

Modern Differential Geometries

By

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In considering the relationship between various spaces recently constructed as generalizations of Riemann spaces, I have come to certain conclusions published elsewhere in detail, of which this is a brief survey. Roughly, the work seems to bear the same relation to the usual differential geometries, affine or otherwise, as does that of Erl. Emmy Noether and Berwald, to the Riemannian geometry; the investigations of Douglas are probably a special case. In the following, the tensor summation convention is used, and differentiation to x^h and x^k is respectively denoted by a comma and a semicolon with the corresponding subscript to follow.

A K-space is defined by the integral curves of

$$(1) \quad \ddot{x}^i + \bar{a}^i(x, \dot{x}, t) = 0 \quad i=1 \dots n$$

which are assumed to be such that for some part of the (x, t) domain, there is one and only one regular solution, for a given set of values x . When one tries to deduce these equations from a variational principle

$$(2) \quad \delta \int f(x, \dot{x}, t) dt = 0$$

one is at once led to the following equations for the integrand f , the metric of the K-space:

$$(3) \quad \ddot{x}^i \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} - \dot{x}^i \frac{\partial^2 f}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 f}{\partial \dot{x}^j \partial t} + \frac{\partial f}{\partial x^j} = 0$$

with the subsidiary condition for non-triviality,

$$(3a) \Delta = \left| \frac{\partial^2 f}{\partial x^i \partial x^j} \right| \neq 0$$

There is not, in general, a non-trivial solution for the most general set of α^i . But if, one postulates that any at least twice differentiable function $\psi(f)$ be also a possible metric with f , one is led on substitution to

$$(4) \frac{\partial f}{\partial t} = -\alpha^i \frac{\partial f}{\partial x^i} + x^i \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial t} = 0$$

This corresponds to a conservation principle, and means precisely that the integrand f must be a constant along the 'paths,' defined by (1). Differentiating (4) successively to x^k and substituting in (1) leads to a first order system for f :

$$(5) \frac{\partial \alpha^i}{\partial x^k} \frac{\partial f}{\partial x^i} - 2 \frac{\partial f}{\partial x^i} = 0$$

The first compatibility condition for (4) & (5) is seen to be

$$(6) \epsilon^i \frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial t} = 0 \quad \epsilon^i = \alpha^i - \frac{1}{2} x^k \frac{\partial \alpha^i}{\partial x^k}$$

This makes it clear that the spaces for which the α^i are homogeneous of degree two in x are simpler than the rest, inasmuch as (4) and (1) are consequences of (5), provided f does not contain t explicitly.

The differential invariants of the space are to be had from the usual conditions of compatibility. The partial derivatives $\frac{\partial f}{\partial x^i}$ being eliminated by means of (5) whenever possible, we have

$$(4) (5) \quad P^i \frac{\partial f}{\partial x^i} = 0$$

$$(5) \quad R'_{jk} \frac{\partial f}{\partial x^t} = 0$$

where

$$(7) \quad P'_j = a^r \alpha^t_{,r}; \quad r; \quad j - \alpha^r \alpha^t_{,r}; \quad j - \frac{1}{2} \alpha^t_{,k} \alpha^k_{,j} - \frac{\partial}{\partial t} \alpha^t_{,j} + 2\alpha^t_{,j}$$

$$R'_{jk} = \frac{2}{3} \left[\frac{\partial}{\partial x^j} P'_k - \frac{\partial}{\partial x^k} P'_j \right]$$

These are the generalized Riemannian curvature tensors for the K-space.

As a special case, for instance, we find that with the usual co-efficients of affine connection, a term of the form

$${}^{(n)}A_{i_1 i_2 \dots i_n} x^{i_1} x^{i_2} \dots x^{i_n}$$

enters into an invariant f , if and only if ${}^{(n)}A_{i_1 \dots i_n}$ is a tensor with vanishing covariant derivative. This settles the problem of the expansion of f as a polynomial in x . Further, we have the Hamilton's principle directly from (3) :

$$(8) \quad \frac{d}{dt} \left(x^t \frac{\partial f}{\partial x^t} - f \right) + \frac{\partial f}{\partial t} = 0$$

So that, for an f not containing t explicitly, but homogeneous of any order except the first in x , or even obeying

$$x^t \frac{\partial f}{\partial x^t} - f = \phi(f) = 0$$

we have the property $\frac{df}{dt} = 0$ also.

