Difference sets and frequently hypercyclic weighted shifts

Frédéric Bayart, Imre Ruzsa

DIFFERENCE SETS AND FREQUENTLY HYPERCYCLIC WEIGHTED SHIFTS

FRÉDÉRIC BAYART, IMRE Z. RUZSA

Abstract. We solve several problems on frequently hypercyclic operators. Firstly, we characterize frequently hypercyclic weighted shifts on $\ell^p(\mathbb{Z})$, $p \geq 1$. Our method uses properties of the difference set of a set with positive upper density. Secondly, we show that there exists an operator which is $\mathcal{U}$-frequently hypercyclic, yet not frequently hypercyclic and that there exists an operator which is frequently hypercyclic, yet not distributionally chaotic. These (surprizing) counterexamples are given by weighted shifts on c_0. The construction of these shifts lies on the construction of sets of positive integers whose difference sets have very specific properties.

1. Introduction

Let X be a Banach space and let $T \in \mathfrak{L}(X)$ be a bounded operator on X. T is called *hypercyclic* provided there exists a vector $x \in X$ such that its orbit $O(x,T) = \{T^n x;\ n \geq 0\}$ is dense in X. x is then called a hypercyclic vector for T and we shall denote by $HC(T)$ the set of T-hypercyclic vectors. The study of hypercyclic operators is a branch of linear dynamics, a very active field of analysis. We refer to the books [4] and [12] to learn more on this subject.

In 2005, the first author and S. Grivaux have introduced in [2] a refinement of the notion of hypercyclicity, called *frequent hypercyclicity*. For an operator to be frequently hypercyclic, one ask now that not only there exists a vector with a dense orbit, but moreover that this orbit visits often each nonempty open subset. To be more precise, let us introduce the following definitions and notations. For $A \subset \mathbb{Z}_+$, we denote by $A(n) = \{a \in A;\ a \leq n\}$. The lower density of A is defined by

$$\underline{d}(A) = \liminf_{n \to +\infty} \frac{\#A(n)}{n},$$

and the upper density of A is defined by

$$\bar{d}(A) = \limsup_{n \to +\infty} \frac{\#A(n)}{n}.$$

We will also use corresponding definitions and notations for subsets of $\mathbb{Z}$. For instance, for $A \subset \mathbb{Z}$ and $n \in \mathbb{Z}_+$, $A(n) = \{a \in A;\ |a| \leq n\}$.

Definition 1. $T \in \mathfrak{L}(X)$ is called *frequently hypercyclic* provided there exists a vector $x \in X$, called a *frequently hypercyclic vector for* T, such that, for any $U \subset X$ open and nonempty, $\{n \in \mathbb{Z}_+;\ T^n x \in U\}$ has positive lower density. We shall denote by $FHC(T)$ the set of frequently hypercyclic vectors for T.

This notion has then been investigated by several authors, see for instance [3], [5], [8], [10], [11], [16]. In particular, it has many connections with ergodic theory. Of course, one may also investigate the corresponding notion replacing lower by upper density, leading to the following definition introduced in [16].

Definition 2. $T \in \mathfrak{L}(X)$ is called *$\mathcal{U}$-frequently hypercyclic* provided there exists a vector $x \in X$, called a *$\mathcal{U}$-frequently hypercyclic vector for T*, such that, for any $U \subset X$ open and nonempty, $\{n \in \mathbb{Z}_+;\ T^n x \in U\}$ has positive upper density. The set of $\mathcal{U}$-frequently hypercyclic vectors for T will be denoted by $\mathcal{U}FHC(T)$.

Several basic problems remain open regarding these two refinements of hypercyclicity. In this paper, we solve several of these problems. We begin by studying the frequently (resp. the $\mathcal{U}$-frequently) hypercyclic weighted shifts on ℓ^p, $p \geq 1$. Let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive real numbers and let $p \geq 1$. The bilateral weighted shift $B_{\mathbf{w}}$ on $\ell^p(\mathbb{Z})$ is defined by $B_{\mathbf{w}}(e_n) = w_n e_{n-1}$, where (e_n) is the standard basis of $\ell^p(\mathbb{Z})$. Hypercyclicity of $B_{\mathbf{w}}$ has been characterized by H. Salas in [14]. In Section 3, we will prove the following characterization of frequently hypercyclic weighted shifts.

Theorem 3. *Let $p \in [1, +\infty)$ and let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive real numbers. The following assertions are equivalent.*

(i) $B_{\mathbf{w}}$ is frequently hypercyclic on $\ell^p(\mathbb{Z})$;
(ii) $B_{\mathbf{w}}$ is $\mathcal{U}$-frequently hypercyclic on $\ell^p(\mathbb{Z})$;
(iii) The series $\sum_{n\geq 1} \frac{1}{(w_1\cdots w_n)^p}$ and $\sum_{n<0}(w_{-1}\cdots w_n)^p$ are convergent.

A similar result holds for the unilateral weighted shift $B_{\mathbf{w}}$ on $\ell^p(\mathbb{Z}_+)$, which is defined by $B_{\mathbf{w}}(e_n) = w_n e_{n-1}$ for $n \geq 1$ and by $B_{\mathbf{w}}(e_0) = 0$.

Theorem 4. *Let $p \in [1, +\infty)$ and let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}_+}$ be a bounded sequence of positive real numbers. The following assertions are equivalent.*

(i) $B_{\mathbf{w}}$ is frequently hypercyclic on $\ell^p(\mathbb{Z}_+)$;
(ii) $B_{\mathbf{w}}$ is $\mathcal{U}$-frequently hypercyclic on $\ell^p(\mathbb{Z}_+)$;
(iii) The series $\sum_{n\geq 1} \frac{1}{(w_1\cdots w_n)^p}$ is convergent.

That $\sum_{n\geq 1} \frac{1}{(w_1\cdots w_n)^p} < +\infty$ implies the frequent hypercyclicity of $B_{\mathbf{w}}$ on $\ell^p(\mathbb{Z}_+)$ is known since [2]. A necessary condition for $B_{\mathbf{w}}$ to be frequently hypercyclic was given in [2] and this condition was improved in [11]. This last condition is the starting point of the present work. We show how to combine this condition with a result on the difference set of a set with positive upper density to prove that (ii) implies (iii) in Theorem 4.

We then investigate frequently hypercyclic weighted shifts on c_0. It is worth noting that the previous theorems cannot be extended to c_0. Indeed, in [3], a frequently hypercyclic backward weighted shift $B_{\mathbf{w}}$ is exhibited on $c_0(\mathbb{Z}_+)$ such that the sequence $(w_1\cdots w_n)^{-1}$ *does not* converge to 0 (in fact, one may require that $w_1\cdots w_n = 1$ for infinitely many n). Nevertheless, we will give in Section 4 a characterization of frequently and $\mathcal{U}$-frequently hypercyclic weighted shifts on c_0. This characterization is necessarily more difficult than that of Theorem 4. However, it will be efficient to give later in the paper nontrivial examples and counterexamples of frequently hypercyclic weighted shifts on $c_0(\mathbb{Z})$.

Interestingly, the weighted shift $B_{\mathbf{w}}$ constructed in [3] give several counterexamples in the theory of frequently hypercyclic operators:

- $B_{\mathbf{w}}$ is frequently hypercyclic, yet neither chaotic nor topologically mixing;
- $B_{\mathbf{w}}$ does not admit any nonzero invariant Gaussian measure.

In Section 5 and 6, we show that weighted shifts on c_0 can help to solve further problems on frequently hypercyclic operators. For instance, there are no examples in the literature of $\mathcal{U}$-frequently hypercyclic operators which are not frequently hypercyclic. Weighted shifts on $c_0(\mathbb{Z}_+)$ give an example.

