International Journal of Computer and Communication Technology

Volume 8 | Issue 3 Article 1

January 2022

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Recommended Citation

Verma, Shailesh; Ghosh, Aniruddha; Ganguly, Anindita; and Chakraborty, Nilotpal (2022) "ANALYSIS OF TIME-DELAYED NON-LINEAR EQUATIONS USING HF FUNCTIONS," *International Journal of Computer and Communication Technology*: Vol. 8 : Iss. 3 , Article 1.
DOI: 10.47893/IJCCT.2022.1420
Available at: https://www.interscience.in/ijcct/vol8/iss3/1

ANALYSIS OF TIME-DELAYED NON-LINEAR EQUATIONS USING HF FUNCTIONS

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Abstract - **The paper deals with the analysis of non-linear time delayed differential equations solved using HF functions. The analysis is first performed on Mackey-Glass Equation, which is a standard model for quantitative characterization of chaotic dynamics. The procedure is then performed on a generalized Human respiratory control model, where for different simulation parameters the analysis of Cheyne-Stokes Breathing is done. Both models are simulated in MATLAB. The graphs thus generated are used to provide suitable conclusions.**

Keywords: cheyne-stokes, mackey-glass, Hill function, ventilation, chemo-reflex, MATLAB

1. INTRODUCTION

Mackey-Glass Equation is an infinite dimensional system, unlike other dynamical systems, such as the Lorenz equation and the Rossler equation, which are low dimensional. Though there has been no proof of chaos in the Mackey-Glass equation, advanced studies are undergoing in understanding the properties of this delayed differential equations, such as equation (1), that contain both exponential decay and non-monotonic delayed feedback. Originally, Mackey and Glass together presented equations having form like that of equation (1) to demonstrate the emergence of complex dynamics in physiological control systems with the help of bifurcations in the dynamics. The paper extends this research to a generalised model of Human Ventilation System. The non-linear time delayed equations are solved using Hybrid functions (HF) [3]. These are piecewise functions composed of the Right-handed triangular function (RHTF) & Sample and hold function (SHF). Approximate equations are obtained and inputting such as suitable algorithms the graphs are studied to analyse following equations. The method is easy to implement and yields very accurate results.

1.1 Mackey-Glass Equation [2]

The Mackey-Glass equation is a non-linear differential equation which is time delayed and is stated as:

$$\frac{dx}{dt} = \beta \frac{x_\tau}{1 + x_\tau^n} - \gamma x \tag{1}$$

Where τ, n, β, γ are real numbers and n is positive, and x_τ signifies the value of the x at time $(t-\tau)$. τ is the delay.

1.2 Modeling of the Lungs [1]

In normal conditions, the level of CO_2 in the arterial blood controls breathing almost exclusively. In fact, ventilation is highly sensitive to c, i.e. the partial pressure of CO_2 in arterial blood. We use Hill function, of the form below, to describe the dependence of the ventilation V on c. Here $V_{\max}$ signifies the maximum ventilation possible, whereas the parameter m and the Hill coefficient n, both are positive constants whose values are obtained from experimental data.

$$V = V_{\max} \frac{c^n(t-\tau)}{m^n + c^n(t-\tau)} \tag{2}$$

It is assumed that the exclusion of CO_2 from the blood is proportional to the CO_2 level in the blood and the product of the ventilation. Let p signify the constant metabolic production rate of CO_2 in the body. Then the CO_2 level dynamics is modeled by:

$$\frac{dc(t)}{dt} = p - kVc(t) = p - kV_{\max} \frac{c^n(t-\tau)}{m^n + c^n(t-\tau)} c(t)$$
(3)

Where, k is a positive parameter which is also obtained from experimental data. Here τ, delay time, is the time between the monitoring by the chemo-receptors in the brainstem & oxygenation of the blood in the lungs. Qualitative characteristics of both normal and abnormal breathing can be reproduced by this equation.

1.3 Cheyne-Stokes Breathing

Cheyne-Stokes respiration (CSR) is one of the several types of unusual breathing with repeated apneas. They were first reported in patients with heart failure or stroke, later it was recognized both in other diseases and as a component of the sleep apnea syndrome. Cheyne-Stokes breathing is an anomalous pattern of breathing recognized by progressively deeper and sometimes faster breathing. It is followed by a gradual decrease that results in a temporary stop in breathing, also called an apnea. This model is repetitive, with each cycle usually taking 30 seconds to 2 minutes.

