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We re-analyse the perturbative radiative corrections to the Higgs mass within the Standard Model in the light of the Taylor-Lagrange renormalization scheme. This scheme naturally leads to completely finite corrections, depending on an arbitrary scale. The formulation avoids very large individual corrections to the Higgs mass. This illustrates the fact that the so-called fine-tuning problem in the Standard Model is just an artefact of the regularization scheme. It should therefore not lead to any physical interpretation in terms of the energy scale at which new physics should show up, nor in terms of a new symmetry. We analyse the intrinsic physical scales relevant for the description of these radiative corrections.

I. INTRODUCTION

The experimental tests of the Standard Model of particle physics are entering a completely new era with the first pp collisions at LHC (CERN) in the TeV energy range and the discovery of a Higgs-like scalar boson [1]. It is commonly admitted that experimental evidences showing deviations from predictions within the SM should be interpreted as signs of new physics, thereby relegating the SM to the status of an effective field theory (EFT). We know that the renormalizable SM is at most valid up to an energy scale $\Lambda_{eff} < M_P$, where M_P is the Planck mass ($\simeq 10^{19} GeV$), for quantum gravitational effects become relevant at that scale. However the way to treat the SM as an effective theory at a much lower energy scale of a yet unknown more fundamental theory is rather elusive and demands certain careful guidance from past experiences.

In any physical process, the theoretical consistency requirement of the Standard Model demands that any characteristic intrinsic momentum, denoted by Λ_k, which is relevant for the description of this process should be less or equal to Λ_{eff}. Otherwise, new contributions of order $(\Lambda_k/\Lambda_{eff})^n$ to the Lagrangian of the Standard Model should start to be sizeable. At tree level, this momentum can be defined for instance by any typical kinematical variables of the process, like $\sqrt{Q^2}$ in Deep Inelastic Scattering (DIS). It is thus under complete control. However, beyond tree level,

one has to deal with internal momenta in loop contributions, that may be large.

How large are they really? To answer this question, one has to enter into the renormalization procedure in order to define the physical amplitudes in terms of the bare ones calculated from the original Lagrangian of the Standard Model.

While physical observables should be independent of the renormalization scheme which is considered, the use of a particular scheme (together with a regularization procedure) may lead to unphysical interpretations when non-observable bare quantities are concerned. We shall address in this article the particular case of radiative corrections to the Higgs mass.

Beyond tree level, the first radiative corrections to the Higgs mass one has to consider in the Standard Model are radiative corrections from a $t\bar{t}$ loop as well as Higgs and W, Z bosons loops. Using a *naïve cut-off* to regularize the bare amplitudes, these corrections immediately lead to a quadratic dependency of the regulated amplitudes on the cut-off scale Λ_C [2]. The (square of the) physical mass, M_H, defined at the pole of the two-body Green's function, can be schematically written as

$$M_H^2 = M_0^2 + b\,\Lambda_C^2 + \dots , \qquad (1)$$

where M_0 is the bare mass of the Higgs particle, and b is a combination of the top quark, W, Z bosons and Higgs masses. As it is, this equation has not much

physical interest. It is just a definition of the bare mass as a function of the physical mass, in the spirit of the renormalization theory.

The so-called fine-tuning problem arises if one wants to give some kind of physical interpretation to the bare mass M_0. Since Λ_C should be much larger than any characteristic energy scale relevant for the description of the theoretical physical amplitude, a large cancellation between M_0^2 and $b\,\Lambda_C^2$ should be enforced by hand - hence the name fine-tuning - unless b is zero (the so-called Veltman condition [3]), or M_H is very large, which we now know is not the case.

Our concern in the following is twofold. We shall first illustrate the above general remarks by the explicit calculation of the radiative corrections to the Higgs mass in a finite field theory based on the Taylor-Lagrange Renormalization Scheme (TLRS) [4], in leading order of perturbation theory. Then we compare the results with standard procedures using a naïve cut-off or Dimensional Regularization (DR). Since BPHZ can be recovered from TLRS, our conclusions do also apply to BPHZ [5]. In addition, in analogy with DR, TLRS also contains an explicit arbitrary scale. Secondly, we shall analyse our results in terms of the characteristic scale Λ_k, and compare its value for both types, finite or infinite, of renormalization schemes. This may in turn have important consequences for the determination of the relevant momentum and/or energy scales at which new physics should show up. We stick in our study to the Standard Model. The discussion of how the gauge hierachy problem is formulated in TLRS when grand unified theories are concerned is beyond the scope of the present article.

The plan of the article is the following. The overall settings for a consistent formulation of an EFT are first discussed in Sec. II. Then we recall in Sec. III the general features of TLRS. We apply this scheme to radiative corrections to the Higgs mass in the Standard Model in Sec. IV. We discuss our results in the light of the fine-tuning problem in Sec. V, and draw our conclusions in Sec. VI.