Theorem 5. *There exists a bounded sequence $\mathbf{w} = (w_n)_{n\geq 1}$ of positive real numbers such that*

(i) *$B_{\mathbf{w}}$ is $\mathcal{U}$-frequently hypercyclic on $c_0(\mathbb{Z}_+)$;*
(ii) *$B_{\mathbf{w}}$ is not frequently hypercyclic on $c_0(\mathbb{Z}_+)$.*

Another problem that weighted shifts on c_0 can solve is related to distributionally chaotic operators. The notion of distributional chaos was introduced by B. Schweizer and J. Smítal in [15]. Let $f : X \to X$ be a continuous map on a metric space X. For each $x, y \in X$ and each $n \in \mathbb{N}$, the *distributional function* $F^n_{xy} : \mathbb{R}_+ \to [0, 1]$ is defined by

$$F^n_{xy}(\tau) := \frac{1}{n}\mathrm{card}\big\{0 \leq i \leq n-1;\ d(f^i(x), f^i(y)) < \tau\big\}.$$

Moreover define

$$F_{xy}(\tau) := \liminf_{n\to+\infty} F^n_{xy}(\tau) \text{ and } F^*_{xy}(\tau) := \limsup_{n\to+\infty} F_{xy}(\tau).$$

f is said *distributionally chaotic* if there exists an uncountable set $\Gamma \subset X$ and $\varepsilon > 0$ such that, for every $\tau > 0$ and each pair of distinct points $x, y \in \Gamma$, we have

$$F_{xy}(\varepsilon) = 0 \text{ and } F^*_{xy}(\tau) = 1.$$

When f is a linear map acting on a Banach space X, the notion of distributional chaos is related to the existence of distributionally irregular vectors.

Definition 6. Given $T \in \mathfrak{L}(X)$ and $\varepsilon > 0$, a vector $x \in X$ is a *distributional irregular vector* for T if there exists $A, B \subset \mathbb{N}$ with $\bar{d}(A) = \bar{d}(B) = 1$ such that

$$\lim_{n\to+\infty,\ n\in A} T^n x = 0 \text{ and } \lim_{n\to+\infty,\ n\in B} \|T^n x\| = +\infty.$$

It is shown in [7] that T is distributionally chaotic if and only if T admits a distributional irregular vector.

In [7], the following question is asked: are there frequently hypercyclic operators which are not distributionally chaotic? We answer this question thanks to weighted shifts on $c_0(\mathbb{Z})$.

Theorem 7. *There exists a frequently hypercyclic weighted shift on $c_0(\mathbb{Z})$ which is not distributionally chaotic.*

We conclude this paper in Section 7 by miscellaneous problems and results on frequently hypercyclic operators. In particular, we show that if T is invertible and frequently hypercyclic, then T^{-1} is $\mathcal{U}$-frequently hypercyclic.

Let us mention that a common feature of Theorems 3 to 7 is their interaction with additive number theory. To prove Theorems 3 and 4, we need a property of the difference set of a set with positive upper density. The proofs of Theorems 5 and 7 need both the construction of big sets of integers such that their difference sets are sparse enough.

2. A result on difference set

Let $A \subset \mathbb{Z}_+$ be a set with positive upper density. A well-known result of Erdös and Sarközy (see for instance [13]) ensures that the difference set $A - A$ is syndetic (namely it has bounded gaps). Our first result is a strenghtening of this property. Let us introduce, for $k \in \mathbb{Z}_+$,

$$B_k = A \cap (A - k).$$

Erdös and Sarközy proved that the set of integers k such that B_k is nonempty is syndetic. We shall prove that the set of integers k such that B_k has a big upper density is also syndetic. For convenience, we formulate this for subsets of $\mathbb{Z}$.

Theorem 8. *Let $A \subset \mathbb{Z}$ be a set with positive upper density, let $\delta = \bar{d}(A)$ and let $\varepsilon \in (0,1)$. For any $k \in \mathbb{Z}$, let $B_k = A \cap (A - k)$ and let $\delta_k = \bar{d}(B_k)$. Let also $F = \{k;\ \delta_k > (1-\varepsilon)\delta^2\}$. Then F is syndetic.*

Proof. We select a sequence (n_i) such that

$$\#A(n_i)/(2n_i + 1) \to \delta.$$

Then we select by a usual diagonal procedure a subsequence (m_i) of (n_i) such that for every $k \in \mathbb{Z}$ the limit

$$\eta_k = \lim \#B_k(m_i)/(2m_i + 1)$$

exists. Observing that $B_k = -k + B_{-k}$, one knows that $\eta_k = \eta_{-k}$. Moreover, $\eta_k \leq \delta_k$ and we shall in fact prove that

$$\mathbf{F} = \{k; \eta_k > (1-\varepsilon)\delta^2\}$$

is syndetic. Let R be a (finite) set with the property that

$$\eta_{k-l} \leq (1-\varepsilon)\delta^2$$

for all $k, l \in R$, $k \neq l$. We will see that the cardinality of such sets is uniformly bounded. We set $r = \#R$ and we put

$$f(x) = \#\{k \in R; x \in A - k\}.$$

We have

$$f(x)^2 = \#\{k, l \in R; x \in (A-k) \cap (A-l)\} = \#\{k, l \in R; x + k \in A \cap (A + k - l)\}.$$

Clearly

$$\sum_{|x| \leq m} f(x) = \sum_{k \in R} \#\big\{x \in \{-m, \dots, m\};\ x \in A - k\big\} = r\#A(m) + O(1),$$

hence

$$\frac{1}{2m_i + 1} \sum_{|x| \leq m_i} f(x) \to r\delta.$$

Similarly

$$\sum_{|x| \leq m} f(x)^2 = \sum_{k,l \in R} \#B_{k-l}(m) + O(1),$$

hence

$$\frac{1}{2m_i + 1} \sum_{|x| \leq m_i} f(x)^2 \to \sum_{k,l \in R} \eta_{k-l} \leq r\delta + (1-\varepsilon) r(r-1)\delta^2.$$

Using the inequality of arithmetic and square means we get

$$(r\delta)^2 \leq r\delta + (1-\varepsilon)r(r-1)\delta^2,$$

hence

$$r \leq \frac{1-\delta(1-\varepsilon)}{\delta\varepsilon}.$$

Now select a maximal set R (take 0, then the integer n with the smallest absolute value which can be added and so on). This procedure stops after a finite number of steps. Maximality means that for every integer n there is a $k \in R$ such that $\eta_{n-k} > (1-\varepsilon)\delta^2$, that is, $n-k \in \mathbf{F}$, which means that $\mathbf{F} + R = \mathbb{Z}$. This amounts to say that $\mathbf{F}$ is syndetic. □

Remark. The previous theorem is reminiscent from Khintchine's recurrence theorem which says the following: for any invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$, for any $\varepsilon > 0$ and any $A \in \mathcal{B}$, the set $\{n \in \mathbb{Z};\ \mu(A \cap T^n A) \geq \mu(A)^2 - \varepsilon\}$ is syndetic. It turns out that we may deduce Theorem 8 from Khintchine's recurrence theorem using Furstenberg's correspondence principle (see [9]), exactly as Furstenberg deduced the Szemerédi's theorem on arithmetic progressions from his extension of the classical Poincaré's recurrence theorem (we refer to [6] and to [9] for details).

One may prove Khintchine's recurrence theorem using the uniform version of von Neumann's ergodic theorem. One can also find in [6] a combinatorial proof of this theorem, which does not match exactly the proof of Theorem 8. To keep a self-contained exposition, we have chosen to give a complete and elementary proof of Theorem 8.

From this, we can deduce a result on series which is the key for the application to frequently hypercyclic weighted shifts.

Corollary 9. *Let $(\alpha_n)_{n\in\mathbb{Z}}$ be a sequence of nonnegative real numbers such that $\sum_n \alpha_n = +\infty$. Suppose that there exists some $C > 0$ such that either $\alpha_n \geq C\alpha_{n-1}$ for every $n \in \mathbb{Z}$ or $\alpha_{n-1} \geq C\alpha_n$ for every $n \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a set with positive upper density and let, for $n \in A$,*

$$\beta_n = \sum_{m\in A} \alpha_{m-n}.$$

Then

$$\limsup_{n\to+\infty} \frac{1}{2n+1} \sum_{|m|\leq n, m\in A} \beta_m = +\infty.$$

In particular, the sequence $(\beta_n)_{n\in A}$ cannot be bounded.

