Ellipsoidal Wave-Functions

By

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Introduction.

The wave equation

$$\Delta\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0$$

occurs widely and normal functions suitable for different boundary conditions have been studied for a long time. In this paper an attempt has been made to obtain and study the properties of normal functions suitable for boundary conditions over ellipsoids or other central quadrics. The normal functions bear a relation to Lamé functions similar to that existing between Mathieu functions and the circular functions. During the course of this work, which was undertaken at widely separated intervals and completed by 1929, a memoir by F. Moeglich, dealing partially with the problem of obtaining functions which could be used for ellipsoidal boundaries, was published in 1927. The normal functions obtained by him are functions of two variables. His method simplifies much of the preliminary work and does not raise the question regarding the existence of solutions for nonlinear integral equations, which looms prominently in the present work.

Also neither the methods employed nor the results obtained bring out the analogy with the Mathieu and Lamé functions as, *e.g.*, the various species of normal functions corresponding to the species of Lamé functions.

The first section deals with the derivation of the differential equation in algebraical form and its uniformisation. The boundary conditions are also specified. In the form involving Jacobean elliptic functions the fundamental differential equation is

$$\frac{d^2U}{d\xi^2} + (a_0 - a_1 k^2 sn^2\xi - n^2 k^4 sn^4\xi)U = 0$$

where n is a constant and a_0 and a_1 have to be characteristic constants.

The next section is devoted to properties common to all the characteristic functions. It is seen that the solutions can be written in the form

$$(sn\xi)^{\sigma_1} \ (cn\xi)^{\sigma_2} \ (dn\xi)^{\sigma_3} \ \psi(sn^2\xi)$$

where ψ is an integral function of $sn\xi$ and σ_1, σ_2, and σ_3 have values equal to 0 or 1. The functions are therefore continuously differentiable for finite values of $sn\xi$. From the usual form of the second solution of a second order differential equation in the normal form the symmetry character of the second solution is found.

The second half of the section is concerned with orthogonal relations. By the usual methods it is shown that the characteristic constants are real so long as we deal with real Cartesian space. The wave functions can be normalised. The linear independence of the characteristic functions follows easily. A third orthogonal relation that will prove of use is also given.

Section III deals with integrals connected with the equations. From an analogue to Whittaker's integral for a wave equation, the integral equations of the characteristic functions are deduced. The integral equations are non-linear. The convention is therefore made that the solutions are always dealt with in their normal form.

Then the actual form of the nuclei for the four species and eight types are given. By ordinary methods the integral equation can be solved for small values of n.

The method of stationary phase gives us the asymptotic expressions for large values of $sn\xi$. It follows simply from the asymptotic expressions for large and positive values of $sn\xi$ that the second solution behaves differently from the characteristic functions.

The work involved is rather heavy for the calculation of the functions. The method of Horn used by Jeffreys was applied to the present problem for large values of n. It is possible that this is the first application of the method for a differential equation with two characteristic constants. The method adopted in the section requires a slight explanation. The condition for determining the various constants that occur is first given and at the end a review is made by comparing the asymptotic solution obtained in this section with the one got by using the method of stationary phase. This is possible as the asymptotic expressions have a common region of validity. Considering formally, the Horn and Jeffreys method is a re-arrangement of the Hamburger approximation about the irregular point. Probably this formal relation may be extended so as to facilitate the identification of solutions in their variant forms. For large values of n^2 no approximations of the third and fourth species have been derived.

I. The fundamental system of confocal quadrics which define the elliptic co-ordinates are taken here to be

$$1 = x^2/(\lambda - e_3) + y^2/(\lambda - e_2) + z^2/(\lambda - e_1)$$

where

e_1, e_2, e_3 are real, $e_1 > e_2 > e_3$ and $e_1 + e_2 + e_3 = 0$.

This choice saves symbols. The quadric is an ellipsoid, a hyperboloid of one sheet or a hyperboloid of two sheets according as $\lambda > e_1$, $e_1 > \lambda > e_2$, or $e_2 > \lambda > e_3$. The co-ordinates corresponding to the three quadrics are denoted by λ, μ, ν. The ranges of values given above for them are useful. As is well-known, the following relation holds:

$$x^2 = (\lambda - e_3)(\mu - e_3)(\nu - e_3)/(e_3 - e_1)(e_3 - e_2)$$

$$y^2 = (\lambda - e_2)(\mu - e_2)(\nu - e_2)/(e_2 - e_1)(e_2 - e_3)$$

$$z^2 = (\lambda - e_1)(\mu - e_1)(\nu - e_1)/(e_1 - e_2)(e_1 - e_3).$$

Denoting for shortness

$$\Delta_\lambda^2 = (\lambda - e_1)(\lambda - e_2)(\lambda - e_3)$$

and similar symbols for other variables (μ, ν) and $\underset{\lambda\mu\nu}{\gtreqless}$ for the sum of cyclically permuted terms, the wave equation

$$\Delta\phi - 1/c^2 . \partial^2\phi/\partial t^2 = 0$$

reduces to

$$-4/(\lambda-\mu)(\mu-\nu)(\nu-\lambda). \left[\underset{\lambda\mu\nu}{\gtreqless} \Delta_\lambda (\mu-\nu)\,\partial(\Delta_\lambda\, \partial\phi/\partial\lambda)\,/\,\partial\lambda \right]$$

$$-1/c^2 . \partial^2\phi/\partial t^2 = 0. \qquad \dots \quad (1.0)$$

Without loss of generality it is assumed that ϕ is proportional to $exp(ipt)$ and so we replace $-1/c^2 . \partial^2\phi/\partial t^2$ by $p^2\phi^2/c^2$.

The equation (1·0) becomes

$$\gtreqless \Delta_\lambda (\mu-\nu)\,\partial(\Delta_\lambda\, \partial\phi/\partial\lambda)/\partial\lambda - p^2(\lambda-\mu)(\mu-\nu)(\nu-\lambda)\phi/4c^2 = 0. \quad \dots \quad (1.1)$$

The equation is separable and we may assume as usual that the solution is of form $\Lambda(\lambda)\,M(\mu)\,N(\nu)$. The three functions $\Lambda(\lambda)$, $M(\mu)$ and $N(\nu)$ satisfy the same differential equation as for Λ; the equation is

$$\Delta_\lambda\, d(\Delta_\lambda\, d\Lambda/d\lambda)/d\lambda = (a_0 + a_1\lambda - p^2\lambda^2/4c^2)\Lambda \qquad \dots \quad (1.2)$$

where a_0 and a_1 are arbitrary constants. A priori their values are not known. They have to be restricted by a choice as in the case of Mathieu or Lamé functions. In the problems that we come across ϕ is of the nature of a varying potential or like quantity. The simplest assumption would be that it is single valued in space. The derivatives of ϕ are of the nature of velocity or force or the like. As we deal with finite magnitudes of these quantities, and much less frequently with infinite values, we make the following restrictions. ϕ is a one valued function of x, y, z with bounded derivatives everywhere in the finite region. This would mean that $\wedge$ has to be one-valued in $(\lambda-e_1)^{\frac{1}{2}}$, $(\lambda-e_2)^{\frac{1}{2}}$ and $(\lambda-e_3)^{\frac{1}{2}}$ with one or more factors of the type $(\lambda-e_1)^{\frac{1}{2}}$, $(\lambda-e_2)^{\frac{1}{2}}$ and $(\lambda-e_3)^{\frac{1}{2}}$. It will be seen later that apart from these factors the main function is an integral function of any of the three quantities

$$(\lambda-e_1)^{\frac{1}{2}},\ (\lambda-e_2)^{\frac{1}{2}};\ \text{and}\ (\lambda-e_3)^{\frac{1}{2}}$$

The equation (1.2) written in full is

$$(\lambda-e_1)(\lambda-e_2)(\lambda-e_3)[d^2\wedge/d\lambda^2+1/2\{1/(\lambda-e_1)+1/(\lambda-e_2)+1/(\lambda-e_3)\}\,d\wedge/d\lambda]$$
$$-(a_0+a_1\lambda-p^2\lambda^2/4c^2)\wedge=0.$$

This equation has singularities at e_1, e_2 and e_3 with exponents 0 and 1/2, and an irregular singularity at infinity. The equation is therefore a confluent form of differential equation with six regular singularities.*

The form of the equation suggests that

$$\wedge=(\lambda-e_1)^{\sigma_1/2}(\lambda-e_2)^{\sigma_2/2}(\lambda-e_3)^{\sigma_3/2}.\ \wedge_1(\lambda)$$

where σ_1, σ_2, σ_3 could be 0 or 1, and $\wedge_1(\lambda)$ is some function of λ which is one valued and bounded and which in a variant form will be shown to be integral in $(\lambda-e_1)^{\frac{1}{2}}$, $(\lambda-e_2)^{\frac{1}{2}}$ and $(\lambda-e_3)^{\frac{1}{2}}$ and the like.

* *Cf.* L. Ince., Ordinary Differential Equations, p. 502.

As every finite point except $\lambda = e_1$, e_2 or e_3 is an ordinary point of the differential equation no such point could be a multiple zero of Λ_1 unless the function is identically zero, which case we bar out. The finite zeros of Λ_1 have to be different from e_1, e_2 or e_3 as otherwise the exponents at these points could not be 0 or 1/2.

The presence of the radicals in the algebraic form of the differential equation prevents an easy handling of it. The limiting or boundary conditions are combrous for use. This difficulty can be overcome by uniformising the variables. The two possible forms of the equations corresponding to the Weirstrassian and the Jacobean elliptic functions are both easily derivable and could be solved in a more elegant way.