2. NUMERICAL METHOD TO SOLVE MACKEY GLASS EQUATIONS:

The Mackey-Glass function can be re-written as:

$$\frac{dx}{dt} = \beta \frac{x(t-\tau)}{1+(x(t-\tau))^n} - \gamma x(t)$$
(4)

Where, β, γ, n, τ are real numbers and the delayed function of $x(t)$ is given by $x(t-\tau)$. In this section we are going to formulate an approximate numerical method in the HF domain to solve the above equation.

In the HF domain the unknown function $x(t)$ is represented as:

$$x(t) \approx C_{SX}^T S_m(t) + C_{TX}^T T_m(t) \qquad (5)$$

Here C_{SX}^T and C_{TX}^T are computed as $C_{SXi}=x(ih)$ and $C_{TXi}=x((i+1)h)-x(ih)$.

Assume, that $x(t)=x_r$ for $t\leq 0$ so the delayed function $x(t-\tau)$ can be written in HF domain as:

$$x(t-\tau) \approx C_{SX\tau}^T S_m(t) + C_{TX\tau}^T T_m(t)$$
(6)

where, for a value $\mu=\tau/h$,the elements of $C_{SX\tau}^T$ and $C_{TX\tau}^T$ are computed as:

$$c_{SX\tau i} = \begin{cases} x_r & i\leq\mu \\ x((i-\mu)h) & \mu<i<m \end{cases}$$

$$c_{TX\tau i} = \begin{cases} 0 & ,i\leq\mu \\ x((i-\mu+1)h)-x((i-\mu)h) & ,\mu<i<m \end{cases}$$

Using HF Theorems and (6) we can write

$$x(t-\tau)^n \approx \left(C_{SX\tau}^T\right)^n S_m(t) + \left(C_{TX\tau}^T\right)^n T_m(t)$$
(7)

For simplicity let us assume

$$g(t) = \frac{x(t-\tau)}{1+\left(x(t-\tau)\right)^n}$$

The HF approximate of $g(t) \approx C_{SG}^T S_m(t) + C_{TG}^T T_m(t)$

where, C_{SG}^T and C_{TG}^T are computed as:

$$c_{SGi} = \frac{c_{SX\tau i}}{1+\left(c_{SX\tau i}\right)^n}$$

$$c_{TGi} = \frac{c_{TX\tau i}}{1+\left(c_{TX\tau i}\right)^n}$$

Integrating both sides of equation (4) we can write:

$$x(t) - x(0) = \beta\int_0^t \left(\frac{x(p-\tau)}{1+\left(x(p-\tau)\right)^n}\right)dp - \gamma\int_0^t x(p)dp$$

$$= \beta\int_0^t g(p)dp - \gamma\int_0^t x(p)dp \qquad t \in (0,T)$$

Using (5), (6), (7) and HF theories the HF estimate of the above equation is

$$C_{SX}^T S_m(t) + C_{TX}^T T_m(t) - C_{SX0}^T S_m(t)$$

$$= \left(\beta\left(C_{SG}^T + \frac{C_{TG}^T}{2}\right) - \gamma\left(C_{SX}^T + \frac{C_{TX}^T}{2}\right)\right)\left(P1 S_m(t) + h \mathrm{I}_m T_m(t)\right)$$

Equating the coefficients of $S_m(t)$ and $T_m(t)$ from the above equation

$$C_{SX}^T - C_{SX0}^T = \beta\left(C_{SG}^T + \frac{C_{TG}^T}{2}\right)P1 - \gamma\left(C_{SX}^T + \frac{C_{TX}^T}{2}\right)P1 \qquad (8)$$

$$C_{TX}^T = \beta\left(C_{SG}^T + \frac{C_{TG}^T}{2}\right)h\,\mathrm{I}_m - \gamma\left(C_{SX}^T + \frac{C_{TX}^T}{2}\right)h\,\mathrm{I}_m \qquad (9)$$

