The local h-vector of the cluster subdivision of a simplex

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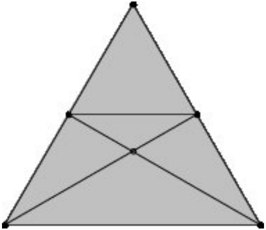
Basic Definitions

An abstract simplicial complex is a *geometric subdivision* Γ of the simplex 2^V if it has a geometric realization which subdivides the simplex.

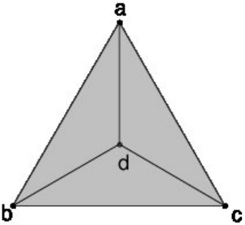
Example:



Basic Definitions

A simplicial complex is called *flag* if every minimal non-face of Γ has at most two elements.

Example of a non-flag subdivision:



$\{a, b, c\}$ is a minimal non-face

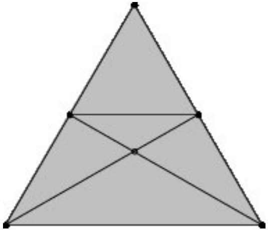
Basic Definitions

Let f_i be the number of the i-dimensional faces of a simplicial complex Γ.

f-vector: $f(\Gamma) = (f_0, \ldots, f_{d-1})$

f-polynomial: $f(\Gamma, x) = f_0 + f_1 x + \cdots + f_{d-1} x^{d-1}$

Example:



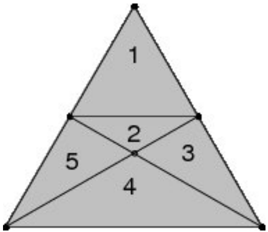
$$f(\Gamma, x) = 6 + 10x + 5x^2$$

Basic Definitions

The *h-vector* $h(\Gamma) = (h_0, h_1, \ldots, h_d)$ and the *h-polynomial* $h(\Gamma, x) = h_0 + h_1 x + \cdots + h_d x^d$ are defined by

$$h(\Gamma, x) = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}, \text{ where } f_{-1} = 0.$$

Example:



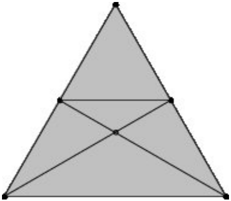
$$h(\Gamma, x) = 1 + 3x + x^2$$

Basic Definitions

For a geometric subdivision Γ of the simplex 2^V the *local h-polynomial* $\ell_V(\Gamma, x)$ of Γ with respect to V is defined as follows:

$$\ell_V(\Gamma, x) = \sum_{i=0}^{d} \ell_i \; x^i = \sum_{F \subseteq V} (-1)^{d-|F|} h(\Gamma_F, x).$$

Example:



$$\ell_V(\Gamma, x) = 1 + 3x + x^2 - (1+x) - (1+x) - 1 + 1 + 1 + 1 - 1$$
$$\ell_V(\Gamma, x) = x + x^2$$

Basic Definitions

Theorem (Stanley)

The local h-polynomial $\ell_V(\Gamma, x)$ has nonnegative and symmetric coefficients, equivalently $\ell_i \geq 0$ and $\ell_i = \ell_{d-i}$ for every $0 \leq i \leq d$.

Thus the *local γ-polynomial* $\xi_V(\Gamma, x)$ of Γ with respect to V can be uniquely defined by

$$\ell_V(\Gamma, x) = (1+x)^d \, \xi_V\left(\Gamma, \frac{x}{(1+x)^2}\right) = \sum_{i=0}^{\lfloor d/2 \rfloor} \xi_i x^i (1+x)^{d-2i}.$$

Example:

$$\ell_V(\Gamma, x) = x + x^2 = x(1+x) \Rightarrow \xi_V(\Gamma, x) = x$$

Motivation

Conjecture (Athanasiadis)

For every flag geometric subdivision Γ of the simplex 2^V we have $\xi_V(\Gamma) \geq 0$.