Proof. We keep the notations of the previous theorem, which we apply with $\varepsilon = 1/2$. We first show that

$$\sum_{n\in F} \alpha_n = +\infty, \tag{2.1}$$

using only that F is syndetic. Indeed, write $F = (f_n)_{n\in\mathbb{Z}}$ in increasing order with $f_0 = \min\{f \in F;\ f \geq 0\}$. There exists some $M > 0$ such that $f_{i+1} - f_i \leq M$ for every $i \in \mathbb{Z}$. Assuming first that $\alpha_n \geq C\alpha_{n-1}$ for every n, we get

$$\alpha_{f_j} \geq \frac{\max(1, C^M)}{M} \sum_{f_{j-1}<i\leq f_j} \alpha_i,$$

and (2.1) follows by summing this for all j. If $\alpha_{n-1}\geq C\alpha_n$ for every n, then we write

$$\alpha_{f_j}\geq \frac{\max(1,C^M)}{M}\sum_{f_j\leq i<f_{j+1}}\alpha_i,$$

and (2.1) follows also by summation. Now, consider the sum

$$s_i=\sum_{|n|\leq m_i,n\in A}\beta_n.$$

This can be rewritten as

$$s_i=\sum_{|n|\leq m_i,n,m\in A}\alpha_{m-n}.$$

We group this sum according to the value of $k=m-n$, and keep only those terms where $k\in F$, $|k|<l$ for some fixed l. We get

$$s_i\geq \sum_{k\in F,|k|<l}\alpha_k\#B_k(m_i).$$

We divide by $2m_i+1$ and let $i\to\infty$. We get

$$\begin{aligned}\limsup_{i\to+\infty}\frac{1}{2m_i+1}\sum_{|n|\leq m_i,n\in A}\beta_n &\geq \sum_{k\in F,|k|<l}\alpha_k\lim_{i\to+\infty}\frac{\#B_k(m_i)}{2m_i+1}\\ &\geq \sum_{k\in F,|k|<l}\alpha_k\eta_k\\ &\geq \frac{\delta^2}{2}\sum_{k\in F,|k|<l}\alpha_k,\end{aligned}$$

and this can be arbitrarily large by (2.1).

□

3. Frequently hypercyclic weighted shifts on ℓ^p

In this section, we prove Theorem 3. The proof of Theorem 4 is similar but simpler. We first prove that *(ii)* implies *(iii)*. Thus we start with a $\mathcal{U}$-frequently hypercyclic weighted shift $B_{\mathbf{w}}$ on $\ell^p(\mathbb{Z})$ and let x be a $\mathcal{U}$-frequently hypercyclic vector for $B_{\mathbf{w}}$. Let

$$A=\big\{n\in\mathbb{Z}_+;\ \|B_{\mathbf{w}}^n x-e_0\|_p\leq 1/2\big\},$$

which has positive upper density. Let $m\in A$. Then $|w_1\cdots w_m x_m-1|\leq 1/2$ so that $|w_1\cdots w_m x_m|\geq 1/2$. Now, for any $n\in A$, we can also write

$$\begin{aligned}\frac{1}{2^p} &\geq \|B_{\mathbf{w}}^n x-e_0\|_p^p\\ &\geq \sum_{m\in A,m<n}(w_m\cdots w_1w_0\cdots w_{m-n+1})^p|x_m|^p+\sum_{m\in A,m>n}(w_m\cdots w_{m-n+1})^p|x_m|^p\\ &\geq \sum_{m\in A,m<n}(w_0\cdots w_{m-n+1})^p|w_1\cdots w_m x_m|^p+\sum_{m\in A,m>n}\frac{|w_1\cdots w_m x_m|^p}{(w_1\cdots w_{m-n})^p}.\end{aligned}$$

Putting this together, we get that for any $n \in A$,

$$\left\{\begin{array}{rcl}\displaystyle\sum_{m\in A, m<n} (w_0\cdots w_{m-n+1})^p & \leq & 1\\ \displaystyle\sum_{m\in A, m>n} \frac{1}{(w_1\cdots w_{m-n})^p} & \leq & 1.\end{array}\right.$$

Firstly, we set $\alpha_n = 0$ provided $n \leq 0$ and $\alpha_n = \frac{1}{(w_1\cdots w_n)^p}$ provided $n > 0$. Because $(w_n)_{n\in\mathbb{Z}}$ is bounded, $\alpha_n \geq C\alpha_{n-1}$ for every $n \in \mathbb{Z}$. Suppose that $\sum_{n\geq 1} \frac{1}{(w_1\cdots w_n)^p} = +\infty$. Then by Corollary 9, the sequence $(\beta_n)_{n\in A}$ is unbounded, where

$$\beta_n = \sum_{m\in A} \alpha_{m-n} = \sum_{m\in A, m>n} \alpha_{m-n} = \sum_{m\in A, m>n} \frac{1}{(w_1\cdots w_{m-n})^p}.$$

This is a contradiction. Secondly, set $\alpha_n = 0$ provided $n \geq 0$ and $\alpha_n = (w_0\cdots w_{-n+1})^p$ provided $n < 0$. Because $(w_n)_{n\in\mathbb{Z}}$ is bounded, $\alpha_n \leq C\alpha_{n-1}$ for any $n \in \mathbb{Z}$. Suppose that $\sum_{n<0}(w_0\cdots w_n)^p = +\infty$. Then by Corollary 9, (β_n) is unbounded where

$$\beta_n = \sum_{m\in A} \alpha_{m-n} = \sum_{m\in A, m<n} \alpha_{m-n} = \sum_{m\in A, m<n} (w_0\cdots w_{m-n})^p.$$

This is also a contradiction, since $w_0\cdots w_{m-n} \leq Cw_0\cdots w_{m-n+1}$.

Let us now show that the condition is sufficient. This follows from a standard application of the frequent hypercyclicity criterion of [8], which we recall for convenience:

Theorem 10. *Let $T \in \mathfrak{L}(X)$, where X is a separable Banach space. Assume that there exists a dense set $\mathcal{D} \subset X$ and a map $S : \mathcal{D} \to \mathcal{D}$ such that*

(1) $\sum T^n(x)$ and $\sum S^n(x)$ converge unconditionally for any $x \in \mathcal{D}$;
(2) $TS = I$ on $\mathcal{D}$.

Then T is frequently hypercyclic.

In our situation, we define S by $S(e_n) = w_{n+1}^{-1}e_{n+1}$ and let $\mathcal{D}$ be the set of finitely supported sequences. It is easy to check that $\sum_{n<0}(w_{-1}\cdots w_{-n})^p < +\infty$ implies that the series $\sum_n B_{\mathbf{w}}^n x$ is unconditionally convergent for any $x \in \mathcal{D}$. In the same vein, the condition $\sum_{n\geq 1} \frac{1}{(w_1\cdots w_n)^p} < +\infty$ implies that $\sum_n S^n x$ is unconditionally convergent for any $x \in \mathcal{D}$. Thus, $B_{\mathbf{w}}$ is frequently hypercyclic.

Our result implies the following interesting corollary.

Corollary 11. *Let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ be a bounded and bounded below sequence of positive real numbers. Then $B_{\mathbf{w}}$ is frequently hypercyclic on $\ell^p(\mathbb{Z})$ if and only if $B_{\mathbf{w}}^{-1}$ is frequently hypercyclic on $\ell^p(\mathbb{Z})$.*

4. Frequently hypercyclic weighted shifts on c_0

In this section, we give a characterization of (invertible) frequently hypercyclic weighted shifts on $c_0(\mathbb{Z}_+)$ and $c_0(\mathbb{Z})$. Because of [3], we know that we cannot expect a statement as clean as Theorem 4. Nevertheless, it will be useful in the forthcoming examples. We begin with invertible bilateral weighted shifts.

Theorem 12. *Let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ be a bounded and bounded below sequence of positive integers. Then $B_{\mathbf{w}}$ is frequently hypercyclic (resp. $\mathcal{U}$-frequently hypercyclic) on $c_0(\mathbb{Z})$ if and only if there exist a sequence $(M(p))$ of positive real numbers tending to $+\infty$ and a sequence (E_p) of subsets of $\mathbb{Z}_+$ such that*

(a) For any $p \geq 1$, $\underline{d}(E_p) > 0$ (resp. $\bar{d}(E_p) > 0$);
(b) For any $p, q \geq 1$, $p \neq q$, $(E_p + [-p,p]) \cap (E_q + [-q,q]) = \emptyset$;
(c) $\lim_{n\to+\infty,\ n\in E_p} w_1 \cdots w_n = +\infty$;
(d) For any $p, q \geq 1$, for any $n \in E_p$ and any $m \in E_q$ with $n \neq m$,

$$\begin{cases} w_1 \cdots w_{m-n} & \geq & M(p)M(q) & \textit{provided } m > n \\ w_{m-n+1} \cdots w_0 & \leq & \dfrac{1}{M(p)M(q)} & \textit{provided } m < n. \end{cases}$$

Proof. We first observe that we may replace "there exists a sequence $(M(p))$" by "for any sequence $(M(p))$" in the statement of the previous theorem. Indeed, if properties (a) to (d) are true for some sequence $(M(p))$, then they are also satisfied for any sequence $(M(p))$, considering instead of (E_p) a subsequence of (E_p) if necessary.

