THE STUDY OF THE DURATION OF CONTACT OF A PIANOFORTE STRING WITH A HARD HAMMER STRIKING NEAR THE END.*

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ABSTRACT. In this paper, a critical study has been made of the different existing theories of the vibration of the pianoforte string when struck by a hard hammer near the end. The values of the duration of contact have been calculated from the theoretical formulae of Kaufmann and of Kar-Ghosh as well as by the direct graphical method which, although involving enormous labour in evaluating λ, is more accurate than that of Kar-Ghosh. Taking into account the probable error of observation in determining experimentally the value of the duration of contact of such a small order of magnitude, the agreement between the calculated values and the available experimental data may be considered to be sufficiently close. In view of this close agreement, it is apparent that the idea of the series, as originally introduced by Raman and Banerjee to explain the theory of the struck string, should be abandoned.

The duration of contact plays an important part in the acoustics of the pianoforte string. The amplitude of different harmonics depends on the nature of the pressure exerted by the hammer on the string as well as on the duration of time for which the pressure acts. In practice, the hammer generally strikes near one end of the string to elicit the fundamental and other important harmonics more strongly. The position of the struck point depends of course upon the mass-ratio of the hammer and the string. So the correct determination of the duration of contact in such cases both experimentally and theoretically is of great value.

In this paper it is proposed to study the case of a string struck by a hard hammer and to critically examine the different existing theories on the subject.

In his well known theory of the pianoforte string, Kaufmann[1] has assumed that the shorter segment of the struck string behaves like a rigid rod and remains straight during the time of contact. This, however, is not the case, since Banerjee and Ganguli[2] have shown that the shorter segment vibrates during the time of contact. According to Kaufmann,[1] the displacement of the struck point during contact is given by

$$Y_0 = \frac{v_0}{c\lambda} \cdot e^{-\frac{\rho c t}{2m}} \sin \lambda c t, \qquad \ldots \qquad \ldots \qquad (1)$$

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$$\text{where } \lambda = \sqrt{\frac{\rho}{ma} - \frac{\rho^2}{4m^2}} \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

and ρ—linear density of the string, v_0—initial velocity of the hammer, a—length of the shorter segment of the string, c—velocity of the transverse wave propagation along the string, m—mass of the hammer + 1/3 the mass of the shorter segment of the string.

The duration of contact 'Φ' is the lowest root of the pressure equation and is obtained by equating $\frac{d^2 y_0}{dt^2}$ to zero. It is given by

$$\Phi = \frac{1}{c\lambda} \tan^{-1}\left(\frac{\lambda}{\frac{\rho}{2m} - \frac{1}{a}}\right); \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

when $\frac{\rho}{m} < \frac{1}{a}$,

$$\Phi = \frac{\pi}{c\lambda}, \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (4)$$

where λ is given by equation (2).

Starting from Rayleigh's theory of the loaded string, Raman and Banerjee[3] have tried to develop a theory of the struck string and have obtained an expression for the pressure exerted by the hammer in the form of a series. This series has been shown later on by Kar [4] to be divergent. Kar and Ghosh [5] have, however, pointed out that only the first term of the series should explain the acoustics of the struck string. The expression for the displacement of the struck point as given by Kar is

$$Y_0 = \frac{2v_0}{\lambda c} \frac{\sin \lambda ct}{\frac{\rho}{m}(a \operatorname{cosec}^2 \lambda a + b \operatorname{cosec}^2 \lambda b) + 1}, \qquad \ldots \qquad \ldots \quad (5)$$

where λ is the lowest root of the equation—

$$m \lambda \sin \lambda a . \sin \lambda b = \rho \sin \lambda l, \qquad \ldots \qquad \ldots \qquad \ldots \quad (6)$$

where $l = a + b =$ the length of the string.

Obviously it is not an easy task to evaluate λ from the equation (6); even the graphical method involves enormous labour. So in their generalised theory of the vibration of the struck string which has gained currency in current literature

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and 227 (1934)], Kar and Ghosh have approximately solved equation (6) for λ and found that (when $b \longrightarrow l \cdot a$)

$$\lambda = \sqrt{\frac{\rho}{ma} - \frac{\rho^2}{3m^2}}, \qquad \ldots \qquad \ldots \qquad \ldots \quad (7)$$

where m stands for the mass of the hammer only. It is interesting that with this value of λ, equation (5) becomes

$$Y_0 = \frac{v_0}{c\lambda} \sin \lambda\, ct. \qquad \ldots \qquad \ldots \qquad \ldots \quad (8)$$

On equating $\frac{d^2 y_0}{dt^2}$ to zero, we get for the duration of contact

$$\Phi = \frac{\pi}{c\lambda}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (9)$$

The value of the duration of contact obtained from Kaufmann's theory is very nearly the same as that given by Kar-Ghosh (*vide* tables I-V). The small difference in the value of λ in the two cases may evidently be attributed to the fact that Kaufmann's formula is based on the assumption of rigidity for the shorter segment of the string, whereas Kar-Ghosh's theory is free from any such assumption.