- Its validity implies the validity of Gal's Conjecture and the monotonicity property for the γ-vector.
- It is proven in dimension 3 and for iterated edge subdivisions.

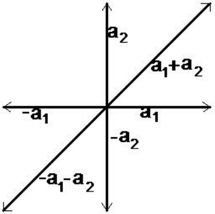
Main Results

- For every root system Φ the local γ-vector of the cluster subdivision $\Gamma(\Phi)$ is nonnegative.

- Combinatorial interpretations to the entries of the local γ-vector of the barycentric subdivision.

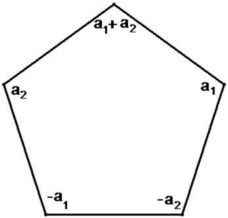
Cluster Subdivision

Given a root system Φ, the *cluster complex* $\Delta(\Phi)$ is a simplicial complex on the vertex set $\Phi_{\geq -1}$ of almost positive roots, having faces defined by a compatibility relation.
Example for type A_2:

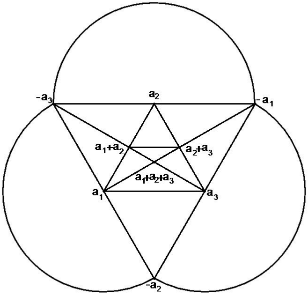


$\Phi = \{a_1, a_2, a_1 + a_2, -a_1, -a_2, -a_1 - a_2\}$ $\Pi = \{a_1, a_2\}$
$\Phi^+ = \{a_1, a_2, a_1 + a_2\}$ $\Phi_{\geq -1} = \{a_1, a_2, a_1 + a_2, -a_1, -a_2\}$

Cluster Subdivision



The cluster complex of type A_2



The cluster complex of type A_3

The *positive cluster complex* $\Delta^+(\Phi)$ is the restriction of $\Delta(\Phi)$ on the positive roots Φ^+. It naturally defines a geometric subdivision of the simplex, the *cluster subdivision* $\Gamma(\Phi)$.

Cluster Subdivision

Theorem (Athanasiadis, Tzanaki)

$$h(\Delta_+(\Phi), x) = \begin{cases} \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} x^i, & \text{if } \Phi = A_n \\ \sum_{i=0}^{n} \binom{n}{i}\binom{n-1}{i} x^i, & \text{if } \Phi = B_n \text{ or } C_n \\ \sum_{i=0}^{n} \left(\binom{n}{i}\binom{n-2}{i} + \binom{n-2}{i-2}\binom{n-1}{i}\right) x^i, & \text{if } \Phi = D_n \end{cases}$$

Cluster Subdivision

For the type A_n the h-polynomial is equal to the Narayana polynomial $C_n(x)$.

$$C_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ 1 + 3x + x^2, & \text{if } n = 3 \\ 1 + 6x + 6x^2 + x^3, & \text{if } n = 4 \\ 1 + 10x + 20x^2 + 10x^3 + x^4, & \text{if } n = 5 \\ 1 + 15x + 50x^2 + 50x^3 + 15x^4 + x^5, & \text{if } n = 6 \end{cases}$$

The coefficient of x^i, $0 \le i \le n$, is the number of $\pi \in \mathrm{NC}^A(n)$ which have $n - i$ blocks.

Cluster Subdivision

Let I be an n-element index set and $\Pi = \{a_i : i \in I\}$. The local h-polynomial $\ell_I(\Gamma(\Phi), x)$ is given by

$$\ell_I(\Gamma(\Phi), x) \;=\; \sum_{J \subseteq I} (-1)^{|I \smallsetminus J|} \; h(\Delta_+(\Phi_J), x),$$

where Φ_J is the standard parabolic root subsystem of Φ corresponding to J.
Example for $\Phi = A_3$:

$$\sum_{i=0}^{3} \ell_i(A_3) x^i = C_3(x) - C_2(x) - C_1(x) \cdot C_1(x) - C_2(x)$$