We just prove the frequently hypercyclic case, the $\mathcal{U}$-frequently hypercyclic one being completely similar. We first assume that $B_{\mathbf{w}}$ is frequently hypercyclic and we let $x \in FHC(B_{\mathbf{w}})$. Let us fix $\rho > 1$ such that $\rho^{-1} \leq w_k \leq \rho$ for any $k \in \mathbb{Z}$. Let us also consider a sequence (ω_p) of positive real numbers such that $\omega_1 = 2$ and, for any $p \geq 2$, $\omega_p > 4\omega_{p-1}\rho^{2p+1}$. We set

$$E_p = \left\{ n \in \mathbb{Z}_+;\ \|B_{\mathbf{w}}^n x - \omega_p(e_{-p} + \cdots + e_p)\|_\infty < \frac{1}{p} \right\}.$$

Since x belongs to $FHC(B_{\mathbf{w}})$, E_p has positive lower density. Let $p \neq q$ and let us show that $(E_p + [-p,p]) \cap (E_q + [-q,q]) = \emptyset$. By contradiction, let us assume that $(n,s,m,t) \in E_p \times [-p,-p] \times E_q \times [-q,q]$ with $n+s = m+t$. Without loss of generality, we may assume $p < q$. $w_{s+1} \cdots w_{n+s} x_{n+s}$ is the s-th coefficient of $B_{\mathbf{w}}^n x$. Its modulus is smaller than $2\omega_p$. Similarly, $w_{t+1} \cdots w_{m+t} x_{m+t}$ is the t-th coefficient of $B_{\mathbf{w}}^m x$. Its modulus is greater than $\omega_q/2$. Moreover, $w_{s+1} \cdots w_{n+s} x_{n+s}$ and $w_{t+1} \cdots w_{m+t} x_{m+t}$ differ by at most $(2q+1)$ coefficients of the sequence $\mathbf{w}$. Hence,

$$\left(\frac{1}{\rho}\right)^{2q+1} \leq \frac{w_{s+1} \cdots w_{n+s} |x_{n+s}|}{w_{t+1} \cdots w_{m+t} |x_{m+t}|} \leq 2\omega_p \times \frac{2}{\omega_q}. \tag{4.1}$$

This contradicts the definition of (ω_n).

Moreover, pick $n \in E_p$ and look at the 0-th coefficient of $B_{\mathbf{w}}^n x$. It is equal to $w_1 \cdots w_n x_n$ and its modulus cannot be less than $\omega_p/2$. Since $x \in c_0(\mathbb{Z})$, we get that $w_1 \cdots w_n$ tends to $+\infty$ when n goes to infinity, $n \in E_p$. Fix another $m \in E_q$, $m \neq n$ and look at the $(n-m)$-th coefficient of $B_{\mathbf{w}}^m x$. This coefficient is equal to $w_{n-m+1} \cdots w_n x_n$ and its modulus is less than $1/q$ (recall that $|n-m| > q$). If $n > m$, then since $w_1 \cdots w_n x_n \geq \omega_p/2$, we can deduce

$$w_1 \cdots w_{n-m} \geq \frac{w_1 \cdots w_n |x_n|}{w_{n-m+1} \cdots w_n |x_n|} \geq q\frac{\omega_p}{2}.$$

Similarly, if $n < m$, then

$$w_{n-m+1} \cdots w_0 \leq \frac{w_{n-m+1} \cdots w_n |x_n|}{w_1 \cdots w_n |x_n|} \leq \frac{2}{q\omega_p}.$$

This shows (d) with $M(p)=p$.

We now show that the condition is sufficient. As pointed out above, we may assume that, for any $p\geq 1$, $M(p)\geq \rho^{4p}$. We set

$$E_p'=E_p\backslash\big\{n\in\mathbb{N};\ w_1\cdots w_n\leq \rho^{4p}\big\}.$$

E_p' is a cofinite subset of E_p, hence it has positive lower density. We write $E_p'=(n_k^p)_{k\geq 0}$ in an increasing order and we set $F_p=(n^p_{(2p+1)k})_{k\geq 0}$. F_p has positive lower density and $|n-m|>2p$ provided n,m are two distinct elements of F_p.

Let $(y(p))_{p\geq 1}$ be a dense sequence in $c_0(\mathbb{Z})$ such that the support of $y(p)$ is contained in $[-p,-p]$ and such that $\|y(p)\|_\infty\leq \rho^p$. We define $x\in\mathbb{C}^{\mathbb{N}}$ by setting

$$x_k=\begin{cases}\frac{1}{w_{s+1}\cdots w_{n+s}}y_p(s) & \text{for } k=n+s,\ n\in F_p,\ |s|\leq p\\ 0 & \text{otherwise.}\end{cases}$$

This definition is not ambiguous because of (b) and the definition of F_p. We claim that x belongs to $c_0(\mathbb{Z})$. Indeed, let $\varepsilon>0$. For $p\geq 1$ and $n\in F_p$, $|s|\leq p$,

$$|x_k| \ \leq\ \frac{\rho^{2p}}{w_1\cdots w_n}\times\rho^p\leq \rho^{-p}\leq\varepsilon \tag{4.2}$$

provided p is greater than some $p_0\geq 1$. Now, fix $p\leq p_0$. Then by (c), x_k goes to zero when k goes to $+\infty$, k staying in $F_p+[-p,-p]$.

We then show that x is a frequently hypercyclic vector for $B_{\mathbf{w}}$. It is sufficient to prove that, for any $p\geq 1$ and any $n\in F_p$, $\|B^n_{\mathbf{w}}x-y(p)\|_\infty\leq\varepsilon(p)$ with $\varepsilon(p)\to 0$ as p goes to $+\infty$. We observe that

$$\|B^n_{\mathbf{w}}x-y(p)\|_\infty=\sup_{s\notin[-p,p]}|w_{s+1}\cdots w_{n+s}x_{n+s}|.$$

The terms which appear in the sup-norm are nonzero if and only if $n+s=m+t$, for some $m\in E_q$, $q\geq 1$, and $t\in[-q,q]$. We distinguish two cases. First, if $m>n$, then we write

$$w_{s+1}\cdots w_{n+s}x_{n+s}=\begin{cases}\dfrac{w_1\cdots w_t}{w_1\cdots w_{m-n+t}}y_t(q) & \text{if } t\geq 1\\[2ex] \dfrac{1}{w_{t+1}\cdots w_0}\times\dfrac{1}{w_1\cdots w_{m-n+t}}y_t(q) & \text{if } t\leq 0.\end{cases}$$

Now, $w_1\cdots w_t\leq\rho^q$ if $t\geq 0$, $(w_{t+1}\cdots w_0)^{-1}\leq\rho^q$ if $t<0$, so that in both cases

$$|w_{s+1}\cdots w_{n+s}x_{n+s}|\leq\frac{\rho^{2q}}{w_1\cdots w_{m-n+t}}\leq\frac{\rho^{2q}\rho^p}{w_1\cdots w_{m-n}}\leq\frac{\rho^{2q}\rho^p}{\rho^{4q}\rho^{4p}}\leq\rho^{-3p}. \tag{4.3}$$

Second, if $m<n$, then we write

$$\begin{aligned}w_{s+1}\cdots w_{n+s}x_{n+s} &= w_{m-n+t+1}\cdots w_t w_{t+1}\cdots w_{m+t}x_{m+t}\\ &= w_{m-n+t+1}\cdots w_t y_t(q)\\ &= \begin{cases} w_{m-n+t+1}\cdots w_0 w_1\cdots w_t y_t(q) & \text{if } t\geq 1\\ \dfrac{w_{m-n+t+1}\cdots w_0}{w_{t+1}\cdots w_0}y_t(q) & \text{if } t\leq 0.\end{cases}\end{aligned}$$

We conclude as before. □

We turn to unilateral weighted shifts. A similar statement holds.