In tables I-IV, the values of the duration of contact calculated from the formulae of Kaufmann and Kar-Ghosh have been compared with the corresponding experimental results. Four sets of experimental data, which are available for the duration of contact with different striking distances, have been taken and may be considered quite sufficient for the present investigation. The corresponding curves are given in figs. 1 to 4 and to avoid confusion they have been marked—Experimental, Kaufmann and Kar-Ghosh—in each figure.

TABLE I.

[$l = 90$ cm., $m = 4{\cdot}4$ gm., $l\rho = 3{\cdot}175$ gm., $\frac{2l}{c} = 1/128$, $\rho = 0{\cdot}0353$ gm.]

Striking distance 'a' in cm.	Duration of contact 'Φ' in sec. $\times 10^3$			
	Experimental	Kaufmann (3)	Kaufmann (4)	Kar-Ghosh
8	3·55	4·03	4·387	4·352
12	4·64	4·86	5·423	5·359
16	5·70	5·55	6·220	6·224
20	6·41	6·33	7·127	6·997

[Data taken from Kaufmann—Ann. d. Phys. U. Che. B. 54, table II, p. 696 (1895).]

TABLE II.

[$l=240$ cm., $m=46$ gm., $l\rho=16{\cdot}4$ gm., $\frac{2l}{c}=0{\cdot}0356$, $\rho=0{\cdot}0683$ gm.]

Striking distance 'a' in cm.	Duration of contact 'Φ' in sec. $\times 10^3$				
	Experimental	Kaufmann (3)	Kaufmann (4)	Kar-Ghosh	Graphical
15	2·62	2·245	2·356	2·350	2·458
20	2·63	2·576	2·726	2·716	2·780
25	3·56	2·868	3·055	3·042	3·076
32	3·69	3·234	3·466	3·447	3·404
40	4·32	3·590	3·890	3·861	3·793
50	4·32	3·991	4·367	4·328	4·054
60	4·45	4·351	4·804	4·752	4·324
80	5·11	4·984	5·591	5·516	4·698
100	5·90	5·546	6·305	6·200	4·969

[Data from Banerjee-Ganguli—Phil. Mag., Vol. 7 (1929).]

TABLE III.

[$l=240$ cm., $m=13{\cdot}4$ gm., $l\rho=16{\cdot}4$ gm., $\frac{2l}{c}=0{\cdot}0356$, $\rho=0{\cdot}0683$ gm.]

Striking distance 'a' in cm	Duration of contact 'Φ' in Sec. $\times 10^3$			
	Experimental	Kaufmann (3)	Kaufmann (4)	Kar-Ghosh
15	1·23	1·179	1·292	1·279
20	1·42	1·503	1·503	1·485
25	1·51	1·500	1·692	1·666
30	1·70	1·640	1·865	1·834
35	1·79	1·760	2·030	1·988
40	2·05	1·890	2·184	2·137
45	2·11	2·000	2·335	2·281
50	2·33	2·090	2·477	2·412
55	2·48	2·170	2·616	2·541
60	2·66	2·284	2·751	2·667

[Data taken from Banerjee-Ganguli—Phil. Mag., Vol. 7 (1929).]

TABLE IV.

[$l=600$ cm., $m=21\cdot21$ gm., $l\rho=29\cdot7$ gm., $2l/c=0\cdot04316$, $\rho=0\cdot0495$ gm.]

Striking distance 'a' in cm.	Duration of contact 'ϕ' in sec. × 10^3			
	Experimental	Kaufmann (3)	Kaufmann (4)	Kar-Ghosh
30	1·1035	1·208	1·317	1·308
40	1·3620	1·385	1·532	1·517
50	1·4335	1·538	1·724	1·702
60	1·4835	1·678	1·901	1·871
70	1·5885	1·805	2·066	2·031
80	1·7655	1·924	2·223	2·180
90	1·8255	2·035	2·373	2·321
100	1·9345	2·141	2·518	2·458

(Data taken from Ghosh, *Ind. Jour. of. Phys.*, Vol. VII, Part V, table I, p. 373.)

