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Steady-state rarefaction waves in relativistic magnetized flows. Theory and application to gamma-ray burst outflows

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We investigate the characteristics of a relativistic magnetized fluid flowing around a corner. If the outflow is faster than the fast-magnetosonic speed (or sound speed for a non-magnetized fluid) the non-smooth boundary induces a rarefaction wave propagating in the body of the flow. The subsequent expansion is accompanied with a very efficient increase of the flow bulk speed and Lorentz factor. We apply this “rarefaction acceleration mechanism” to the Collapsar model of gamma-ray bursts, in which a relativistic jet initially propagates in the interior of the progenitor star, before crossing the stellar surface with a simultaneous drop in the external pressure support. We integrated the steady-state equations using a special set of partial solutions, called r – self similar. The use of these solutions degrades the system of the complex, non-linear, 2nd order partial differential equations into a system of two 1st order ordinary differential equations whose integration is straightforward. For the conditions expected in a GRB, a fully analytical solution can also be obtained. The aim of this work is to give insight to the results of recent time-depended numerical simulations and show that rarefaction is a plausible mechanism for these phenomena.

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1. Introduction

If a flow passes a vertex of an angle and its velocity exceeds the velocity of the fastest propagating disturbance (sonic speed for an unmagnetized plasma, or fast magnetosonic speed in a magnetized fluid) the information of the non-smooth boundary propagates in the body of the outflow via a weak discontinuity called rarefaction wave[1]. In general three regions are formed: the unperturbated flow, the perturbated/rarefied region and the void space. Due to their importance and application in a wide range of phenomena, rarefaction waves have been extensively studied in various physical regimes and the theory of the classical hydrodynamical waves is presented in many textbooks; see [1] for example. It has been proposed lately by Komissarov et al. [2] that rarefaction operates well in relativistic magnetized gamma-ray burst (GRB) outflows. While in most magnetohydrodynamic (MHD) jet acceleration mechanisms the acceleration and final Lorentz factor is strongly correlated with the opening angle of the jet ($\gamma\theta \sim 1$), the acceleration associated with the rarefaction makes it possible for this product to become $\gamma\theta \gg 1$ being in agreement with the panchromatic breaks in the afterglow light curves.

In our study we investigate analytically the results that time – depended numerical simulations obtained [2, 3] considering the steady state problem. We describe the outflow using the relativistic MHD equations (*section 2*) and we solve the resulting expressions using a special set of similar solutions called r self-similar solutions (*section 3*). Under this set the system of the 2nd order partial and highly nonlinear equations degrades to a system of a couple 1st order ordinary differential equations, which is easily integrated using a common numerical algorithm (results are given in *section 4*). For the conditions expected in GRB outflows (cold, ultra – relativistic) we can proceed further and obtain purely analytical solutions (*section 5*). Finally we consider an application of our model at the Collapsar scenario of GRB (*section 6*).

2. Equations – Assumptions

The special relativistic steady state MHD equations in a covariant and vector form are:

$$\left(\rho u^{\nu}\right)_{,\nu} = 0 \Rightarrow \quad \vec{\nabla}\cdot\left(\gamma\rho\,\vec{v}\right) = 0 \tag{1}$$

$$T^{\mu\nu}_{\ ,\nu} = 0 \xrightarrow[\text{use of Maxwell eqs}]{\text{Spatial Components}} \gamma\rho\left(\vec{v}\cdot\vec{\nabla}\right)\left(\gamma h\vec{v}\right) + \vec{\nabla}p - \frac{J^0\vec{E} + \vec{J}\times\vec{B}}{c} = 0 \tag{2}$$

$$u_{\mu}T^{\mu\nu}_{\ ,\nu} = 0 \Rightarrow \quad \vec{v}\cdot\vec{\nabla}\left(\frac{P}{\rho^{\hat{\Gamma}}}\right) = 0 \tag{3}$$

where P, ρ are the pressure and density, $w = e + p$ the enthalpy density ($e = P\big/\left(\hat{\Gamma}-1\right)$ the internal energy density), $h = w\big/\rho c^2$ its normalized value and $\hat{\Gamma}$ the usual polytropic index; all the above quantities are defined/measured in the comoving system. On the other hand quantities

[1] Weak discontinuity is a discontinuity on the derivatives of the quantities describing the outflow rather than on the quantities itself

$u^{\nu}, \vec{v}$ are the 4-velocity and the conventional spatial velocity, $\vec{E}, \vec{B}$ are the electric and magnetic field and $J^0/c, \vec{J}$ the charge and current density, all measured at any frame.