$$+ C_1(x) + C_1(x) + C_1(x) - C_0(x)$$

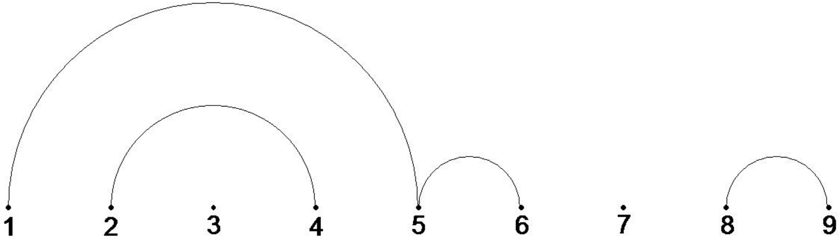
Cluster Subdivision - Type A

$$\sum_{i=0}^{n} \ell_i(\Phi)x^i = \begin{cases} 0, & \text{if} \quad n=1 \\ x, & \text{if} \quad n=2 \\ x+x^2, & \text{if} \quad n=3 \\ x+4x^2+x^3, & \text{if} \quad n=4 \\ x+8x^2+8x^3+x^4, & \text{if} \quad n=5 \\ x+13x^2+29x^3+13x^4+x^5, & \text{if} \quad n=6 \\ x+19x^2+73x^3+73x^4+19x^5+x^6, & \text{if} \quad n=7 \\ x+26x^2+151x^3+266x^4+151x^5+26x^6+x^7, & \text{if} \quad n=8 \end{cases}$$

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi)x^i = \begin{cases} 0, & \text{if} \quad n=1 \\ x, & \text{if} \quad n=2,3 \\ x+2x^2, & \text{if} \quad n=4 \\ x+5x^2, & \text{if} \quad n=5 \\ x+9x^2+5x^3, & \text{if} \quad n=6 \\ x+14x^2+21x^3, & \text{if} \quad n=7 \\ x+20x^2+56x^3+14x^4, & \text{if} \quad n=8 \end{cases}$$

Cluster Subdivision - Type A

Nested and nonnested singletons in $\mathrm{NC}^A(n)$:



The singleton block $\{3\}$ is nested, while $\{7\}$ is nonnested.

Cluster Subdivision - Type A

Proposition

For the root system Φ of type A_n the following hold:

- $\ell_i(\Phi)$ *is equal to the number of partitions* $\pi \in \mathrm{NC}^A(n)$ *with* i *blocks, such that every singleton block of* π *is nested,*
- $\xi_i(\Phi)$ *is equal to the number of partitions* $\pi \in \mathrm{NC}^A(n)$ *which have no singleton block and a total of* i *blocks.*

Moreover, we have the explicit formulas

$$\xi_i(\Phi) = \begin{cases} 0, & \text{if } i = 0 \\ \dfrac{1}{n-i+1}\dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } 1 \le i \le \lfloor n/2 \rfloor \end{cases}$$

and

$$\ell_i(\Phi) = \sum_{j=1}^{i} \frac{1}{n-j+1}\binom{n}{j}\binom{n-j-1}{j-1}\binom{n-2j}{i-j}\binom{n-2j}{i-j}.$$

Cluster Subdivision - Type A

For the combinatorial interpretation of the local γ-polynomial given by

$$\ell_V(\Gamma, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \xi_i x^i (1+x)^{d-2i}$$

an equivalence relation in $\mathrm{NC}^A(n)$ is defined.
Example:

$\{1,3\},\{2\},\{4,5,6\}$ $\{1,3\},\{2\},\{4,6\},\{5\}$ $\{1,2,3\},\{4,6\},\{5\}$ $\{1,2,3\},\{4,5,6\}$

Cluster Subdivision - Type B

$$\sum_{i=0}^{n} \ell_i(\Phi)x^i = \begin{cases} 2x, & \text{if } n=2 \\ 3x+3x^2, & \text{if } n=3 \\ 4x+14x^2+4x^3, & \text{if } n=4 \\ 5x+35x^2+35x^3+5x^4, & \text{if } n=5 \\ 6x+69x^2+146x^3+69x^4+6x^5, & \text{if } n=6 \\ 7x+119x^2+427x^3+427x^4+119x^5+7x^6, & \text{if } n=7 \end{cases}$$

