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Continued Fractions Associated with Ellipsoidal Wave-Functions.

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In a previous memoir [1] the fundamental equation for an ellipsoidal wave-function in the Jacobean elliptic form has been derived as

$$d^2U/d\xi^2 + (a_0 - a_1k^2sn^2\xi - n^2k^4sn^4\xi)U = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $U(\xi)$ is of form

$$(sn\xi)^{\sigma_1}(cn\xi)^{\sigma_2}(dn\xi)^{\sigma_3}\psi(sn^2\xi)$$

ψ being an integral function of $sn\xi$ and a_1 and a_0 are characteristic constants.

By the transformation $sn\xi = v$ the above equation reduces to

$$(1-v^2)(1-k^2v^2)d^2U/dv^2 - v(1+k^2-2k^2v^2)dU/dv$$
$$+ (a_0 - a_1k^2v^2 - n^2k^4v^4)U = 0 \quad \ldots \quad (2).$$

We may use either of these equations for the purposes of this paper. The values of σ throughout this paper are 0 or 1.

[1] See 'Ind. Journal of Physics,' Vol. IX, p. 45 *et seq.*, 1934.

I. Let us assume that

$$U(\xi)=\Sigma A_\nu\ (sn\xi)2\nu+\sigma\ ;$$

then

$$(2\nu+2+\sigma)(2\nu+1+\sigma)A_{\nu+1}-\{(1+k^2)(2\nu+\sigma)^2-a_0\}A_\nu$$
$$+\{(2\nu-2+\sigma)(2\nu-1+\sigma)-a_1\}k^2A_{\nu-1}-n^2k^4A_{\nu-2}=0 \quad \ldots \quad (3)$$

$$(4+\sigma)(3+\sigma)A_2-\{(1+k^2)(2+\sigma)^2-a_0\}A_1+\{\sigma(1+\sigma)-a_1\}k^2A_0=0 \quad \ldots \quad (4)$$

$$(2+\sigma)(1+\sigma)A_1-\{(1+k^2)\sigma^2+a_0\}A_0=0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

If we put

$$B_\nu\ =A_{\nu+1}/A\nu$$

then

$$(2\nu+2+\sigma)(2\nu+1+\sigma)B_\nu\ -\{(1+k^2)(2\nu+\sigma)^2-a_0\}$$
$$+\{(2\nu-2+\sigma)(2\nu-1+\sigma)-a_1\}k^2/B_{\nu-1}-n^2k^4/B_{\nu-1}B_{\nu-2}=0 \quad \ldots \quad (6)$$

It is seen that the value of B_ν tends for large values of ν either to unity or zero. As $U(\xi)$ is an integral function of $sn\xi$ the value of B_ν must necessarily be zero ultimately.

The value of B_ν can be put in the form of a continued fraction of Fürstenau's type.

$$B_\nu\ =n^2k^4/[\{(2\nu+2+\sigma)(2\nu+3+\sigma)-a_1\}k^2$$
$$-\{(1+k^2)(2\nu+4+\sigma)^2a_0\}B_\nu\ +\ (2\nu+5+\sigma)(2\nu+6+\nu)B_{\nu+1}B_{\nu+2}]\ldots \quad (7)$$

The values of B_0 and B_1 obtained in the form of continued fractions have to be equal to their values given by the equations

$$(\sigma+4)(\sigma+3)B_1-\{(1+k^2)(2+\sigma)^2-a_0\}+\{\sigma(\sigma+1)-a_1\}k^2/B_0=0$$
$$(\sigma+2)(\sigma+1)B_0=(1+k^2)\sigma^2-a_0$$

Just as in the problems dealt by Kelvin, Darwin and Goldstein, it is necessary to point out that the coefficients A can be determined with the relations 3, 4 and 5 only if we know the value of a_0 and a_1 exactly and we make no approximation at any stage.[2] Otherwise the value of B_ν will inevitably tend to unity instead of zero. Similar remarks will apply to functions of other species also.