Employing HF properties, (8) can be written as

$$C_{SX}^{T} = \left(\beta C_{SG}^{T} - \gamma C_{SX}^{T}\right) P + \frac{h}{2}\left(\beta C_{SG0}^{T} - \gamma C_{SX0}^{T}\right) + C_{SX0}^{T}$$

$$\sum_{i=0}^{m-1} c_{sxi} - \sum_{i=0}^{m-1}\left(\beta c_{SGi} - \gamma c_{sXi}\right) P = \frac{h}{2}\sum_{i=0}^{m-1}\left(\beta c_{SX0} - \gamma c_{TX0}\right) + C_{SX0}^{T} = \text{constant}$$

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

From the above expression, the *(i+1)*th and i^{th} iteration is given as

$$c_{SXi+1} - \left(\beta c_{SGi+1} - \gamma c_{SXi+1}\right) P = c_{SXi} - \left(\beta c_{SGi} - \gamma c_{SXi}\right) P$$

(10)

$$c_{SXi+1} - \frac{h}{2}\left(\beta c_{SGi+1} - \gamma c_{SXi+1}\right) - h\sum_{j=0}^{i}\left(\beta c_{SGj} - \gamma c_{SXj}\right)$$

$$= c_{SXi} - \frac{h}{2}\left(\beta c_{SGi} - \gamma c_{SXi}\right) - h\sum_{j=0}^{i-1}\left(\beta c_{SGj} - \gamma c_{SXj}\right)$$

$$c_{SXi+1} - \frac{h}{2}\left(\beta c_{SGi+1} - \gamma c_{SXi+1}\right) = c_{SXi} + \frac{h}{2}\left(\beta c_{SGi} - \gamma c_{SXi}\right)$$

$$c_{SXi+1} = \frac{\left(\frac{2}{h} - \gamma\right) c_{SXi} + \beta\left(c_{SGi+1} + c_{SGi}\right)}{\left(\frac{2}{h} + \gamma\right)}$$

(11)

Equation (11) is a recursive formula to find the coefficients of SHF.

Similarly, (9) can also be written as

$$C_{TX}^{T} = \beta\left(C_{TG}^{T} P1 + C_{TG0}^{T} + \frac{hC_{TG}^{T}}{2}\right) - \gamma\left(C_{TX}^{T} P1 + C_{TX0}^{T} + \frac{hC_{TX}^{T}}{2}\right)$$

$$= \left(\beta C_{TG}^{T} - \gamma C_{TX}^{T}\right) P + \left(\beta C_{TG0}^{T} P - \gamma C_{TX0}^{T}\right)$$

(12)

Solving, Equation (12) we find a recursive formula similar to (11) given as:

$$c_{TXi+1} = \frac{\left(\frac{2}{h} - \gamma\right) c_{TXi} + \beta\left(c_{TGi+1} + c_{TGi}\right)}{\left(\frac{2}{h} + \gamma\right)}$$

(13)

Both the recursive formulas are helpful as they convert a non-linear differential equation into a set of algebraic equations in the HF domain. Combining (11) and (13) we can find the value of the unknown function *x(t)* in the HF domain.

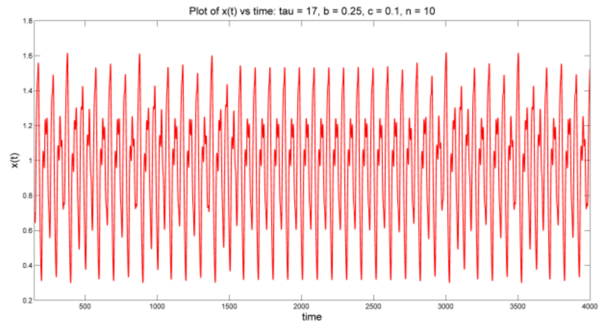

Fig. 1: Plot of *x(t)* vs time, τ = 17, *b* = 0.25, *c* = 0.1, *n*=10.

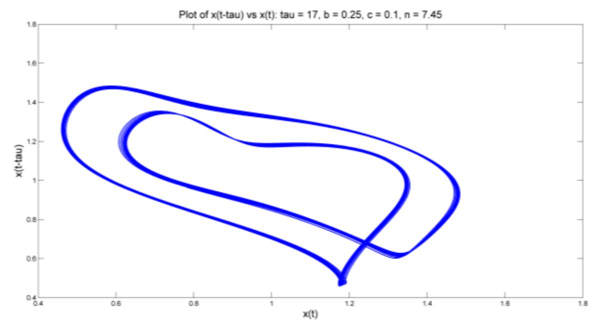

Fig.2: Plot of *x(t)* vs *x(t-τ),* τ = 17, *b* = 0.25, *c* = 0.1, *n*=7.45.