Theorem 13. *Let $\mathbf{w}=(w_n)_{n\in\mathbb{Z}_+}$ be a bounded sequence of positive integers. Then $B_{\mathbf{w}}$ is frequently hypercyclic (resp. $\mathcal{U}$-frequently hypercyclic) on $c_0(\mathbb{Z}_+)$ if and only if there exist a sequence $(M(p))$ of positive real numbers tending to $+\infty$ and a sequence (E_p) of subsets of $\mathbb{Z}_+$ such that*
(a) For any $p\geq 1$, $\underline{d}(E_p)>0$ (resp. $\bar{d}(E_p)>0$);
(b) For any $p,q\geq 1$, $p\neq q$, $(E_p+[0,p])\cap(E_q+[0,q])=\emptyset$;
(c) $\lim_{n\to+\infty,\ n\in E_p+[0,p]} w_1\cdots w_n=+\infty$;
(d) For any $p,q\geq 1$, for any $n\in E_p$ and any $m\in E_q$ with $m>n$, for any $t\in\{0,\dots,q\}$,

$$w_1\cdots w_{m-n+t}\geq M(p)M(q).$$

Proof. The proof is more or less a rephrasing of the proof of Theorem 12. We have to take into account that $\mathbf{w}$ is not necessarily bounded below. This was used at several places:

- to prove that x belongs to c_0; this remains true because we have a stronger assumption (c).
- to obtain inequalities (4.1), (4.2) and (4.3). This is settled by the stronger assumption (d) and by adjusting the values of ω_p and $M(p)$. For instance, we may choose

$$\omega_p \geq 4\omega_{p-1}\rho^{p+1}\times\frac{1}{\min(1,\inf(w_t^{p+1};\ t\in[0,p]))}$$
$$M(p)\geq \rho^{4p}\times\frac{1}{\min(1,\inf(w_t^{2p};\ t\in[0,p]))}.$$

The details are left to the reader. □

5. A $\mathcal{U}$-frequently hypercyclic operator which is not frequently hypercyclic

We turn to the proof of Theorem 5. It requires careful constructions. We first build sequences of integers with positive upper density and additional properties. These sequences allow us to define our weight $\mathbf{w}$. We then conclude by showing that $B_{\mathbf{w}}$ is not frequently hypercyclic and by applying Theorem 13 to show that $B_{\mathbf{w}}$ is $\mathcal{U}$-frequently hypercyclic. The rest of this section is devoted to these constructions.

5.1. **The sequences of integers.** We shall construct sets of integers $(E_p)_{p\geq 1}$ and sequences of integers $(a_r)_{r\geq 1}$, $(b_r)_{r\geq 1}$ satisfying the following properties:

(S1): For any $r\geq 1$, $a_{r+1}\geq b_r+2r+1$, $b_r\geq ra_r$ and $b_r>r^2(2r+1)$;
(S2): For any $p\geq 1$, $\overline{d}(E_p)>0$;
(S3): For any $p\geq 1$, $E_p\subset b_p\mathbb{N}$;
(S4): For any $p,q\geq 1$ with $p\neq q$ and any $(n,m)\in E_p\times E_q$ with $m>n$, then

$$\begin{aligned} m-n &> p\\ m-n &\notin \bigcup_{r\geq 1}[a_r-(r+1)-q;b_r+r+p+q];\\ n &\notin \bigcup_{r\geq 1}[a_r-(r+1)-p;b_r+2r]. \end{aligned}$$

The construction of these sequences is done by induction. Precisely, at Step r, we construct integers a_r, b_r, $N_{p,r}$ for $p\leq r$ and subsets E_p^r of $b_p\mathbb{N}$ for $p\leq r$ such that

- $a_r \geq b_{r-1} + 2(r-1) + 1$, $b_r \geq r a_r$ and $b_r > r^2(2r+1)$;
- For any $p \leq r$,
$$\#E_p^r(N_{p,r}) \geq \frac{1}{2b_p} N_{p,r}.$$
- For any $p, q \in \{1, \dots, r\}$ with $p \neq q$ and any $(n, m) \in E_p^r \times E_q^r$ with $m > n$, then
$$\begin{array}{rcl} m-n & > & p \\ m-n & \notin & \displaystyle\bigcup_{\rho=1}^{r} [a_\rho - (\rho+1) - q; b_\rho + \rho + p + q]; \\ n & \notin & \displaystyle\bigcup_{\rho=1}^{r} [a_\rho - (\rho+1) - p; b_\rho + 2\rho]. \end{array}$$
- For any $p < r$, $E_p^{r-1} \subset E_p^r$.

Provided this construction has been done, it is enough to set $E_p = \bigcup_{r \geq p} E_p^r$. The initialization of the induction is very easy. One just sets for instance $a_1 = 1$, $b_1 = 4$, $N_{1,1} = 8$ and $E_1^1 = \{8\}$. Let us explain how to proceed with Step $r+1$ provided the construction has been done until Step r. Let a_{r+1} be any integer such that
$$a_{r+1} \geq \left\{ \begin{array}{l} b_r + 2r + 1 \\ \max\big(n + p;\ p \leq r,\ n \in E_p^r\big) + (r+2). \end{array} \right.$$
Next we set $b_{r+1} = \max\big((r+1)a_{r+1}, (r+1)^2(2r+3) + 1\big)$. In particular, it is clear that if $(n, m) \in E_p^r \times E_q^r$ with $m > n$ and $p \neq q \in \{1, \dots, r\}$, then
$$\begin{array}{rcl} m-n & > & p \\ m-n & \notin & \displaystyle\bigcup_{\rho=1}^{r+1} [a_\rho - (\rho+1) - q; b_\rho + \rho + p + q]; \\ n & \notin & \displaystyle\bigcup_{\rho=1}^{r+1} [a_\rho - (\rho+1) - p; b_\rho + 2\rho]. \end{array}$$
Let us now define E_1^{r+1}. We first set
$$M_{1,r+1} = b_{r+1} + 3(r+1) + \max(E_p^r;\ p \leq r)$$
and we consider $N_{1,r+1} \geq M_{1,r+1}$ such that
$$\#\big([M_{1,r+1}; N_{1,r+1}] \cap b_1\mathbb{N}\big) \geq \frac{1}{2b_1} N_{1,r+1}.$$
We then set $E_1^{r+1} = E_1^r \cup ([M_{1,r+1}; N_{1,r+1}] \cap b_1\mathbb{N})$. In particular, if m belongs to $E_1^{r+1} \backslash E_1^r$ and $n \in E_p^r$ with $m > n$ and $2 \leq p \leq r$, then
$$\left\{ \begin{array}{rcl} m-n & \geq & b_{r+1} + 3(r+1) \geq b_{r+1} + (r+1) + p + 1 > p \\ m & \geq & b_{r+1} + 3(r+1) > b_{r+1} + 2(r+1). \end{array} \right.$$
The construction of the other sets E_p^{r+1}, $2 \leq p \leq r+1$, follows exactly the same lines, defining first
$$M_{p,r+1} = b_{r+1} + 3(r+1) + \max(E_q^m;\ 1 \leq q \leq m \leq r \text{ or } 1 \leq q \leq p-1, m = r+1).$$
The remaining details are left to the reader. We point out that, for $p \neq q$, $E_p + [0, p]$ does not intersect $E_q + [0, q]$.

5.2. **The weight.** We first define a weight $\mathbf{w}^0$ whose behaviour is adapted to the sequence (a_r). Precisely, for $n\geq 1$, w_n^0 is defined by

- $w_n^0=2$ provided $n\notin\bigcup_{r\geq 1}[a_r-r;b_r+r]$;
- $w_{a_r-r}^0$ is the (very small) positive real number such that $w_1^0\cdots w_{a_r-r}^0=1$;
- $w_n^0=1$ otherwise.

The main interest of $\mathbf{w}^0$ is that the product $w_1^0\cdots w_n^0$ is rather large when n belongs to a difference set E_p-E_q, with $p\neq q$, or to a set E_p, whereas $w_1^0\cdots w_n^0=1$ if $n\in\bigcup_{r\geq 1}[a_r-r;b_r+r]$. We then define, for each $p\geq 1$, a weight $\mathbf{w}^p$ which is suitable for the difference set E_p-E_p. Indeed, let

- $w_n^p=2$ provided $n=b_pk+u$, with $k\geq 1$ and $u\in\{-(p-1),\dots,0\}$;
- $w_n^p=\frac{1}{2^p}$ provided $n=b_pk+p+1$, $k\geq 1$;
- $w_n^p=1$ otherwise.