The invariants of an elliptic function (Weirstrassian) are so determined that the semi-periods $\omega_1, \omega_2, \omega_3$; $(\omega_1+\omega_2+\omega_3=0)$ satisfy $\wp(\omega_r)=e_r$; $(r=1, 2 \text{ or } 3)$. The variables α, β, γ are found so that $\lambda=\wp(\alpha)$; $\mu=\wp(\beta)$; and $\nu=\wp(\gamma)$. The equation (1.1) becomes

$$\sum_{\alpha\beta\gamma} [\wp(\beta)-\wp(\gamma)]\ \partial^2\phi/\partial\alpha^2$$

$$-p^2\{\wp(\beta)-\wp(\gamma)\}\{\wp(\gamma)-\wp(\alpha)\}\{\wp(\alpha)-\wp(\beta)\}\phi/c^2=0. \qquad \dots \quad (1.3)$$

This equation gives rise to a one-variable equation in the form

$$d^2 \mathrm{A}/d\alpha^2+\{p^2/c^2.[\wp(\alpha)]^2-a_1\wp(\alpha)-a_0\}\mathrm{A}=0. \qquad \dots \quad (1.4)$$

where a_1, a_0 are arbitrary constants which vary according to the form of the equation employed.

The Jacobean form is obtained by taking

$$\left.\begin{aligned} \lambda&=\wp(\alpha)=e_1-(e_1-e_3)\ k^2\ sn^2\xi \\ \mu&=\wp(\beta)=e_1-(e_1-e_3)\ k^2\ sn^2\eta \\ \nu&=\wp(\gamma)=e_1-(e_k-e_3)\ k^2\ sn^2\zeta \end{aligned}\right\} \qquad \dots \quad (1.5)$$

where $k^2=(e_1-e_2)/(e_1-e_3)$;

and $k'^2=(e_2-e_3)/(e_1-e_3)$

which is possible if $\alpha=(\xi+iK')/\sqrt{(e_3-e_1)}$

with similar relations for β and γ. iK' is the quarter period in the usual notation.*

As the explicit relations between the Cartesian and the Jacobean elliptic co-ordinates may be useful they are collected below :

$$\left.\begin{aligned} x&=q/k'\; dn\xi\; dn\eta\; dn\zeta \\ y&=-iqk^2/k'\; cn\xi\; cn\eta\; cn\zeta \\ z&=-iqk^2\; sn\xi\; sn\eta\; sn\zeta \end{aligned}\right\} \quad \ldots \quad (1.6)$$

where $q=\underset{+}{\sqrt{}}(e_1-e_3)$.

The equation (1.1) reduces to

$$\sum_{\xi\eta\zeta} (sn^2\eta-sn^2\zeta)\, \partial^2\phi/\xi\partial^2 + n^2k^4(sn^2\xi-sn^2\eta)\,(sn^2\eta-sn^2\zeta)\,(sn^2\zeta-sn^2\xi)\phi=0 \quad \ldots \quad (1.7)$$

where $n=pq/c$. The corresponding single variable equation is

$$d^2U/d\xi^2+(a_0-a_1k^2\; sn^2\xi-n^2k^4\; sn^4\xi)U=0 \quad \ldots \quad (1.8)$$

This is the generalised Lamé equation as denoted by L. Ince. When $n=0$ it passes to the usual Lamé form and with proper conditions leads to Lamé functions.

The limiting conditions for the arbitrary constants are that they should be so chosen that

(*i*) U or Λ is a doubly periodic function of ξ or α as the Jacobean or the Weirstrassian form is used.

(*ii*) U or Λ has bounded derivatives at all points except possibly at $\xi=iK'$ or $2K+iK'$ and congruent points in the Jacobean form or $\alpha=0$ in the Weirstrassian form.

* See, *e.g.*, Whittaker and Watson, Modern Analysis, p. 501.

We may confine ourselves to the Jacobean form as being simpler to deal with in spite of the fact that the formulae are unsymmetrical. When $n=0$ the above conditions would lead to the Lamé functions, with $a_1=l(l+1)$ where l is an integer and a_0 has one of $2l+1$ discrete characteristic values. In the memoir by Moeglich* already referred to he has proved the existence of the characteristic constants with the help of linear integral equations for the above equation. And they correspond to the characteristic constants of Lamé functions.

II. It is necessary to have and to utilize the general properties of the functions which can be derived without evaluating their particular values.

A comparison with the algebraic form or a simple examination of the Jacobean form of the equation shows that the form of the solution should be

$$U(\xi)=(sn\xi)^{\sigma_1}\,(cn\xi)^{\sigma_2}\,(dn\xi)^{\sigma_3}\,\psi(sn^2\xi),$$

where σ_1, σ_2 or σ_3 may be 0 or 1, and the nature of ψ is to be determined. By the limitations imposed already its derivatives exist at all points with the possible exception of points congruent to $\xi=i\mathrm{K}'$ or $\xi=2\mathrm{K}+i\mathrm{K}'$.

As the differential equation is unchanged by changing ξ to $-\xi$, $U(-\xi)$ is also a solution of the same differential equation. Hence it is enough if we consider solutions of type

$$(sn\xi)^{\sigma_1}(cn\xi)^{\sigma_2}(dn\xi)^{\sigma_3}\psi(sn^2\xi). \qquad \ldots \qquad \ldots \quad (2.1)$$

If the characteristic constants a_1, a_0 be real and n^2 is real ψ may be taken to be a real function of the argument, *i.e.*, $sn^2\xi$.

Let us consider the solution valid about the point $\xi=0$. Here ψ $(sn^2\xi)$ can be expanded as a power series in terms of

* F. Moeglich, Annalen der Physik, Band 83, p. 609; Beugungserscheinungen an Körpern von ellipsoidischer Gestalt, 1927.

$sn^2\xi$. Ordinarily this power series ceases to converge as the singular points are approached.

As $\psi(sn^2\xi)$ is unchanged when ξ is replaced by $(2K-\xi)$, $K=\xi$ is a point of symmetry for the function and as the derivatives of $U(\xi)$ and hence ψ exist unless $| sn\xi |$ is infinite ;

so $d\psi/d\xi=0$; at $\xi=K$.

Similarly $d\psi/d\xi=0$ at $\xi=K+iK'$.

Any solution of the differential equation (1.8) about the point $\xi=K$ may be written as

$$AF_1(1-\theta^2)+B\sqrt{(1-\theta^2.)}F_2(1-\theta^2)$$

where θ denotes $sn\xi$ and $F_1\,F_2$ are power series of their arguments.

Hence $$\psi(\theta^2)=\theta^{-\sigma_1}(1-\theta^2)^{-\sigma_2/2}(1-k^2\theta^2)^{-\sigma_3/2}\times[AF_1(1-\theta^2)+B\sqrt{(1-\theta^2)}F_2(1-\theta^2)].$$

As $$d\psi/d\xi=\sqrt{(1-\theta^2)(1-k^2\theta^2)}.d\psi/d\theta$$

and $$d\psi/d\xi=0 \text{ at } \xi=K \text{ or } \theta=1,$$

it follows that $A=0$ if $\sigma_2=1$

$B=0$ $\sigma_2=0$,

i.e., ψ considered as a function of θ or $sn\xi$ has no singularity at $\theta=1$ or $\xi=K$. Similarly ψ has no singularity at $\xi=K+iK'$ or $\theta=1/k$. But ψ can have no other singular points for finite values of $| sn\xi |$. Hence $\psi\,(sn^2\xi)$ is an integral function of $sn\xi$. ψ can similarly be considered as integral functions of $cn\xi$ or $dn\xi$. This property is useful.

A second solution of the differential equation could be taken to be as

$$U(\xi)\int^{\xi}dt\Big/\Big\{U(t)\Big\}^2.$$

It follows easily that this would have the form

$$(sn\xi)^{1-\sigma_1}\,(cn\xi)^{1-\sigma_2}\,(dn\xi)^{1-\sigma_3}\,\psi_1(sn^2\xi). \qquad \ldots \quad (2.2)$$

where ψ_1 is not necessarily an integral function of $sn^2\xi$. But from the form of the solution, it follows that when ξ is real, the origin $(\xi=0)$ is a point of symmetry for one solution, while it is the point of anti-symmetry for the other solution of the differential equation. Also when $\xi=\mathrm{K}+i\sigma$, *i.e.*, when ξ lies along a line parallel to the imaginary axis through $\xi=\mathrm{K}$, the point $\xi=\mathrm{K}$ is a point of symmetry for one solution and the point of anti-symmetry for the other.

Orthogonal relations. Corresponding to the two characteristic constants of the differential equation two important orthogonal relations are obtainable.

For clearness $\mathrm{U}(\xi \mid a_1\, a_0)$ will be written to show that $a_1\, a_0$ are the two characteristic constants.

Let us consider the case, when the constant a_1 is the same for two characteristic functions and the other constants are a_0 and a'_0.

The respective functions are $\mathrm{U}(\xi \mid a_1\, a_0)$ and $\mathrm{U}(\xi \mid a_1\, a'_0)$.

