by two planes. The singular solution takes the form:

$$\mathbf{u}_s=\sum_{i=1}^{8}\mathbf{v}_i-\sum_{j=1}^{8}\mathbf{e}_j\ , \tag{2}$$

where $\mathbf{v}_i$ is the singular solution in the trihedral corner with vertex *i* and $\mathbf{e}_j$ is the singular solution in the dihedral corner with edge *j*. The solutions $\mathbf{v}_i$ are given in [5]. The solutions $\mathbf{e}_j$ are the well-known Goodier-Taylor solution or the shear flow for a dihedral corner where the moving wall slides in the direction parallel to the edge.

Extraction of all the discontinuities with (1), (2) from the velocity field allowed us to satisfy the boundary conditions with the accuracy of within 10^{-4}. Fig.1 shows the streamlines in the plane of symmetry when only the top wall is moving constantly, while the other walls are fixed. We can see that the streamlines in the corner eddies are spirals. The same effect was observed in a circular cylinder [2]. In fig.2 we can see that even a small displacement from the symmetry plane gives a really 3D streamline. It approaches a symmetry plane at one angle and moves away at the opposite one.

Periodic points. Structure of manifolds

According to Brauer's fixed point theorem, any continuous incompressible mapping of a closed domain into itself in a finite-dimensional Euclidean space has a fixed point. Obviously, such a fixed point of Poincare map will be a periodic point of flow. The location and type of periodic points determine the nature, intensity and quality of mixing.

In terms of Poincare map, a periodic point with period T is a fixed point. If the Poincare map is defined as $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$, then for a periodic point with period T we have $\mathbf{P} = f(\mathbf{P})$. In 2D flows, a periodic point can be classified as elliptic, hyperbolic or parabolic depending on the field structure in a vicinity of the periodic point. In 3D flows, the classification of periodic points is much more complicated [7]. Let some material particle close to point $\mathbf{P}$ be in position $\mathbf{P} + d\mathbf{x}_0$ at the initial moment of time. After a period, this material particle moves into the position $\mathbf{P} + d\mathbf{x}_1 = f(\mathbf{P} + d\mathbf{x}_0)$. After linearization, we obtain $d\mathbf{x}_1 = \mathbf{F} d\mathbf{x}_0$ where $\mathbf{F} = \partial f(\mathbf{x}) / \partial \mathbf{x} \,|\, \mathbf{x} = \mathbf{P}$ is the Jacobi matrix at point $\mathbf{P}$. The properties of stability of mapping f are determined by the eigenvalues of matrix $\mathbf{F}$. The eigenvalues are the roots of the equation $|\mathbf{F} - \lambda \mathbf{E}| = 0$, which in the three-dimensional case takes the following form:

$$\lambda^3 - J_1 \lambda^2 + J_2 \lambda - J_3 = 0, \qquad (3)$$

where the three invariants of the matrix are defined as follows $J_1 = \mathrm{tr}(\mathbf{F})$, $2J_2 = \mathrm{tr}^2(\mathbf{F}) - \mathrm{tr}(\mathbf{F}^2)$, $J_3 = \det(\mathbf{F})$.

All the elements of matrix $\mathbf{F}$ were calculated numerically for the Stokes problem in a cylinder. The condition of fluid incompressibility $J_3 = 1$ was used to control accuracy of the numerical calculations. The numerical error did not exceed 10^{-3} near the surface of the cylinder and took much smaller values inside the calculation area. Obviously, equation (3) always has one real positive root. Let it be λ_1. Then equation (3) can be represented as follows

$$(\lambda - \lambda_1)\left(\lambda^2 + (\lambda_1 - J_1)\lambda + \frac{1}{\lambda_1}\right) = 0. \qquad (4)$$