This is not ambiguous since $b_p>2p+1$. Notice that $w_1^p\cdots w_n^p=2^p$ provided $n\in b_p\mathbb{N}+[0,p]$ whereas $w_1^p\cdots w_n^p=1$ is equal to 1 outside $\bigcup_{k\geq 1}[b_pk-(p-1);b_pk+p]$.

The weight $\mathbf{w}$ combines the properties of all $\mathbf{w}^p$. It is defined by setting by induction on $n\geq 1$

$$w_1\cdots w_n=\max(w_1^p\cdots w_n^p;\ p\geq 0).$$

$\mathbf{w}$ is well-defined. Indeed, let $n\geq 1$ and let $r\geq 1$ be such that $n\in[b_r,b_{r+1})$. Then $w_1^p\cdots w_n^p=1\leq w_1^0\cdots w_n^0$ provided $p\geq r+2$, so that $b_p-p\geq b_{r+1}$. Moreover $\mathbf{w}$ is bounded by 2, since its definition easily implies that, for any $n\geq 1$, $w_n\leq\max(w_n^p;\ p\geq 0)\leq 2$.

We shall point out several important facts regarding $\mathbf{w}$ which come from the properties of $\mathbf{w}^0$ and $\mathbf{w}^p$, $p\geq 1$. The products $w_1\cdots w_n$ and $w_1\cdots w_{n-m}$ for $(n,m)\in E_p\times E_q$, $m>n$, are large. Indeed, let r be the unique integer such that $n\in[b_r+2r+1;a_{r+1}-(r+2)-p]$. Then, for any $s\in\{0,\dots,p\}$, the definition of $\mathbf{w}$ ensures that

$$\begin{aligned}w_1\cdots w_{n+s}&\geq w_1^0\cdots w_{n+s}^0\\&\geq w_1^0\cdots w_{b_r+r}^0w_{b_r+r+1}^0\cdots w_n^0w_{n+1}^0\cdots w_{n+s}^0\\&\geq 1\cdot 2^r\cdot 1.\end{aligned}\tag{5.1}$$

Moreover, if $p\neq q$, there exists some $\rho\geq 1$ such that $m-n$ belongs to $[b_\rho+\rho+p+q+1;a_{\rho+1}-(\rho+2)-q-1]$, so that, for any $t\in\{0,\dots,q\}$,

$$w_1\cdots w_{m-n+t}\geq w_1^0\cdots w_{m-n+t}^0\geq 2^{p+q}.\tag{5.2}$$

If $p=q$, then

$$w_1\cdots w_{m-n+t}\geq w_1^p\cdots w_{m-n+t}^p\geq 2^p.\tag{5.3}$$

On the contrary, $w_1\cdots w_n$ is often small. Indeed, observe that for $p\geq 1$, $w_1^p\cdots w_n^p\leq 2^p$ for any $n>1$ and that $w_1^0\cdots w_n^0=1$ provided $n\in\bigcup_{r\geq 1}[a_r,b_r]$. Hence, if n belongs to $\bigcup_{r\geq 1}[a_r,b_r]$ and satisfies $w_1\cdots w_n>2^p$, then there exists $q>p$ such that $n\in b_q\mathbb{N}+[-q,q]$.

5.3. $B_{\mathbf{w}}$ **is not frequently hypercyclic.** Assume on the contrary that $B_{\mathbf{w}}$ is frequently hypercyclic. Then there exists $E\subset\mathbb{N}$ with $\underline{d}(E)>0$ such that $w_1\dots w_n\to+\infty$ when $n\to+\infty$, $n\in E$. In particular, for any $p\geq 1$,

$$F_p=\left\{n\in E;\ w_1\cdots w_n>2^p\right\}$$

is a cofinite subset of E. It has the same lower density. Now, let $r \geq 1$ and let $n \in F_p \cap [0, b_r]$. Then either $n \leq a_r$ or there exists $q > p$ such that n belongs to $b_q\mathbb{N} + [-q, q]$. This yields

$$\#F_p(b_r) \leq a_r + b_r \times \sum_{q>p} \frac{2q+1}{b_q}.$$

Since a_r/b_r goes to zero, this implies

$$\underline{d}(E) = \underline{d}(F_p) \leq \sum_{q>p} \frac{1}{q^2}.$$

Since p is arbitrary, $\underline{d}(E) = 0$ and $B_{\mathbf{w}}$ cannot be frequently hypercyclic.

5.4. **$B_{\mathbf{w}}$ is $\mathcal{U}$-frequently hypercyclic.** This follows from an application of Theorem 13 for the sets E_p defined above and from the work of Subsection 5.2. Indeed, condition (c) of this theorem follows from (5.1) whereas condition (d) is a consequence of (5.2) and (5.3), setting $M(p) = 2^{p/2}$.

6. Frequent hypercyclicity vs distributional chaos

We turn to the proof of Theorem 7. We follow the same kind of proof.

6.1. **The sequences of integers.** For $a > 1$, $\varepsilon > 0$ and $u \in \mathbb{N}$, we set

$$I_u^{a,\varepsilon} = [(1-\varepsilon)a^u, (1+\varepsilon)a^u].$$

Lemma 1. *There exist $a > 1$ and $\varepsilon > 0$ such that $\bar{d}(\bigcup_{u\geq 1} I_u^{a,4\varepsilon}) < 1$ and, for any $u > v \geq 1$,*

$$I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset,\ I_u^{a,2\varepsilon} - I_v^{a,2\varepsilon} \subset I_u^{a,4\varepsilon}.$$

Proof. It is easy to check that, for any $u > v \geq 1$, $I_u^{a,2\varepsilon} - I_v^{a,2\varepsilon} \subset I_u^{a,4\varepsilon}$ as soon as, for any $u \geq 2$,

$$(1-2\varepsilon)a^u - (1+2\varepsilon)a^{u-1} \geq (1-4\varepsilon)a^u.$$

This condition is satisfied provided

$$\frac{2\varepsilon a}{1+2\varepsilon} \geq 1.$$

Moreover, let us also assume that $(1+4\varepsilon)/(1-4\varepsilon) < a$. Then

$$\begin{aligned}\bar{d}\left(\bigcup_{u\geq 1} I_u^{a,4\varepsilon}\right) &\leq \lim_{k\to+\infty} \frac{8\varepsilon(1+\cdots+a^k)}{(1+4\varepsilon)a^k}\\ &\leq \frac{8\varepsilon a}{(1+4\varepsilon)(a-1)},\end{aligned}$$

and this is less than 1 provided ε is small enough and a is large enough. Observe also that this choice of a and ε guarantees that $I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset$ for any $u \neq v$. ☐

From now on, we fix $a > 1$ and $\varepsilon > 0$ satisfying the conclusions of the previous lemma. We then consider an increasing sequence of positive integers $(b_p)_{p\in\mathbb{N}}$ such that

$$\sum_{p\geq 1} \bar{d}(b_p\mathbb{N} + [-2p, 2p]) + \bar{d}\left(\bigcup_{u\geq 1} I_u^{a,4\varepsilon}\right) < 1.$$

Observe that $b_p \geq 4p$ for any $p \geq 1$.

We also consider a partition of $\mathbb{N}$ into $\bigcup_{p\geq 1} A_p$ where each A_p is syndetic. For instance, we may set $A_p = 2^{p-1}\mathbb{N}\backslash 2^p\mathbb{N}$. We finally set

$$E_p = \bigcup_{u\in A_p} I_u^{a,\varepsilon} \cap b_p\mathbb{N}.$$

Lemma 2. *For any $p \geq 1$, $\underline{d}(E_p) > 0$.*

Proof. Let (n_k) be an increasing enumeration of A_p and let $M > 0$ be such that $n_{k+1} - n_k \leq M$. Then

$$\begin{aligned}\underline{d}(E_p) &\geq \liminf_{k\to+\infty} \frac{\#E_p\big((1+\varepsilon)a^{n_k}\big)}{a^{n_{k+1}}}\\ &\geq \liminf_{k\to+\infty} \frac{2\varepsilon a^{n_k}}{b_p a^{n_k+M}} > 0.\end{aligned}$$

□

Deleting a finite number of elements in A_p if necessary, we may and shall assume that for any $u \in A_p$, $I_u^{a,\varepsilon} + [-2p, 2p] \subset I_u^{a,2\varepsilon}$. Since $I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset$ whenever $u \neq v$ and since $b_p \geq 4p$, we get the following lemma.