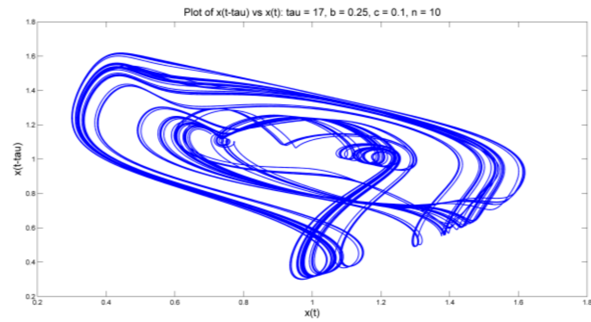

Fig.3: Plot of *x(t)* vs *x(t-τ),* τ = 17, *b* = 0.25, *c* = 0.1, *n* = 10.

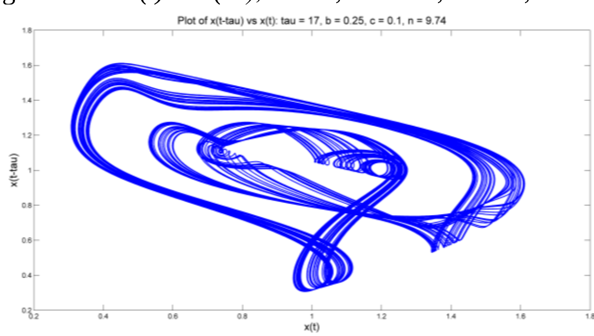

Fig. 4: Plot *x(t)* vs *x(t-τ),* τ=17, *b*=0.25, *c*=0.1, *n*=9.74.

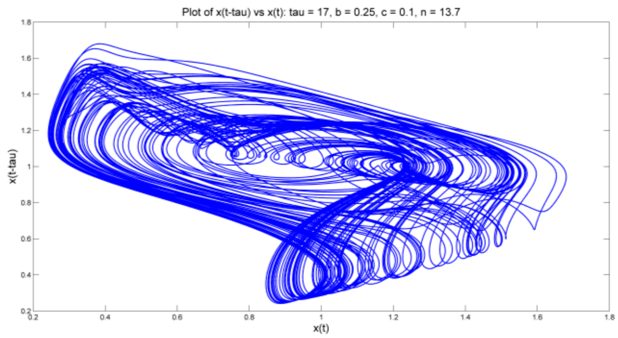

Fig. 5: Plot of x(t) vs x(t-tau), τ=17, b=0.25, c=0.1, n=13.7

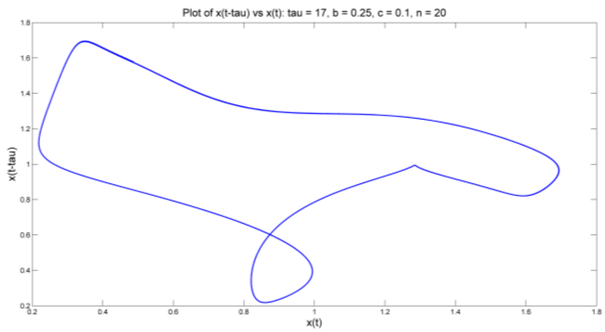

Fig. 6: Plot of $x(t)$ vs $x(t-\tau)$, $\tau = 17$, $b = 0.25$, $c = 0.1$, n=20.