The electromagnetic properties of the fluid are governed by Maxwell's equations

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0 \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = 0 \end{cases} \text{ and } F^{\mu\nu}_{\ ,\nu} = -\frac{4\pi}{c} j^{\mu} \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{4\pi}{c} J^0 \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{J} \end{cases} \tag{4}$$

and Ohm law determines the coupling between matter and emf. For a perfectly conducting flow:

$$\vec{J} = \sigma\left(\vec{E} + \vec{v}/c\times\vec{B}\right) \xrightarrow{\sigma\to\infty} \vec{E} = -\frac{\vec{v}}{c}\times\vec{B} \tag{5}$$

In what follows, we assume planar symmetry $\left(\partial/\partial y = 0\right)$ and degrade our problem in 2-dim, as in most analytical works ([1, 4]). This limits our application in collapsars at distances small comparing to the radius of the star, at least in the transverse direction. We also assume that the magnetic field is in the $\hat{y}$ direction, ie. its component on the poloidal $\left(\hat{x}, \hat{z}\right)$ plane is negligible, but this doesn't affect the validity of our application (see *section 6*).

The manipulation of equations is done via the following quantities:

$$\Psi = \iint_{\Omega} \gamma\rho\vec{v}_p \cdot d\vec{S}\ ,\ \ \sigma = -\frac{EB_y}{4\pi\rho hc\gamma^2 v_p}\ \ ,\ \ h = 1 + \frac{\hat{\Gamma}}{\hat{\Gamma}-1}\frac{P}{\rho c^2} \tag{6}$$

where Ψ is the poloidal stream-function per transverse length; integration is done on a rectangular surface Ω at the transverse $\left(x, y\right)$ plane and as in the classical case is used to label every poloidal streamline. Quantity σ is the magnetization function, defined as the ratio of the energy flux carried by the electromagnetic field over the energy flux carried by the matter. The above expression of enthalpy assigns to a polytropic gas, where $P = Q\rho^{\hat{\Gamma}}$, a result of eq.(3).

A partial integration of the equations yields the following integrals of motion:

$$\mu\left(\Psi\right) = \gamma h - \frac{EB_y}{4\pi\gamma\rho c v_p}\ \ ,\ \ \varphi\left(\Psi\right) = \frac{E}{\gamma\rho v_p}\ \ ,\ \ L\left(\Psi\right) = \gamma h v_y\ \ ,\ \ Q\left(\Psi\right) = \frac{P}{\rho^{\hat{\Gamma}}} \tag{7}$$

where μ is the total energy to energy-mass flux ratio, φ an integral related to the electric field and L is related to the the angular momentum.

Using the above integrals two expressions remain: a purely algebraic equation, the Bernoulli equation (coming from the identity $\gamma^2 = 1 + \gamma^2\left(v_p/c\right)^2 + \gamma^2\left(v_y/c\right)^2$):

$$\frac{\mu^2}{h^2\left(\sigma+1\right)^2} = \frac{L^2}{h^2c^2} + \left(\frac{\varphi^2\left|\vec{\nabla}\Psi\right|}{4\pi ch\sigma}\right)^2 + 1 \tag{8}$$

and a partial differential, highly nonlinear equation that assigns to the projection of the energy-momentum equation normal to the poloidal streamlines (transfield equation)

$$\frac{\varphi^2}{\sigma}\left[(1+\sigma)\nabla^2\Psi - \vec{\nabla}\Psi\cdot\vec{\nabla}\ln \hat{u}\vec{\nabla}\Psi\ \hat{u}\right] + \frac{\left|\vec{\nabla}\Psi\right|^2}{2}\frac{d\varphi^2}{d\Psi} - \frac{1}{2}\frac{\vec{\nabla}\Psi}{\left|\vec{\nabla}\Psi\right|^2}\cdot\vec{\nabla}\left(\frac{4\pi\mu c}{\varphi}\frac{\sigma}{\sigma+1}\right)^2 - \frac{\hat{\Gamma}-1}{\hat{\Gamma}}\vec{\nabla}\left[16\pi^2\sigma c^2\frac{h(h-1)}{\varphi^2}\right]\cdot\frac{\vec{\nabla}\Psi}{\left|\vec{\nabla}\Psi\right|^2} = 0 \tag{9}$$

It seems hopeless to seek for the general solution of this equation and so we focus on the quest for partial solutions that describe the phenomenon.