Lemma 3. *Let $p, q \geq 1$, $n \in E_p$, $m \in E_q$ with $n \neq m$. Then $|n - m| > 2\max(p, q)$.*

In particular, $(E_p + [-p, p]) \cap (E_q + [-q, q]) = \emptyset$ if $p \neq q$.

6.2. **The weight.** As in Section 5.2, we will define several weights: weights $\mathbf{w}^p$ such that $w_{m-n+1}^p \cdots w_0^p$ is small when $m < n$ belong to the same E_p, and weights $\mathbf{w}^{u,v}$, $u > v$, such that $w_{m-n+1}^{u,v} \cdots w_0^{u,v}$ is small when m belongs to $I_v^{a,\varepsilon}$ and n belongs to $I_u^{a,\varepsilon}$. Elsewhere, they will be large to ensure that $B_{\mathbf{w}}$ cannot be distributionally chaotic.

We begin with $\mathbf{w}^p$, $p \geq 1$. We set $\mathbf{w}^p = (w_k^p)$ any sequence of positive integers such that

$$w_{-k+1}^p \cdots w_0^p = \begin{cases} 1 & \text{provided } k \notin b_p\mathbb{N} + [-2p, 2p] \\ \frac{1}{2^p} & \text{provided } k \in b_p\mathbb{N}, \end{cases}$$

$$\begin{aligned} \tfrac{1}{2} \leq w_k^p \leq 2 &\quad \text{for any } k \in \mathbb{Z} \\ w_k^p = 2 &\quad \text{for any } k \geq 1. \end{aligned}$$

Let us now define $\mathbf{w}^{u,v}$ for $u > v$. Let $p, q \geq 1$ such that $u \in A_p$ and $v \in A_q$. We set $\mathbf{w}^{u,v} = (w_k^{u,v})$ any sequence of positive real numbers such that

$$w_{-k+1}^{u,v} \cdots w_0^{u,v} = \begin{cases} 1 & \text{provided } k \notin I_u^{a,4\varepsilon} \\ \min\left(\frac{1}{2^{2p}}, \frac{1}{2^{2q}}\right) & \text{provided } k \in I_u^{a,\varepsilon} - I_v^{a,\varepsilon}, \end{cases}$$

$$\begin{aligned} \tfrac{1}{2} \leq w_k^{u,v} \leq 2 &\quad \text{for any } k \in \mathbb{Z} \\ w_k^{u,v} = 2 &\quad \text{for any } k \geq 1. \end{aligned}$$

It is possible to construct such a weight because

$$I_u^{a,\varepsilon} - I_v^{a,\varepsilon} + [-2\max(p,q), 2\max(p,q)] \subset I_u^{a,2\varepsilon} - I_v^{a,2\varepsilon} \subset I_u^{a,4\varepsilon}.$$

We finally define our weight $\mathbf{w}$ by setting inductively w_{-n} for $n > 0$ with the relation

$$w_{-n+1} \cdots w_0 = \min_{p,u,v}(w_{-n+1}^p \cdots w_0^p, w_{-n+1}^{u,v} \cdots w_0^{u,v})$$

and by letting $w_k = 2$ for $k \geq 1$. $\mathbf{w}$ is well-defined because, for a fixed $n \geq 1$, $w^p_{-n+1}\cdots w^p_0 = 1$ and $w^{u,v}_{-n+1}\cdots w^{u,v}_0 = 1$ provided p and u are large enough. Moreover, the definition of $\mathbf{w}$ easily implies that $\frac{1}{2} \leq w_k \leq 2$ for any $k \in \mathbb{Z}$, so that the weighted shift $B_{\mathbf{w}}$ is bounded and invertible on $c_0(\mathbb{Z})$.

6.3. **$B_{\mathbf{w}}$ is not distributionally chaotic.** We verify that the product $w_{-n+1}\cdots w_0$ is not small very often. Indeed, let $A = \mathbb{N}\backslash\left(\bigcup_p (b_p\mathbb{N} + [-2p, 2p]) \cup \bigcup_u I_u^{a,4\varepsilon}\right)$. Then our choices of a, ε and (b_p) tell us that $\underline{d}(A) > 0$. Moreover, by the construction of our weight, $w_{-n+1}\cdots w_0 = 1$ provided $n \in A$. Pick now $x \in c_0(\mathbb{Z})$, $x \neq 0$ and let k be such that $x_k \neq 0$. If $n - k$ belongs to A, then $\|B_{\mathbf{w}}^{n-k}x\|_\infty \geq |x_k| > 0$ so that x cannot be a distributional irregular vector for $B_{\mathbf{w}}$. Therefore, $B_{\mathbf{w}}$ is not distributionally chaotic.

6.4. **$B_{\mathbf{w}}$ is frequently hypercyclic.** We apply Theorem 12. The only thing that we do not have verified yet is property (d). Thus, let $n \in E_p$, $m \in E_q$ with $m < n$. If $p = q$, then $m - n \in b_p\mathbb{N}$ so that

$$w_{m-n+1}\cdots w_0 \leq \frac{1}{2^{2p}}.$$

If $p \neq q$, then there exists $u > v$ such that $n \in I_u^{a,\varepsilon}$ and $m \in I_v^{a,\varepsilon}$. Thus,

$$w_{m-n+1}\cdots w_0 \leq \min\left(\frac{1}{2^{2p}}, \frac{1}{2^{2q}}\right) \leq \frac{1}{2^{p+q}}.$$

Hence,

$$w_{m-n+1}\cdots w_0 \leq \frac{1}{M(p)M(q)}$$

with $M(p) = 2^p$. If $m > n$, then we just observe that $m - n \geq p + q$ to conclude

$$w_1\cdots w_{m-n} \geq 2^{p+q} = M(p)M(q).$$

7. Final comments and open questions

The work of Section 6 shows that a frequently hypercyclic operator does not need to be distributionally chaotic. However, it admits plenty of half distributional irregular vectors!

Proposition 14. *Let $T \in \mathfrak{L}(X)$ be frequently hypercyclic. Then there exists a residual subset $\mathcal{R}$ of X such that any vector $y \in \mathcal{R}$ has a distributional unbounded orbit, namely there exists $B \subset \mathbb{N}$ such that $\overline{d}(B) = 1$ and $\lim_{n\to+\infty, n\in B}\|T^n y\| = +\infty$.*

Proof. By the work of [7], it is sufficient to find $\varepsilon > 0$, a sequence $(y_k) \subset X$ and an increasing sequence (N_k) in $\mathbb{N}$ such that $\lim_k y_k = 0$ and

$$\#\{1 \leq j \leq N_k;\ \|T^j y_k\| > \varepsilon\} \geq \varepsilon N_k.$$

Let $x \in FHC(T)$ and let $\eta > 0$ be such that

$$\underline{d}(\{n \in \mathbb{N}; \|T^n x\| > 1\}) > \eta.$$

We set $\varepsilon = \eta/2$. For any $k \geq 1$, let $p_k > 0$ be such that $\|T^{p_k}x\| < 1/k$. We set $y_k = T^{p_k}x$. This p_k being fixed, we may find N_k as large as we want such that

$$\#\{1 \leq n \leq N_k;\ \|T^n y_k\| > 1\} = \#\{p_k + 1 \leq n \leq N_k + p_k;\ \|T^n x\| > 1\} \geq \frac{\eta N_k}{2}.$$

$\square$

This proposition has several interesting corollaries. First of all, a frequently hypercyclic operator is "almost" distributionally chaotic.

Corollary 15. *Let $T \in \mathfrak{L}(X)$ be frequently hypercyclic and assume that there exists a dense set $X_0 \subset X$ such that $T^n x \to 0$ for any $x \in X$. Then T is distributionally chaotic.*

Proof. By [7], an operator with a distributional unbounded orbit and such that $T^n x \to 0$ for any x in a dense subset X_0 of X is distributionally chaotic. And we have just proved that a frequently hypercyclic operator has a distributional unbounded orbit. □

Our example of a frequently hypercyclic operator which is not distributionally chaotic was a bilateral weighted shift. This would be impossible with a unilateral weighted shift.