From their differential equations it follows

$$\iint\left\{\mathrm{U}(\xi \mid a_1 a'_0)\,\partial^2\mathrm{U}(\xi \mid a_1 a_0)/\partial\xi^2-\mathrm{U}(\xi \mid a_1 a^0)\,\partial^2\mathrm{U}(\xi \mid a_1 a'_0)/\partial\xi^2\right\}$$

$$\int\mathrm{U}(\xi \mid a_1 a_0)\,\mathrm{U}(\xi \mid a_1 a'_0)\,d\xi=0 \qquad \ldots \quad (2.3)$$

where the integrals are taken over the same range. The first integral vanishes when the initial and final limits of integration differ by a whole number of periods which do not contain points congruent to $i\mathrm{K}'$ or $2\mathrm{K}+i\mathrm{K}'$. Hence if $a_0 \neq a'_0$

$$\int\mathrm{U}(\xi \mid a_1 a_0)\,\mathrm{U}(\xi \mid a_1 a'_0)\,d\xi=0,$$

the range of integration being from $\xi=\xi_1$ to $\xi=\xi_1+4m\mathrm{K}+4m'i\mathrm{K}'$ where m and m' are integers. Preferably we may take the whole range of integration to be from $-2\mathrm{K}$ to $+2\mathrm{K}$.

Now let us consider the case when the characteristic constants are all different, and let $a_1\, a_0$ and $a'_1 a'_0$ be the constants. Now the product function $\mathrm{U}(\xi \mid a_1 a_0)\mathrm{U}(\eta \mid a_1 a_0)$ satisfies the differential equation

$$\{\partial^2/\partial\xi^2-\partial^2/\partial\eta^2\}\mathrm{U}(\xi \mid a_1a_0)\mathrm{U}(\eta \mid a_1a_0)$$
$$-\{a_1k^2+n^2k^4(sn^2\xi+sn^2\eta)\}(sn^2\xi-sn^2\eta)\mathrm{U}(\xi)\mathrm{U}(\eta)=0.$$

Hence it follows that

$$\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\eta \mid a'_1a'_0)[\mathrm{U}(\xi \mid a'_1a'_0)\partial^2\mathrm{U}(\xi \mid a_1a_0)/\partial\xi^2$$
$$-\mathrm{U}(\xi \mid a_1a_0)\partial^2\mathrm{U}(\xi \mid a'_1a'_0)/\partial\xi^2]$$
$$-\mathrm{U}(\xi \mid a_1a_0)\mathrm{U}(\xi \mid a'_1a'_0)[\mathrm{U}(\eta \mid a'_1a'_0)\partial^2\mathrm{U}(\eta \mid a_1a_0)/\partial\eta^2$$
$$-\mathrm{U}(\eta \mid a_1a_0)\partial^2\mathrm{U}(\eta \mid a'_1a'_0/\partial\eta^2)]$$
$$-k^2(a_1-a'_1)(sn^2\xi-sn^2\eta)\mathrm{U}(\xi \mid a_1a_0)\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\xi \mid a'_1a'_0)\mathrm{U}(\eta \mid a'_1a'_0)=0.$$

Integrating with respect to ξ and η so that the limits of integration of each of these integrals differ by a whole number of periods, the first two terms integrate to zero. Of course, these ranges exclude points congruent to $i\mathrm{K}'$ or $2\mathrm{K}+i\mathrm{K}'$. The ranges for the two integrals need not be the same.

Hence

$$(a_1-a'_1)\int\int(sn^2\xi-sn^2\eta)\mathrm{U}(\xi \mid a_1a_0)\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\xi \mid a'_1a'_0)\times$$
$$\mathrm{U}(\eta \mid a'_1a'_0)d\xi d\eta=0.$$

The anti-symmetry of the integrand with respect to ξ and η prevents us from taking congruent ranges for the two integrals. The integration ranges may be mutually perpendicular to each other, not passing through $i\mathrm{K}'$ or $2\mathrm{K}+i\mathrm{K}'$. The convenient choice is for ξ from $-2\mathrm{K}$ to $+2\mathrm{K}$, and for η from

$K-2iK'$ to $K+2iK'$. If $a_1 \neq a'_1$ the integral is zero over these ranges. If $a_1 = a'_1$ but $a_0 \neq a'_0$ from (2.3) it follows that the integral is still zero. Hence

$$\iint (sn^2\xi - sn^2\eta) U(\xi \mid a_1 a_0) U(\eta \mid a_1 a_0) U(\xi \mid a'_1 a'_0) U(\eta \mid a'_1 a'_0) d\xi d\eta = 0 \ldots (2.4)$$

if $a_1 \neq a'_1$; or if $a_1 = a'_1$ but $a_0 \neq a'_0$.

From the above equation it can easily be deduced that in the infinitesimally narrow strip where n^2 and $sn^2\xi$ are real, the characteristic constants are real.

If possible, let a_1 and a_0 be complex characteristic values and let $\bar{a}_1$ and $\bar{a}_0$ be their imaginary conjugates. From the differential equation it follows that if ψ_1 and ψ_2 are real functions of their arguments such that

$$U(\xi \mid a_1 a_0) = (sn\xi)^{\sigma_1} (cn\xi)^{\sigma_2} (dn\xi)^{\sigma_3} \{\psi_1(sn^2\xi) + i\psi_2(sn^2\xi)\},$$

the differential equation with constants $\bar{a}_1$ and $\bar{a}_0$ has a solution

$$U(\xi \mid \bar{a}_1 \bar{a}_0) = (sn\xi)^{\sigma_1} (cn\xi)^{\sigma_2} (dn\xi)^{\sigma_3} \{\psi_1(sn^2\xi) - i\psi_2(sn^2\xi)\},$$

and as this function satisfies the conditions of a characteristic function, the constants $\bar{a}_1$ and $\bar{a}_0$ are also characteristic constants. From (2.4) it follows that

$$(a_1 - \bar{a}_1) \iint (sn^2\xi - sn^2\eta) U(\xi \mid a_1 a_0) U(\eta \mid a_1 a_0) U(\xi \mid \bar{a}_1 \bar{a}_0) U(\eta \mid \bar{a}_1 \bar{a}_0) d\xi d\eta = 0.$$

$-2K < \xi < 2K$; and η is from $K-2iK'$ to $K+2iK'$;

but
$$U(\xi \mid a_1 a_0) U(\xi \mid \bar{a}_1 \bar{a}_0) = (sn\xi)^{2\sigma_1} (cn\xi)^{2\sigma_2} (dn\xi)^{2\sigma_3} + \{[\psi_1(sn^2\xi)]^2 + [\psi_2(sn^2\xi)]^2\},$$

this quantity has a constant sign (positive) in the range $-2K < \xi < 2K$. Similarly, $U(\eta \mid a_1 a_0) U(\eta \mid \bar{a}_1 \bar{a}_0)$ preserves the same sign throughout its range, positive, if $\sigma_2 = 0$ and negative, if $\sigma_2 = 1$, and $(sn^2\xi - sn^2\eta)$ is negative throughout the ranges of ξ and η. Hence the value of the integrand is real and has the same sign throughout the ranges of integrations. The factor i is introduced owing to the fact that the range of integration for η is parallel to the imaginary axis. It follows that unless the value of the integrand is everywhere zero $(a_1 - \bar{a}_1) = 0$ or a_1 is real. We definitely bar out the null solution. Similarly from the other orthogonal relation it can be proved that a_0 is also real.

Let us consider the integral

$$\iint (sn^2\xi - sn^2\eta)\{U(\xi \mid a_1 a_0) U(\eta \mid a_1 a_0)\}^2 d\xi d\eta$$

taken over the ranges $-2K < \xi < 2K$ and η from $K - 2iK'$ to $K + 2iK'$.

As before

$$\{U(\xi \mid a_1 a_0)\}^2 = (sn\xi)^{2\sigma_1} (cn\xi)^{2\sigma_2} (dn\xi)^{2\sigma_3} [\psi(s^2\xi)]^2$$

ψ being a real function of $sn^2\xi$ has a positive sign in the range of ξ. And $\{U(\eta \mid a_1 a_0)\}^2$ is also real in the integration range for η and preserves the same sign positive, if $\sigma_2 = 0$ and negative, if $\sigma_2 = 1$, and $(sn^2\xi - sn^2\eta)$ is negative and hence as before the integral cannot be zero unless the characteristic functions identically vanish. This last possibility we ignore. As the value of the integral is not zero its value could be fixed arbitrarily. Owing to the presence of an imaginary factor it would not be possible to normalise the integral to unity and yet preserve the real nature of $\psi(sn^2\xi)$ in the required range. The normalisation which commends itself most is the one when the value of the integral is $\pm i$. The ambiguity in signs is necessary as $cn\eta$ is purely imaginary on the line $K + i\sigma$. This causes no confusion.

The ranges of integration being the narrow strip $sn^2\xi$ and n^2 real, it is not possible to study the characteristic constants when $sn^2\xi$ is complex. The work would be prohibitive. All the quadrics which are real in the Cartesian co-ordinates are fully accounted for by the real values of $sn^2\xi$. Of course this would be no reason to treat the complex values of $sn^2\xi$ with indifference. In fact, it is only by considering the asymptotic nature of the characteristic functions when $sn^2\xi$ is large and real, *i.e.*, when the quadric has all its axes imaginary, that we can deduce the difference between the two solutions of the differential equation.

The corresponding orthogonal relations are well known for Lamé functions and form the basis of Liouville and Klein expansion of an arbitrary function in terms of Lamé products.