A parallelism, or, to be accurate, two such, can be derived on the assumption that there exists a derivate $D(u)$ which

is a vector with u , such that (1) are obtained directly from $D(x)=0$. Even under a general law of vector transformation

$$\bar{u}^i = F_j^i u^j \quad (\bar{x} \leftarrow \rightarrow x) \quad [F_j^i] \neq 0$$

the conditions above lead at once to

$$(9) \quad D(u)^i = \bar{u}^i + \gamma_k^i u^k + \epsilon^i \quad \epsilon^i = \alpha^i - \gamma_k^i \dot{x}^k$$

Moreover, ϵ^i is also a vector, and the law of transformation of the γ_k^i is given by

$$(10) \quad \bar{\gamma}_k^i F_j^k + \frac{d}{dt} F_j^i = F_k^i \gamma_j^k$$

There is a "restricted" derivate, obtained by leaving off the vector ϵ^i .

$$(11) \quad \mathcal{D}(u)^i = \bar{u}^i + \gamma_k^i u^k$$

This is also a contravariant vector with u . It can further be defined for covariant vectors as well, and will be a covariant vector itself, with the proper law of transformation for γ_k^i .

$$\mathcal{D}(u)_i = \bar{u}_i + \gamma_j^i u^j \quad \bar{u}_j = \phi_j^i u_i \text{ etc.}$$

The process obviously gives a derivate for tensors of any rank, as in the usual case. Parallelism would be defined as the vanishing of the derivate, along the particular curve in question.

The determination of the γ_k^i requires further assumption, of which the simplest is that the equations of variation of (1) can be written as

$$(12) \quad \ddot{u}^i + u^k \frac{\partial \alpha^i}{\partial \dot{x}^k} + u^r \frac{\partial \alpha^i}{\partial x^r} = \begin{cases} D^2(u)^i + \phi^i(x, \dot{x}, t) \\ \mathcal{D}^2(u)^i + \psi^i(x, \dot{x}, t) \end{cases}$$

This gives, on the assumption that u is parallel displaced along the base, either in the general or the restricted sense,

$$\gamma_k^i = \frac{1}{2} \frac{\partial u^i}{\partial x^k}$$

It is found as a trivial consequence that a covariant derivative independent of the direction exists only in spaces with symmetric affine connections, under the assumptions of this paper. Furthermore, laws of vector transformation that involve the vector in the co-efficients F_j^i are in general not feasible.

There remain certain obvious consequences of the theories to be noted. First, the laws of vector transformation can be assimilated to those of a group of matrices. Again, as the formula (10) shows, all the γ_k^i can be made to vanish if there exists a transformation for which the co-efficients are equal to a family of vectors, which is *biparallel* along the base, *i. e.*, whose restricted derivatives all vanish along the given curve. The transformation then allows a reduction of the equations of variation (12) to the normal form

$$(13) \quad \ddot{u}^i = -\frac{1}{2} P_j^i u^j$$

This is particularly useful for the discussion of conjugate foci, and problems of stability. Now the said transformation can always be shown to exist, for the ordinary laws of vector transformation in an infinity of ways. Stability thus comes to the investigation of the roots of the equation

$$(14) \quad \left\| \delta_j^i \lambda + \frac{1}{2} P_j^i \right\| = 0$$

which must all be real and negative for complete stability. Furthermore, P , being a mixed tensor, these roots are known to be invariants of any non-singular point transformation.

In closing, I point out that affine relativity theories of the unified gravitational and electromagnetic fields can be constructed, to include the work done recently by Paolo Straneo, Einstein and Meyer, by means of assuming a restricted derivate, with the corresponding "bipaths" as derivable from a metric, while the vector that represents the residual co-efficients is capable of being interpreted as one related to the electromagnetic forces. The foregoing immediately gives the curvatures of the affine space.

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