3. r-self similar solutions

The method of self-similarity has been applied and extended both in relativistic and magnetized fluids. As a relative example we refer here to [5] where a semi-analytical solution to jet acceleration has been obtained. The basic concept of this method is to assume a form $r^{F_i} f(\theta)$ for all the flow quantities and determine exponents F_i in such a way that the variables r,θ are separable. Every such set will lead to a class of partial solutions; in our case we find:

$$\Psi = -r^F f(\theta) \qquad \sigma=\sigma(\theta) \qquad h=h(\theta) \qquad L,\mu = const$$
$$Q = const\ \Psi^{\frac{1-F}{F}(\hat{\Gamma}-1)} \qquad \varphi^2 = const\ \Psi^{\frac{1-F}{F}} \tag{10}$$

Notice that the above choice is the most general; the cases considered in [1, 4] in which the flow depends solely on the angle theta, correspond to a specific choice of $F=1$. Under these forms partial eq.(9) degrades to an ordinary one:

$$\frac{dh}{d\theta}=\frac{N}{D}$$
$$N = \frac{(F-1)(\hat{\Gamma}-1)}{2\tan(\theta-\vartheta)}\left[\frac{y^2}{\sin^2(\theta-\vartheta)} - 2\frac{\hat{\Gamma}-1}{\hat{\Gamma}}(h-1)h\,\sigma - \left(\frac{\sigma}{1+\sigma}\right)^2\mu^2\right] \tag{11}$$
$$D = -\left[\frac{1}{\sigma}+\frac{1}{\sin^2(\theta-\vartheta)}\right]\left(\frac{1}{h-1}-\frac{\hat{\Gamma}-1}{h}\right)y^2 + h\sigma(1+\sigma)(\hat{\Gamma}-1) + \left(\frac{\sigma}{1+\sigma}\right)^2\left(\frac{1}{h-1}-\frac{\hat{\Gamma}-1}{h}\right)\mu^2$$

In the above expression we have introduced angle $\vartheta : \tan\vartheta = v_x/v_z$, identical with χ used in [1, 4], and choose to study our solution in terms of

$$y = \frac{const}{4\pi c}F\,f^{1/F} \Rightarrow \frac{dy}{d\theta} = \frac{y}{\tan(\theta-\vartheta)} \tag{12}$$

instead of $f(\theta)$ purely for algebraic reasons. Except these two ordinary differential equation the system is completed with two other algebraic equations

$$\sigma = \frac{\varphi^2}{4\pi h}\left(\frac{\hat{\Gamma}-1}{\hat{\Gamma}}\frac{h-1}{Q}c^2\right)^{1/(\hat{\Gamma}-1)} \tag{13}$$

and the Bernoulli

$$\frac{y^2}{\sin^2(\theta-\vartheta)} = \left(\frac{\sigma}{1+\sigma}\mu\right)^2 - \sigma^2\left[h^2 + \left(\frac{L}{c}\right)^2\right] \qquad (14)$$

The procedure is to give the physical quantities at the base of the flow $(\theta_t = const)$, calculate the integrals and solve the system for (h,σ,y,ϑ) quantities. Using the known values of these quantities one is able to determine all the physical quantities along a reference streamline and by similarity along every other streamline. The integration of this system is easy and can be performed by a simple numerical algorithm (we used Matlab and its Adams Algorithm).

4. Numerical results

As we stated in our introduction the main purpose of our study is to apply the solutions at the GRB-Collapsar model. Due to the uncertainty on the magnitude of the magnetic field we present two limiting cases. In the first one we consider a cold magnetized flow (*Cld Model*), while in the second an unmagnetized, purely hydrodynamic flow (*Ht Model*). In both cases we assume the same initial Lorentz factor $\gamma_i = 100$ and the same total energy to mass flux ratio ($\mu = 500$). The initial inclination of the streamlines was taken in $\hat{z}$-direction ($\vartheta_i = 0$).