Finally, we need a third orthogonal relation. Considering the differential equations for

$$U(\xi \mid a_1 a'_0;\ U(\eta \mid a_1 a_0);\ U(\zeta \mid a_1 a_0)$$

the $a_1\, a_0$; $a_1\, a'_0$ being two pairs of characteristic constants with one common member a_1 ; and writing for simplicity

$$F(\xi,\ \eta,\ \zeta)=U(\xi \mid a_1 a'_0)U(\eta \mid a_1 a_0)U(\zeta \mid a_1 a_0)$$

we have

$$(sn^2\eta-sn^2\zeta)\frac{\partial^2 F}{\partial\xi^2}+(sn^2\zeta-sn^2\xi)\frac{\partial^2 F}{\partial\eta^2}+(sn^2\xi-sn^2\eta)\frac{\partial^2 F}{\partial\zeta^2}$$

$$+n^2k^4(sn^2\xi-sn^2\eta)(sn^2\eta-sn^2\zeta)(sn^2\zeta-sn^2\xi)F$$

$$+(a'_0-a_0)(sn^2\eta-sn^2\zeta)F=0. \qquad \ldots \quad (2.5)$$

Let $f(\xi\eta\zeta)$ be a symmetrical, doubly periodic bounded and continuously differentiable but not necessarily separable solution of (1.7), *i.e.*,

$$\sum_{\xi\eta\zeta}(sn^2\eta-sn^2\zeta)\frac{\partial^2 f}{\partial\xi^2}$$

$$+n^2k^4(sn^2\xi-sn^2\eta)(sn^2\eta-sn^2\zeta)(sn^2\zeta-sn^2\xi)f=0. \qquad \ldots \quad (17b\ \text{is})$$

By the usual process it is seen that

$$(a'_0-a_0)\iiint F(\xi\eta\zeta)f(\xi\eta\zeta)(sn^2\eta-sn^2\zeta)d\xi d\eta d\zeta=0 \qquad \ldots \quad (2.6)$$

ranges

$$-2K \leqq \xi \leqq 2K;\ -2K \leqq \eta \leqq 2K \text{ and } \zeta \text{ from } K-2iK' \text{ to } K+2iK'.$$

Hence if $a'_0 \neq a_0$ the integral is zero.

The first two orthogonal relations provide us with proofs of linear independence of the various combinations of the characteristic functions.

For a given a_1 and different values of a_0 all being characteristic constants, the functions $U(\xi \mid a_1 a_0)$ are all linearly independent. It may be recalled that for a given finite value of a_1 the number of possible a'_0s are finite. If the linear independence did not exist we would have

$$U(\xi \mid a_1 a_0) = \sum_{a'_0}{}' A(a_1 a'_0) U(\xi \mid a_1 a'_0)$$

Σ' denotes that in the summation with respect to a_0', $a_0' = a_0$ is excluded. Multiplying by $U(\xi \mid a_1 a_0'')$ and integrating for ξ over a whole number of periods as, *e.g.*, $-2K$ to $+2K$ it is found that $A(a_1\, a'_0) = 0$ for all values of a'_0, whence the theorem follows.

Let

$$U(\xi \mid a_1 a_0) U(\eta \mid a_1 a_0)$$

$$= \sum_{a'_1 a'_0}{}' A(a'_1 a'_0) U(\xi \mid a'_1 a'_0) U(\eta \mid a'_1 a'_0)$$

Σ' denoting that in the summation, $a_1 = a'_1$ and $a_0 = a'_0$ are excluded, and that only a finite number of a_1 and a_0s are taken.

Multiplying by

$$(sn^2\xi - sn^2\eta) U(\xi \mid a''_1 a''_0) U(\eta \mid a'_1 a''_0)$$

and integrating between the limits $-2K < \xi < 2K'$
and η between $K - 2iK'$ to $K + 2iK'$
we obtain $A(a'_1\, a'_0) = 0$ for all values of a'_1 and a'_0 included in the summation.

Let $f(\xi\eta\zeta)$ be defined as before and let

$$i\iint f(\xi\eta\zeta)(sn^2\eta - sn^2\zeta)\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\zeta \mid a_1a_0)d\eta d\zeta = \pm\mathrm{B}(\xi \mid a_1a_0)$$

$$-2\mathrm{K} \leq \eta \leq 2\mathrm{K} \quad \text{and } \zeta \text{ is from } \mathrm{K}-2i\mathrm{K}' \text{ to } \mathrm{K}+2i\mathrm{K}'$$

it follows that over the same range

$$\iint\{f(\xi\eta\zeta) - \mathrm{B}(\xi \mid a_1a_0)\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\zeta \mid a_1a_0)\}(sn^2\eta - sn^2\zeta)$$

$$\times\, \mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\zeta \mid a_1a_0)d\eta d\zeta = 0.$$

The symmetry of $f(\xi\eta\zeta)$ and a repeated application of (2.6) shows that $\mathrm{B}(\xi \mid a_1a_0) = \mathrm{A}(a_1a_0)\mathrm{U}(\xi \mid a_1a_0)$. The $\pm$ are used as before to keep the functions real; $-$ve if there be a factor of type $cn\xi$ in $\mathrm{U}(\xi \mid a_1a_0)$ and $+$ve otherwise.

It is possible to break up $f(\xi\eta\zeta)$ as a sum of terms like $\mathrm{A}(a_1a_0)\mathrm{U}(\xi \mid a_1a_0)\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\zeta \mid a_1a_0)$ and a remainder which is orthogonal to any finite number of characteristic functions $\mathrm{U}(\eta \mid a_1a_0)\mathrm{U}(\zeta \mid a_1a_0)$ as we like.

If $f(\xi\eta\zeta)$ be representable as

$$(sn\xi)^{\sigma_1}(cn\xi)^{\sigma_2}(dn\xi)^{\sigma_3} \times \text{a function of } sn^2\xi;$$

σ_1, σ_2 or σ_3 being 0 or 1, all the $\mathrm{A}(a_1a_0)$'s which do not correspond to the particular characteristic functions with the same factor

$$(sn\xi)^{\sigma_1}(cn\xi)^{\sigma_2}(dn\xi)^{\sigma_3}$$

are easily seen to be zero. As it is possible to arrange any $f(\xi\eta\zeta)$ as a sum of functions of the above form, the ambiguity due to $\pm$ signs need not trouble us.

III. (*a*) *Integral Equations.** The integral equations for the characteristic functions, which are naturally more complicated than usual, may now be deduced.

* Communicated first at the Bangalore Session of the Conference of the Indian Mathematical Society, April, 1926.

The method adopted here is analogous to the one used by E. T. Whittaker in the case of Mathieu functions.* We start from a modified expression of his general solution for the wave equation †

$$\Delta\phi - 1/c^2 . \frac{\partial^2\phi}{\partial t^2} = 0$$

when the solution is bounded at the origin. Whittaker's solution is

$$\iint F(x \cos u \cos v + y \cos u \sin v + z \sin u + ct;\ u;\ v)\,du\,dv.$$

As we are using the Jacobean elliptic functions, the modified expression can be put as

$$\iint F\left(\frac{1}{k'} . x\,dnu\,dnv + \frac{ik}{k'}\,y\,cnu\,cnv + kz\,snu\,snv + ct;\ u;\ v\right) du\,dv \qquad \ldots \quad (3.0)$$

ranges

$$-2K < u < 2K;\ v \text{ from } K - 2iK' \text{ to } K + 2iK'.$$

The moduli k and k' are at our disposal. It may be supposed that they are equal to the values used in the previous sections.

If the solution of the wave equation in Jacobean elliptic functional form of the last section be a particular case of this integral, *i.e.*, if

$$U(\xi|a_1a_0)U(\eta|a_1a_0)U(\zeta|a_1a_0)exp(ipt)$$

a particular value of this integral, it may be supposed without much sophistication that in the above integral the form of $F(X, u, v)$ may be $exp(iX/c)$ $S(u, v)$, where $S(u, v)$ is some function of u and v only.

* E. T. Whittaker, Proc. of the Vth International Congress of Mathematics, 1912, Cambridge, also Modern Analysis. Whittaker and Watson, pp. 407 *et seq.*

† E. T. Whittaker, Math. Annalen, 1902, Vol. 57, pp. 353 *et seq.*

Replacing the values of x, y, z by their equivalents in terms of $\xi\eta\zeta$ we obtain.

$$U(\xi|a_1a_0)U(\eta|a_1a_0)U(\zeta|a_1a_0)$$

$$=\iint exp\left[in\left\{\frac{1}{k'^2}\, dnu\; dnv\; dn\xi\; dn\eta\; dn\zeta\right.\right.$$

$$+\frac{k^3}{k'^2}\, cnu\; cnv\; cn\xi\; cn\eta\; cn\zeta$$

$$\left.\left.-ik^3 snu\; snv\; sn\xi\; sn\eta\; sn\zeta\right\}\right]S(u,v)dudv \qquad \ldots \quad (3.1)$$

The integration limits for u and v are the same as before. As $\xi\eta\zeta$ are independent of each other, any two of them may assume arbitrary values. Omitting all constant factors we get:

for

$$\eta=K;\ \zeta=0$$

$$U(\xi)=\iint exp(in/k'\, dn\xi\, dnu\, dnv)S(u,v)dudv\,;$$

for

$$\eta=-K+iK';\ \zeta=0$$

$$U(\xi)=\iint exp(nk^2/k'\, cn\xi\, cnu\, cnv)S(u,v)dudu;$$

for

$$\eta=K+iK';\ \zeta=K.$$

$$U(\xi)=\iint exp(nk^2 sn\xi\, snu\, snv)S(u,v)dudv.$$

These three integrals suggest to us the types met with. We may replace the various exponential functions by cos, sin, cosh, and sinh functions, according to the requirements of symmetry about the points 0 and K. Though it is possible to deal individually with the three integrals, a slightly generalised form simplifies much of the work. It is evident that every one of the exponential or cos, sin, cosh, and sinh functions is a symmetrical in ξuv, doubly periodic, bounded and continuously differentiable (except at points congruent to iK' and $2K+iK'$) solution of

$$\sum_{\xi uv} (sn^2 u - sn^2 v) \frac{\partial^2 f}{\partial \xi^2}$$

$$+ n^2 k^4 (sn^2\xi - sn^2 u)(sn^2 u - sn^2 v)(sn^2 v - su^2\xi) f = 0.$$

The exponential and circular functional solutions of the integrals satisfy exactly the same conditions postulated for f on page 58. The additional condition of symmetry about the point 0 and K may be introduced as it is useful.