II. INTRODUCTORY REMARKS ON THE MEANING OF EFT

We are familiar with the EFT approach from the early days of Quantum Electrodynamics (QED), where a simple example of EFT valid at very low energies, $E_\gamma \ll m_e$, with m_e the electron mass, is provided by the Euler-Heisenberg Lagrangian [6]:

$$\mathcal{L}_{eff} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a}{m_e^4}(F^{\mu\nu}F_{\mu\nu})^2$$
$$+ \frac{b}{m_e^6}F^{\mu\nu}F_{\nu\sigma}F^{\sigma\rho}F_{\rho\mu} + O(\frac{F^6}{m_e^8}). \quad (2)$$

It is important to emphasize some key features from this EFT analysis: *i)* the fundamental Gauge, Lorentz, Charge Conjugation and Parity invariance severely constrain the possible forms of EFT terms, *ii)* the success of the full non-perturbative Euler-Heisenberg expression of $\mathcal{L}_{eff}$ is tied up to the subtraction of the (infinite) free-field effective action. It was further realized that the other (logarithmically divergent) subtraction terms can be seen as an embryonic recognition of charge renormalization. *iii)* in this low-energy regime, all the information of the full *renormalized* QED dynamics is embodied in the values of the effective couplings $a = \frac{\alpha^2}{36}$ and $b = \frac{7\alpha^2}{90}$, with $\alpha = \frac{e^2}{4\pi}$.

The EFT describes then the low-energy physics, to a given accuracy ϵ, in terms of a finite set of parameters. The successful ab-initio determination of these effective couplings is tied up to the renormalizibility of the full theory. To explore a domain of higher energies still in EFT terms one has first to include operators of higher dimensions and then re-adjust the values of the initial effective couplings through renormalization group (RG)-equations.

The consistency of the procedure is imposed by *matching conditions* around the energy threshold region where physical predictions should be identical, to a given accuracy, in the full and effective theories. In this way if an energy domain shows up where the matching conditions cannot be fulfilled with the desired accuracy, the problematic is twofold: either the successive addition of operators of higher dimensions is not any more feasible due to fast increasing values of the effective couplings with energy -a verifiable situation- or, more speculative, the full initial theory is itself a renormalizable EFT of a deeper fundamental description of the physics reality involving new degrees of freedom in this energy domain.

Obviously, since the 1940's, the accumulation of experimental evidences in particle physics continuously supported the progression towards the SM formulation. At the same time the seminal works of Euler-Heisenberg heavily influenced similar successful EFT developppments in non-abelian gauge field, SM and string theories, as discussed in [7]. In this respect it is often argued that, although effective field theo-

ries contain an infinite number of terms, renormalizability is not an issue since, at a given order in the energy expansion, the low-energy theory is specified by a finite number of couplings and allows then for an order-by-order renormalization. From the works of Euler-Heisenberg and followers and the above discussion this is evidently not the case: an EFT-SM approach cannot be considered without taking into account the full renormalizibility of the underlying theory in the determination of the effective couplings even in the supposed energy threshold region for the onset of new physics.

More recently the necessity for a mass-independent renormalization scheme has clearly been emphasized by A. Pich [8]: if one cuts without renormalizing loop integrals at a scale Λ where new physics is supposed to become important, one obtains erroneous dependences for mass corrections which are not suppressed by powers of $\frac{1}{\Lambda}$. This makes it impossible to reach converging mass contributions from higher orders of perturbation theory so that the whole approach breaks down. On the contrary, a mass independent renormalization scheme yields results where mass corrections are suppressed by powers of $\frac{\mu}{\Lambda}$ where μ is the RG scale parameter. In addition, for a gauge invariant theory like the SM, gauge invariance excludes renormalization schemes violating this invariance, e.g. all cut-off methods.

We shall consider in what follows two kinds of renormalization schemes, depending on whether the renormalization of the bare amplitudes is finite or infinite. The latter is the one which is widely used in standard perturbation theory *à la* Feynman. In this scheme, the choice of a regularization procedure is a necessary prerequisite to give a mathematical sense to a-priori divergent bare amplitudes. In the literature two regularization methods are mainly used: *i)*The first one exhibits a very large mass scale, denoted by Λ_C. This mass scale is either a naïve cut-off in (Euclidean) four-momentum space, or the mass of Pauli-Villars (PV) particles in the PV regularization scheme. This explicit mass scale should be much larger then any characteristic energy, or momentum, scale relevant in the calculation of the theoretical physical amplitude. *ii)*The second one, the so-called dimensional regularization (DR) procedure, amounts to extending the space-time dimension D away from 4. The divergences of the original amplitudes show up as singularities in $\varepsilon = 4 - D$, with $\varepsilon > 0$. In this case, the bare amplitudes depend on a finite, and arbitrary, mass scale μ (the so-called "unit of mass" [10]).

Using these regularization procedures, any bare amplitude is thus made finite. However, the original divergences are recovered in the limiting process $\Lambda_C \to \infty$ or $\epsilon \to 0$. In this sense the renormalization schemes using these regularizations procedures are called "infinite". Of course, all fully renormalized amplitudes are indeed equivalently finite irrespective of the regularization method which has been used to derive them, provided the symmetries of the system are conserved.

The prototype of a finite renormalization scheme is the well-known BPHZ procedure. In this scheme, any bare amplitude is made finite by subtracting as many terms as necessary from the Taylor expansion at zero external momenta of the integrand. All Feynman integrals being convergent, no further regularization is required. In this context we shall focus on the relevance of the TLRS. This scheme originates from the well known observation that the divergences of bare amplitudes can be traced back to the violation of causality, originating from ill-defined products of distributions at the same point [9, 11]. The correct mathematical treatment, known since a long time, is to consider covariant fields as Operator Valued Distribution (OPVD), these distributions being applied on test functions with well-defined properties [12–14]. These considerations led to the development and applications of TLRS [15, 16].