Model	$\hat{\Gamma}$	F	$u_{y,i}$	h_i	σ_t	Model	$\hat{\Gamma}$	F	$u_{y,i}$	h_i	σ_t
Cld	5/3	1.1	0	1	4	*Ht*	4/3	1.1	0	5	0

Fig. 1 shows the shape of the streamlines and the Lorentz factor for both models:

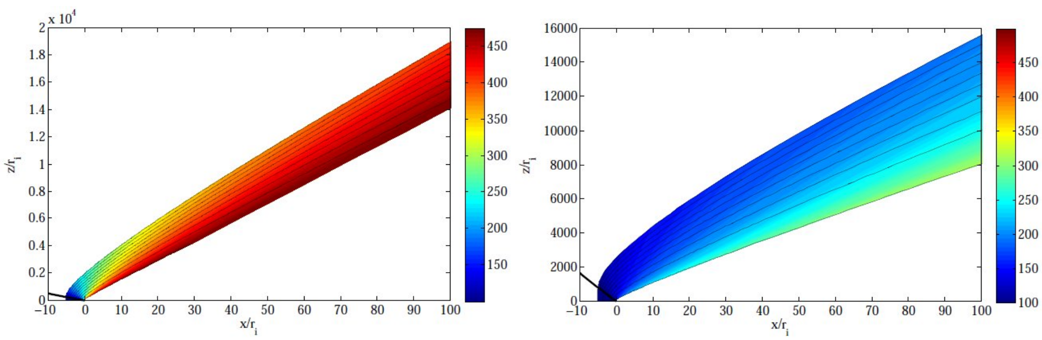

fig. 1: The shape of the poloidal streamlines as produced by self-similarity and Lorentz factor evolution (color) for *Cld* (left) and *Ht* model (right).

Note that we used an arbitrary distance r_i to scale the results; we choose here the initial $\hat{x}$-distance of the reference streamline. The three regions are shown: at the beginning of the outflow we have the unperturbated region separated from the rarefied region via a rarefaction wave (black line). The perturbated region extends up to a maximum angle $\theta_{\max}$.

In the magnetized case, the acceleration is faster and the flow tends to be more collimated. In *Fig. 2* we show how Lorentz factor evolves along a line. In order to quantify this figure we mention that rarefaction occurs at $50\,r_i$ in the magnetic dominated model and $300\,r_i$ at the thermal one. In both cases almost all of the energy is converted to kinetic and the Lorentz factor reaches 90% of its maximum value ($\gamma_f \sim \mu$) at $10^4\,r_i$ in the *Cld model* and $2\cdot 10^7\,r_i$ in the *Ht*

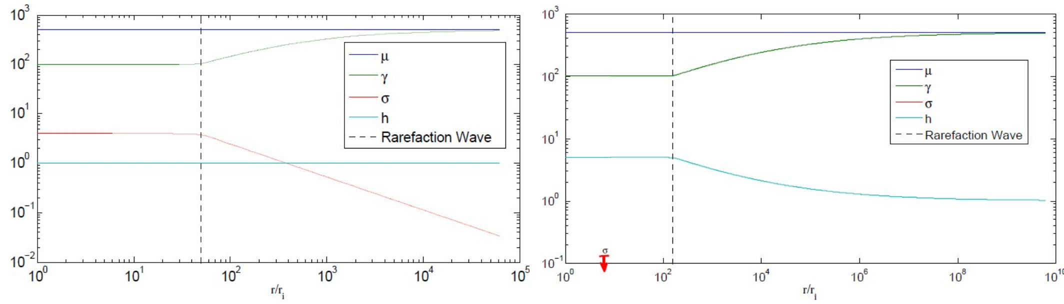

fig. 2: The "energetic" aspect along the reference line for both models *Cld* (left) and *Ht* (right). We plotted Lorentz factor (green), magnetization parameter (red) and specific enthalpy (cyan) in contrast with the total energy to energy-mass flux ratio (blue). In the diagram we exhibit also where rarefaction begins (dashed line).

Finally in the diagrams below we sketch the evolution of various quantities of the outflow as a function of y (monotonically decreasing function along a line)[2]. Notice that all quantities tend to zero as expected.