We may also write

$$(sn^2 u - sn^2 v) S(u, v)$$

instead of $S(u, v)$.

Hence the integral can be written as*

$$U(\xi) = \int\int f(\xi uv)(sn^2 u - sn^2 v) S(u, v)\, du dv$$

the limits of integration are as before

$$-2K < u < 2K \text{ and for } v,\ K - 2iK' \text{ to } K + 2iK'.$$

As

$$d^2 U(\xi | a_1 a_0)/d\xi^2 - (n^2 k^4 sn^4 \xi + a_1 k^2 sn^2 \xi - a_0) U(\xi | a_1 a_0) = 0$$

and supposing that the conditions of differentiation under the integral sign are satisfied we get

$$\int\int (sn^2 u - sn^2 v) S(u, v) \{ \partial^2 f / \partial \xi^2 - (n^2 k^4 sn^4 \xi + a_1 k^2 sn^2 \xi - a_0) f \} du dv = 0.$$

Utilising the differential equation satisfied by $f(\xi uv)$ this may be written as :

$$\int\int S(u,v) [(sn^2 v - sn^2 \xi) \{ \partial^2 f / \partial u^2 - (n^2 k^4 sn^4 u + a_1 k^2 sn^2 u - a_0) f \}$$

$$+ (sn^2 \xi - sn^2 u) \{ \partial^2 f / \partial v^2 - (n^2 k^4 sn^4 v + a_1 k^2 sn^2 v - a_0) f \}]$$

$$\times du dv = 0.$$

* *Cf.* L. Ince., *loc. cit.*, p. 197.

And by partial integration we have

$$\iint f(\xi uv)[(sn^2v - sn^2\xi)\,(\partial^2/\partial u^2 - n^2k^4sn^4u - a_1k^2sn^2u + a_0)S(u,v)$$
$$+ (sn^2\xi - sn^2u)\,(\partial^2/\partial v^2 - n^2k^4sn^4v - a_1k^2sn^2v + a_0)\,S(u,v)] \times dudv.$$

$$+ \int dv\,\left(sn^2v - sn^2\xi\right)\left(S\,\partial f/\partial u - f\,\frac{\partial S}{\partial u}\right)\Big|_u$$

$$+ \int du\,\left(sn^2\xi - sn^2u\right)\left(S\,\frac{\partial f}{\partial v} - f\,\frac{\partial S}{\partial v}\right)\Big|_v$$

$$= 0.$$

This equation can be satisfied if in particular $S(u, v)$ is doubly periodic, bounded function of type $U_1(u)\ U_2(v)$ where

$$d^2U_1(u)/du^2 - (n^2k^4sn^4u + a_1k^2sn^2u - a_0)\ U_1(u) = 0$$

and

$$d^2U_2\ (v)/dv^2 - (n^2k^4sn^4v + a_1k^2sn^2v - a_0)\ U_2(v) = 0.$$

As a_1 and a_0 are characteristic constants, if it be assumed that two characteristic solutions cannot exist for the same pair of characteristic constants, $U_1(u)$ and $U_2(v)$ may be identified with $U(u \mid a_1a_0)$ and $U(v \mid a_1a_0)$. Otherwise we can also proceed as follows:

The symmetry character of $f(\xi uv)$ is the same as that of $U(\xi \mid a_1a_0)$ for the variable ξ. (It is assumed that we use $f(\xi uv)$ in the cos, sin, etc., form). Hence if we assume that $U_1(u)$ is equal to $AU(u|a_1a_0) + BV\ (u|a_1a_0)$ where $V\ (u|a_1a_0)$ is the second solution of the differential equation, and as the ranges are symmetrical about 0 and K, and the symmetry characters are different for the two solutions at each of these points, it is seen easily that the second solution need not appear in the integrals. So we obtain an integral equation in the form

$$U(\xi|a_1a_0) = \text{const} \iint U(u|a_1a_0)\ U(v \mid a_1a_0)\ (sn^2u - sn^2v)\ f(\xi uv)dudv \ldots \quad (3.8)$$

the ranges of integration which have to be different for the two variables are as before

$$-2K < u \leq 2K\ ; \quad v \text{ from } K-2iK' \text{ to } K+2iK'.$$

The integral equation is not necessarily the most general one possible, but it is sufficient for our purpose. The constant in the integral equation is indefinite as the latter is nonlinear. The value of the constant may be made definite by assuming that if U (ξ) is a possible solution, then the followiug relation should be satisfied :

$$i \iint \{U(u)\ U(v)\}^2\ (sn^2 u - sn^2 v)\ du dv = \pm 1.$$

In other words it may be said that the solution should always be used in its normal form. The ambiguity of signs is retained as before to keep the functions real and need not trouble us. Corresponding to this identity the integral equation may be written as

$$U(\xi) = \pm ic \iint f\ (\xi uv)\ (sn^2 u - sn^2 v)\ U(u)\ U(v)\ du dv,$$

where c is a constant of the integral equation. It is found that the integral equation has solutions which satisfy all our previous conditions only for certain discrete values of c.

The integral equation in the Weirstrassian form is easily obtainable as

$$U(\alpha) = \text{const} \iint U(\beta) U(\gamma)\ \{\wp(\beta) - \wp(\gamma)\}\ \phi\ (\alpha\beta\gamma)\ d\beta d\gamma \quad \ldots \quad (3.4)$$

$$\beta \text{ from } (\omega_1 - \omega_2) \text{ to } (\omega_1 + \omega_2)$$

$$\gamma \text{ from } (\omega_2 - \omega_1) \text{ to } (\omega_1 + \omega_2)$$

where ϕ satisfies the following differential equation and where ϕ is symmetrical, doubly periodic and bounded and has bounded derivatives,

$$\sum_{\alpha\beta\gamma} [\wp(\beta) - \wp(\gamma)]\ \partial^2\phi/\partial\alpha^2$$
$$- [\wp(\beta) - \wp(\gamma)]\ [\wp(\gamma) - \wp(\alpha)]\ [\wp(\alpha) - \wp(\beta)]\ p^2\phi/c^2 = 0,$$

i.e., the same wave equation in the Weirstrassian form with which we started.

In both these integral equations the nucleus satisfies, except for a trivial factor, the partial differential equation from which we started. This is analogous to the integral equations for Mathieu and Lamé functions.*

III. (*b*) Till now the integral equation has been treated in a general form. It is easy to specify the nuclei. We have eight and only eight distinct forms corresponding to the four species of Lamé functions or in other words to the various possible modes of symmetry character about the two points 0 and K. As the characteristic function consists of two portions one being

$$(sn\xi)^{\sigma_1}\ (cn\xi)^{\sigma_2}\ (dn\xi)^{\sigma_3}$$

and the other an integral function of $sn^2\xi$ a tabular arrangement of the first factor may be useful.

	$sn\ \xi$,	$sn\ \xi\ cn\ \xi$,	
1,	$cn\ \xi$,	$cn\ \xi\ dn\ \xi$,	$sn\xi\ cn\xi\ dn\xi$
	$dn\ \xi$,	$dn\ \xi\ sn\ \xi$,	

The corresponding integral equations are:

factor 1.

$$U(\xi) = ic \iint \cosh\ (nk^2 sn\xi\ sn\eta\ sn\ \zeta)\ (sn^2\eta - sn^2\zeta)\ U(\eta)\ U(\zeta)\ d\eta d\zeta \quad \ldots \quad (3.51)$$

factor $sn\xi$.

$$U(\xi) = ic \iint \sinh\ (nk^2 sn\xi\ sn\eta\ sn\zeta)\ (sn^2\eta - sn^2\zeta)\ U(\eta)\ U(\zeta)\ d\eta d\zeta \quad \ldots \quad (3.52)$$

factor $cn\ \xi$.

$$U(\xi) = -ic \iint \sinh\ (nk^2/k'.cn\xi\ cn\eta\ cn\zeta)\ (sn^2\eta - sn^2\zeta)\ U(\eta)\ U(\zeta)\ d\eta d\zeta \quad \ldots \quad \ldots \quad (3.53)$$

* E. T. Whittaker, *loc. cit.* and Proc. Lond. Math. Soc. (2), Vol. XIV, pp. 260 *et. seq.* See also Modern Analysis, p. 564.

factor $dn\,\xi$.

$$U(\xi)=ic\iint \sin(n/k'.dn\xi\, dn\eta\, dn\zeta)(sn^2\eta-sn^2\zeta)\,U(\eta)\,U(\zeta)\,d\eta d\zeta \quad \dots \quad (3.54)$$

factor $sn\,\xi\, cn\xi$.

$$U(\xi)=-ic\iint cn\xi\, cn\eta\, cn\zeta \sinh(nk^2 sn\xi\, sn\eta\, sn\zeta)(sn^2\eta-sn^2\zeta)\,U(\eta)\,U(\zeta)\,d\eta d\zeta \quad \dots \quad (3.55)$$

factor $cn\,\xi\, dn\,\xi$.

$$U(\xi)=-ic\iint dn\,\xi\, dn\eta\, dn\zeta \sinh(nk^2/k'.\, cn\,\xi\, cn\eta\, cn\zeta)(sn^2\eta-sn^2\zeta) \times U(\eta)\,U(\zeta)\,d\eta d\zeta \quad \dots \quad (3.56)$$

factor $dn\xi sn\xi$.

$$U(\xi)=ic\iint sn\xi\, sn\eta\, sn\zeta\, sin(n/k'.dn\xi\, dn\eta\, dn\zeta)(sn^2\eta-sn^2\zeta)U(\eta)U(\zeta)\, d\eta d\zeta \quad \dots \quad (3.57)$$

factor $sn\xi cn\xi dn\xi$.

$$U(\xi)=-ic\iint sn\xi\, sn\eta\, sn\zeta\, cn\xi\, cn\eta\, cn\zeta\, \sin(n/k'.dn\xi\, dn\eta\, dn\zeta)(sn^2\eta-sn^2\zeta)\,U(\eta)\,U(\zeta)\,d\eta d\zeta \quad \dots (3.58)$$

In addition every characteristic function satisfies the normalising condition that

$$i\iint(sn^2\eta-sn^2\zeta)[U(\eta)U(\zeta)]^2\,d\eta d\zeta=\pm 1 \quad \dots \quad (3.6)$$

according as the characteristic function has not or has a factor of type $cn\zeta$.