III. THE TAYLOR-LAGRANGE RENORMALIZATION SCHEME

Any quantum field $\phi(x)$ - taken here as a scalar field for simplicity - should be considered as an OPVD [17–20]. This has been known for a long time. However, its full significance for practical calculations was not fully recognized until recently [4, 9, 11, 13–15].

As any distribution, quantum fields should be defined by their application on test functions, denoted by ρ, with well identified mathematical properties [12]. The physical field $\varphi(x)$ is thus defined by [4]

$$\varphi(x) \equiv \int d^4y \, \phi(y)\rho(x-y) \ . \tag{3}$$

If we denote by f the Fourier transform of the test function, we can further write $\varphi(x)$ in terms of creation and destruction operators, leading to

$$\varphi(x) = \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{f(\varepsilon_p^2, \mathbf{p}^2)}{2\varepsilon_p} \left[a_\mathbf{p}^\dagger e^{ip.x} + a_\mathbf{p} e^{-ip.x} \right] , \tag{4}$$

with $\varepsilon_p^2 = \mathbf{p}^2 + m^2$.

From this decomposition, it is apparent that test functions should be attached to each fermion and boson fields. Each propagator being the contraction of two fields should be proportional to f^2. In order to have a dimensionless argument for f, we shall introduce an arbitrary scale Λ to "measure" all momenta. Λ can be any of the masses of the constituents. To deal with massless theories, we shall consider some a-priori arbitrary value. The final expression of any amplitude should be independent of Λ.

As recalled in Ref. [4], the test function f should have peculiar properties. It is chosen as a super regular partition of unity (see [4] for more details on PU's), i.e. a function of finite support which is 1 everywhere except at the boundaries. As a super regular test function (SRTF), it vanishes as well as all its derivatives, at this boundaries, in the UV and in the IR domains.

The boundary conditions of the test function - which in this study is assumed to depend on a one dimensional variable X - should embody a scale invariance inherent, in the UV domain for instance, to the limit $X \to \infty$ since in this limit $\eta^2 X$ also goes to ∞, where η^2 is an arbitrary dimensionless scale. This can be done by considering a running boundary condition for the test function, i.e. a boundary condition which depends on the variable X according to

$$f(X \geq H(X)) = 0 \quad \text{for} \quad H(X) \equiv \eta^2 X g(X) . \quad (5)$$

This condition defines a maximal value, X_{max}, with $f(X_{max}) = 0$ [1]. In order to extend the test function to 1 over the whole space, we shall consider a set of functions $g(X)$, denoted by $g_\alpha(X)$, where by construction α is a real positive number smaller than 1. A typical example of $g_\alpha(X)$ is given in Ref. [4], where it is shown that in the limit $\alpha \to 1^-$, with $\eta^2 > 1$, the running support of the PU test function then stretches over the whole integration domain, $X_{max} \to \infty$ and $f \to 1$. In this limit $g_\alpha(X) \to 1^-$. This running condition is equivalent to having an ultra-soft cut-off [15], i.e. an infinitesimal drop-off of the test function in the asymptotic limit, the rate of drop-off being governed by the arbitrary scale η^2. A similar scale invariance is also present in the IR domain, when $X \to 0$.

With these properties, the TLRS can be summarized as follows, first in the UV domain. Starting from a general amplitude $\mathcal{A}$ written for simplicity in a one

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[1] The square η^2 in (5) is only here for later convenience

dimensional space as

$$\mathcal{A} = \int_0^\infty dX \, T^>(X) \, f(X) , \quad (6)$$

where $T^>(X)$ is a singular distribution, we apply the following general Lagrange formula to $f(X)$, after separating out an intrinsic scale a from the (running) dynamical variable X

$$f^>(aX) = -\frac{X}{a^k k!} \int_a^\infty \frac{dt}{t}(a-t)^k \partial_X^{k+1}\left[X^k f^>(Xt)\right] . \quad (7)$$

This Lagrange formula is valid for any order k, with $k > 0$, since f is chosen as a SRTF. It is therefore equal to its Taylor remainder for any k. After integration by part in (6), and using (7), we can thus express the amplitude $\mathcal{A}$ as

$$\mathcal{A} = \int_0^\infty dX \, \widetilde{T}^>(X) \, f(X) , \quad (8)$$

where $\widetilde{T}^>(X)$ is the so-called extension of the singular distribution $T^>(X)$. In the limit $f \to 1$, it is given by [4]

$$\widetilde{T}^>(X) \equiv \frac{(-X)^k}{a^k k!} \partial_X^{k+1}\left[X T^>(X)\right] \int_a^{\eta^2} \frac{dt}{t}(a-t)^k . \quad (9)$$

The value of k in (9) corresponds to the order of singularity of the original distribution $T^>(X)$ [4]. In practice, it can be chosen as the smallest integer, positive or null, which leads to a non singular extension $\widetilde{T}^>(X)$. If in the absence of the test function $T^>(X)$ leads to a logarithmic divergence in (6), k is 0. It is 1 if the divergence is quadratic. With this choice for k, the extension of $T^>(X)$ is no longer singular due to the derivative in (9), so that we can safely perform the limit $f \to 1$ in (8), and obtain

$$\mathcal{A} = \int_0^\infty dX \, \widetilde{T}^>(X) , \quad (10)$$

which is well defined but depends on the arbitrary dimensionless scale η. This scale is the only remnant of the presence of the test function.