The other two eigenvalues are

$$\lambda_{2,3} = \frac{(J_1 - \lambda_1)}{2} \pm \sqrt{D}, \qquad (5)$$

where the discriminant

$$D = \frac{(J_1 - \lambda_1)^2}{4} - \frac{1}{\lambda_1}. \qquad (6)$$

Eigenvalues λ_2, λ_3 can take either real or complex conjugate values. And the transition occurs at $D = 0$, i.e. at $\lambda_2 = \lambda_3 = \pm 1/\sqrt{\lambda_1}$. It is now obvious that in 3D flows we can distinguish nine types of periodic points, depending on the values of λ_1 and D. Let us describe the periodic point classification in more detail. If $\lambda_1 = 1$, it means that there is no stretching or shrinking in the direction of the corresponding eigenvector. In the canonical coordinate system $(\eta^{(1)}, \eta^{(2)}, \eta^{(3)})$, the axes of which are determined by the eigenvectors, the condition $\lambda_1 = 1$ means that $\eta^{(1)} = f(\eta^{(1)})$. Then for the other two eigenvalues the condition $\lambda_2 \lambda_3 = 1$ is fulfilled, i.e. in the canonical plane $(\eta^{(2)}, \eta^{(3)})$ mapping f satisfies the condition of incompressibility. Thus, the case $\lambda_1 = 1$ is essentially two-dimensional. Such periodic points can be divided into three classes: elliptic, hyperbolic and parabolic. The condition $\lambda_1 = 1$ is fulfilled at the points of periodic lines. Indeed, there is no stretching or shrinking in the tangential direction of the periodic line, i.e. the tangential direction of the periodic line corresponds to the direction of eigenvector $\mathbf{n}_1$. The classification of points of a periodic line is equivalent to the classification of periodic points in 2D flows. Therefore the periodic lines can be divided into elliptic and hyperbolic segments that connect at the parabolic points. It follows from (6) that the point of a periodic line is elliptic if $-1 < J_1 < 3$. In this case $D < 0$, and therefore the eigenvalues $\lambda_{2,3}$ are complex conjugate.

If $J_1 < -1$ or $J_1 > 3$, then $D > 0$, and therefore such a point of the periodic line is hyperbolic. In 2D systems, the elliptic points are surrounded by islands of regular motion. The Poincaré map in such islands is a rotation around the elliptic periodic point. The liquid inside the elliptical region does not leave this region. It becomes isolated from the rest of the liquid. In 3D domains the elliptic segments of the periodic lines are also surrounded by pipe-shaped islands. The hyperbolic points in 2D systems are associated with stable and unstable manifolds, which are the lines along which fluid particles are shrunk or stretched. In the 3D case, the stable and unstable manifolds associated with a hyperbolic segment of the periodic line combine and merge into the surfaces. Thus, the Poincare map in a vicinity of a periodic line is essentially 2D in nature.

If $\lambda_1 \neq 1$, then such a periodic point is isolated.

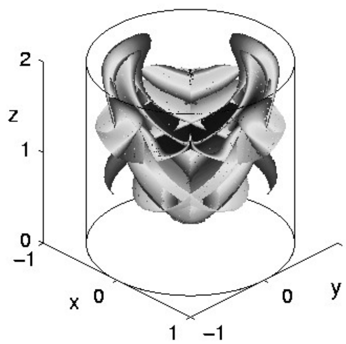

Fig. 3. Stable and unstable manifolds. Problem 1.

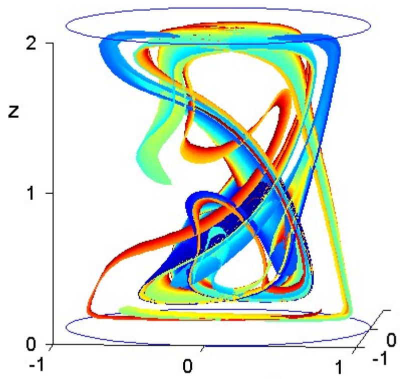

Fig. 4. Stable and unstable manifolds. Problem 2.

The Poincare map in the vicinity of such a periodic point is essentially three-dimensional. The classification of isolated points is much more complicated as compared with the classification of periodic lines. The phase portraits in the canonical coordinate system are given in [7]. In case $D > 0$, all three eigenvalues λ_i are real. Phase portraits in the coordinate planes are two saddles and a node. In the case of $D < 0$, the eigenvalues $\lambda_{2,3}$ acquire complex conjugate values that are responsible for rotation in the plane $(\eta^{(2)}, \eta^{(3)})$. The phase portrait in this plane is a hyperbolic focus. In the degenerate case $D = 0$ we have two equal eigenvalues $\lambda_2 = \lambda_3 = \pm 1/\sqrt{\lambda_1}$. In the plane $(\eta^{(2)}, \eta^{(3)})$, the phase portrait is a degenerate node. Thus, when $\lambda_1 \neq 1$ a periodic point is isolated and can be classified as a hyperbolic node or a hyperbolic focus. It is obvious that the phase portrait of isolated periodic points does not have closed lines or surfaces. Therefore isolated islands of regular motion similar to those formed around the elliptical points of the periodic lines cannot be formed around isolated periodic points. In a flow that has only isolated periodic points and no periodic lines, the liquid must be mixed throughout the flow area.