The limits of integration are same for all the integrals.

$-2K<\eta<2K$; and ζ is from $K-2iK'$ to $K+2iK$.

We might of course make the ranges smaller owing to symmetry considerations.

The integral equations can be solved in the usual fashion for small values of n and $sn^2\xi$. An example is given for the function which reduces to a constant when n is zero.

We assume the following expansions in powers of n:

$$U(\xi) = (8\pi/k^2)^{-\frac{1}{4}}\{1+n^2U_2(\xi)+n^4U_4(\xi)+\dots\}/(1+n^2b_2+n^4b_4+\dots)$$

$$(8\pi c/k^2)^{-1} = (8\pi/k^2)^{-\frac{1}{4}}(1+n^2a_2+n^4a_4+n^6a_6+\dots); \quad \dots \quad (3.7)$$

aft r substituting these values in the two integrals we obtain the following equations :—

$$\{U_2(\xi)+a_2+b_2\}$$

$$=ik^2/8\pi\iint(sn^2\eta-sn^2\zeta)\left\{\frac{k^4sn^2\xi\, sn^2\eta\, sn^2\zeta}{2}+U_2(\eta)+U_2(\zeta)\right\}d\eta d\zeta$$

$$2b_2=ik^2/8\pi\iint(sn^2\eta-sn^2\zeta)\,\{U_2(\eta)+U_2(\zeta)\}d\eta d\zeta.$$

$$U_4(\xi)+(a_2+b_2)U_2(\xi)+a_4+a_2b_2+a_4$$

$$=ik^2/8\pi\iint(sn^2\eta-sn^2\zeta)\left[\frac{k^8sn^4\xi\, sn^4\eta\, sn^4\zeta}{4!}\right.$$

$$+\{U_2(\eta)+U_2(\zeta)\}\frac{k^4\, sn^2\zeta\, sn^2\eta\, sn^2\zeta}{2!}$$

$$\left.+U_4(\eta)+U_2(\eta)\,U_2(\zeta)+U_4(\zeta)\right]d\eta d\zeta.$$

... ...

... ...

These equations lead to the following values provided we assume that $U_2(\xi)$, $U_4(\xi)$, etc., have no constant term :

$$U_2(\xi)=k^2sn^2\xi/6;\; a_2=(1+k^2)/18=b^2.$$

$$U_4(\xi)=k^4sn^4\xi/5!+(1+k^2)k^2.\tfrac{1}{270}.sn^2\xi.$$

... ...

... ...

Much of the work is reduced by using Legendre's equality :

$$EK'+E'K-KK'=\pi/2$$

in the usual notation, or in the integral form as used here

$$i\iint(sn^2\eta-sn^2\zeta)d\eta d\zeta=8\pi/k^2.$$

III. (*c*) : Asymptotic Expansions for large values $|sn\xi|$.

It is possible to obtain the asymptotic expansions for characteristic functions with the help of the integral equations for large values of $|sn\xi|$. At present we shall confine ourselves to real values of $sn^2\xi$. In the integral equations the nucleus is a rapidly oscillating function if $sn^2\xi$ is large and negative. The method of stationary phase introduced by Kelvin would be necessary to evaluate the integral. When $sn^2\xi$ is large and positive we deal with exponentials of real quantities and simpler calculations lead to the asymptotic expression.

For the purpose of this subsection it may be assumed that the values of the characteristic function for small values of $sn^2\xi$ and the values of the characteristic constant are known. The integral limits may conveniently be taken as $0<\eta<K$ and ζ from K to $K+iK'$ and suitable factors of 2 are introduced.

Let us consider the first integral equation

$$U(\xi)=16ic\iint\cosh(nk^2 sn\xi\, sn\eta\, sn\zeta)(sn^2\eta-sn^2\zeta)U(\eta)U(\zeta)d\eta d\zeta.$$

The points $sn\eta=1$ and $sn\zeta=1$, or $sn\zeta=1/k$ are stationary points. Of these it can be seen that owing to the factor $(sn^2\eta-sn^2\zeta)$ the dominant term of the integral would be contributed from the neighbourhood of the point $sn\eta=1$; $sn\zeta=1/k$. The line of integration for ζ may be divided into two sections at $sn^2\zeta=\frac{1+k^2}{2k^2}$, so that the contributions about the two stationary points may be considered separately. The integral is therefore

$$16ic\int_0^K\int_{K+iG}^{K+iK'}\cosh(nk^2 sn\xi\, sn\eta\, sn\zeta)(sn^2\eta-sn^2\zeta)\,U(\eta)\,U(\zeta)d\eta d\zeta$$

$$+16ic\int_0^K\int_{K}^{K+iG}\cosh(nk^2 sn\xi\, sn\eta\, sn\zeta)(sn^2\eta-sn^2\zeta)U(\eta)\,U(\zeta)d\eta d\zeta$$

$$=I_1+I_2,\qquad \text{where } sn^2(K+iG)=(1+k^2)/2k^2.$$

Let us consider the first integral I_1. In the integrand, $(sn^2\eta - sn^2\zeta)U(\eta)U(\zeta)$ is bounded and has limited total fluctuation in the interval of integration $0<\eta<\lambda K$; ζ from $K+iG$ to $K+iK'$, the singularity of the term being at η or $\zeta = iK'$. Let us introduce new variables so that

$$sn\eta = 1 - k'^2\rho^2/2;\ sn\zeta = 1/k.\quad (1-k'^2\zeta^2/2)$$

Then the integrand becomes

$$\cosh\,[nk\,sn\xi(1-k'^2\rho^2/2)(1-k'^2\tau^2/2)]\phi(\rho,\tau)$$
$$\times[(1-k'^2\rho^2/4)\,(1+k^2\rho^2-k^2k'^2\rho^4/4)\,(1-k'^2\tau^2/4)\times$$
$$(1-\tau^2+k'^2\tau^4/4)]^{-\frac{1}{2}}\,d\rho d\tau.$$

where $\phi(\rho, \tau)$ is the function obtained by the above transformation on $(sn^2\eta - sn^2\zeta)U(\eta)U(\zeta)$. The new ranges of integration are

$$0<\rho<\sqrt{2.}/k';\ 0<\tau<{}^1/_2\{1+\sqrt{(1+k^2)/2}\}.$$

It can easily be seen that in the ranges of integration

$$[(1-k^2\rho^2/4)(1+k^2\rho^2-k^2k'^2\rho^4/4)(1-k'^2\tau^2/4)(1-\tau^2+k'^2\tau^4/4)]^{-\frac{1}{2}}$$

is bounded and has limited total fluctuation. Hence the integrand satisfies the condition necessary for the application of the method of stationary phase.*

If further we introduce the variables so that

$$\rho'^2+\tau'^2 = \rho^2+\tau^2-k'^2\rho^2\tau^2/2$$

$$\frac{\partial(\rho,\tau')}{\partial(\rho',\tau')} = 1$$

the integrand can be put in the form

$$\cosh\quad[nk\,sn\xi\{1-\frac{k'^2}{2}\,(\rho'^2+\tau'^2)\}]\ F(\rho',\tau')d\rho'd\tau$$

* G. N. Watson, Bessel Funcutions p. 197.

where $F(\rho'\tau')$ is a bounded function with limited total fluctuation. The actual value of the integral is deducible with the help of Fresnel's integrals

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}\ \pi x^2\, dx = \frac{1}{2\sqrt{2}}.$$

The second integral may also be discussed in a similar fashion.

The leading terms may be written as

$$I_1 \backsim 16c\, k'^2/k^2 U(K)U(K+iK')\Bigg[\frac{\pi k \sin v/k}{2vk'^2}$$

$$+\Big\{1+K^2[k^2/2+U'(K)/U(K)]\Big\}\frac{\sqrt{\pi}}{4K}\ (2k/vk'^2)^{3/2}$$

$$\cos\left\{v/k.\left(1-\frac{k'^2K^2}{2}\right)+\pi/4\right\}$$

$$+\Big\{1-(K'-G)^2\left(\tfrac{1}{2}+U''(K+iK')/U(K+iK')\right)\Big\}\frac{\sqrt{\pi}}{4(K'-G)}\left((2k/vk'^2\right)^{3/2}$$

$$\text{Cos}\left\{v/k.\left(1-\frac{k'^2(K'-G)^2}{2}\right)+\pi/4\right\}\Bigg]$$

$$I_2 \backsim 16\, c\{U(K)\}^2\sqrt{\pi/2}.\ ({}^1/vk'^2)^{3/2}$$

$$\left[K\ \sin\left\{v\left(1-\frac{k'^2K^2}{2}\right)+\pi/4\right\}\right.$$

$$\left.-G\ \sin\left\{v\left(1-\frac{k'^2G^2}{2}\right)+\pi/4\right\}\right]$$

where $iv = nk^2 sn\xi$; and v is supposed to be positive.