The extension of singular distributions in the IR domain can be done similarly [4, 15]. For an homogeneous distribution in one dimension, with $T^<[X/t] = t^{k+1}T^<(X)$, the extension of the distribution in the IR domain is given by

$$\widetilde{T}^<(X) = (-1)^k \partial_X^{k+1}\left[\frac{X^{k+1}}{k!}T^<(X)\ln(\tilde\eta X)\right] , \quad (11)$$

with $\tilde{\eta} = \eta^2 - 1$. The usual singular distributions in the IR domain are of the form $T^<(X) = 1/X^{k+1}$. In that case $\widetilde{T}^<(X)$ reads

$$\widetilde{T}^<(X) = \frac{(-1)^k}{k!} \partial_X^{k+1} \ln(\tilde{\eta}X) \ , \qquad (12)$$

where the derivative should be understood in the sense of distributions. Doing this, the extension $\widetilde{T}^<(X)$ is nothing else than the pseudo-function (Pf) of $1/X^{k+1}$ [4, 12, 15]

$$\widetilde{T}^<(X) = \text{Pf}\left(\frac{1}{X^{k+1}}\right). \qquad (13)$$

The extension $\widetilde{T}^<(X)$ differs from the original distribution $T^<(X)$ only at the $X = 0$ singularity.

For massive theories with a mass scale M, it is easy to translate this arbitrary dimensionless scale η to an arbitrary "unit of mass" $\mu = \eta M$. For massless theories, one can identify similarly an arbitrary unit of mass $\mu = \eta\Lambda$, as we shall see in the next section. This unit of mass, analogous to the unit of mass of DR [10], should be kept arbitrary in order to check that physical observables are indeed independent of μ (or η) after proper renormalization, order by order in perturbation theory.

Note that we do not need to know the explicit form of the test function in the derivation of the extended distribution $\widetilde{T}^>(X)$. We only rely on its mathematical properties and on the running construction of the boundary conditions.

IV. APPLICATION TO THE FINE-TUNING PROBLEM

In leading order of perturbation theory, the radiative corrections to the Higgs mass in the Standard Model are shown in Fig. 1. We have left out, for simplicity, all contributions coming from ghosts and Goldstone bosons. Each diagram in this figure gives a contribution to the self-energy $-i\Sigma(p^2)$, where p is the four-momentum of the external particle, and we have

$$M_H^2 = M_0^2 + \Sigma(M_H^2) \ . \qquad (14)$$

Using a naïve cut-off to regularize the amplitudes, these radiative corrections lead to the well known mass
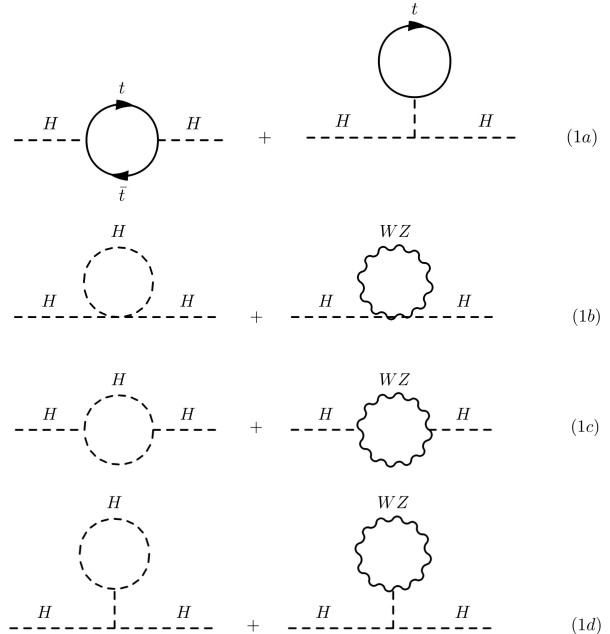

FIG. 1. Radiative corrections to the Higgs mass in the Standard Model. For simplicity, we have not shown contributions from ghosts or Goldstone bosons.

correction

$$M_H^2 = M_0^2 + \frac{3\Lambda_C^2}{8\pi^2 v^2}\left[M_H^2 + 2M_W^2 + M_Z^2 - 4m_t^2\right] + \dots \ , \qquad (15)$$

where $m_t, M_{W,Z}$ and M_H are the masses of the top quark, W, Z and Higgs bosons respectively, and v is the vacuum expectation value of the Higgs potential in the Standard Model. The dots include logarithmic corrections in Λ_C as well as contributions independent of Λ_C in the large Λ_C limit.

The calculation of the four different types of contributions shown in Fig. 1 is very easy in TLRS. Let us first illustrate the calculation of the simple Higgs loop contribution in Fig. 1.b. In Euclidean space one has

$$-i\Sigma_{1b,H} = -\frac{3iM_H^2}{2v^2} \int_0^\infty \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + M_H^2} f\left(\frac{k_E^2}{\Lambda^2}\right) \ , \qquad (16)$$

where k_E^2 is the square of the four-momentum k. As already mentioned in Sec. III, Λ is an arbitrary momentum scale. The test function f provides the necessary (ultra-soft) cut-off in the calculation of the integral.