3. NUMERICAL METHOD TO SOLVE EQUATIONS OF CHEYNE-STOKES RESPIRATION

At first we introduce the non-dimensional quantities

$$x = \frac{c}{m}, t^* = \frac{pt}{m}, \tau^* = \frac{pT}{m}, \alpha = \frac{mbV_{\max}}{p}, V^* = \frac{V_{\max}}{V}$$

And the model of equation (3) changes to:

$$\frac{dx(t)}{dt} = 1 - \alpha x(t) \frac{x^n(t-\tau)}{1 + x^n(t-\tau)}$$

(14)

For notational simplicity we have removed the asterisk signs from the parameters. Cheyne-Stokes breathing is often seen in patients who have increased delay times between the blood oxygenation in the lungs and chemo-receptors stimulation in the brainstem, and also has increased sensitivity to CO_2. In this section we are going to formulate an approximate numerical method in the HF domain to solve the above equation. In the HF domain the unknown function $x(t)$ may be represented as:

$$x(t) \approx C_{SX}^T S_m(t) + C_{TX}^T T_m(t)$$

(15)

Where, C_{SX}^T and C_{TX}^T are computed as $c_{SXi}=x(ih)$ and $c_{TXi}=x((i+1)h)-x(ih)$.

Assume, that $x(t)=x_r$ for $t\leq 0$, so the delayed function $x(t-\tau)$ can be written in HF domain as:

$$x(t-\tau) \approx C_{SX\tau}^T S_m(t) + C_{TX\tau}^T T_m(t)$$

(16)

Where, for a value $\mu = \frac{\tau}{h}$, the elements of $C_{SX\tau}^T$ and $C_{TX\tau}^T$ are computed as

$$c_{SX\tau i} = \begin{cases} x_r & i \leq \mu \\ x((i-\mu)h) & \mu < i < m \end{cases}$$

$$c_{TX\tau i} = \begin{cases} 0 & ,i \leq \mu \\ x((i-\mu+1)h) - x((i-\mu)h) & ,\mu < i < m \end{cases}$$

Using HF Theorems and (6) we can write

$$x(t-\tau)^n \approx \left(C_{SX\tau}^T\right)^n S_m(t) + \left(C_{TX\tau}^T\right)^n T_m(t) \qquad (17)$$

For simplicity let us assume

$$g(t) = \frac{\left(x(t-\tau)\right)^n}{1+\left(x(t-\tau)\right)^n}$$

The HF approximate of $g(t) \approx C_{SG}^T S_m(t) + C_{TG}^T T_m(t)$

where, C_{SG}^T and C_{TG}^T are computed as:

$$c_{SGi} = \frac{\left(c_{SX\tau i}\right)^n}{1+\left(c_{SX\tau i}\right)^n} \text{ and } c_{TGi} = \frac{\left(c_{TX\tau i}\right)^n}{1+\left(c_{TX\tau i}\right)^n}.$$

Integrating both sides of (14) we can write

$$x(t) - x(0) = \int_0^t 1dp - \alpha \int_0^t \left(x(p) \frac{x(p-\tau)^n}{1+\left(x(p-\tau)\right)^n} \right) dp$$

$$= t - \alpha \int_0^t x(p) g(p) dp \qquad t \in (0,T)$$

Using (16), (17) and HF theories the HF estimate of the above equation is :

$$C_{SX}^T S_m(t) + C_{TX}^T T_m(t) - C_{SX0}^T S_m(t)$$
$$= t - \left(\alpha \left(C_{SG}^T + \frac{C_{TG}^T}{2} \right) \left(C_{SX}^T + \frac{C_{TX}^T}{2} \right) \right) \left(P1S_m(t) + hI_m T_m(t) \right)$$

Employing HF properties, we can derive,

$$C_{SX}^T = \left(\alpha C_{SG}^T C_{SX}^T\right) P + \frac{h}{2}\left(\alpha C_{SG0}^T C_{SX0}^T\right) + \sum_{i=0}^{m} ih$$

$$\sum_{i=0}^{m-1} c_{sxi} - \sum_{i=0}^{m-1}\left(\alpha c_{SGi} c_{sXi}\right)P - P = \frac{h}{2}\sum_{i=0}^{m-1}\left(\alpha c_{SX\,0} c_{TX\,0}\right) = \text{constant}$$

Where,

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Now using the same procedure as in Mackey-Glass, the recursive formula to find the coefficients of SHF is given as:

$$c_{SXi+1} = \frac{2 + \left(\frac{2}{h} - \alpha c_{SGi}\right) c_{SXi}}{\left(\frac{2}{h} + \alpha\left(c_{SGi+1}\right)\right)}$$

(18)

And the recursive formula for RHTF is given as:

$$c_{TXi+1} = \frac{2 + \left(\frac{2}{h} - \alpha c_{TGi}\right) c_{TXi}}{\left(\frac{2}{h} + \alpha\left(c_{TGi+1}\right)\right)}$$