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi)x^i = \begin{cases} 2x, & \text{if } n=2 \\ 3x, & \text{if } n=3 \\ 4x+6x^2, & \text{if } n=4 \\ 5x+20x^2, & \text{if } n=5 \\ 6x+45x^2+20x^3, & \text{if } n=6 \\ 7x+84x^2+105x^3, & \text{if } n=7 \\ 8x+140x^2+336x^3+70x^4, & \text{if } n=8 \end{cases}$$

The Dynkin diagram for type B is of the form

Cluster Subdivision - Type B

Proposition

For the root system Φ of type B_n the following hold:

- $\ell_i(\Phi)$ *is equal to the number of partitions* $\pi \in \mathrm{NC}^B(n)$ *with no zero block and* i *pairs* $\{B, -B\}$ *of nonzero blocks, such that every positive singleton block of* π *is nested,*
- $\xi_i(\Phi)$ *is equal to the number of partitions* $\pi \in \mathrm{NC}^B(n)$ *which have no zero block, no singleton block and a total of* i *pairs* $\{B, -B\}$ *of nonzero blocks.*

Moreover, we have the explicit formula

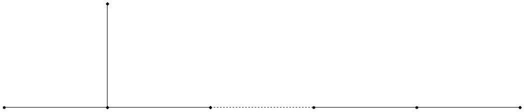
$$\xi_i(\Phi) = \begin{cases} 0, & \text{if } i = 0 \\ \binom{n}{i}\binom{n-i-1}{i-1}, & \text{if } 1 \le i \le \lfloor n/2 \rfloor. \end{cases}$$

Cluster Subdivision - Type D

$$\sum_{i=0}^{n} \ell_i(\Phi)x^i = \begin{cases} 2x + 6x^2 + 2x^3, & \text{if } \ n = 4 \\ 3x + 18x^2 + 18x^3 + 3x^4, & \text{if } \ n = 5 \\ 4x + 40x^2 + 80x^3 + 40x^4 + 4x^5, & \text{if } \ n = 6 \\ 5x + 75x^2 + 250x^3 + 250x^4 + 75x^5 + 5x^6, & \text{if } \ n = 7 \end{cases}$$

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi)x^i = \begin{cases} 2x + 2x^2, & \text{if } \ n = 4 \\ 3x + 9x^2, & \text{if } \ n = 5 \\ 4x + 24x^2 + 8x^3, & \text{if } \ n = 6 \\ 5x + 50x^2 + 50x^3, & \text{if } \ n = 7 \\ 6x + 90x^2 + 180x^3 + 30x^4, & \text{if } \ n = 8 \end{cases}$$

The Dynkin diagram for type D is of the form



Cluster Subdivision - Type D

Proposition

For the root system Φ of type D_n we have

$$\ell_I(\Gamma(\Phi), x) = (n-2)\cdot xC_{n-1}(x).$$

Moreover, we have the explicit formulas

$$\ell_i(\Phi) \; = \; \begin{cases} 0, & \textit{if } \; i=0 \\[2ex] \dfrac{n-2}{i}\dbinom{n-1}{i-1}\dbinom{n-2}{i-1}, & \textit{if } \; 1 \leq i \leq n \end{cases}$$

and

$$\xi_i(\Phi) \; = \; \frac{n-2}{i}\binom{2i-2}{i-1}\binom{n-2}{2i-2}, \textit{ for } \; 1 \leq i \leq \lfloor n/2 \rfloor.$$