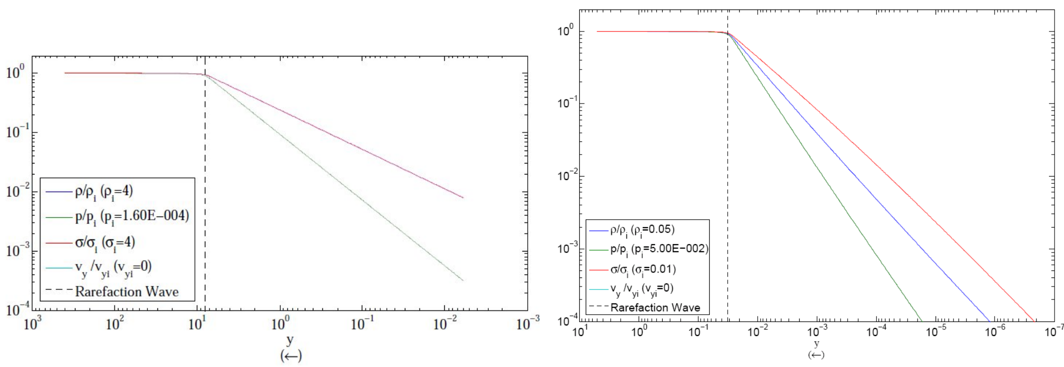

***fig. 3:** Varius physical quantities of the outflow for Cld (left) and Ht model (right).*

5.Asymptotic analysis – Cold outflow

In some asymptotic cases equations (11)-(14) are simplified and fully analytical solutions are obtained. Here we present the cold approximation ($h \to 1$), under which equations become:

$$\frac{d\sigma}{d\theta} = \frac{F-1}{2}\frac{1}{\tan(\theta-\vartheta)}\frac{\sigma^4\left[1+(L/c)^2\right]}{y^2-\sigma^3\left[1+(L/c)^2\right]} \tag{15}$$

$$\frac{d\vartheta}{d\theta} = \frac{(F-1)}{2}\frac{\sigma}{1+\sigma}\frac{\left[1+(L/c)^2\right]}{\frac{y^2}{\sigma^2}-\sigma\left[1+(L/c)^2\right]}\left\{1-\frac{\sigma}{\left(\frac{\mu}{1+\sigma}\right)^2-\left[1+(L/c)^2\right]}\right\} \tag{16}$$

while eq.(12) remains unaltered. Notice that we changed the variable of integration $h \leftrightarrow \sigma$ using eq.(13) and we chose the differential form of Bernoulli for convenience.

Combining equations for σ, y we obtain:

[2] This choice is because the scalling law derived at the next section is in terms of this quantity rather than r.

$$\frac{dy}{d\sigma} = \frac{2}{F-1} y \frac{y^2 - \sigma^3 \left[1+(L/c)^2\right]}{\sigma^4 \left[1+(L/c)^2\right]} \tag{17}$$

which is integrated analytically:

$$y^2 \cong \frac{3F+1}{4}\left[1+(L/c)^2\right]\sigma^3 \tag{18}$$

The asymptotic behaviour $y \sim \sigma^{3/2}$ can be used in eq.(15) to find the rarefied area:

$$\theta_{\max} \cong \theta_i + \frac{\sqrt{3F+1}}{2\mu}\sqrt{1+(L/c)^2}\left(3\sigma_i^{1/2} + \sigma_i^{3/2}\right) \tag{19}$$

In the above relationship θ_i is the angle where the wave occurs. This angle is easily calculated by the vanishing of the denominator of eq.(15):

$$\frac{y^2}{\sigma^2} - \sigma\left[1+\left(\frac{L}{c}\right)^2\right] \Rightarrow \quad \sin\theta_i = \sqrt{\sigma\frac{c^2-u_p^2}{u_p^2}} \overset{u_p \ll u_y}{=} \frac{\sqrt{\sigma}}{\gamma_i} \tag{20}$$

6. Conclusions – Collapsar Application

As we already stated magnetic acceleration in GRBs suffers by the $\gamma\theta \sim 1$ problem and rarefaction was proposed by Komissarov et. al. [2] as a solution to this problem. Rarefaction occurs as the outflow breaks the star envelope and propagates from an area of external pressure (interior of the star) to an area of zero pressure. In GRB we expect that the poloidal magnetic field at the point of rarefaction is negligible since, as most MHD models indicate, any possible poloidal field at the beginning of the outflow soon will end up as an azimouthial field (bending of wires, [5]). In our study the results we obtained above refer to an area close to the breaking of the envelope and to a small region compared to the radius of the progenitor star. From the results presented in *section 4* and as a conclusion we might state that rarefaction is indeed a plausible mechanism for the GRB acceleration. Though this is the case in both situations, in magnetically dominated plasmas the acceleration is completed in much shorter distances.

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