(19)

Both the recursive formulas are helpful as they convert a nonlinear differential equation into a set of algebraic equations in the HF domain. Combining (18) and (19) we can find the value of the unknown function *x(t)* in the HF domain. Using MATLAB the graphs thus obtained are:

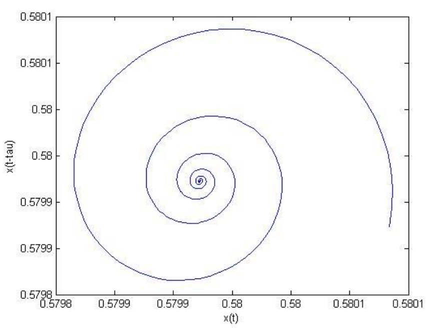

Fig.7: Plot of *x(t)* vs *x(t-τ)*, *α=28*, *n* =5, *τ* = 0.15.

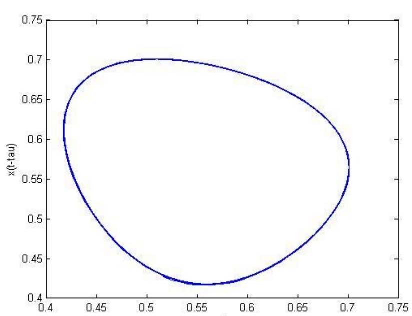

Fig.8: Plot of *x(t)* vs *x(t-τ)*, *α=28*, *n* =5, *τ* = 0.25.

4. ERROR ANALYSIS

The proposed algorithm is compared with a well-established computational method (4th order RK method) to study its efficacy.

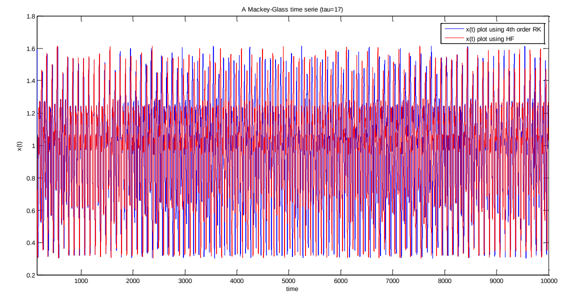

Fig9: Plot of the solution using HF (in red) and RK method (in blue)

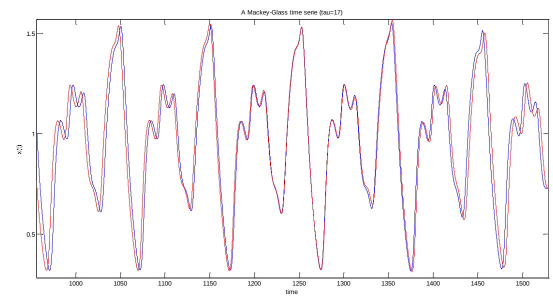

Fig10: Zoomed in view of the plot in Fig 9

Serial no	X(t) using HF	X(t) using 4th order RK	Error%
1	1.537	1.508	-1.92
2	1.065	1.047	-1.72
3	0.033	0.029	-1.38
4	1.240	1.2456	0.45
5	1.528	1.528	0
6	0.3252	0.3252	0
7	0.9827	0.9816	-0.11
8	0.638	0.642	0.62
9	1.492	1.508	1.06
10	0.6507	0.662	1.7

Fig11: Table showing the error in calculating the solution using HF taking, τ = 17, b = 0.25, c = 0.1, n=10.

As per the figures and the table, the proposed algorithm didn't vary much with the results obtained from RK method. The maximum error is less than 2% which shows the efficacy of the developed algorithm.

5. CONCLUSION

Hybrid function is emerging as an effective tool. Complicated equations can be easily solved with its least computational burden. From plots we see if there is increase in either the steepness of the CO_2 response or the delay time, the steady states become unstable. Also low amplitude oscillations or high amplitude oscillations having distinct apnea are observed. A slight deviation of less than 2% was observed in the solution obtained through the developed method with respect to a well-established method shows the efficacy of the developed algorithm. In future works we intend to look deeper into the error analysis and come up with modifications in the developed algorithm in order to obtain more accurate results.

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