TABLE V.

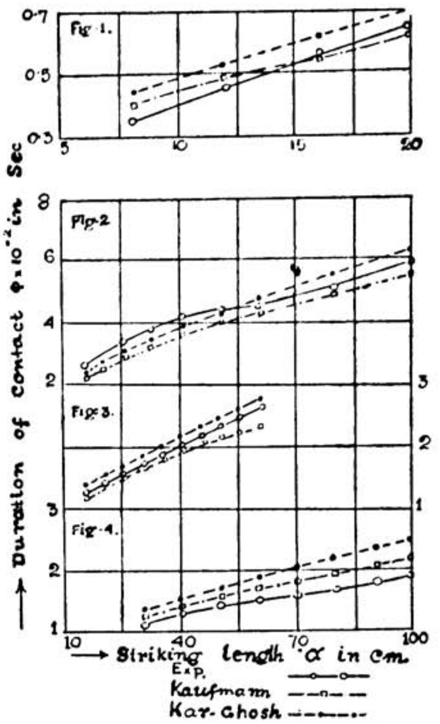
[$l=92$ cm., $l\rho=3\cdot25$ gm., $a=4\cdot5$ gm., $2l/c=1/128$, $\rho=0\cdot0353$ gm.]

Mass 'm' of the hammer in gm.	Duration of contact 'ϕ' in sec. × 10^3.			
	Experimental	Kaufmann (3)	Kaufmann (4)	Kar-Ghosh
4·4	3·047	2·984	3·227	3·174
9·0	4·101	4·320	4·564	4·477
11·4	5·078	4·875	5·125	5·125

(Data taken from Kaufmann—*Ann. d. Phy. U. Chem. Band*, 54 (1895), table III, p. 696.)

In table V the experimental and calculated values of the duration of contact, for the same striking distance but for different masses of the hammer, have been given.

In the last column of table II, we give the values of the duration of contact calculated from the lowest root of λ as obtained directly from equation (6) by the graphical method. It may be noted here that although the graphical method of finding λ is very tedious, yet it is undoubtedly more accurate than the approximate method of Kar-Ghosh.



A critical study of the experimental and theoretical values as given in the adjoining tables (I-V), reveals the following noteworthy features :—

(*i*) The theoretical values obtained from Kaufmann's formula (3) are systematically less than those of Kar-Ghosh, although to a small extent.

(*ii*) The values obtained from Kaufmann's approximate formula (4) are practically the same as those of Kar-Ghosh.

(*iii*) In tables I, IV and V, Kaufmann's values are in better agreement with the experimental results where the linear density ρ of the string is comparatively small, whereas, in tables II and III where ρ is relatively large, Kar-Ghosh's values are in closer agreement with the experimental results.

(*iv*) The values obtained by the graphical method as given in table II are generally in close agreement with Kar-Ghosh's values when the striking distance is relatively small.

It may, therefore, be observed that the agreement between the experimental and theoretical values is sufficiently close to testify to the fact that the theoretical formulae of both Kaufmann and Kar-Ghosh reasonably explain the acoustics of the struck string.

Before concluding, we think it worth while to make a passing reference to the work recently published by Iyengar [6] on the subject. In his paper Mr. Iyengar has attempted to establish the convergency of the series originally introduced by Raman and Banerjee as referred to before. In so doing, he has broken up the series into three component series and thus the expression for the pressure comes out as

$$P=\frac{2\rho v_0 c}{\pi}\left[\sum_1^\infty \frac{\sin\frac{n\pi}{a}\cdot ct}{n}+\sum_1^\infty \frac{\sin\frac{n\pi}{b}\cdot ct}{n}+\sum_1^\infty K_n\right], \qquad \ldots \quad (10)$$

where $K_n<\frac{\beta}{n}$, β being a suitably chosen constant. Of these component series the first two are obviously convergent, while the third one is divergent, although, curiously enough, he thinks it to be slowly convergent for no other reason than that it is less than the divergent series $\sum_1^\infty \frac{1}{n}$.

It is, however, apparent from the closeness of agreement of Kar-Ghosh's theoretical values with the experimental as also with the graphical values, that the idea of introducing the series should be abandoned. It should be noted in this connection that the problem of the pianoforte string is essentially different from that of the vibration of a string to which a load is permanently attached It is only in the latter case that this idea of the series may be supported.

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