Cluster Subdivision

For the exceptional types we have

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i = \begin{cases} (m-2)x, & \text{if } \Phi = I_2(m) \\ 8x, & \text{if } \Phi = H_3 \\ 42x + 40x^2, & \text{if } \Phi = H_4 \\ 10x + 9x^2, & \text{if } \Phi = F_4 \\ 7x + 35x^2 + 13x^3, & \text{if } \Phi = E_6 \\ 16x + 124x^2 + 112x^3, & \text{if } \Phi = E_7 \\ 44x + 484x^2 + 784x^3 + 120x^4, & \text{if } \Phi = E_8. \end{cases}$$

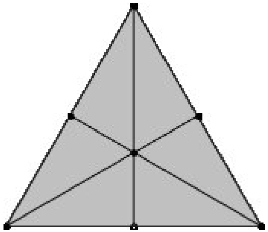
Corollary

For every root system Φ the local γ-vector of $\Gamma(\Phi)$ is nonnegative.

Barycentric Subdivision

Vertices of $\mathrm{sd}(2^V)$: $F \subseteq V$
Faces of $\mathrm{sd}(2^V)$: Chains $F_1 \subset F_2 \subset \ldots \subset F_n$ of subsets of V

Example:



Barycentric Subdivision

Theorem (Stanley)

$$\ell_V(\mathrm{sd}(2^V), x) \;=\; \sum_{w \in \mathcal{D}_n} x^{\mathrm{ex}(w)},$$

where $\mathcal{D}_n$ is the set of derangements (permutations with no fixed points) in $\mathcal{S}_n$ and $\mathrm{ex}(w) = |\{i : w(i) > i\}|$.

This polynomial, known as the derangement polynomial $d_n(x)$ of order n, has been studied by

- Brenti (1990)
- Stembridge (1992)
- Zhang (1995)
- Chen, Tang, Zhao (2009).

Barycentric Subdivision

For the first few values of n we have

$$d_n(x) \;=\; \begin{cases} x, & \text{if} \quad n = 2 \\ x + x^2, & \text{if} \quad n = 3 \\ x + 7x^2 + x^3, & \text{if} \quad n = 4 \\ x + 21x^2 + 21x^3 + x^4, & \text{if} \quad n = 5 \\ x + 51x^2 + 161x^3 + 51x^4 + x^5, & \text{if} \quad n = 6. \end{cases}$$

Barycentric Subdivision

Theorem

Let $(\xi_0, \xi_1, \ldots, \xi_{\lfloor n/2 \rfloor})$ *be the local* γ*-vector of the barycentric subdivision* $\mathrm{sd}(2^V)$ *of the* $(n-1)$*-dimensional simplex* 2^V*. Then* ξ_i *is equal to each of the following:*

(i) *the number of permutations* $w \in \mathcal{S}_n$ *with* i *runs and no run of length one,*

(ii) *the number of derangements* $w \in \mathcal{D}_n$ *with* i *excedances and no double excedance,*

(iii) *the number of permutations* $w \in \mathcal{S}_n$ *with* i *descents and no double descent, such that every left to right maximum of* w *is a descent.*

Barycentric Subdivision

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i x^i \;=\; \begin{cases} x, & \text{if} \quad n = 2, 3 \\ x + 5x^2, & \text{if} \quad n = 4 \\ x + 18x^2, & \text{if} \quad n = 5 \\ x + 47x^2 + 61x^3, & \text{if} \quad n = 6 \end{cases}$$

For example we have the following permutations in $\mathcal{S}_4$ with no run of length one

$$\begin{array}{ccc} 1234 & 13.24 & 14.23 \\ 23.14 & 24.13 & 34.12. \end{array}$$

Such permutations have been studied by Gessel.

Open Problems

- A more conceptual proof for the cluster subdivision of type D in the spirit of those of type A and B.
- Uniform interpretations for $\ell_i(\Phi)$ and $\xi_i(\Phi)$ for all types Φ.
- Real-rootness for the local h-polynomial and the local γ-polynomial of the cluster subdivision.
- The local h-polynomial and the local γ-polynomial of the barycentric subdivision of an arbitrary subdivision of the simplex.

Thank you all for your attention!

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