Corollary 16. *A frequently hypercyclic unilateral weighted shift is distributionally chaotic.*

Proof. The orbit of any vector with a finite support goes to zero, so that we may apply Corollary 15. □

Thanks to Proposition 14, we can solve another open question of [7].

Definition 17. Let $T \in \mathfrak{L}(X)$. We say that the $\mathbb{T}$-eigenvectors of T are *perfectly spanning* if, for any countable set $D \subset \mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$, the linear span of $\bigcup_{\lambda \in \mathbb{T}\setminus D} \ker(T - \lambda)$ is dense in X.

The next corollary extends a result of [7] from Hilbert spaces to general Banach spaces.

Corollary 18. *Let $T \in \mathfrak{L}(X)$ be such that its $\mathbb{T}$-eigenvectors are perfectly spanning. Then T is distributionally chaotic.*

Proof. By [5], T is frequently hypercyclic. By [1], there exists a dense set $X_0 \subset X$ such that $T^n x \to 0$ for all $x \in X_0$. Thus we may apply Corollary 15. □

A striking difference between hypercyclic operators and frequently hypercyclic operators is the comparison of the size of $HC(T)$ and $FHC(T)$. Whereas $HC(T)$ is always residual when it is nonempty, it was shown (see for instance [4]) that, for many frequently hypercyclic operators, $FHC(T)$ is of first category. It turns out that $FHC(T)$ is always meagre.

Corollary 19. *Let $T \in \mathfrak{L}(X)$ be frequently hypercyclic. Then $FHC(T)$ is a set of first category.*

Proof. A vector with a distributional unbounded orbit cannot be a frequently hypercyclic vector. By Proposition 14, a frequently hypercyclic operator admits a residual subset of vectors with distributionally unbounded orbit. □

We now turn to an interesting problem regarding frequently hypercyclic operators.

Question. Let $T \in \mathfrak{L}(X)$ be frequently hypercyclic and invertible. Is T^{-1} invertible?

In view of this paper, it is natural to study whether a bilateral weighted shift on c_0 could be a counterexample. It we look at Theorem 12, this does not seem impossible; indeed, because of (c), the conditions on the right part and on the left part of $B_{\mathbf{w}}$ are not symmetric. However, it could be possible that this condition is superfluous.

Question. Let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ be a bounded and bounded below sequence of positive integers such that conditions (a), (b) and (d) of Theorem 12 are satisfied. Does it automatically satisfy conditions (a), (b), (c) and (d), maybe for another family (E_p) of subsets of $\mathbb{N}$?

At least, we are able to give a partial positive answer to our first question.

Proposition 20. *Let $T \in \mathfrak{L}(X)$ be frequently hypercyclic and invertible. Then T^{-1} is $\mathcal{U}$-frequently hypercyclic.*

Proof. Let $x \in FHC(T)$, let (U_k) be a basis of open subsets of X and let

$$\delta_k = \underline{d}\big(\{n \in \mathbb{N} :\ T^n x \in U_k\}\big).$$

We set, for $k, N \geq 1$ and $n \geq N$,

$$U_{k,N,n} = \big\{y \in X;\ \#\{1 \leq j \leq n;\ T^{-j}y \in U_k\} \geq \delta_k n/2\big\}.$$

$U_{k,N,n}$ is clearly open. Moreover, $\bigcap_{k,N\geq 1}\bigcup_{n\geq N} U_{k,N,n}$ contains $\mathcal{U}FHC(T^{-1})$. We intend to apply Baire's theorem and we prove that $\bigcup_{n\geq N} U_{k,N,n}$ is dense for any $k, N \geq 1$. Let $V \subset X$ be open and nonempty and let $N_k \geq N$ be such that, for any $n \geq N_k$,

$$\#\{0 \leq j \leq n;\ T^j x \in U_k\} \geq \frac{\delta_k n}{2}.$$

There exists $n \geq N_k$ such that $T^n x \in V$. Let us set $y = T^n x$. Then

$$\#\{0 \leq j \leq n;\ T^{-j}y \in U_k\} = \#\{0 \leq j \leq n;\ T^j x \in U_k\}.$$

In particular, $y \in \bigcup_{n\geq N} U_{k,N,n} \cap V$. By Baire's theorem, $\mathcal{U}FHC(T^{-1})$ is residual, hence nonempty. □

An easy modification of the previous argument yields the following interesting corollary, to be compared with Corollary 19.

Proposition 21. *Let $T \in \mathfrak{L}(X)$ be $\mathcal{U}$-frequently hypercyclic. Then $\mathcal{U}FHC(T)$ is residual.*

Proof. Let $x \in \mathcal{U}FHC(T)$, let (U_k) be a basis of open subsets of X and let

$$\delta_k = \bar{d}\big(\{n \in \mathbb{N} :\ T^n x \in U_k\}\big).$$

We set, for $k, N \geq 1$ and $n \geq N$,

$$U_{k,N,n} = \big\{y \in X;\ \#\{1 \leq j \leq n;\ T^j y \in U_k\} \geq \delta_k n/2\big\},$$

which is open. Moreover, it is easy to see that any iterate $T^p x$ belongs to $\bigcap_{k,N\geq 1}\bigcup_{n\geq N} U_{k,N,n}$. Since these iterates are dense in X, $\bigcap_{k,N\geq 1}\bigcup_{n\geq N} U_{k,N,n}$ is a dense G_δ-set, which is contained in $\mathcal{U}FHC(T)$. □

Acknowledgements. We thank Pr. Vitaly Bergelson for fruitful discussions.

References

[1] F. Bayart and S. Grivaux. Hypercyclicity and unimodular point spectrum. *J. Funct. Anal.*, 226:281–300, 2005.

[2] F. Bayart and S. Grivaux. Frequently hypercyclic operators. *Trans. Amer. Math. Soc.*, 358:5083–5117, 2006.

[3] F. Bayart and S. Grivaux. Invariant Gaussian measures for operators on Banach spaces and linear dynamics. *Proc. Lond. Math. Soc.*, 94:181–210, 2007.

[4] F. Bayart and É. Matheron. *Dynamics of linear operators*, volume 179 of *Cambridge Tracts in Math.* Cambridge University Press, 2009.

[5] F. Bayart and É. Matheron. Mixing operators and small subsets of the circle. *preprint*, 2011. arXiv:1112.1289.

[6] V. Bergelson. Ergodic Ramsey theory: an update. In *Ergodic theory of $\mathbb{Z}^d$ actions (Warwick, 19931994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, 1996.

[7] N.C. Bernardes, A. Bonnila, V. Müller and A. Peris. Distributional chaos for linear operators. *preprint*, 2013.

[8] A. Bonilla and K.-G. Grosse-Erdmann. Frequently hypercyclic operators and vectors. *Ergodic Theory Dynamical Systems*, 27:383–404, 2007.

[9] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. d'Analyse Math.*, 31:204–256, 1977.

[10] S. Grivaux. A new class of frequently hypercyclic operators. *Indiana Univ. Math. J.*, 60:1177–1201, 2011.

[11] K.-G. Grosse-Erdmann and A. Peris. Frequently dense orbits. *C. R. Acad. Sci. Paris*, 341:123–128, 2005.

[12] K.-G. Grosse-Erdmann and A. Peris. *Linear chaos.* Universitext, Springer, 2011.

[13] I. Ruzsa. On difference-sequences. *Acta Arith.*, 25:151–157, 1973/74.

[14] H. N. Salas. Hypercyclic weighted shifts. *Trans. Amer. Math. Soc.*, 347(3):993–1004, 1995.

[15] B. Schweizer and J. Smítal. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.*, 344(2):737–754, 1994.

[16] S. Shkarin. On the spectrum of frequently hypercyclic operators. *Proc. Amer. Math. Soc.*, 137:123–134, 2009.

Clermont Université, Université Blaise Pascal, Laboratoire de Mathematiques, BP 10448, F-63000 CLERMONT-FERRAND - CNRS, UMR 6620, Laboratoire de Mathématiques, F-63177 AUBIERE

Alfréd Rényi Institute of Mathematics, Budapest, Pf. 127, H-1364 Hungary

E-mail address: Frederic.Bayart@math.univ-bpclermont.fr

E-mail address: ruzsa@renyi.hu