Both the integrals are taken up to $v^{-\frac{3}{2}}$. The dominant term, of order v^{-1},

$$= 8\pi c\ U(K)\ U(K+iK')\ \frac{\sin v/k}{vk}.$$

We can also find the asymptotic value of the characteristic function when $sn\xi$ is real and large. As far as physical applications are concerned it is unnecessary as it corresponds to a quadric with all its axes imaginary. But it proves useful for the demonstration that the differential equations have only one characteristic solution. It is sufficient to derive only the dominant terms.

Let $sn\xi$ be positive. In the integral

$$16ic\iint \cosh\,(nk^2\, sn\xi\; sn\eta\; sn\zeta)\;(sn^2\eta - sn^2\zeta)\;U(\eta)\;U(\zeta)\;d\eta d\zeta.$$

$nk^2\, sn\xi\; sn\eta\; sn\zeta$ has a maximum value equal to $nk\, sn\xi$. We can write the cosh expression as a sum of two exponentials. The term with the factor $exp(-nk^2\; sn\xi sn\eta\; sn\zeta)$ is seen to be of a far smaller order than the term with the factor

$$exp(nk^2\; sn\xi\; sn\eta\; sn\zeta).$$

In

$$U(\xi) = 16ic\; exp\;(nksn\xi)\iint exp(-nksn\xi)\times$$
$$\cosh\,(nk^2\; sn\xi\; sn\eta sn\zeta)\;(sn^2\eta - sn^2\zeta)\;U(\eta)\;U(\zeta)\;d\eta d\zeta$$

the important contribution is from the neighbourhood of $\eta = K$; $\zeta = K + iK'$ and very simple calculation gives the leading term as

$$\frac{32C}{k^2}\quad U(K)U(K+iK')\quad \frac{exp(nksn\xi)}{nk\; sn\xi}$$

When $sn\xi$ is real and negative we use the other exponential term which we neglected in the above case, and we obtain a similar dominant term.

It is sufficient to give only the dominant terms for the other fuctions when $sn^2\xi$ is large and negative.

factor $sn\xi \quad iv = nk^2 sn\xi$.

$$16\; ck'^2/k^2.\;\; U(K)U(K+iK')\frac{\pi k\;\; \sin\;\; v/k}{2vk'^2}\,.$$

Factor cn $\quad \frac{\sin (nkcn\xi)}{ncn\xi}$ omitting all constants

Factor dn $\quad \mathrm{Sin}\,(ndn\xi/k)/ndn\xi$

Factor cn $\quad cn\xi \sin v\, K_2/4 \sin (v - vK^2/4 - \pi/4)/v^{\frac{3}{2}}$

where $\quad v = nksn\xi.$

The other functions are not given here as they are quite similar. When $sn^2\xi$ is large and positive, *i.e.*, ξ is on the line iK' it would be convenient to express all elliptic functions in terms of $sn\xi$ as $|\, sn\xi \,| \backsim |\, cn\xi \,| \backsim \frac{1}{k} \,|\, dn\xi \,|$. For simplicity, we may suppose that $sn\xi$ is positive, without losing the generality.

The characteristic functions of the first species have the asymptotic form leaving off all constant factors:

$$exp\,(nk\; sn\xi)/sn\xi.$$

Second species:

$$exp\,(nk\; sn\xi)/sn\xi.$$

Third species:

$$exp\,(nk\; sn\xi)/\sqrt{sn\xi}\;.$$

Fourth species:

$$exp\,(nk\; sn\xi).$$

Let us consider the value of the function on the line $iK' + \epsilon$. Let $\epsilon > 0$, and be small. All the functions have the dominant term given by an expression of type $exp(nksn\xi)$. $(sn\xi)^{-s}$ where s is some constant depending on the species. The corresponding asymptotic expression for the second solution of the differential equation would be given by

$$V(\xi) = U(\xi) \int_{iK'+0}^{\xi} dt/\{U(t)\}^2$$

where $+0$ is put to show distinctly that $sn\xi$ is large and positive. The behaviour when $sn\xi$ is large and negative could be obtained from considerations of symmetry. The order of the above integral can be obtained as

$$V(\xi) \backsim \text{const. } exp\ (-nk\ sn\xi)\ .\ (sn\xi)^{2 \cdot s}.$$

This asymptotic expression for the second solution is bounded as $sn\xi$ tends to $+\infty$ while all the characteristic functions become infinite for the same values of $sn\xi$. Hence the second solution of the differential equation must be falling in a separate category, quite apart from the characteristic functions.

IV. Asymptotic expansion when n is large.

The methods developed by Horn and Jeffreys* for the determination of the asymtotic solutions of differential equations have been so fruitful in many instances that it appeared worth while to apply similar methods in the present investigation, and obtain asymptotic expressions for large values of n^2. It may be remarked incidentally that the calculation involved in this method is considerably less than for the other methods.

In $d^2U/d\xi^2 - (n^2k^4sn^4\xi + a_1k^2sn^2\xi - as)U = 0$

we assume

$$U = exp\ (nX).\ Y\ \{1 + f_1/n + f_2/n^2 + \ldots\ldots\ldots\} \qquad \ldots\ (4.10)$$

where X ; Y ; f_1 ; f_2 ;......are functions of ξ only. We also assume that

$$a_1 = \alpha_{-2}n^2 + \alpha_{-1}n + \alpha_0 + \alpha_1/n + \alpha_2/n^2 \ldots\ldots\ldots\ldots$$

$$a_0 = \beta_{-2}n^2 + \beta_{-1}n + \beta_0 + \beta_1/n + \beta_2/n^2 \ldots\ldots\ldots$$

(4.11 & 4.12)

* Horn, Math. Annalen, Vol. 52, p. 342, 1899. Jeffreys, Proc. Lond. Math. Soc., Vol. XXIII (Ser. 2), p. 428; see also Goldstein, Trans. Phil Soc. of Cambridge, 1927.

Substituting in the differential equation and comparing the coefficients of the powers of n, we have the following equations :

$$X'^2 - k^4 sn^4\xi - \alpha_{-2}k^2 sn^2\xi + \beta_{-2} = 0; \quad (4.21)$$

$$2X'Y' + YX'' - (\alpha_{-1}k^2 sn^2\xi - \beta_{-1})Y = 0; \quad (4.22)$$

$$Y'' + 2X'Yf'_1 - (\alpha_0 k^2 sn^2\xi - \beta_0)Y = 0; \quad (4.23)$$

where the primes denote derivatives with respect to ξ. The first equation is solved by choosing α_{-2} and β_{-2} so that X is doubly periodic. The other equations are then treated similarly.

From (4.11)

$$X' = \pm \int \sqrt{(k^4 sn^4\xi + \alpha_{-2}k^2 sn^2\xi - \beta_2)}\,d\xi.$$

It is seen that X has three and only three forms when it could be doubly periodic. Each of the three forms leads to an asymptotic expression and to corresponding constants

For $X = \pm k\, sn\xi$;

$$\left.\begin{array}{l} X'^2 - k^4 sn^4\xi + k^2(1+k^2)sn^2\xi - k^2 = 0 \\ \text{So } \alpha_{-2} = -(1+k^2);\ \beta_{-2} = -k^2 \end{array}\right\} \quad \ldots \quad (4.31)$$

For $X = \pm ikcn\xi$.

$$\left.\alpha_{-2} = -1; \qquad \beta_{-2} = 0 \right\} \quad \ldots \quad (4.32)$$

For $X = \pm idn\xi$

$$\left.\alpha_{-2} = -k^2; \quad \beta_{-2} = 0 \right\} \quad \ldots \quad (4.33$$

From (4.22)

$$log\ (YX'^{\frac{1}{2}}) - \tfrac{1}{2}\int(\alpha_{-1}k^2 sn^2\xi - \beta_{-1})d\xi/X' = \text{const.} \quad \ldots \quad (4.41)$$

We may choose the most appropriate constant by its simplicity. Its value multiplies the whole function by a constant, and hence would not be very important for the form of the asymptotic expression.