Any isolated periodic hyperbolic point is associated with the surface or the line called manifolds. The stable manifold $W^s(\mathbf{P})$ is a set of points that approach a periodic point $\mathbf{P}$ in the process of Poincare map. That is for any point $\mathbf{x} \in W^s(\mathbf{P})$ we have $f^n(\mathbf{x}) \to \mathbf{P}$ at $n \to \infty$. An unstable manifold $W^u(\mathbf{P})$ is a set of points that tend to $\mathbf{P}$ in the reverse process, i.e. for an arbitrary point $\mathbf{x} \in W^u(\mathbf{P})$ we have $f^{-n}(\mathbf{x}) \to \mathbf{P}$ at $n \to \infty$. In a two-dimensional case, stable and unstable manifolds are two lines that intersect infinitely many. Since there are infinitely many points of intersection of the stable and unstable manifolds, they concentrate when approaching the periodic point. The manifolds form a complex layered structure. Due to this phenomenon the mixing of fluid represents the elongation of a liquid spot along the unstable manifold and multiple folding when passing through homoclinic points. Such elongation and folding of a liquid spot is called

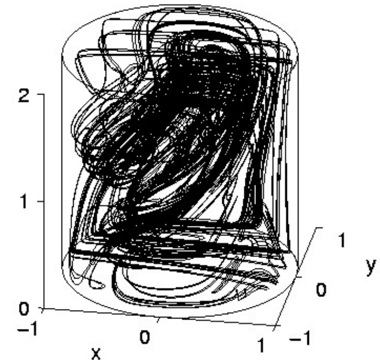

Fig. 5. Unstable manifold. Problem 3.

a horseshoe map. It is obvious that in three-dimensional flows with periodic lines, the mixing mechanism is similar to that in two-dimensional flows. In such a quasi-two-dimensional mapping the liquid does not mix along the periodic line. The elongation and folding of the fluid elements occurs only in the plane perpendicular the periodic line. Although this horseshoe map also leads to a complex layered structure, but the presence of only a two-dimensional mixing mechanism does not allow us to obtain a high-quality homogeneous mixture in the entire three-dimensional region of the flow.

The above theoretical considerations are illustrated by the example of three possible periodic flows generated in a finite cylinder [7]. In all three cases, the flows in the cylinder are generated by the periodic motion of the cylinder's lids. The lateral surface of the cylinder is stationary while the end faces move periodically. Three typical protocols of the lids motion were considered in [7]: 1) Problem 1. Only the upper end wall moves. The bottom wall remains stationary. The motion of the upper wall is zigzag. During the first half of the period, the upper end wall moves with a constant velocity in the direction of the x-axis, during the second half of the period in the direction of the y-axis. 2) Problem 2. The upper and lower walls move alternately in opposite directions parallel to the X-axis. 3) Problem 3. The protocol of motion of the end walls consists of three different steps used in Problems 1 and 2. The flows in the first two problems have periodic lines, while the flow in problem 3 has only isolated periodic points. Thus, according to the theory described above, the flows in problems 1 and 2 have 2D nature. Although the mixing mode may be chaotic, the passive impurity spot cannot be mixed throughout the area due to its two-dimensional nature. In addition, in such a flow there may be zones of regular mixing. The islands around the elliptical segments of the periodic lines are formed, which also prevent the good mixing. The flow of Problem 3 has only isolated periodic points and therefore the islands of regular motion cannot be formed around such points. The passive impurity spot should be mixed throughout the flow area.

The stable and unstable manifolds of periodic points and lines for Problems 1-3 are shown in fig. 3-5. The results of numerical calculation confirm the theoretical reasonings outlined above. In the flows of Problems 1-2 the passive impurity spot will not move throughout the domain. The flow of Problem 3, where only isolated periodic points exist, provides effective mixing of fluid.

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