After an evident change of variable, we get

$$\Sigma_{1b,H} = \frac{3M_H^4}{32\pi^2 v^2} \int_0^\infty dX \frac{X}{X+1} f\left(\frac{M_H^2}{\Lambda^2}X\right) \qquad (17)$$

$$= \frac{3M_H^4}{32\pi^2 v^2} \int_0^\infty dX \left(1 - \frac{1}{X+1}\right) f\left(\frac{M_H^2}{\Lambda^2}X\right) \ .$$

The first term under the integral, with no intrinsic scale, can be reduced to a pseudo-function, using (12). Indeed, with $Z = 1/X$, we have

$$\int_0^\infty dX f(X) = \int_0^\infty \frac{dZ}{Z^2} f\left(\frac{M_H^2}{\Lambda^2}\frac{1}{Z}\right) \qquad (18)$$

$$= \int_0^\infty dZ \ \mathrm{Pf}\left(\frac{1}{Z^2}\right)$$

$$= -\frac{1}{Z}\bigg|^\infty = 0 \ .$$

The notation $f(u)|^a$ simply indicates that $f(u)$ should be taken at the value $u = a$, the lower limit of integration being taken care of by the definition of the pseudo-function. This result is reminiscent of the property $\int d^D \mathbf{p}(\mathbf{p}^2)^n = 0$, for any n, in DR [18].

The self-energy thus writes

$$\Sigma_{1b,H} = -\frac{3M_H^4}{32\pi^2 v^2} \int_0^\infty dX \frac{1}{X+1} f\left(\frac{M_H^2}{\Lambda^2}X\right) \ . \quad (19)$$

The constant factor M_H^2/Λ^2 in the argument of the test function has no physical meaning since it can be absorbed by a rescaling of the arbitrary dimensionless scale η. This can be easily seen by applying the Lagrange formula (7) with the intrinsic scale $a = M_H^2/\Lambda^2$ and $k = 0$. It can thus safely be removed [2].

We can now apply the Lagrange formula for $k = 0$. Using the boundary condition on the support of the test function

$$Xt \leq H(X) = \eta^2 X g_\alpha(X) \ , \qquad (20)$$

we finally get, in the limit $f \to 1$

$$\Sigma_{1b} = -\frac{3M_H^4}{32\pi^2 v^2} \int_0^\infty dX \partial_X \left(\frac{X}{X+1}\right) \int_1^{\eta^2} \frac{dt}{t}$$

$$= -\frac{3M_H^4}{32\pi^2 v^2} \ln\left(\eta^2\right) \ . \qquad (21)$$

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[2] This could also be done more directly by choosing a particular value for Λ.

It is easy to see that using a naïve cut-off on k_E^2 one would have obtained, in the large Λ_C limit

$$\Sigma_{1b,H}^C = \frac{3M_H^2}{32\pi^2 v^2} \left[\Lambda_C^2 - M_H^2 \ln\left(\frac{\Lambda_C^2}{M_H^2}\right)\right] \ . \qquad (22)$$

For completeness, we recall below the result of the direct calculation of (16) in DR

$$\Sigma_{1b,H}^{DR} = \frac{3M_H^4}{32\pi^2 v^2} \left[-\frac{2}{\varepsilon} + c - \ln\left(\frac{\mu^2}{M_H^2}\right)\right] \ , \qquad (23)$$

where $c = \gamma_E - 1 - \ln 4\pi$ and γ_E is the Euler constant.

We can proceed further to the calculation of the $\bar{t}t$ polarization correction indicated in Fig. 1.a (left diagram). This one depends explicitly on the (square of the) momentum p of the external particle. Similarly to (16), in the Euclidean space, and using Feynman parametrization, one gets

$$-i\Sigma_{1a}(p^2) = \frac{12im_t^2}{v^2} \int_0^1 dx \int_0^\infty \frac{d^4 k_E}{(2\pi)^4}$$

$$\frac{k_E^2 - x(1-x)p_E^2 - m_t^2}{[k_E^2 + x(1-x)p_E^2 + m_t^2]^2} f\left(\frac{k_E^2}{\Lambda^2}\right) \ , \quad (24)$$

We thus have,

$$\Sigma_{1a}(p^2) = -\frac{3m_t^2}{4\pi^2 v^2} \int_0^1 dx \ M^2(x,p^2) \int_0^\infty dX \ X$$

$$\frac{X-1}{(X+1)^2} f\left[X\frac{M^2(x,p^2)}{\Lambda^2}\right] \qquad (25)$$

with $M^2(x,p^2) = x(1-x)p_E^2 + m_t^2$ and $X = k_E^2/M^2(x,p^2)$. The integral over X, which we shall call I, can be decomposed into three parts, $I = I_1 + I_2 + I_3$, with

$$I_1 = \int_0^\infty dX f\left[X\frac{M^2(x,p^2)}{\Lambda^2}\right] \ ,$$

$$I_2 = -3 \int_0^\infty dX \frac{1}{X+1} f\left[X\frac{M^2(x,p^2)}{\Lambda^2}\right] \ , \quad (26)$$

$$I_3 = 2 \int_0^\infty dX \frac{1}{(X+1)^2} f\left[X\frac{M^2(x,p^2)}{\Lambda^2}\right] \ . \qquad (27)$$