If $X = \pm k sn\xi$

Then

$$\log\,[Y(cn\xi dn\xi)^{\frac{1}{2}}]$$

$$= \frac{1}{2kk'^2}\,[\alpha_{-1}k^2 - \beta_{-1})\,\log\{(1+sn\xi)/cn\xi\}$$

$$-(\alpha_{-1}-\beta_{-1})k\,\log\,\{(1+ksn\xi)/dn\xi\}]$$

Hence

$$Y = (cn\xi\,dn\xi)^{-\frac{1}{2}}\left(\frac{1+sn\xi}{cn\xi}\right)^{\pm(l+\frac{1}{2})}\left(\frac{1+ksn\xi}{dn\xi}\right)^{\mp(m+\frac{1}{2})} \quad \ldots$$

provided

$$\left.\begin{aligned} \alpha_{-1}k^2-\beta_{-1} &= 2kk'^2(l+\tfrac{1}{2}) \\ \alpha_{-1}-\beta_{-1} &= 2k'^2(m+\tfrac{1}{2}) \end{aligned}\right\} \quad \ldots \quad (4.43)$$

It is easily seen that Y is doubly periodic and symmetrical or anti-symmetrical about the various points only if l and m are integers. Let Y_1 and Y_2 be the values of Y with +ve and −ve signs. We have for large values of n compared with f_1.

$$U(\xi) \backsim A\,\exp\,(nk\,sn\xi).Y_1 + B\,\exp(-nksn\xi)Y_2$$

$$Y_1(\xi) = Y_2(-\xi)$$

If $U(\xi) = \pm U(-\xi)$; then $B = \pm A$.

$$U(\xi) \backsim A\,[Y_1\,\exp(nksn\xi) \pm Y_2\,\exp(-nksn\xi)]$$

giving the even and odd functions. The solutions fail completely at $\xi = K$ or $K+iK'$ and Stoke's phenomena occur at these points.

We may obtain the next approximation as follows:

From (4.23) we have

$$2f'_1 + Y''/YX' - (\alpha_0 k^2 sn^2\xi - \beta_0)/X' = 0$$

$$\text{Or } 2f_1 + Y'/YX' + \int \{Y'/Y.\ (Y'/Y + X''X) - \alpha_0 k^2 sn^2\xi + \beta_0\} \frac{d\xi}{X'} = 0 \quad \text{... (4.51)}$$

We leave off the constant of integration as before. The above partial integration simplifies the calculation.

In particular for $X = \pm k sn\xi$

we have:

$$2f_1 + 1/k.[(l+\tfrac{1}{2})/cn^2\xi - (m+\tfrac{1}{2})k/dn^2\xi]$$
$$\pm sn\xi(dn^2\xi + k^2 cn^2\xi)/2k\ cn^2\xi dn^2\xi$$
$$\pm sn\xi/2k.[(l^2+l)/cn^2\xi) + (m^2+m)/dn^2\xi]$$
$$\pm 1/kk'^2.\{\beta_0 - 2k(l+\tfrac{1}{2})(m+\tfrac{1}{2}) - (\alpha_0+\tfrac{1}{2})k^2 + \tfrac{1}{2}k'^2(l^2+l+\tfrac{1}{2})\}$$
$$\times \log\{(1+sn\xi)/cn\xi\}$$
$$\pm 1/k'^2.\{\beta_0 - 2k(l+\tfrac{1}{2})(m+\tfrac{1}{2}) - (\alpha_0+\tfrac{1}{2}) - \tfrac{1}{2}k'^2(m^2+m+\tfrac{1}{2})\}$$
$$\times \log\{(1+ksn\xi)/dn\xi\}$$
$$= 0.$$

We determine α_0 and β_0 as follows: Let f_{1+} and f_{1-} be the values corresponding to $nksn\xi$ and $-nksn\xi = X$ respectively. Then to a second approximation the asymptotic expansion is (neglecting terms of order $1/n^2$)

$$U(\xi) \backsim A\ \exp(nksn\xi)(1+f_{1+}/n)/Y + B\ \exp\ (-nksn\xi)(1+f_{1-}/n)/Y.$$

Consider the solution for points on the line $K+i\sigma$ where σ is real and $K' > |\sigma|$. On the half line below the real axis the constants may be taken as A' and B'. We find the relation between the two sets of constants by the symmetry about K. But it will be found that it is not possible to have non-zero values of A, B, A' and B' with the symmetrical property unless the logarithmic term $\log\{(1+sn\xi)/cn\xi\}$ has its co-efficient zero. Hence we equate the co-efficient to zero. Again considering the symmetry about $K+iK'$ for points lying parallel to the real axis the co-efficient of the other logarithmic term is found to be zero. The required criteria are that the co-efficients of the logarithmic terms must be equated to zero. Similarly in the higher stages of approximations also we successively put the co-efficients of logarithmic terms to zero and also determine the successive approximations of the characteristic constants.

Hence

$$\bar{\alpha}_0+1/2= -\tfrac{1}{2}(l^2+l+m^2+m)$$

$$\beta_0= +(\alpha_0+1/2)+2k(l+1/2)(m+1/2)+1/2k'^2(m^2+m+1/2).$$

The case of $X= \pm ksn\xi$ has been given in more detail than the other two cases as it appears to be the more important one. The other two functions can be written down without much explanation

For $X= \pm ikcn\xi$

$$Y=(sn\xi dn\xi)^{-\frac{1}{2}}\left(\frac{1+cn\xi}{sn\xi}\right)^{\pm(l+1/2)}\left(\frac{k'+ikcn\xi}{dn\xi}\right)^{\mp(m+1/2)} \quad \ldots \quad (4.44)$$

$$\beta_{-1}= -2ik(l+1/2);\ \alpha_{-1}=2k'(m+1/2)-2ik(l+1/2) \quad \ldots \quad (4.45)$$

$$\beta_0=2(l+1/2)(m+1/2)ikk'-1/2(l^2+l+k^2+1/2)$$

$$\alpha_0= -(l^2+l)/2-(m^2+m)/2$$

$$f_1=i/2k[(l+1/2)/sn^2\xi-(m+1/2)ikk'/dn^2\xi]$$

$$\pm cn\xi(dn^2\xi-k^2sn^2\xi)/4iksn^2\xi dn^2\xi$$

$$\mp cn\xi[(l^2+l)/sn^2\xi-(m^2+m)k^2/dn^2\xi]/2ik$$

for $X= \pm idn\xi.$

$$Y=(sn\xi cn\xi)^{-\frac{1}{2}}\left(\frac{1+dn\xi}{ksn\xi}\right)^{\pm(l+1/2)}\left(\frac{k'+dn\xi}{kcn\xi}\right)^{\pm(m+1/2)}$$

$$\beta_{-1}= -2ik^2(l+1/2);\ \alpha_{-1}= -2ik'(m+1/2)-2i(l+1/2)$$

$$f_1=i\{(l+1/2)/sn^2\xi-(m+1/2)k'/cn^2\xi\}/2k^2$$

$$\pm dn\xi(cn^2\xi - sn^2\xi)/4ik^2 sn^2\xi cn^2\xi$$

$$\mp dn\xi\{(l^2+l)/sn^2\xi - (m^2+m)/cn^2\xi\}/4ik^2$$

$$\beta_0 = 2k'(l+1/2)(m+1/2) - 1/2(l^2+l+3/2)$$

$$a_0 k^2 = -(l^2+l+k^2)/2 - (m^2+m)/2.$$

There is an important point yet to be noticed. In the choice for X it has been said that it should be doubly periodic, and we obtained three possible values for it. Given the value of X the rest of the steps follow simply from considerations of symmetry or doubly periodic property of the resulting functions. Consider $U(\xi)$ for a given value of n. Its asymptotic expression for large values of $sn^2\xi$ has been found by the application of the method of stationary phase. In every case we found that we had the asymptotic expression in terms of $nksn\xi$, $nkcn\xi$ or $ndn\xi$. Hence the method of stationary phase would be quite applicable even when $|sn\xi|$ is large. The Horn-Jeffreys approximation gives the asymptotic expressions in terms of the same values for large values of n. Hence the asymptotic expression derived from large values of n becomes better applicable for large values of $|sn\xi|$. Hence for sufficiently large values of $|sn\xi|$ and n the asymptotic expression derived from the method of stationary phase must approximate to that obtained in this section for large values of n. And at least the dominant terms must be identical.

Leaving off all constant factors the dominant term obtained by the method of stationary phase is

$$\begin{matrix}\cos\\ \sin\end{matrix}(ink\,sn\xi)/nk\,sn\xi$$

when $sn^2\xi$ is large and negative, and

$$\exp(nk\,sn\xi)/n\,sn\xi; \quad \exp(-nk\,sn\xi)/n\,sn\xi$$

when $sn\xi$ is large and positive, and negative respectively. To the same order of approximation, the expression derived in this section is

$$\exp(+nk\,sn\xi)/sn\xi.$$

The other functions like those having a factor $cn\xi$ and $dn\xi$ give dominant terms with expressions of form

$$\exp\,(\pm ink\; cn\xi)/cn\xi$$

$$\exp\,(\pm indn\xi)/dn\xi.$$

As

$$|sn\xi| \sim |cn\xi| \sim |\,dn\xi\,|\,/k$$

when $sn^2\xi$ is large these expressions are not different from those already obtained. We may start with any of the three possible values for the expression X, namely,

$$= \pm ksn\xi, \quad \pm ikcn\xi \quad \text{or} \quad \pm idn\xi.$$

All the three possible values for X are doubly periodic functions of ξ. These three expressions exhaust the possible values for X. The relations between these three possible asymptotic expressions to the other functions obtained in this essay can be obtained by comparing the asymptotic expressions obtained for large values of n and $sn^2\xi$ in this and previous section.

From the three possible asymptotic forms it can easily be seen that the characteristic constants of the differential equation and asymptotic expressions are real only when (*i*) when n is purely real $X = \pm ksn\xi$, (*ii*) when n is purely imaginary $X = \pm idn\xi$. The third case when $X = \pm ikcn\xi$ the characteristic constants are always complex.

But we know that in the infinitesimally narrow strip when n^2 and $sn^2\xi$ are real the characteristic constants are real. Hence we need consider only $X = \pm ksn\xi$ when n^2 is positive and $X = \pm idn\xi$ when n^2 is negative. In such cases the asymptotic expressions are also real.

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