II. Let

$$U(\xi)=\Sigma A_\nu\ (sn\xi)^{2\nu+\sigma}.cn\xi\ ;$$

then

$$(2\nu+2+\sigma)(2\nu+1+\sigma)A_{\nu+1}-\{(2\nu+1+\sigma)^2+(2\nu+\sigma)^2k^2-a_0\}A_\nu$$
$$+\{(2\nu-1+\sigma)(2\nu+\sigma)-a_1\}k^2.A_{\nu-1}-n^2k^4A_{\nu-2}=0 \quad \dots \quad (8)$$

$$(4+\sigma)(3+\sigma)A_2-\{(3+\sigma)^2+(2+\sigma)^2k^2-a_0\}A_1$$
$$+\{(1+\sigma)(2+\sigma)-a_1\}k^2.A_0=0 \quad \dots \quad \dots \quad (9)$$

and $(1+\sigma)(2+\sigma)A_1\{(1+\sigma)^2+\sigma^2k^2-a_0\}A_0=0 \quad \dots \quad \dots \quad (10)$

putting $B_\nu=A_{\nu+1}/A_\nu$ we get

$$B_\nu=n^2k^4/[\{(2\nu+\sigma+3)(2\nu+4+\sigma)-a_1\}k^2-\{(2\nu+5+\sigma)^2+(2\nu+4+\sigma)^2k^2$$
$$-a_0\}B_{\nu+1}+(2\nu+6+\sigma)(2\nu+5+\sigma)B_{\nu+1}B_{\nu+2}] \quad \dots \quad (11)$$

III. If

$$U(\xi)=\Sigma A_\nu\ (sn\xi)^{2\nu+\sigma}.dn\xi\ ;$$

then

$$(2\nu+2+\sigma)(2\nu+1+\sigma)A_{\nu+1}-\{(2\nu+1+\sigma)^2k^2+(2\nu+\sigma)^2-a_0\}A_\nu$$
$$+\{(2\nu+\sigma)(2\nu-1+\sigma)-a_1\}k^2.A_{\nu-1}-n^2k^4A_{\nu-2}=0 \quad \dots \quad (12)$$

$$(4+\sigma)(3+\sigma)A_2-\{(3+\sigma)^2k^2+(2\nu+\sigma)^2-a_0\}A_1$$
$$+\{(2+\sigma)(1+\sigma)-a_1\}k^2A_0=0 \quad \dots \quad (13)$$

and $(2+\sigma)(1+\sigma)A_1-\{(1+\sigma)^2k^2+\sigma^2-a_0\}A_0=0 \quad \dots \quad (14)$

If

$$B_\nu=A_{\nu+1}/A_\nu\ ;$$

[2] See the references in Lamb's Hydrodynamics, pp. 335, sixth edition, 1932. See also Goldstein on Mathieu Functions. Trans. of the Cambridge Phil. Soc., Vol. XXXIII, No. XI, pp. 303-36, 1927.

then

$$B_\nu = n^2k^4/[\{(2\nu+4+\sigma)(2\nu+3+\sigma)-a_1\}k^2-\{(2\nu+5+\sigma)^2k^2+(2\nu+4+\sigma)^2$$
$$-a_0\}B_{\nu+1}+(2\nu+6+\sigma)(2\nu+5+\sigma)B_{\nu+1}B_{\nu+2}]. \quad \ldots \quad (15)$$

IV. If

$$U(\xi)=\Sigma A_\nu\ (sn\xi)^{2\nu+\sigma}\ cn\xi.dn\xi\ ;$$

then

$$(2\nu+2+\sigma)(2\nu+1+\sigma)A_{\nu+1}-\{(2\nu+1+\sigma)^2(1+k^2)-a_0\}A_\nu$$
$$+\{(2\nu+\sigma)(2\nu+1+\sigma)-a_1\}k^2A_{\nu-1}-n^2k^4A_{\nu-2}=0 \quad \ldots \quad (16)$$

$$(4+\sigma)(2+\sigma)A_2-\{(3+\sigma)^2(1+k^2)-a_0\}A_1$$
$$+\{(2+\sigma)(1+\sigma)-a_1\}k^2A_0=0 \quad \ldots \quad (17)$$

and $$(2+\sigma)(1+\sigma)A_1-\{(1+\sigma)^2(1+k^2)-a_0\}A_0=0 \quad \ldots \quad (18)$$

if $B_\nu = A_{\nu+1}/A_\nu$ then it is equal to

$$n^2k^4/[\{(2\nu+5+\sigma)(2\nu+4+\sigma)-a_1\}k^2-\{(2\nu+6+\sigma)^2(1+k^2)-a_0\}B\nu+1$$
$$+(2\nu+6+\sigma)(2\nu+5+\sigma)B_{\nu+1}B_{\nu+2}] \quad \ldots \quad (19)$$

All the possible species and types of characteristic functions are included in the above as in each instance σ could be either one or zero.