The first one is zero according to (18). The second one can be calculated following the derivation of (19), with an intrinsic scale $M^2(x,p^2)/\Lambda^2$ in the Lagrange formula (7) for $k = 0$. This gives, using $\Lambda = M_H$,

$$I_2 = \ln\left[\eta^2 \frac{M_H^2}{M^2(x,p^2)}\right] \ , \qquad (28)$$

while the third one is trivial and gives $I_3 = 2$ in the limit $f \to 1$.

The self-energy correction from the $\bar{t}t$ polarization diagram is thus

$$\Sigma_{1a}(p^2) = -\frac{3m_t^2}{4\pi^2 v^2} \int_0^1 dx\; M^2(x,p^2)$$
$$\left[-3 \ln\left[\eta^2 \frac{M_H^2}{M^2(x,p^2)}\right] + 2 \right] \;. \quad (29)$$

It is interesting to calculate directly (25) without decomposing the integral I. To do that, we should apply the Lagrange formula for $k = 1$. We thus get for this integral, with an intrinsic scale $a = M^2(x,p^2)/M_H^2$,

$$I = -\int_0^\infty dX\; X \partial_X^2 \left[\frac{X^2(X-1)}{(X+1)^2} \right] \frac{1}{a} \int_a^{\eta^2} \frac{dt}{t}(a-t) \;. \quad (30)$$

The self energy calculated in this way, and denoted by $\overline{\Sigma}_{1a}(p^2)$, is given by

$$\overline{\Sigma}_{1a}(p^2) = -\frac{3m_t^2}{4\pi^2 v^2} \int_0^1 dx\; M^2(x,p^2)$$
$$\left[\frac{3\eta^2 M_H^2}{M^2(x,p^2)} - 3 \ln\left[\eta^2 \frac{M_H^2}{M^2(x,p^2)}\right] - 3 \right] \;. \quad (31)$$

Comparing (29) and (31), we see that the calculation of the extension of a singular distribution is not unique. However, the self-energies differ either by a true constant (which depends on the arbitrary scale η, and is thus irrelevant in the calculation of the physical mass of the Higgs particle and more generally of any physical observable), or by a redefinition of η. They are thus said to be almost equivalent in the sense that they give identical physical, i.e. fully renormalized, amplitudes.

Using a naïve cut-off to calculate the self-energy (24), one would have obtained

$$\Sigma_{1a}^C(p^2) = -\frac{3m_t^2}{4\pi^2 v^2} \int_0^1 dx\; M^2(x,p^2)$$
$$\left[\frac{\Lambda_C^2}{M^2(x,p^2)} - 3 \ln\left[\frac{\Lambda_C^2}{M^2(x,p^2)}\right] + 2 \right] \;. \quad (32)$$

For completeness, we recall below the result in DR

$$\Sigma_{1a}^{DR}(p^2) = -\frac{3m_t^2}{4\pi^2 v^2} \int_0^1 dx\; M^2(x,p^2)$$
$$\left[-\frac{6}{\varepsilon} + 3c - 3 \ln\left[\frac{\mu^2}{M^2(x,p^2)}\right] + 1 \right] \;. \quad (33)$$

We can already see from these results that TLRS and DR lead to a similar p^2-dependent logarithmic term, with the identification $\eta^2 = \mu^2/M_H^2$. They both depend on a completely arbitrary constant. The quadratic and logarithmic divergent terms using a cut-off procedure are transmuted in TLRS into contributions depending on the arbitrary dimensionless scale η.

The other contributions to the radiative corrections to the Higgs mass indicated in Fig. 1 can be calculated similarly. The final correction to the bare mass can thus be written schematically as

$$M_H^2 = M_0^2 + \bar{a}\, \eta^2 + \bar{b} \ln[\eta^2] + cte \;. \quad (34)$$

This should be compared with the corrections indicated in Eqs. (1,15) when using a naïve cut-off. We emphasize again that η in (34) has no reasons whatsoever to be very large. It is a completely arbitrary real dimensionless parameter bigger than one. The exact expressions of $\bar{a}$ and $\bar{b}$ are of no physical interest since physical observables should be independent of η. They are completely finite and depend only on the masses of the top quark, W, Z and Higgs bosons and on the vacuum expectation value v of the Higgs field.

Away from the on mass shell condition $p^2 = M_H^2$, the constant term includes p^2-dependent logarithmic corrections which give rise to the well-known running of the mass. These corrections are identical in all three schemes, as expected.

V. DISCUSSION

We shall discuss our results in terms of the various scales appearing in the calculation of the radiative corrections to the Higgs mass, and more generally, to any physical observable. Three of them are of physical origin, and depend on the dynamical content of the underlying theory and on the kinematical conditions of the physical process under consideration, one is of mathematical origin, and two are completely arbitrary and are linked to the renormalization procedure.

The first physical scale is of course Λ_{eff} which defines the domain of validity of the underlying theory. It fixes, in the Standard Model, the energy scale above which new physics should show up. In that case, the Standard Model Lagrangian, $\mathcal{L}_{SM}$, should be supplemented by effective operators, and one should work

with the following effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \sum_{i>0} \frac{O_i}{(\Lambda_{eff})^i} \equiv \mathcal{L}_{SM} + \Delta\mathcal{L} , \quad (35)$$

where O_i is a set of local operators of dimension $(mass)^{i+4}$, compatible with the symmetries of the system. This has been examplified in Eq.(2) for the case of QED. For a given physical process, these new contributions are proportional to $(\Lambda_k/\Lambda_{eff})^i$. These operators originate from integrating out from the action the degrees of freedom of mass greater than Λ_{eff}.

The second one corresponds to the kinematical scale defined by the physical process under consideration. It can be for instance $\sqrt{Q^2}$ in DIS. We shall call it Λ_Q for simplicity. From a phenomenological point of view, in a bottom-up approach, Λ_{eff} should correspond to the value of Λ_Q for which theoretical predictions within the Standard Model are not corroborated by experimental results. The last physical scale we have already mentioned is the characteristic momentum relevant for the calculation of a given amplitude, called Λ_k. As we shall see below, it is intimately linked to the regularization procedure which are used. Generally speaking, we should expect $\Lambda_Q < \Lambda_k < \Lambda_{eff}$.

The mathematical scale is simply the cut-off Λ_C used in the calculation of any integral over momenta in internal loops. As mentioned above, it can not have any physical interpretation. It should be chosen large enough, and one should check that any physical observable is independent — within a given accuracy ϵ - of the exact value of Λ_C. In the literature, this mathematical scale is often taken equal to Λ_{eff}. We prefer here to separate clearly both scales since one has a physical interpretation while the other has not. As we shall explain below, this distinction is of particular interest for finite renormalization schemes like TLRS.

The last two scales are related to the renormalization procedure. The first one is the arbitrary scale η introduced in (5) in TLRS. It is the analog of the arbitrary mass scale μ of DR, with $\mu \simeq \eta\Lambda$. The second one is the mass scale, called R, which is chosen to fix the bare parameters of the original Lagrangian in terms of physical measurable quantities[3]. This is the so-called renormalization point. It appears

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[3] The scale R may in fact correspond to a set of many different scales, if one chooses for instance to fix the coupling constants at different momentum scales for all external particles [21].

of course in both finite or infinite renormalization schemes. These two scales are closely related to the RG analysis, in the sense that all physical observables should be independent of both η (or μ) and R.

In order to determine Λ_k from a quantitative point of view, we shall proceed in the following way. Writing the self-energy as

$$\Sigma(p^2) = \int_0^{\Lambda_C^2} dk_E^2 \ \sigma(k_E^2, p^2) , \quad (36)$$

we shall define the characteristic momentum Λ_k by requiring that the reduced self-energy defined by

$$\bar{\Sigma}(p^2) = \int_0^{\Lambda_k^2} dk_E^2 \ \sigma(k_E^2, p^2) \quad (37)$$

differs from $\Sigma(p^2)$ by ϵ in relative value, i.e. with the constraint

$$\frac{\bar{\Sigma}(p^2)}{\Sigma(p^2)} = 1 - \epsilon , \quad (38)$$

provided we have $|\bar{\Sigma}(p^2)| < |\Sigma(p^2)|$. In the Standard Model, ϵ can be taken of the order of 1%.

We show in Fig. 2 the characteristic scale Λ_k calculated for two typical expressions of the self-energy of the Higgs particle, as a function of Λ_C. The first expression is the bare one given by $\Sigma(M_H^2)$ in (14), while the second one is the fully (on-shell) renormalized amplitude, i.e. with both mass and wave function renormalization, defined by [21]

$$\Sigma_R(p^2) = \Sigma(p^2) - \Sigma(M_H^2) - (p^2 - M_H^2) \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=M_H^2}$$
$$(39)$$

and calculated at two different values of p^2, $p^2 = -10 \ M_H^2$ and $p^2 = -100 \ M_H^2$. The calculation is done using a typical contribution to the self energy, namely (32) for the cut-off regularization scheme, while (30) is taken for the calculation in TLRS, with an upper limit for the X integral given by $\Lambda_C^2/M^2(x,p^2)$.

Note that the derivation summarized in Sec. III to calculate the extension of the distribution $T(X)$ in the UV domain is valid for any test function with finite support. Since $\widetilde{T}(X)$ in (8) is not singular anymore, all corrections from the finite support of f in the upper limits of the integrals over X or t give a correction of order $1/X_{max} = M_H^2/\Lambda_C^2$.

The results indicated in Fig. 2 exhibit two very different behaviors. If one considers first the calculation

of the bare amplitude, the use of a naïve cut-off regularization scheme does not allow to identify any characteristic momentum Λ_k. Since Λ_k is always very close to Λ_C, all momentum scales up to Λ_C are involved in the calculation of the bare self-energy. This is indeed a trivial consequence of the fact that the renormalization of the bare amplitude is infinite in that case. This would also be the case in any infinite renormalization scheme using DR (in the limit of dimension 4). However, using TLRS, we can clearly identify a characteristic momentum Λ_k, since it reaches a constant value for Λ_C large enough. Note also that in this renormalization scheme, we can choose a value of Λ_C which is rather arbitrary, as soon as it is much larger than any mass or external momentum of the constituents. It can even be infinite, since it does not have any physical meaning, the only requirement being that physical amplitudes should be independent, within an accuracy ϵ, of the precise value of Λ_C. This behavior is typical of finite renormalization schemes.
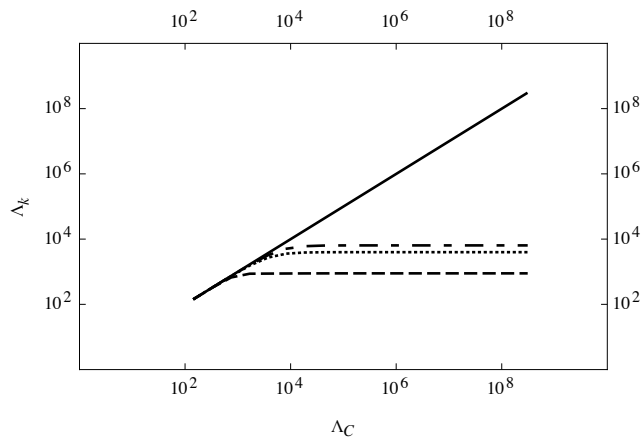

FIG. 2. Characteristic momentum scale Λ_k calculated from the self-energy contribution $\bar{\Sigma}(M_H^2)$ defined from the condition (38), in two different schemes: with a naïve cut-off (solid line) and using TLRS (dashed line). The calculation is done for $M_H = 125$ GeV, with $\eta^2 = 100$. We also show on this figure Λ_k calculated with the fully renormalized self-energy (39) for $p^2 = -10\ M_H^2$ (dotted line) and $p^2 = -100\ M_H^2$ (dash-dotted line).

If we now consider the characteristic momentum scale relevant for the description of the fully renormalized amplitude Σ_R, we can also identify a finite value for Λ_k since it saturates at sufficiently large values of Λ_C compared to the typical masses and external momenta of the system. This behavior is extremely similar to the result obtained in the above analysis of the bare amplitude Σ using TLRS. This is indeed not surprising since the fully renormalized amplitude is also completely finite. It depends only slightly on the external kinematical condition Λ_Q (given here by $\sqrt{-p^2}$). In any case, the characteristic momentum scale is of the order of 10 times Λ_Q, and, what is more important, it is independent of Λ_C. One can check that Σ_R is of course identical in all renormalization schemes.

VI. CONCLUSIONS

We have analyzed in this article the fine-tuning problem in the Standard Model of particle physics in the light of the recently proposed Taylor-Lagrange renormalization scheme. Since this scheme leads naturally to completely finite bare amplitudes - in contrast to a naïve cut-off regularization scheme which leads to quadratic divergences - the so-called fine-tuning problem can not have any physical reality in this scheme.

In order to understand in more quantitative details the differences between the various schemes, we have analysed the bare amplitudes, as well as the fully renormalized ones, in terms of the characteristic momentum scale relevant for the description of radiative corrections to the Higgs mass. In the case of the bare amplitudes, we find that this characteristic momentum scale is finite and independent of the cut-off Λ_C, provided it is large enough, when using TLRS, while it is as large as the cut-off scale in a cut-off regularization scheme. This forcludes any physical analysis in terms of a characteristic momentum scale when one uses a naïve cut-off - or using DR near four-dimensional space-time - on the bare amplitudes. The reason resides in an infinitely large bare amplitude for both regularization schemes.

On the contrary, we can clearly identify a characteristic relevant momentum scale when using TLRS. This scale is finite, while the value of Λ_C can be very large, independently of the precise value of Λ_{eff}. In that case Λ_k is completely determined by the dynamics of the underlying theory, as it should, and not by the mathematical properties of an ill-defined integral. It should only satisfy the consistency condition $\Lambda_k < \Lambda_{eff}$. This condition can be interpreted in two different ways. If Λ_{eff} is known (top-down approach), one should *verify* this condition in order to check the consistency of the effective theory. If Λ_{eff} is not known (bottom-up approach), one should use this condition to *induce* a lower limit on Λ_{eff}, given

by Λ_k. This is the case of the Standard Model.

As expected, the equivalence between the use of different schemes preserving gauge symmetry is restored if one analyses the fully renormalized amplitudes. In that case, one finds that the characteristic momentum scale is equivalent for all such schemes, and it is of the order of the masses, or external momenta, of the constituents of the system.

A remarkable feature of TLRS is that the identification of this characteristic momentum scale in the calculation of any amplitude can be done already at the level of the bare amplitude, in four physical space-time dimensions. This is at variance with both the usual DR or cut-off regularization procedures.

Our analysis of radiative corrections to the Higgs mass in the Standard Model has shown that the characteristic momentum scale is of the order of the typical mass scale given, on the mass shell, by the physical Higgs mass. Moreover, once the various relevant scales have been clearly identified, one can not give, on the sole consideration of radiative corrections to the Higgs mass, any information on the energy/momentum scale at which new physics should show up since this characteristic momentum is independent of Λ_{eff}. It also does not rely on any new symmetry.

ACKNOWLEDGEMENTS

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